SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Our apologies for omitting correct solutions to problem 3466 by George Apostolopoulos, Messolonghi, Greece, and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

3475. [2009 : 463, 465] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be an equilateral triangle with side length a, and let P be a point on the line BC such that $AP = 2x > a$. Let M be the midpoint of AP. If $\dfrac{BM}{x} = \dfrac{BP}{a} = \alpha$ and $\dfrac{CM}{x} = \dfrac{CP}{a} = \beta$, find x, α, and β.

Solution by Joe Howard, Portales, NM, USA.

We show that $x = \dfrac{a}{\sqrt{2}}$, $\alpha = \dfrac{1+\sqrt{5}}{2}$, and $\beta = \dfrac{-1+\sqrt{5}}{2}$ without assuming that $\triangle ABC$ is equilateral; rather we assume that $AB = BC = a$, and that $AP = 2x > AC$ with C between B and P. The equations for α and β then imply that

$$\alpha \;=\; \frac{BP}{a} \;=\; \frac{a+CP}{a} \;=\; 1+\frac{CP}{a} \;=\; 1+\beta\,. \tag{1}$$

Stewart's theorem applied to $\triangle ABP$ and cevian BM yields

$$AB^2\cdot PM + BP^2\cdot MA \;=\; AP(BM^2 + PM\cdot MA)\,,$$

or

$$a^2x + a^2\alpha^2x \;=\; 2x(x^2\alpha^2 + x^2)\,,$$

whence,

$$a \;=\; \sqrt{2}x\,. \tag{2}$$

Stewart's theorem applied to $\triangle BMP$ and cevian MC yields

$$BM^2\cdot CP + MP^2\cdot BC \;=\; BP(CM^2 + BC\cdot CP)\,,$$

or

$$\alpha^2x^2\cdot a\beta + x^2\cdot a \;=\; a\alpha(\beta^2x^2 + a\cdot a\beta)\,.$$

From (2) this equation becomes $\alpha^2\beta + 1 = \alpha(\beta^2 + 2\beta)$, which upon using the relation in (1) becomes $(\beta^2 + 2\beta + 1)\beta + 1 = (\beta+1)(\beta^2 + 2\beta)$, and reduces to

$$\beta^2 + \beta - 1 \;=\; 0\,.$$

Since $\beta > 0$, we conclude that $\beta = \dfrac{-1+\sqrt{5}}{2}$. Finally, using (1) again, we get that $\alpha = \dfrac{1+\sqrt{5}}{2}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2 solutions); MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; JOEL SCHLOSBERG, Bayside, NY, USA; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incorrect submission.

Applying Stewart's theorem to $\triangle ACP$ with cevian CM, we deduce that $a = AC$, so that $\triangle ABC$ is necessarily equilateral and, consequently, the conditions stated in the problem are consistent. Although most of the submitted solutions were based on Stewart's theorem and the cosine law, a few exploited other notable features of the configuration such as the cyclic quadrilateral $ABCM$ and the similar triangles BMP and ACP.

Bataille observed that his result provides a simple way to construct line segments having lengths $\frac{1+\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2}$ given an equilateral triangle ABC with sides of unit length: construct a unit segment CQ perpendicular to AC at C; the circle with centre A and radius $AQ = \sqrt{2}$ intersects BC at P. The segments BP and CP have the desired lengths.

3476. [2009 : 463, 466] *Proposed by Michel Bataille, Rouen, France.*

Let ℓ be a line and O be a point not on ℓ. Find the locus of the vertices of the rectangular hyperbolas centred at O and tangent to ℓ. (A hyperbola is rectangular if its asymptotes are perpendicular.)

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Choose coordinates so that O is the origin and the equation of ℓ is $x = 1$. Let $\mathcal{R}$ denote the family of all rectangular hyperbolas with centre O that are tangent to ℓ, and let $A : (1, 0)$ be the foot of the perpendicular from O to ℓ. We will see that the desired locus is the hyperbola with the equation $x^2 - y^2 = 1$.

Lemma. The area of any triangle formed by the asymptotes of a rectangular hyperbola together with one of its tangents equals the square of the distance between the centre of the hyperbola and one of its vertices.

It is easier to provide a proof of this familiar result than to look up a reference: We introduce coordinates so that the rectangular hyperbola has equation $xy = a^2$; the tangent to this hyperbola at the point $(ak, a/k)$ is $y = -\dfrac{1}{k^2}x + \dfrac{2a}{k}$, which forms with the asymptotes $x = 0$, $y = 0$ a triangle whose vertices are $(0, 0)$, $(0, 2a/k)$ and $(2ak, 0)$, and whose area (for every k) is the constant $2a^2$. Since the vertices of $xy = a^2$ are (a, a) and $(-a, -a)$, the distance from the centre $(0, 0)$ to a vertex is $2a^2$.

Returning to the main problem, let V be the vertex of the branch for which $x > 0$ of any rectangular hyperbola of $\mathcal{R}$, and define $\theta = \angle AOV$. Because OV bisects the right angle formed by the two asymptotes, the area of the triangle these asymptotes form with the tangent ℓ is

$$OV^2 \;=\; \frac{1}{2}\sec(45^\circ - \theta)\sec(45^\circ + \theta) \;=\; \frac{1}{\cos 2\theta}\,.$$

Hence, V lies on the curve whose polar equation is $r^2 = \dfrac{1}{\cos 2\theta}$. We recognize this as the rectangular hyperbola $x^2 - y^2 = 1$ of $\mathcal{R}$ that has A as its vertex. Because $-45^\circ < \theta < 45^\circ$, it is clear that every point of this hyperbola is a vertex of some member of $\mathcal{R}$, whence the locus of vertices is the entire hyperbola.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; and the proposer.

3477. [2009 : 463, 466] *Proposed by Michel Bataille, Rouen, France.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2y^2\big(f(x+y) - f(x) - f(y)\big) \;=\; 3(x+y)f(x)f(y)$$

for all real numbers x and y.

Similar solutions by Edward J. Barbeau, University of Toronto, Toronto, ON and Salvatore Ingala, student, Scuola Superiore di Catania, University of Catania, Catania, Italy.

We prove that the only two solutions are the zero function, $f(x) = 0$, $x \in \mathbb{R}$, and the cube function $f(x) = x^3$, $x \in \mathbb{R}$. Clearly these two functions are solutions.

If f is the zero-function, then we are done, so henceforth we assume that f is not the zero function.

We first show that $f(0) = 0$.

Assume for the sake of contradiction that $f(0) \neq 0$. Setting $y = 0$ yields $0 = 3xf(x)f(0)$. Thus, $f(x) = 0$ for all $x \neq 0$. Then setting $x = 1$, $y = -1$ we obtain $f(0) = 0$, a contradiction.

Thus, $f(0) = 0$.

Now setting $y = -x$ yields

$$x^4[-f(x) - f(-x)] \;=\; 0\,,$$

and since $f(0) = 0$ we have $f(-x) = -f(x)$ for all x.

Setting $y = x$ for $x \neq 0$ yields

$$f(2x) \;=\; \frac{6}{x^3}f(x)^2 + 2f(x)\,.$$

Now picking $z \neq 0$ and setting $x = 2z$, $y = -z$ yields

$$4z^4\left[2f(z) - f(2z)\right] \;=\; 3zf(2z)f(-z) \;=\; -3zf(2z)f(z)\,.$$

Hence,

$$4z^3\left(\frac{6}{z^3}f(z)^2\right) \;=\; 3f(z)\left(\frac{6}{z^3}f(z)^2 + 2f(z)\right)\,.$$

Thus,

$$24z^3f(z)^2 \;=\; 18f(z)^3 + 6z^3f(z)^2\,,$$

or

$$z^3f(z)^2 \;=\; f(z)^3\,.$$

Therefore, for all $z \neq 0$, either $f(z) = 0$ or $f(z) = z^3$. We show that if f is not the zero function, then $f(z) = z^3$ for all z.

Suppose, for the sake of contradiction, that $f(z) = 0$ for some $z \neq 0$.

By putting $y = z$ in the original equation we get that

$$x^2z^2[f(x+z) - f(x)] \;=\; 0\,,$$

thus $f(x+z) = f(x)$ for all $x \neq 0$.

Since f is not the zero function, there exists an $x \neq 0$ so that $f(x) \neq 0$. Then

$$x^3 \;=\; f(x) \;=\; f(x+z)\,.$$

which contradicts the fact that $f(x+z) = 0$ or $f(x+z) = (x+z)^3$.

Thus, $f(z) \neq 0$ for all $z \neq 0$, and hence $f(z) = z^3$ for all $z \neq 0$.

This completes the proof.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incorrect solution and one incomplete solution were submitted.

3478. [2009 : 463, 466] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a and b be positive real numbers. Prove that

$$\frac{a}{b} + \frac{b}{a} + \sqrt{1 + \frac{2ab}{a^2+b^2}} \;\geq\; 2 + \sqrt{2}\,.$$

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let $x = \frac{a}{b}$, then

$$\frac{a}{b} + \frac{b}{a} + \sqrt{1 + \frac{2ab}{a^2+b^2}} \;=\; x + \frac{1}{x} + \frac{x+1}{\sqrt{x^2+1}}\,.$$

Let $f : (0,\infty) \to (0,\infty)$ be given by $f(x) = x + \frac{1}{x} + \frac{x+1}{\sqrt{x^2+1}}$. From the derivative

$$\begin{aligned} f'(x) &= 1 - \frac{1}{x^2} + \frac{1-x}{(x^2+1)^{3/2}} \\ &= (x-1)\left(\frac{x+1}{x^2} - \frac{1}{(x^2+1)^{3/2}}\right) \\ &= (x-1)\frac{(x+1)\left(x^2+1\right)^{3/2} - x^2}{x^2\left(x^2+1\right)^{3/2}} \\ &= (x-1)\frac{x^8+2x^7+4x^6+6x^5+5x^4+6x^3+4x^2+2x+1}{x^2\left(x^2+1\right)^{3/2}\left(x^2+(x+1)\left(x^2+1\right)^{3/2}\right)} \end{aligned}$$

we see that $f'(x) \le 0$ for $x \in (0,1]$ while $f'(x) \ge 0$ for $x \in [1,\infty)$. Therefore, $f(1) = 2+\sqrt{2}$ is the absolute minimum value of f on $(0,\infty)$. Equality occurs if and only if $\frac{a}{b} = 1$, that is, if and only if $a = b$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USAROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; AARON ESSNER, MARK FARRENBURG, and LUKE E. HARMON, students, Southeast Missouri State University, Cape Girardeau, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; DUNG NGUYEN MANH, Student, Hanoi University of Technology, Hanoi, Vietnam; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Tucson, AZ, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; PANOS E. TSAOUSSOGLOU, Athens, Greece; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; LUKE WESTBROOK, student, Southeast Missouri State University, Cape Girardeau, MO, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. One incomplete solution was submitted and two submitted solutions were disqualified.

3479. *Proposed by Jonathan Schneider, student, University of Toronto Schools, Toronto, ON.*

The real numbers x, y, and z satisfy the system of equations

$$\begin{aligned} x^2 - x &= yz + 1\,, \\ y^2 - y &= xz + 1\,, \\ z^2 - z &= xy + 1\,. \end{aligned}$$

Find all solutions (x, y, z) of the system and determine all possible values of $xy + yz + zx + x + y + z$ where (x, y, z) is a solution of the system.

Solution by George Apostolopoulos, Messolonghi, Greece.

By subtracting the second equation from the first one we obtain

$$\begin{aligned} & x^2 - y^2 - x + y \;=\; z(y - x) \\ \iff\;& (x + y)(x - y) - (x - y) + z(x - y) \;=\; 0 \\ \iff\;& (x - y)(x + y + z - 1) \;=\; 0 \\ \iff\;& x \;=\; y \quad \text{or} \quad x + y + z \;=\; 1\,. \end{aligned}$$

Similarly, we deduce that $x = z$ or $x + y + z = 1$, and that $y = z$ or $x + y + z = 1$.

Thus, if $x + y + z \neq 1$, then $x = y = z$ and the first given equation yields that $x = -1$. By symmetry we deduce that

$$(x, y, z) \;=\; (-1, -1, -1) \tag{1}$$

is the only solution.

Otherwise $x + y + z = 1$, and we now classify the solutions in this case.

Substituting $z = 1 - x - y$ into the first given equation and simplifying yields

$$x^2 + (y - 1)x + (y^2 - y - 1) \;=\; 0\,,$$

hence by the quadratic solution formula

$$x \;=\; \frac{(1 - y) \pm \sqrt{-3y^2 + 2y + 5}}{2}\,.$$

Since x is a real number, we must have $D = -3y^2 + 2y + 5 \geq 0$, which holds if and only if $y \in [-1, \frac{5}{3}]$. For y in this range we obtain

$$z \;=\; 1 - x - y \;=\; \frac{(1 - y) \mp \sqrt{D}}{2}\,,$$

so that the solutions are

$$(x, y, z) \;=\; \left(\frac{(1 - y) \pm \sqrt{D}}{2}, y, \frac{(1 - y) \mp \sqrt{D}}{2}\right)\,, \quad y \in \left[-1, \frac{5}{3}\right]\,. \tag{2}$$

Finally, in either case (1) or (2), it is a straightforward calculation to show that $xy + yz + zx + x + y + z = 0$.

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; EDWARD J. BARBEAU, University of Toronto, Toronto, ON; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; BAO CHANGJIN, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CALVIN DENG, student, William G. Enloe High School, Cary, North Carolina, USA; OLIVER GEUPEL, Brühl, NRW, Germany; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. One incorrect solution and four incomplete solutions were submitted.

3480. *Proposed by Bianca-Teodora Iordache, Carol I National College, Craiova, Romania.*

Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n \;\geq\; a_1 a_2 \cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Prove that $a_1^n + a_2^n + \cdots + a_n^n \geq a_1^{n-1} a_2^{n-1} \cdots a_n^{n-1}$. Find a necessary and sufficient condition for equality to hold.

Comments by Oliver Geupel, Brühl, NRW, Germany and Albert Stadler, Herrliberg, Switzerland.

Oliver Geupel comments that there is a duplication of problem 3480 with problem 880 in the *College Mathematical Journal*, 2008, May issue.

Albert Stadler comments that problem 3480 has already been published by the same author as Aufgabe 1251 in *Elemente der Mathematick*, 2008, issue 1. The solution appeared in the same journal in 2009, issue 1.

Solutions were received from ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; DUNG NGUYEN MANH, Student, Hanoi University of Technology, Hanoi, Vietnam; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3481. [2009 : 464, 466] *Proposed by Joe Howard, Portales, NM, USA.*

Let $\triangle ABC$ have at most one angle exceeding $\frac{\pi}{3}$. If $\triangle ABC$ has area F and side lengths a, b, and c, prove that

$$(ab + bc + ca)^2 \;\geq\; 4\sqrt{3} \cdot F\left(a^2 + b^2 + c^2\right).$$

Composite of similar solutions by Michel Bataille, Rouen, France and Oliver Geupel, Brühl, NRW, Germany.

The given inequality actually holds for all triangles. Recall first the Hadwiger-Finsler Inequality

$$a^2 + b^2 + c^2 \;\geq\; 4\sqrt{3}F + (a-b)^2 + (b-c)^2 + (c-a)^2\,, \tag{1}$$

two proofs of which are given in *Problem-Solving Strategies* by A. Engel, Springer, 1998; pp. 173-4. We rewrite (1) as

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \;\geq\; 4\sqrt{3}F\,. \tag{2}$$

Note that

$$\begin{aligned}
&(ab + bc + ca)^2 \\
&\quad= \Big((a^2 + b^2 + c^2) - (ab + bc + ca)\Big)^2 \\
&\qquad - (a^2 + b^2 + c^2)^2 + 2(ab + bc + ca)(a^2 + b^2 + c^2) \\
&\quad= \frac{1}{4}\Big((a-b)^2 + (b-c)^2 + (c-a)^2\Big)^2 \\
&\qquad + \Big(2(ab + bc + ca) - (a^2 + b^2 + c^2)\Big)(a^2 + b^2 + c^2) \\
&\quad\geq \Big(2(ab + bc + ca) - (a^2 + b^2 + c^2)\Big)(a^2 + b^2 + c^2)\,.
\end{aligned} \tag{3}$$

The result now follows from (2) and (3).

It is easy to see that equality holds if and only if $a = b = c$; that is, if and only if $\triangle ABC$ is equilateral.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; KEE-WAI LAU, Hong Kong, China; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3482. [2009 : 464, 467] *Proposed by José Luis Díaz-Barrero and Josep Rubió-Massegú, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $a_n \neq 0$ and $p(z) = \sum\limits_{k=0}^{n} a_k z^k$ be a polynomial with complex coefficients and zeros z_1, z_2, $\ldots$, z_n, such that $|z_k| < R$ for each k. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{R^2 - |z_k|^2}} \;\geq\; \frac{2}{R^2}\left|\frac{a_{n-1}}{a_n}\right|\,.$$

When does equality occur?

Solution by Michel Bataille, Rouen, France.

For $x \in (0, R^2)$, we have $x(R^2-x) \le \left(\frac{x+R^2-x}{2}\right)^2$ (by the AM-GM Inequality). Hence,

$$\frac{1}{\sqrt{R^2-x}} \;\ge\; \frac{2}{R^2}\sqrt{x}\,,$$

which still holds if $x = 0$. Note that equality holds if and only if $x = R^2 - x$, that is, $x = \frac{R^2}{2}$. It follows that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{R^2-|z_k|^2}} \;\ge\; \frac{2}{R^2}\sum_{k=1}^{n}|z_k|\,. \tag{1}$$

By the triangle inequality,

$$\left|\sum_{k=1}^{n} z_k\right| \;\le\; \sum_{k=1}^{n}|z_k| \tag{2}$$

and $\sum\limits_{k=1}^{n} z_k = \frac{a_{n-1}}{a_n}$ (Vieta's formula), hence

$$\sum_{k=1}^{n} \frac{1}{\sqrt{R^2-|z_k|^2}} \;\ge\; \frac{2}{R^2}\left|\frac{a_{n-1}}{a_n}\right|\,, \tag{3}$$

as required.

Equality holds in (1) if and only if $|z_k| = \frac{R}{\sqrt{2}}$ for $k = 1, 2, \dots, n$ and in (2) if the nonzero z_k's have the same argument. As a result, equality in (3) is equivalent to $z_1 = z_2 = \cdots = z_n = \frac{R}{\sqrt{2}} \cdot e^{i\theta}$ for some $\theta \in\ [0, 2\pi)$ that is, when $p(z)$ is of the form $a_n\left(z - \frac{R}{\sqrt{2}} \cdot e^{i\theta}\right)^n$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposers. Three incomplete solutions were submitted.

3483. [2009 : 464, 467] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let $n \ge 3$ be an integer and let $x_1, x_2, \dots, x_n$ be positive real numbers. Prove that

$$\left(\frac{x_1}{x_2}\right)^{n-2} + \left(\frac{x_2}{x_3}\right)^{n-2} + \cdots + \left(\frac{x_n}{x_1}\right)^{n-2} \;\ge\; \frac{x_1+x_2+\cdots+x_n}{\sqrt[n]{x_1x_2\cdots x_n}}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let
$$N \;=\; \frac{(n-3)n}{2}+\sum_{k=1}^{n-1} k \;=\; (n-2)n\,.$$
The arithmetic mean of the following N numbers
$$\underbrace{1,1,1,\dots,1}_{\frac{(n-3)n}{2}\text{ numbers}}\,,\;\underbrace{\left(\frac{x_1}{x_2}\right)^{n-2},\dots,\left(\frac{x_1}{x_2}\right)^{n-2}}_{n-1\text{ numbers}}\,,$$
$$\underbrace{\left(\frac{x_2}{x_3}\right)^{n-2},\dots,\left(\frac{x_2}{x_3}\right)^{n-2}}_{n-2\text{ numbers}}\,,\;\dots\,,\;\underbrace{\left(\frac{x_{n-1}}{x_n}\right)^{n-2}}_{\text{one number}}$$
is not less than their geometric mean; that is
$$\begin{aligned}&\frac{(n-3)n}{2}+\sum_{k=1}^{n-1}(n-k)\left(\frac{x_k}{x_{k+1}}\right)^{n-2}\\ &\qquad\ge\; N\left[\frac{x_1^{(n-2)(n-1)}}{\prod\limits_{k=2}^{n} x_k^{n-2}}\right]^{1/N} \;=\; \frac{Nx_1}{\sqrt[n]{x_1x_2\cdots x_n}}\,.\end{aligned}$$
We obtain n variants of this inequality by cyclic shifts of the index k. Adding these n inequalities and using the AM–GM Inequality again yields
$$\begin{aligned}&\frac{(n-3)n^2}{2}+\frac{(n-1)n}{2}\left[\left(\frac{x_1}{x_2}\right)^{n-2}+\left(\frac{x_2}{x_3}\right)^{n-2}+\cdots+\left(\frac{x_n}{x_1}\right)^{n-2}\right]\\ &\ge\; N\left(\frac{x_1+x_2+\cdots+x_n}{\sqrt[n]{x_1x_2\cdots x_n}}\right)\\ &\ge\; \frac{(n-3)n}{2}\cdot\frac{n\sqrt[n]{x_1x_2\cdots x_n}}{\sqrt[n]{x_1x_2\cdots x_n}}+\frac{(n-1)n}{2}\cdot\frac{x_1+x_2+\cdots+x_n}{\sqrt[n]{x_1x_2\cdots x_n}}\\ &=\; \frac{(n-3)n^2}{2}+\frac{(n-1)n}{2}\cdot\frac{x_1+x_2+\cdots+x_n}{\sqrt[n]{x_1x_2\cdots x_n}}\,.\end{aligned}$$
The desired inequality follows immediately.

Equality holds if and only if $x_1=x_2=\cdots=x_n$.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; JOE HOWARD, Portales, NM, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3484★. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let N be a positive integer with decimal expansion $N = a_1a_2\dots a_r$, where r is the number of decimal digits and $0 \le a_i \le 9$ for each i, except for a_1, which must be positive. Let $s(N) = a_1 + a_2 + \cdots + a_r$. Find all pairs (N,p) of positive integers such that $\big(s(N)\big)^p = s\big(N^p\big)$.

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

We relabel the decimal expansion as $N = a_{r-1}a_{r-2}\dots a_1a_0$, so that $N = a_{r-1}10^{r-1} + a_{r-2}10^{r-2} + \cdots + a_1 10^1 + a_0 10^0$. Then for any positive integer p we have

$$N^p \;=\; \sum_{n=0}^{p(r-1)} q_n 10^n\,,$$

where

$$q_n \;=\; \sum_{\substack{(k_1,\dots,k_p)\in\{1,\dots,r\}^p\\ k_1+\cdots+k_p=n}} a_{k_1}a_{k_2}\cdots a_{k_p}\,.$$

We call the number N *carry-free of order* p, or CF-p for short, if all the numbers q_0, q_1, …, $q_{p(r-1)}$ are less than 10. This means that no carries occur in the $p-1$ multiplications needed to compute N^p.

If for positive integers $M = \sum b_k 10^k$ and $N = \sum c_j 10^j$ the long multiplication of MN is free of carries, then we have $s(MN) = s(M)s(N)$, as is verified by considering the polynomial multiplication $\left(\sum b_k x^k\right)\left(\sum c_j x^j\right)$ and substituting $x = 10$. On the other hand, the digit sum decreases by 9 each time a carry occurs.

Therefore, a pair (N,p) is a solution if and only if N is CF-p.

We observe that if N is CF-$(p+1)$, then N is CF-p, and furthermore decreasing any (positive) digit of a CF-p integer leaves a CF-p integer.

Proposition 1 For $p \ge 5$, the number N is CF-p if and only if $N = 10^{r-1}$.

Proof: Clearly, $N = 10^{r-1}$ is CF-p.

Conversely, assume that N is CF-p. Then N is CF-5 with each digit either zero or one. If N contains the digit one at least twice, then zeros can be eliminated so that $a_0 = a_{r-1} = 1$, and we obtain

$$\begin{aligned} q_{2r-2} \;&=\; \left|\{(k_1,\dots,k_5)\in\{1,\dots,r\}^5 \mid k_1+\cdots+k_5 = 2r+3\}\right| \\ &\ge\; \binom{5}{2} a_0^3 a_{r-1}^2 = 10\,, \end{aligned}$$

a contradiction. ∎

Proposition 2 The number N is CF-4 if and only if it is CF-5 or $N = 10^k + 10^\ell$ where k and ℓ are distinct nonnegative integers.

Proof: The number $N = 10^k + 10^\ell$ is CF-4, since $q_n \leq \max_m \binom{4}{m} = 6$ for each n.

Conversely, let N be CF-4. Then each digit of N is either zero or one. Assume that the digit one occurs at least thrice, say $a_0 = a_t = a_{r-1} = 1$, $1 < t < r - 1$. Then, $q_{2r+t-2} \geq \binom{4}{2}\binom{2}{1}a_0 a_t a_{r-1}^2 = 12$, a contradiction. ∎

Proposition 3 If N is CF-3, then $a_i \in \{0, 1, 2\}$ for each digit a_i of N, and N contains the digit 2 if and only if $N = 2 \cdot 10^{r-1}$.

Proof: Since $3^3 > 9$, it follows that $a_i \in \{0, 1, 2\}$ for each i.

Clearly, $N = 2 \cdot 10^{r-1}$ is CF-3.

Conversely, assume that the digit 2 occurs in the CF-3 number N with at least one more digit of N being nonzero.

Then a number of the form $N_1 = 2 \cdot 10^{r-1} + 1$ would also be CF-3. However, for N_1 we have $q_{2r-2} = \binom{3}{2} \cdot 2 \cdot 2 \cdot 1 = 12$, a contradiction. ∎

Proposition 4 If N is CF-2, then $a_i \in \{0, 1, 2, 3\}$ for each digit a_i of N, and if N contains the digit 3, then all other digits of N are zero or one.

Proof: Since $4^2 > 9$, we have $a_i \in \{0, 1, 2, 3\}$ for each i. The second statement follows from the fact that if $N = 3 \cdot 10^{r-1} + 2$, then $q_{r-1} = 12$. ∎

Clearly, any positive integer N is CF-1, and this observation completes our characterization.

One hankers for a more explicit description of the solutions when $p = 3$ and N consists of 0's and 1's, or when $p = 2$. In each of these cases Geupel exhibited infinitely many positive integers N that are solutions, and infinitely many that are not.

Richard I. Hess, Rancho Palos Verdes, CA, USA determined all solutions (N, p) with N having at most 5 digits.

The proposer indicated that the special case when $p = 2$ and N is a two-digit integer had been posed by him earlier in the October 2007 issue of the journal School Science Mathematics.

3485. [2009 : 465, 465] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let x, y, z be positive real numbers in the interval $[0, 1]$. Prove that

$$\frac{x}{y+z+1} + \frac{y}{x+z+1} + \frac{z}{x+y+1} + (1-x)(1-y)(1-z) \;\leq\; 1\,.$$

Comment by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

This is Problem 5 from the USA Mathematical Olympiad, 1980. A solution can be found here: `http://www.artofproblemsolving.com/Resources/Papers/MildorfInequalities.pdf`.

Solutions were received from ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of

Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; ALBERT STADLER, Herrliberg, Switzerland; PANOS E. TSAOUSSOGLOU, Athens, Greece; TITU ZVONARU, Comănești, Romania; and the proposer. One incomplete solution was submitted.

Arslanagic; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Howard also noted that the problem is the same as Problem 5 of the USAMO.

3486. [2009 : 465, 467] *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b, and c be positive real numbers. Prove that

$$\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab} \;\le\; \frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\,.$$

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

The inequality to be proved is

$$\sum_{\text{cyclic}}\left(1-\frac{a^2}{a^2+bc}\right) \;\le\; \frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\,,$$

or

$$\frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}+\sum_{\text{cyclic}}\frac{a^2}{a^2+bc} \;\ge\; 3\,.$$

Applying the Cauchy–Schwarz inequality, $\|u\|^2\,\|v\|^2 \ge (u\cdot v)^2$, to

$$\begin{aligned} u &= \left(\frac{a}{\sqrt{a^2+bc}},\frac{b}{\sqrt{b^2+ca}},\frac{c}{\sqrt{c^2+ab}}\right)\,,\\ v &= \left(\sqrt{a^2+bc},\sqrt{b^2+ca},\sqrt{c^2+ab}\right)\,,\end{aligned}$$

we deduce that

$$\begin{aligned}\sum_{\text{cyclic}}\frac{a^2}{a^2+bc} &\ge \frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca}\\ &= \frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)}\,.\end{aligned}$$

Hence, it suffices to show that

$$\frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} + \frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)} \;\ge\; 3\,. \tag{1}$$

Since the inequality is homogeneous, we may assume that $a+b+c=1$. Then $(a,b,c)\cdot(b,c,a)\le a^2+b^2+c^2=1-2(ab+ac+bc)$, so $3(ab+ac+bc)\le 1$. Thus, there is some $0\le x\le 1$ such that $ab+bc+ca=\dfrac{1-x^2}{3}$. Now, it has been shown (see *Mathematical Reflections*, issue 2, 2007, "On a class of three–variable inequalities", by Vo Quoc Ba Can) that

$$abc \;\le\; \frac{(1-x)^2(1+2x)}{27}\,,$$

and thus, (1) follows from

$$\frac{1}{2+x^2} \;+\; \frac{1}{2}\sqrt[3]{\frac{1+x}{(1-x)(1+2x)}} \;\ge\; 1\,,$$

which is equivalent to

$$\frac{1}{8}\frac{1+x}{(1-x)(1+2x)} \;-\; \left(1-\frac{1}{2+x^2}\right)^3 \;\ge\; 0\,. \tag{2}$$

After some algebra, we see that (2) will follow from

$$x^2(16x^6-7x^5+41x^4-18x^3+30x^2-12x+4) \;\ge\; 0\,.$$

Evidently this last inequality holds, since it is the same as

$$x^2\left[16x^6+(41x^4-7x^5)+(18x^2-18x^3)+4(3x^2-3x+1)\right] \;\ge\; 0\,,$$

the quadratic $3x^2-3x+1$ takes only positive values, and $0\le x\le 1$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incorrect solution was submitted.

3487★. [2009 : 465, 467] *Proposed by Neven Jurič, Zagreb, Croatia.*

Does the following hold for every positive integer n?

$$\sum_{k=0}^{n-1}(-1)^k\frac{1}{2n-2k-1}\binom{2n-1}{k} \;=\; (-1)^{n-1}\frac{16^n}{8n\binom{2n}{n}}\,.$$

Solution by George Apostolopoulos, Messolonghi, Greece, modified and expanded by the editor.

The answer is Yes. We start off by setting $l=2n-1-k$. Then $2n-2k-1=2n-2(2n-1-l)-1=-(2n-2l-1)$ and l takes the values $2n-1$, $2n-2$, $\dots$, n as k takes the values 0, 1, 2, $\dots$, $n-1$; respectively.

Hence,

$$\begin{aligned}(-1)^k \frac{1}{2n-2k-1}\binom{2n-1}{k} &= (-1)^{2n-1-l}\frac{1}{-(2n-2l-1)}\binom{2n-1}{l} \\ &= (-1)^l \frac{1}{2n-2l-1}\binom{2n-1}{l}.\end{aligned}$$

Thus, if S denotes the sum on the left side of the given identity, then we deduce that

$$\begin{aligned}2S &= \sum_{k=0}^{n-1}(-1)^k \frac{1}{2n-2k-1}\binom{2n-1}{k} + \sum_{l=n}^{2n-1}(-1)^l \frac{1}{2n-2l-1}\binom{2n-1}{l}; \\ S &= \frac{1}{2}\sum_{k=0}^{2n-1}(-1)^k \frac{1}{2n-2k-1}\binom{2n-1}{k} \\ &= \frac{1}{2(2n-1)}\sum_{k=0}^{2n-1}(-1)^k \frac{\frac{1}{2}-n}{\frac{1}{2}-n+k}\binom{2n-1}{k}. \qquad (1)\end{aligned}$$

We now prove a lemma.

Lemma For any fixed non-integer constant α,

$$\sum_{k=0}^{n}(-1)^k \frac{\alpha}{\alpha+k}\binom{n}{k} = \frac{n!}{\prod_{i=1}^{n}(\alpha+i)}. \qquad (2)$$

Proof: Let A_n denote the left side of the identity to be proved.

Then,

$$\begin{aligned}A_n &= \sum_{k=0}^{n}(-1)^k \frac{\alpha}{\alpha+k}\binom{n}{k} \\ &= 1+\sum_{k=1}^{n-1}(-1)^k \frac{\alpha}{\alpha+k}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)+(-1)^n\frac{\alpha}{\alpha+n} \\ &= A_{n-1}+\sum_{k=1}^{n}(-1)^k \frac{\alpha}{\alpha+k}\binom{n-1}{k-1} \\ &= A_{n-1}+\frac{\alpha}{n}\sum_{k=0}^{n}(-1)^k \frac{k}{\alpha+k}\binom{n}{k} \\ &= A_{n-1}+\frac{\alpha}{n}\left(\sum_{k=0}^{n}(-1)^k\left(1-\frac{\alpha}{\alpha+k}\right)\binom{n}{k}\right) \\ &= A_{n-1}+\frac{\alpha}{n}\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}-\sum_{k=0}^{n}(-1)^k \frac{\alpha}{\alpha+k}\binom{n}{k}\right) \\ &= A_{n-1}-\left(\frac{\alpha}{n}\right)A_n,\end{aligned}$$

since $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (1-1)^n = 0$. Hence, we obtain the recurrence relation

$$A_n = \left(\frac{n}{\alpha+n}\right) A_{n-1}\,. \tag{3}$$

Now $A_1 = 1 - \dfrac{\alpha}{\alpha+1} = \dfrac{1}{\alpha+1}$, so the recurrence (3) yields (2). ∎

Before continuing our proof, we follow the usual practice and use the notation $(2n-1)!!$ for the product $(2n-1)(2n-3)\cdots 3\cdot 1$ and $(2n-2)!!$ for the product $(2n-2)(2n-4)\cdots 4\cdot 2$.

It is easily seen that $\dfrac{(2n-2)!}{(2n-3)!!} = (2n-2)!! = 2^{n-1}(n-1)!$ and $(2n-1)!! = \dfrac{(2n)!}{2^n n!}$.

To complete our proof we take $\alpha = \dfrac{1}{2} - n$ and replace n by $2n-1$ in the lemma. Then we have

$$\begin{aligned}
&\sum_{k=0}^{2n-1}(-1)^k \frac{\frac{1}{2}-n}{k+\frac{1}{2}-n}\binom{2n-1}{k} \\
&= \frac{(2n-1)!}{\prod_{i=1}^{2n-1}\left(\frac{1}{2}-n+i\right)} = \frac{(2n-1)!}{\left(\frac{3}{2}-n\right)\left(\frac{5}{2}-n\right)\cdots\left(\frac{4n-1}{2}-n\right)} \\
&= \frac{(-1)^{2n-1}2^{2n-1}(2n-1)!}{(2n-3)!!(-1)(-3)\cdots(-(2n-1))} \\
&= \frac{(-1)^{n-1}2^{2n-1}(2n-1)!}{(2n-3)!!(2n-1)!!}\,.
\end{aligned} \tag{4}$$

From (1) and (3) we then have

$$\begin{aligned}
S &= \frac{(-1)^{n-1}2^{2n-1}(2n-1)!}{2(2n-1)(2n-1)!!(2n-3)!!} = \frac{(-1)^{n-1}2^{2n-2}(2n-2)!}{(2n-1)!!(2n-3)!!} \\
&= \frac{(-1)^{n-1}2^{2n-2}(2n-2)!!}{(2n-1)!!} = \frac{(-1)^{n-1}2^{3n-3}(n-1)!}{(2n-1)!!} \\
&= \frac{(-1)^{n-1}2^{3n-3}n!n!}{n(2n-1)!!n!} = \frac{(-1)^{n-1}2^{4n-3}(n!)^2}{n(2n)!} \\
&= \frac{(-1)^{n-1}16^n}{8n\binom{2n}{n}}\,,
\end{aligned}$$

and our proof is complete.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and ALBERT STADLER, Herrliberg, Switzerland.

Stan Wagon gave the usual comment that the proposed identity can by verified by using Mathematica *if one expresses the given sum in terms of the Pochhammer function. Curtis used Gauss' Hypergeometric function and the Gamma function, while Stadler used complex contour integration, the Residue Theorem, and the Gamma function as well. Our featured solution is the only elementary one submitted, though by no means easy.*