PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively BOTH by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Curtis Cooper**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Shing So**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than August 15, 2011.

PROBLEMS

951. *Proposed by Duong Viet Thong, National Economics University, Hanoi City, Vietnam.*

Let $f : [0, 1] \to \mathbb{R}$ be a continuously differentiable function such that

$$\int_0^1 f(x)\,dx = 1; \quad \int_0^1 xf(x)\,dx = 2; \quad \int_0^1 x^2 f(x)\,dx = 3.$$

Prove that for every $t \in [-24, 60]$, there exists a $c \in (0, 1)$ such that $f'(c) = t$.

952. *Proposed by Michel Bataille, Rouen, France.*

Let m and n be nonnegative integers. Show that

$$\sum_{k=0}^{n} \binom{k}{m}\binom{n-k}{k}(-1)^{k-m}\, 2^{n-2k} = \binom{n+1}{2m+1}.$$

953. *Proposed by Tom Beatty, Florida Gulf Coast University, Ft. Myers FL.*

Let $x_0 = 1$ and for natural numbers n, define $x_n \in [0, 1]$ implicitly by the recurrence

$$(1 - x_n)^2 - (1 - x_{n-1})^2 = \frac{x_n + x_{n-1}}{\alpha},$$

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where $\alpha > 1$. Show that

$$\sum_{n=1}^{\infty} x_n = \frac{1}{2}(\alpha - 1).$$

954. *Proposed by Erwin Just (Emeritus), Bronx Community College of the City University of New York, Bronx NY.*

Among twelve identical looking coins there are two counterfeits. The two counterfeits are equally heavier than the true coins and a maximum of four comparisons of sets of coins using a balance scale are permitted. Show how one can determine which coins are counterfeit under these conditions.

955. *Proposed by José Luis Díaz-Barrero, Universitat Politécnica de Catalunya, Barcelona, Spain.*

Let $a, b, c > 0$ and $a + b + c = 3$. Prove that

$$\sum \frac{bc}{\sqrt[4]{a^2 + 3}} \leq \frac{3\sqrt{2}}{2},$$

where the sum is over all cyclic permutations of (a, b, c) and equality occurs when $a = b = c = 1$.

SOLUTIONS

Conditions of convergence for a series

902. (Clarification) *Proposed by Mohsen Soltanifar, K. N. Toosi University of Technology, Tehran, Iran.*

Given a sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers, let $s_n = \sum_{k=1}^{n} a_k$ and

$$I = \sum_{n=1}^{\infty} \frac{a_n^p}{s_n^q}.$$

Find all real values of p and q such that the infinite series I is convergent for all sequences $\{a_n\}_{n=1}^{\infty}$ of positive real numbers.

Solution by Eugene Herman, Grinnell College, Grinnell IA and Paolo Perfetti, Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy (independently).

We will show that I converges for all such sequences $\{a_n\}$ if and only if

$$1 \leq p < q.$$

Case 1. If $q \leq 1$ and p is arbitrary, let $a_n \equiv 1$. Then $s_n = n$ for all $n \in \mathbb{N}$. Thus $a_n^p / s_n^q = 1/n^q$, and so I diverges.

Case 2. If $q > 1 > p$, let $a_n = 1/n$ for all $n \in \mathbb{N}$. Then s_n is the nth harmonic number, which is approximately $\gamma + \ln n$ for large n. Thus, for larger n, $s_n \leq 1 + \ln n$

and so

$$\frac{a_n^p}{s_n^q} \geq \frac{1}{n^p(1+\ln n)^q}.$$

Since

$$\int_1^\infty \frac{1}{x^p(1+\ln x)^q}\,dx = \int_0^\infty \frac{e^u}{e^{pu}(1+u)^q}\,du = \int_0^\infty \frac{e^{u(1-p)}}{(1+u)^q}\,du$$

and $p < 1$, I diverges by the integral test because the integrand of the final integral above is increasing for large u.

Case 3. If $p \geq q > 1$, let $a_n = e^n$ for all $n \in \mathbb{N}$. Then

$$s_n \leq \int_0^{n+1} e^x\,dx = e^{n+1} - 1 < e^{n+1}$$

and

$$\frac{a_n^p}{s_n^q} \geq \frac{(e^n)^p}{(e^{n+1})^q} = \frac{e^{n(p-q)}}{e^q} \geq \frac{1}{e^q}.$$

Therefore, I diverges.

Case 4. If $1 \leq p < q$, we shall show that I converges both when $\sum_n a_n$ converges and when it does not.

Suppose $S = \sum_{n=1}^\infty a_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n = S$. Thus, there exists an $N \in \mathbb{N}$ such that $a_n^p \leq a_n$ (since $p \geq 1$) and $s_n \geq S/2$ whenever $n \geq N$. Therefore,

$$\frac{a_n^p}{s_n^q} \leq \left(\frac{2}{S}\right)^q a_n$$

for $n \geq N$, and hence I converges.

Suppose $\sum_n a_n$ diverges to infinity. Since $a_n \leq s_n$ and $p \geq 1$, we have

$$\frac{a_n^p}{s_n^q} = \left(\frac{a_n}{s_n}\right)^{p-1} \frac{a_n}{s_n^{q-p+1}} \leq \frac{a_n}{s_n^r} \tag{1}$$

where $r = q - p + 1 > 1$. We claim that, for all $n \in \mathbb{N}$,

$$\frac{a_n}{s_n^r} \leq \frac{1}{r-1}\left(\frac{1}{s_{n-1}^{r-1}} - \frac{1}{s_n^{r-1}}\right). \tag{2}$$

Note that inequalities (1) and (2), together with the facts that $s_n \to \infty$ and $r > 1$, imply that I converges, since the expression in parentheses in (2) is a term of a convergent telescoping series. Multiplying inequality (2) by s_n^r/s_{n-1} and replacing a_n by $s_n - s_{n-1}$ yields the equivalent inequality

$$\frac{s_n}{s_{n-1}} - 1 \leq \frac{1}{r-1}\left(\left(\frac{s_n}{s_{n-1}}\right)^r - \frac{s_n}{s_{n-1}}\right).$$

To prove this inequality, it suffices to show that

$$f(x) = \frac{1}{r-1}(x^r - x) - x + 1 \geq 0 \quad \text{for all } x \geq 1.$$

This inequality is true since $f(1) = 0$ and

$$f'(x) = \frac{1}{r-1}(rx^{r-1} - 1) - 1 = \frac{r}{r-1}(x^{r-1} - 1) \geq 0.$$

Also solved by ARKADY ALT, San Jose CA; MICHEL BATAILLE, Rouen, France; TOM JAGER, Calvin C.; GEORGE MATTHEWS, Indianapolis IN; WILLIAM SEAMAN, Albright C.; and the proposer.

One incorrect solution was received.

Equal cevian intercepts

926. *Proposed by Mowaffaq Hajja, Yarmouk University, Irbid, Jordan.*

Let ABC be a triangle in which $AB \neq AC$. When is it possible to draw cevians BB' and CC' that make equal angles with the base BC and equal intercepts BC' and CB' with the sides?

Solution by Lucas Mol (student), Mount Allison University, Sackville, New Brunswick, Canada and Ruthven Murgatroyd, Albany OR (independently).

Since $AB \neq BC$ in $\triangle ABC$, assume that $\angle B > \angle C$. Because the cevians $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ make equal angles of measure α with the base $\overline{BC}$, applying the law of sines to $\triangle BB'C$ and $\triangle CC'B$ yields

$$\frac{BC}{\sin(\alpha + C)} = \frac{B'C}{\sin\alpha} = \frac{BC}{\sin(\alpha + B)},$$

and hence $\sin(\alpha + C) = \sin(\alpha + B)$. It follows from

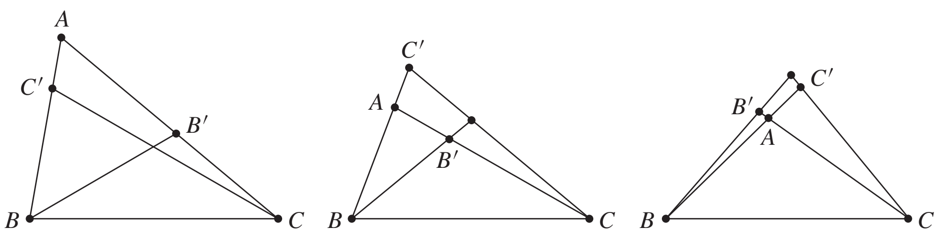
$$\sin\alpha\cos C + \cos\alpha\sin C = \sin\alpha\cos B + \cos\alpha\sin B$$

that

$$\tan\alpha = \frac{\sin B - \sin C}{\cos C - \cos B} = \frac{\cos(\frac{B+C}{2})}{\sin(\frac{B+C}{2})} = \tan\frac{A}{2}.$$

Therefore, $\alpha = \frac{A}{2}$ and $BC' = CB'$.

The three triangles in the following figure illustrate the cases when $\angle\frac{A}{2}$ is less than both $\angle B$ and $\angle C$; $\angle C < \angle\frac{A}{2} < \angle B$; and $\angle C < \angle B < \angle A$.



Also solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State U.; Subramanyam Durbha, Rowan U.; Dmitry Fleischman, Santa Monica CA; Michael Goldenberg, The Ingenuity Project, Baltimore Poly. Inst. and Mark Kaplan, C.C. of Baltimore County (jointly); James Hochschild, Butner NC; Elias Lampakis, Kiparissia, Greece; Joel Schlosberg, Bayside NY; H. T. Tang, Haywood CA; R. S. Tiberio, Natick MA; Michael Vowe, Therwil, Switzerland; Jeff Wagner, Big Sandy Community & Tech. C.; Stuart Witt, Brooklyn NY; and the proposer.

One incorrect solution was received.

An inequality of a function with continuous first derivative

927. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania and Tudorel Lupu (student), Decebal High School, Constanta, Romania.*

Let $f: [a, b] \to \mathbb{R}$ be a differentiable function whose derivative is continuous on the interval $[a, b]$. Prove that

$$\left| \int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^b f(x)\,dx \right| \le \frac{(b-a)^2}{4} \sup_{x\in[a,b]} |f'(x)|.$$

Solution 1 by Chip Curtis, Missouri Southern State University, Joplin MO and Yajing Yang, Farmingdale State College, Farmingdale NY (independently).

Let $I = \int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^b f(x)\,dx$ and $M = \sup_{x\in[a,b]} |f'(x)|$. Then

$$\begin{aligned}
|I| &= \left| \int_a^{\frac{a+b}{2}} \left[f(x) - f\left(\frac{a+b}{2}\right)\right] dx + \int_{\frac{a+b}{2}}^b \left[f\left(\frac{a+b}{2}\right) - f(x)\right] dx \right| \\
&\le \int_a^{\frac{a+b}{2}} \left| f(x) - f\left(\frac{a+b}{2}\right)\right| dx + \int_{\frac{a+b}{2}}^b \left| f\left(\frac{a+b}{2}\right) - f(x)\right| dx \\
&\le \int_a^{\frac{a+b}{2}} M\left(\frac{a+b}{2} - x\right) dx + \int_{\frac{a+b}{2}}^b M\left(x - \frac{a+b}{2}\right) dx \\
&= \frac{(b-a)^2}{4} \cdot M,
\end{aligned}$$

where the second inequality follows from the mean-value theorem.

Solution 2 by Bianca-Teodora Iorache (student), "Carol I" National College, Craiova, Romania and Northwestern University Math Problem Solving Group, Northwestern University, Evanston IL (independently).

Let $I = \int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^b f(x)\,dx$ and $M = \sup_{x\in[a,b]} |f'(x)|$. Rewriting I and integrating the result by parts, we have

$$\begin{aligned}
|I| &= \left| \int_0^{\frac{b-a}{2}} [f(a+x) - f(b-x)]\,dx \right| \\
&= \left| \Big[\{f(a+x) - f(b-x)\}x \Big]_0^{\frac{b-a}{2}} - \int_0^{\frac{b-a}{2}} x[f'(a+x) + f'(b-x)]\,dx \right|
\end{aligned}$$

$$\leq 2M \int_0^{\frac{b-a}{2}} x\,dx$$
$$= \frac{(b-a)^2}{4} \cdot M.$$

Also Solved by ARKADY ALT, San Jose CA; MICHEL BATAILLE, Rouen, France; M. BENITO, Ó. CIAURRI, E. FERNÀNDEZ, and L. RONCAL (jointly), Logroño, Spain; ARIN CHAUDHURI, Morrisville NC; SEI-HYUM CHUN (student), Seoul, Korea; CHARLES DIMINNIE, Angelo State U.; DMITRY FLEISCHMAN, Santa Monica CA; OVIDIU FURDUI, Campia Turzii, Romania; MICHAEL GOLDENBERG, The Ingenuity Project, Baltimore Poly. Inst. and MARK KAPLAN, C.C. of Baltimore County (jointly); EUGENE HERMAN, Grinnell C.; TOM JAGER, Calvin C.; ELIAS LAMPAKIS, Kiparissia, Greece; ROBERT LAVELLE, Iona C.; LONGXIANG LI (student) and LUYUAN YU (jointly), Tianjin U.; MATHRAMZ PROBLEM SOLVING GROUP (2 solutions); GEORGE MATTHEWS, Indianapolis IN; PETER MERCER, Buffalo State C.; MICROSOFT RESEARCH PROBLEMS GROUP; LUCAS MOL and PAM SARGENT (students, jointly), Mount Allison U., NB, Canada; KANDASAMY MUTHUVEL, U. of Wisconsin-Oshkosh; CARLO PAGANO, Università degli Studi di Roma "Tor Vergata"; PAOLO PERFETTI, Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata"; ÁNGEL PLAZA, U. of Las Palmas de Gran Canaria; JOEL SCHLOSBERG, Bayside NY; WILLIAM SEAMAN, Albright C.; NORA THORNBER, Raritan Valley C.C.; THOMAS TURIEL, Schenectady NY; JEFF WAGNER, Big Sandy Community & Tech. C.; STUART WITT, Brooklyn NY; and the proposers.

Conditions for rank(A) = rank(B)

928. *Proposed by Michel Bataille, Rouen, France.*

Let A, B be $n \times n$ complex matrices such that $A^2 = A = AB$. Prove that $B^2 = B = BA$ if and only if rank A = rank B.

Solution by Michael Bacon, Sumter Problem Solving Group, Sumter SC.

Note that the statement for $n \times n$ complex matrices can be replaced by linear transformations $A, B : E \to F$ over a division ring D with $\dim_D E = n$ for a more general setting.

First note that for any linear transformations $M, N : E \to E$ with $\dim_D E = n$ that

$$\operatorname{rank} MN \leq \min(\operatorname{rank} M, \operatorname{rank} N).$$

Since $A = A^2 = AB$, it follows that

$$\operatorname{rank} A = \operatorname{rank} AB \leq \min(\operatorname{rank} A, \operatorname{rank} B) \leq \operatorname{rank} B.$$

Similarly, since $B^2 = B = BA$,

$$\operatorname{rank} B = \operatorname{rank} BA \leq \min(\operatorname{rank} A, \operatorname{rank} B) \leq \operatorname{rank} A.$$

Hence rank A = rank B.

Next, suppose rank A = rank B. Then nullity A = nullity B. Let $x \in \ker B$. Then

$$A(x) = AB(x) = A(Bx) = A(0) = 0,$$

and so

$$\ker B \subseteq \ker A.$$

Since rank A = rank B, by dimension arguments $\ker B = \ker A$.

To show $B^2 = B = BA$, we note that $A^2 = A = AB$ implies that for any $x \in E$, $x - Ax = x - ABx \in \ker A = \ker B$. Thus,

$$0 = B(x - Ax) = Bx = Bx - BAx,$$

and since x is arbitrary, $B = BA$. Similarly,

$$0 = B(x - Ax) = Bx = Bx - BABx = Bx - (BA)Bx = Bx - B^2x,$$

and hence $B = B^2$.

Also solved by RICARDO ALFARO, U. of Michigan-Flint; OSKAR BAKSALARY, Adam Mickiewicz U., Poznań, and GÖTZ TRENKLER, Dortmund, Germany (jointly); PAUL BUDNEY, Sunderland MA; ARIN CHAUDHURI, Morrisville NC; CON AMORE PROBLEM GROUP, Inst. of Curriculum Research, Copenhagen, Denmark; CHIP CURTIS, Missouri Southern State U.; LUZ DEALBA, Drake U.; JAMES DUEMMEL, Bellingham WA; DMITRY FLEISCHMAN, Santa Monica CA; MICHAEL GOLDENBERG, The Ingenuity Project, Baltimore Poly. Inst. and MARK KAPLAN, C.C. of Baltimore County (jointly); EUGENE HERMAN, Grinnell C.; TOM JAGER, Calvin C.; YU-JU KUO, Indiana U. of Pennsylvania; MATHRAMZ PROBLEM SOLVING GROUP; MICROSOFT RESEARCH PROBLEMS GROUP; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP; LUCAS MOL and PAM SARGENT (students, jointly), Mount Allison U.; WILLIAM SEAMAN, Albright C.; HAOHAO WANG and JERZY WOJDYLO (jointly), Southeast Missouri State U.; KEN YANOSKO, Humboldt State U.; LUYUAN YU, Tianjin U.; and the proposer.

One incomplete solution was received.

A constrained inequality

929. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania and Vlad Matei (student), University of Bucharest, Bucharest, Romania.*

Suppose a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that $a^3 + b^3 + c^3 + 3abc \le 6$.

Solution by Eugene Herman, Grinnell College, Grinnell IA.

More generally, let a, b, c be any positive real numbers such that $a^2 + b^2 + c^2 = k$ for some real number $k > 0$. We will show that $a^3 + b^3 + c^3 + 3abc \le 6(k/3)^{3/2}$ with equality if and only if $a = b = c = \sqrt{k/3}$.

The function $f(a, b, c) = a^3 + b^3 + c^3 + 3abc$ is continuous on the compact set (a sphere) given by $g(a, b, c) = a^2 + b^2 + c^2 - k = 0$. Thus f has a maximum value on this domain. At this maximum, grad $f = \lambda$ grad g for some real number λ. That is,

$$3a^2 + 3bc = \lambda 2a, \quad 3b^2 + 3ac = \lambda 2b, \quad 3c^2 + 3ab = \lambda 2c.$$

Multiplying the first of these equations by bc, the second by ac, and the third by ab produces identical right-hand sides. Thus

$$a^2bc + b^2c^2 = b^2ac + a^2c^2 = c^2ab + a^2b^2.$$

Let us assume temporarily that a, b, and c are nonzero. Then from the first of the above equations, we have $abc(a - b) = c^2(a^2 - b^2)$ and so

$$ab = ca + cb \quad \text{or} \quad a = b. \tag{1}$$

By symmetry,

$$bc = ab + ac \quad \text{or} \quad b = c. \tag{2}$$

Combining the first of equations (1) and the first of equations (2) yields $2ac = 0$, a contradiction. Combining the second of equations (1) with the first of equations (2) yields $b^2 = 0$, again a contradiction. Therefore, by symmetry, the only possibility is $a = b = c$. Substituting into $a^2 + b^2 + c^2 = k$ yields $a = b = c = \pm\sqrt{k/3}$ and hence $f(a, b, c) = \pm 6(k/3)^{3/2}$.

It remains only to show that, when one or two of the variables a, b, c are zero, the values of f are strictly smaller than $M = 6(k/3)^{3/2}$. If two of the variables are zero, the remaining one equals $\pm\sqrt{k}$ and the values of f are then $\pm k^{3/2}$, which are smaller than M. If exactly one of the variables is zero, the other two equal each other and are equal to $\pm\sqrt{k/2}$ (shown by an easy use of Lagrange multipliers) and the values of f are then $\pm 2(k/2)^{3/2}$, which are smaller than M.

Also solved by ARKADY ALT, San Jose CA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; BRIAN BEASLEY, Presbyterian C.; JOHN CHRISTOPHER, California State U., Sacramento; CON AMORE PROBLEM GROUP, Inst. of Curriculum Research, Copenhagen, Denmark; TIM CROSS, King Edward's School, Birmingham, UK; CHIP CURTIS, Missouri Southern State U.; DMITRY FLEISCHMAN, Santa Monica CA; FULLERTON COLLEGE MATH ASSOCIATION; MICHAEL GOLDENBERG, The Ingenuity Project, Baltimore Poly. Inst. and MARK KAPLAN, C.C. of Baltimore County (jointly); G.R.A.20 PROBLEM SOLVING GROUP, Rome, Italy; PHILIP GWANYAMA, Northeastern Illinois U.; JAMES HOCHSCHILD, Butner NC; TOM JAGER, Calvin C.; FRANCIS JONES, Huntington U.; GEORGE LABARIA (student), U. of California, Berkeley; ELIAS LAMPAKIS, Kiparissia, Greece; KEE-WAI LAU, Hong Kong, China; WILSON LEE, Fullerton C.; CLARENCE LIENHARD, Mansfield U.; MATHRAMZ PROBLEM SOLVING GROUP; GEORGE MATTHEWS, Indianapolis IN; KIM MCINTURFF, Santa Barbara CA; MICROSOFT RESEARCH PROBLEMS GROUP; SHOELEH MUTAMENI, Morton C.; PETER NÜESCH, Lausanne, Switzerland; PAOLO PERFETTI, Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata"; ÁNGEL PLAZA, U. of Las Palmas de Gran Canaria; PETER SIMONE, U. of Nebraska Medical Center; EARL SMITH (2 solutions), Sun City FL; RON SMITH, Edison State C.; TEXAS STATE UNIVERSITY PROBLEM SOLVERS GROUP; MICHAEL VOWE, Therwil, Switzerland; HAOHAO WANG and JERZY WOJDYLO (jointly), Southeast Missouri State U.; ALBERT WHITCOMB, Castle Shannon PA; and the proposers. A computer generated solution was submitted by STAN WAGON, Macalester C.

Editor's Note. Michel Bataille of Rouen, France showed the following slightly more general version of Problem 929:

$$(a^3 + b^3 + c^3 + 3abc)(a + b + c) \le 9 + 3(ab + bc + ca) \le 6(a + b + c).$$

An application of the Weierstrass Approximation Theorem

930. *Proposed by Ovidiu Furdui, Cluj, Romania.*
Let f be a continuous function on $[0, \infty)$ with $\lim_{x\to\infty} f(x) = L$. Prove that if

$$\int_0^\infty f(x)e^{-nx}\,dx = L/n$$

for all positive integers n, then $f \equiv L$.

Solution by Fullerton College Math Association, Fullerton College, Fullerton CA.

More generally, we prove the following result: If $h : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^1 h(u)u^m du = 0$ for all $m = 0, 1, 2, \ldots$, then $h \equiv 0$.

For any $\epsilon > 0$, by the Weierstrass Approximation Theorem there exists a polynomial p such that $|f(u) - p(u)| < \epsilon$ for all $u \in [0, 1]$. Since $\int_0^1 h(u)u^m = 0$ for all $m = 0, 1, 2, \ldots$, $\int_0^1 h(u)p(u)du = 0$. Thus,

$$\left|\int_0^1 [h(u)]^2\, du\right| = \left|\int_0^1 [h(u)]^2\, du - \int_0^1 h(u)p(u)\, du\right|$$
$$= \left|\int_0^1 h(u)[h(u) - p(u)]\, du\right|$$
$$\leq \epsilon \int_0^1 |h(u)|\, du.$$

Since $\int_0^1 |h(u)|\, du < \infty$ and ϵ is arbitrary, $\int_0^1 [h(u)]^2\, du = 0$, and hence $h \equiv 0$.

Now, let $g(x) = f(x) - L$. Then g is continuous, $\lim_{x\to\infty} g(x) = 0$, and

$$\int_0^\infty f(x)e^{-nx}\, dx = \int_0^\infty g(x)e^{-nx}\, dx + L\int_0^\infty e^{-nx}\, dx = \int_0^\infty g(x)e^{-nx}\, dx + \frac{L}{n}.$$

Making the substitution $u = e^{-x}$ and $du = -e^{-x}\, dx$,

$$\int_0^\infty g(x)e^{-nx}\, dx = \int_0^1 g(-\ln u)u^{n-1}\, du = \int_0^1 h(u)u^m\, du,$$

where $m = n - 1$ and $h : [0, 1] \to \mathbb{R}$ is given by $h(u) = g(-\ln u)$ for $0 < u \leq 1$, and

$$h(0) = \lim_{u\to 0^+} h(u) = \lim_{u\to 0^+} g(-\ln u) = \lim_{x\to\infty} g(x) = 0.$$

Since we have proved that $h \equiv 0$, $g \equiv 0$, and hence $f \equiv L$.

Also solved by Michel Bataille, Rouen, France; Khristo Boyadzhiev, Ohio Northern U.; Paul Budney, Sunderland MA; Bruce Davis, St. Louis C.C. at Florissant Valley; Michele Gallo (student), Università degli Studi di Roma “Tor Vergata”; G.R.A.20 Problem Solving Group, Rome, Italy; Eugene Herman, Grinnell C.; Bianca-Teodora Iorache (student), “Carol I” National C.; Tom Jager, Calvin C.; Santiago de Luxán (student) and Ángel Plaza (jointly), U. of Las Palmas de Gran Canaria; Mathramz Problem Solving Group; George Matthews, Indianapolis IN; Paul Matsumoto; Microsoft Research Problems Group; Northwestern University Math Problem Solving Group; Paolo Perfetti, Dipartimento di Matematica, Università degli Studi di Roma “Tor Vergata”; Izzy Rom, Queen’s C., CUNY; William Seaman, Albright C.; Chikkanna Selvaraj and Suguna Selvaraj (jointly), Penn State U. Sharon; Tony Tam, Calexico CA; and the proposer.