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Control of Quantum Langevin Equations

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Abstract. The problem of controlling quantum stochastic evolutions arises naturally in several different fields such as quantum chemistry, quantum information theory, quantum engineering, etc. In this paper, we apply the recently discovered closed form of the unitarity conditions for stochastic evolutions driven by the square of white noise [9] to solve this problem in the case of quadratic cost functionals (cf. (5.5) below). The optimal control is explicitly given in terms of the solution of an operator Riccati equation. Under general conditions on the system Hamiltonian part of the stochastic evolution and on the system observable to be controlled, this equation admits solutions with the required properties and they can be explicitly described.

1. Introduction

The problem of optimal control of solutions of a quantum Langevin equation with constant coefficients (see Definition 3 below) arises naturally in several different fields such as quantum chemistry, quantum information, quantum engineering, etc. The mathematical formulation of this problem was recently considered in [8, 10] for quantum systems affected by first order white noise. It was preceded by several studies on the quadratic control of the solution of a quantum evolution (see Definition 1 below) driven by first order white noise [24, 25, 26, 27] and the dual Kalman-Bucy filtering problem [1, 26]. A general treatment of the control problem for quantum evolution driven by a general class of quantum noises can be found in [10] with the use of the representation free calculus of [12]. The quadratic form of the control criterion allows the quantum control problem to be solved in analogy with the classical stochastic control problem with the use of quantum stochastic calculus. The statement of the problem is the following: one starts from the Langevin equation for a system observable X (cf. equations (4.3) and (5.4)

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below) and then looks for the coefficients (L, W for first order white noise and D_-, W for the square of white noise) of the martingale terms of this equation, which minimize a given cost functional (cf. equations (4.4) and (5.5) below: in our case the functional is quadratic). Once this problem is solved, using the stochastic limit technique [15], one then looks for a bona fide Hamiltonian interaction which, in the stochastic limit, gives rise to the optimal Langevin equation.

Thus, combining techniques of quantum stochastic control with the stochastic limit of quantum theory, one can find a real physical interaction to concretely realize the optimal quantum evolution. In the classical case the optimal control is constructed via the solution of an operator Riccati differential equation. In the quantum case the equation is replaced by a quantum stochastic Riccati equation which in the problem considered in this paper is reduced to an operator algebraic Riccati equation (cf. equations (4.22) and (5.17)).

In Sec. 2 and 3 we define quantum evolutions, Langevin equations, the first order and the square of white noise stochastic differentials. In Sec. 4 and 5 we review the results on the control of first order white noise evolutions and Langevin equations (extended to include the conservation operator) and we solve the corresponding square of white noise control problem. For both first and second order quantum Langevin equations the theory of quadratic control should be extended to the case of noisy partial observations.

For the white noise approach to stochastic calculus, as well as for the square of white noise calculus we use primarily the results of [9] and also [3, 4, 5, 6, 7, 11] and [13, 14, 16, 18, 19, 20, 21, 22, 23, 28]. For more details on the technical aspects of first order quantum stochastic calculus we refer to [29].

2. Quantum Evolutions and Langevin Equations

DEFINITION 1 Let a_t, $a_t^\dagger$ be the standard Fock white noise of [15] and [16] defined on the Fock space Γ over $L^2(\mathbb{R}_+, \mathbb{C})$ in the case of first order white noise, while for the square of white noise it is over $L^2(\mathbb{R}_+, l_2(\mathbb{N}))$. For each $i = 1, 2, 3, \dots$ and $t \geq 0$ let

$$N_i(t) \;=\; \int_0^t p_i(a_s, a_s^\dagger)\, ds$$

be a noise process on Γ, where $p_i(a_s,\, a_s^\dagger)$ is a polynomial in a_s and $a_s^\dagger$ written in normal order. Let also $\{C_0(t) : t \geq 0\}$ and $\{C_i(t) : t \geq 0\}$ be adapted bounded operator processes on the system Hilbert space $\mathcal{H}$. If

$$h_s \;=\; C_0(s) + \sum_i C_i(s)\, p_i(a_s,\, a_s^\dagger)\,, \tag{2.1}$$

then

$$H_t \;=\; \int_0^t h_s\, ds \;=\; \int_0^t \Big(C_0(s) + \sum_i C_i(s)\, p_i(a_s, a_s^\dagger)\Big)\, ds$$

$$= \int_0^t C_0(s)\,ds + \sum_i C_i(s)\,dN_i(s) \tag{2.2}$$

is called a normally ordered white noise Hamiltonian on $\mathcal{H}\otimes\Gamma$. A unitary evolution is a family $U = \{U_t : t \geq 0\}$ of unitary operators on $\mathcal{H}\otimes\Gamma$ satisfying the normally ordered white noise Schrödinger equation

$$\frac{dU_t}{dt} = -i\,h_t\,U_t\,, \qquad U_0 = 1\,, \tag{2.3}$$

and its adjoint

$$\frac{dU_t^*}{dt} = i\,U_t^*\,h_t^*\,, \qquad U_0^* = 1\,, \tag{2.4}$$

or, equivalently, the quantum stochastic differential equation

$$dU_t = -i\,(dH_t)\,U_t, \qquad U_0 = 1\,, \tag{2.5}$$

and its adjoint

$$dU_t^* = i\,U_t^*\,(dH_t^*)\,, \qquad U_0^* = 1\,. \tag{2.6}$$

DEFINITION 2 Let H and U be as in Definition 1, let X be an observable on the system Hilbert space $\mathcal{H}$, and let $\mathbb{1}$ denote the identity operator on Γ. A quantum flow is a family $j(X) = \{j_t(X) = U_t^*(X\otimes\mathbb{1})U_t : t \geq 0\}$ of operators on $\mathcal{H}\otimes\Gamma$ satisfying the quantum Langevin equation

$$\begin{aligned} dj_t(X) &= dU_t^*\,(X\otimes\mathbb{1})\,U_t + U_t^*\,(X\otimes\mathbb{1})\,dU_t + dU_t^*\,(X\otimes\mathbb{1})\,dU_t \\ &= iU_t^*\,h_t^*\,(X\otimes\mathbb{1})\,U_t\,dt - iU_t^*\,(X\otimes\mathbb{1})\,h_t\,U_t\,dt + U_t^*\,h_t^*\,(X\otimes\mathbb{1})\,h_t\,U_t\,(dt)^2 \\ &= i\,U_t^*\,(h_t^*\,(X\otimes\mathbb{1}) - (X\otimes\mathbb{1})\,h_t)\,U_t\,dt + U_t^*\,h_t^*\,(X\otimes\mathbb{1})\,h_t\,U_t\,(dt)^2 \\ &= j_t(i\,\rho(h_t, X\otimes\mathbb{1}))\,dt + j_t(h_t^*\,(X\otimes\mathbb{1})\,h_t)\,(dt)^2, \end{aligned} \tag{2.7}$$

where $\rho(x,y) := x^*y - yx$ agrees with the usual commutator $[x,y]$ if $x^* = x$.

Equivalently, (2.7) can be written as

$$dj_t(X) = j_t(i\,\rho(dH_t, X\otimes\mathbb{1}))\,dt + j_t(dH_t^*\,(X\otimes\mathbb{1})\,dH_t)\,. \tag{2.8}$$

The $j_t(h_t^*\,(X\otimes\mathbb{1})\,h_t)\,(dt)^2$ term in (2.7) is computed with the use of the following rules (see [17]):

(a) Commute coefficient processes with the white noise functionals appearing in h_t.

(b) Put the result in normal order (i.e $a_t^\dagger a_t$) using the commutation rule $[a_t, a_t^\dagger] = \delta(0)$.

(c) Treat dt as a scalar.

(d) Replace the expression $\delta(0)\,dt$ by 1.

(e) Replace the product of any normally ordered expression times dt^2 by 0.

In the case of the square of white noise, rules (a)–(e) lead to “renormalization” choices such as the subtraction of an infinite constant or the product of two delta functions. Both choices and their consequences have been extensively studied in [4, 5, 6, 7, 9, 11, 14].

The computation of the $j_t(dH_t^* (X \otimes \mathbb{1})\, dH_t)$ term in (2.8) by means of the above rules gives the same result that one obtains when this term is computed with the use of the Itô table for the stochastic differentials appearing in dH_t. In what follows we will use the formalism of (2.8).

3. Quantum Stochastic Differentials

Let $\mathcal{K} = \mathbb{C}$ in the case of first order white noise and $\mathcal{K} = l_2(\mathbb{N})$ for the square of white noise. The Boson Fock space $\Gamma = \Gamma(L^2(\mathbb{R}_+, \mathcal{K}))$ can be defined as the Hilbert space completion of the linear span of the exponential vectors $\psi(f)$ under the inner product

$$\langle \psi(f), \psi(g) \rangle \;=\; e^{\langle f,g \rangle}, \tag{3.1}$$

where $f, g \in L^2(\mathbb{R}_+, \mathcal{K}) \equiv L^2(\mathbb{R}_+) \otimes \mathcal{K}$. For $f \in L^2(\mathbb{R}_+, \mathcal{K})$ and an adjointable linear operator F on $L^2(\mathbb{R}_+, \mathcal{K})$ the annihilation, creation and conservation operators $A(f)$, $A^\dagger(f)$ and $\Lambda(F)$, respectively, are defined on the exponential vectors of Γ by

$$A(f)\psi(g) \;=\; \langle f, g \rangle \psi(g)\,, \tag{3.2}$$

$$A^\dagger(f)\psi(g) \;=\; \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \psi(g + \epsilon f)\,, \tag{3.3}$$

$$\Lambda(F)\psi(g) \;=\; \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \psi(e^{\epsilon F} g)\,, \tag{3.4}$$

where F must be such that the exponential $e^{\epsilon F}$ is defined.

The first order quantum stochastic differentials dA_t, $dA_t^\dagger$, and $d\Lambda_t$ are defined by:

$$dA_t(f) \;=\; A(\chi_{[t,t+dt]} \otimes f)\,, \qquad f \in L^2(\mathbb{R}_+, \mathcal{K})\,, \tag{3.5}$$

$$dA_t^\dagger(f) \;=\; A^\dagger(\chi_{[t,t+dt]} \otimes f)\,, \qquad f \in L^2(\mathbb{R}_+, \mathcal{K})\,, \tag{3.6}$$

$$d\Lambda_t(F) \;=\; \Lambda(\chi_{[t,t+dt]} \otimes F)\,, \qquad F \in \mathcal{D}(\mathcal{K})\,, \tag{3.7}$$

where $\mathcal{D}(\mathcal{K})$ is the space of adjointable operators on $\mathcal{K}$.

Notice that $\chi_{[t,t+dt]}$ is a vector in $A^+(\cdot)$, $A(\cdot)$ and a multiplication operator in $\Lambda(\cdot)$. The multiplication rules for the stochastic differentials are given in the following:

PROPOSITION 1 *The first order white noise Itô table is (see [29])*

	$dA_t^\dagger(f_1)$	$d\Lambda_t(F_1)$	$dA_t(f_1)$	dt
$dA_t^\dagger(f_2)$	0	0	0	0
$d\Lambda_t(F_2)$	$dA_t^\dagger(F_2 f_1)$	$d\Lambda_t(F_2 F_1)$	0	0
$dA_t(f_2)$	$\langle f_2, f_1\rangle dt$	$dA_t(F_1^* f_2)$	0	0
dt	0	0	0	0

The square of white noise quantum stochastic differentials dM_t, dB_t^+, and dB_t^- are defined by

$$dM_t = d\Lambda_t(\rho^+(M)) + dt\,, \tag{3.8}$$

$$dB_t^+ = d\Lambda_t(\rho^+(B^+)) + dA_t^\dagger(e_0)\,, \tag{3.9}$$

$$dB_t^- = d\Lambda_t(\rho^+(B^-)) + dA_t(e_0)\,, \tag{3.10}$$

where for $n,k,l,m \in \{0,1,\dots\}$

$$d\Lambda_{n,k,l}(t) = d\Lambda_t(\rho^+(B^{+^n} M^k B^{-^l}))\,, \tag{3.11}$$

$$dA_m(t) = dA_t(e_m)\,, \tag{3.12}$$

$$dA_m^\dagger(t) = dA_t^\dagger(e_m)\,, \tag{3.13}$$

where we have used the notation

$$dX_t(y) = X(\chi_{[t,t+dt]} \otimes y) \tag{3.14}$$

and the operators $\rho^+(B^{+n} M^k B^{-k})$ are defined by

$$\rho^+(B^{+^n} M^k B^{-^l})\, e_m = \theta_{n,k,l,m}\, e_{n+m-l}\,, \tag{3.15}$$

where e_m, $m = 0,1,2,\dots$ is any orthonormal basis of $l_2(\mathbb{N})$,

$$\theta_{n,k,l,m} := H(n+m-l)\sqrt{\frac{m-l+n+1}{m+1}}\, 2^k (m-l+1)_n (m+1)^{(l)} (m-l+1)^k, \tag{3.16}$$

$H(x)$ is the Heaviside function ($H(x) = 0$ for $x < 0$; $H(x) = 1$ for $x \geq 0$),

$$0^0 = 1\,, \quad (B^+)^n = (B^-)^n = N^n = 0\,, \quad \text{for} \quad n < 0\,,$$

and the "factorial powers" are defined by

$$x^{(n)} = x(x-1)\cdots(x-n+1)\,,$$

$$(x)_n = x(x+1)\cdots(x+n-1)\,,$$

$$(x)_0 = x^{(0)} = 1\,.$$

The Itô multiplication table for $d\Lambda_{n,k,l}(t)$, $dA_m(t)$, and $dA_m^\dagger(t)$ is

$$d\Lambda_{\alpha,\beta,\gamma}(t)\, d\Lambda_{a,b,c}(t) = \sum c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon}\, d\Lambda_{a+\alpha-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda+c}(t) \tag{3.17}$$

$$d\Lambda_{\alpha,\beta,\gamma}(t)\, dA_n^\dagger(t) = \theta_{\alpha,\beta,\gamma,n}\, dA_{\alpha+n-\gamma}^\dagger(t) \tag{3.18}$$

$$dA_m(t)\, d\Lambda_{a,b,c}(t) = \theta_{c,b,a,m}\, dA_{c+m-a}(t) \tag{3.19}$$

$$dA_m(t)\, dA_n^\dagger(t) = \delta_{m,n}\, dt\,, \tag{3.20}$$

where

$$c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon} = \tag{3.21}$$
$$\binom{\gamma}{\lambda}\binom{\gamma-\lambda}{\rho}\binom{\beta}{\omega}\binom{b}{\epsilon}2^{\beta+b-\omega-\epsilon}S_{\gamma-\lambda-\rho,\sigma}a^{(\gamma-\lambda)}(a+\lambda-1)^{(\rho)}(a-\gamma+\lambda)^{\beta-\omega}\lambda^{b-\epsilon},$$

$S_{\gamma-\lambda-\rho,\sigma}$ are the Stirling numbers of the first kind and $\sum$ in (3.17) denotes the finite sum

$$\sum_{\lambda=0}^{\gamma}\sum_{\rho=0}^{\gamma-\lambda}\sum_{\sigma=0}^{\gamma-\lambda-\rho}\sum_{\omega=0}^{\beta}\sum_{\epsilon=0}^{b}.$$

All other products of differentials are equal to zero.

To obtain a concise formulation of the square of white noise evolutions we proceed as follows. Let $\mathcal{D}(\mathcal{K})$ and $\mathcal{B}(\mathcal{H})$ denote, respectively, the spaces of adjointable operators on $\mathcal{K}$ and bounded linear operators on $\mathcal{H}$. The tensor product $\mathcal{B}(\mathcal{H})\otimes\mathcal{K}$ is a pre–Hilbert module with $\mathcal{B}(\mathcal{H})$-valued inner product defined on elementary tensors by

$$(a\otimes\xi\,|\,b\otimes\eta) = a^*b\langle\xi,\eta\rangle\,. \tag{3.22}$$

On $\mathcal{B}(\mathcal{H})\otimes\mathcal{K}$ define linear operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ by

$$\mathcal{A}(a\otimes\xi) = a\otimes A(\xi)\,, \tag{3.23}$$
$$\mathcal{A}^\dagger(a\otimes\xi) = a\otimes A^\dagger(\xi)\,, \tag{3.24}$$

while on $\mathcal{B}(\mathcal{H})\otimes\mathcal{D}(\mathcal{K})$ define a linear operator $\mathcal{L}$ by

$$\mathcal{L}(a\otimes T) = a\otimes\Lambda(T)\,. \tag{3.25}$$

For $\alpha,\beta,\gamma,a,b,c\in\{0,1,2,\dots\}$, if $\{D_{\alpha,\beta,\gamma}\}$ and $\{E_{a,b,c}\}$ are families of operators in $\mathcal{B}(\mathcal{H})$ and

$$D = \sum_{\alpha,\beta,\gamma}D_{\alpha,\beta,\gamma}\otimes\rho^+(B^{+^\alpha}M^\beta B^{-^\gamma})\,,\quad E = \sum_{a,b,c}E_{a,b,c}\otimes\rho^+(B^{+^a}M^bB^{-^c})\,,$$

define the $\circ$-product $D\circ E$ of D and E by

$$D\circ E = \sum_{\alpha,\beta,\gamma}\sum_{a,b,c}\sum c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon}D_{\alpha,\beta,\gamma}E_{a,b,c}\otimes\rho^+(B^{+^{a+\alpha-\gamma+\lambda}}M^{\omega+\sigma+\epsilon}B^{-^{\lambda+c}})\,, \tag{3.26}$$

where $\sum$ and $c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon}$ are as in (3.17) and (3.21). Define also linear operators r and l on $\mathcal{B}(\mathcal{H})\otimes\mathcal{D}(\mathcal{K})$ with values in the space of linear operators on $\mathcal{B}(\mathcal{H})\otimes\mathcal{K}$ by

$$r(D)T = \sum_{n,\alpha,\beta,\gamma}D_{\alpha,\beta,\gamma}\theta_{\alpha,\beta,\gamma,n-\alpha+\gamma}T_{n-\alpha+\gamma}\otimes e_n\,, \tag{3.27}$$

$$l(D)T = \sum_{n,\alpha,\beta,\gamma}T_{n+\alpha-\gamma}\theta_{\gamma,\beta,\alpha,n+\alpha-\gamma}D_{\alpha,\beta,\gamma}\otimes e_n\,, \tag{3.28}$$

where $T = \sum_n T_n \otimes e_n \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{K}$, $n \in \{0, 1, \dots\}$ and θ is as in (3.16). If

$$D_+ = \sum_n D_{+,n} \otimes e_n \,, \tag{3.29}$$

$$D_- = \sum_m D_{-,m} \otimes e_m \,, \tag{3.30}$$

$$D_1 = \sum_{\alpha,\beta,\gamma} D_{1,\alpha,\beta,\gamma} \otimes \rho^+(B^{+^\alpha} M^\beta B^{-\gamma}) \,, \tag{3.31}$$

$$E_1 = \sum_{a,b,c} E_{1,a,b,c} \otimes \rho^+(B^{+^a} M^b B^{-c}) \,, \tag{3.32}$$

where $n, m, \alpha, \beta, \gamma, a, b, c \in \{0, 1, 2, \dots\}$ and $D_{+,n}, D_{-,m}, D_{1,\alpha,\beta,\gamma}, E_{1,a,b,c} \in \mathcal{B}(\mathcal{H})$, then

$$(D_-^* | D_+) = \sum_{m,n} D_{-,m} \, D_{+,n} \langle e_m, e_n \rangle = \sum_n D_{-,n} \, D_{+,n} \tag{3.33}$$

and the square of white noise Itô table takes the following form:

PROPOSITION 2 *The square of white noise Itô table is (see [9]):*

$$d\mathcal{A}_t(D_-) \, d\mathcal{A}_t^\dagger(D_+) = (D_-^* | D_+) \, dt \,, \tag{3.34}$$

$$d\mathcal{L}_t(D_1) \, d\mathcal{L}_t(E_1) = d\mathcal{L}_t(D_1 \circ E_1) \,, \tag{3.35}$$

$$d\mathcal{L}_t(D_1) \, d\mathcal{A}_t^\dagger(D_+) = d\mathcal{A}_t^\dagger(r(D_1) D_+) \,, \tag{3.36}$$

$$d\mathcal{A}_t(D_-) \, d\mathcal{L}_t(E_1) = d\mathcal{A}_t(l(E_1) D_-) \,. \tag{3.37}$$

All other products of stochastic differentials (including dt) are equal to zero.

The differential form of the square of white noise Hamiltonian operator is

$$dH_t = D_0(t) \, dt + d\mathcal{L}_t(D_1) + d\mathcal{A}_t^\dagger(D_+) + d\mathcal{A}_t(D_-) \,. \tag{3.38}$$

4. Control of First Order White Noise Langevin Flows

In the notation of Section 2 and 3, we consider a quantum flow $\{j_t(X) / \, t \in [0, T]\}$ of bounded linear operators on $\mathcal{H} \otimes \Gamma$ defined by $j_t(X) = U_t^* \, X \, U_t$, where $\mathcal{H}$ is a separable Hilbert space, Γ is the Boson Fock space over $L^2(\mathbb{R}_+, \mathbb{C})$, X is a self-adjoint operator on $\mathcal{H}$ identified with its ampliation $X \otimes \mathbb{1}$ to $\mathcal{H} \otimes \Gamma$, and $U = \{U_t \, : \, t \geq 0\}$ is a unitary evolution satisfying on $\mathcal{H} \otimes \Gamma$ a quantum stochastic differential equation of the form

$$dU_t = -\Big(\Big(iH + \frac{1}{2} \, L^* L\Big) \, dt + L^* \, W \, dA_t - L \, dA_t^\dagger + (1 - W) \, d\Lambda_t\Big) \, U_t \,, \; t \in [0, T] \,, \tag{4.1}$$

with adjoint

$$dU_t^* = -U_t^* \Big(\Big(- iH + \frac{1}{2} L^* L\Big) \, dt - L^* dA_t + W^* \, L \, dA_t^\dagger + (1 - W^*) d\Lambda_t\Big), \; t \in [0, T] \tag{4.2}$$

and initial conditions

$$U_0 \;=\; U_0^* \;=\; 1\,,$$

where H, L, W are bounded operators on $\mathcal{H}$ with H self-adjoint and W unitary. These conditions guarantee the existence uniqueness and unitarity of the solution of (4.1), (4.2).

Using the Itô table for first order white noise we can show that the flow $\{j_t(X) : t \in [0,T]\}$ satisfies the quantum stochastic differential equation

$$\begin{aligned} dj_t(X) \;&=\; j_t\Big(i[H,X] - \frac{1}{2}(L^*LX + XL^*L - 2L^*XL)\Big)\,dt \\ &\quad + j_t([L^*,X]\,W)\,dA_t + j_t(W^*\,[X,L])\,dA_t^\dagger + + j_t(W^*\,X\,W - X)\,d\Lambda_t \end{aligned} \tag{4.3}$$

with initial condition

$$j_0(X) \;=\; X\,, \qquad t \in [0,T]\,.$$

DEFINITION 3 On a finite time interval $[0,T]$, the cost functional for the solution of the quantum Langevin equation (4.3) is given by:

$$J_{\xi,T}(L,W) \;=\; \int_0^T \Big[\|j_t(X)\,\xi\|^2 + \frac{1}{4}\|j_t(L^*L)\,\xi\|^2\Big]\,dt + \frac{1}{2}\|j_T(L)\,\xi\|^2, \tag{4.4}$$

where ξ is an arbitrary vector in the exponential domain of $\mathcal{H}\otimes\Gamma$.

Thinking of L and W as controls we interpret the first term of the right hand side of (4.4) as a measure of the size of the flow over $[0,T]$, the second as a measure of the control effort over $[0,T]$ and the third as a "penalty" for allowing the evolution to go on for a long time. We consider the problem of controlling the size of such a flow by minimizing the cost functional $J_{\xi,T}(L,W)$ of (4.4).

THEOREM A *Let $U = \{U_t \,:\, t \geq 0\}$ be a process satisfying the quantum stochastic differential equation*

$$dU_t \;=\; (F\,U_t + u_t)\,dt + \Psi\,U_t\,dA_t + \Phi\,U_t\,dA_t^\dagger + Z\,U_t\,d\Lambda_t\,, \quad U_0 = \mathbb{1}\,, \quad t \in [0,T]\,, \tag{4.5}$$

with adjoint

$$dU_t^* = (U_t^*\,F^* + u_t^*)\,dt + U_t^*\,\Psi^*\,dA_t^\dagger + U_t^*\,\Phi^* dA_t + U_t^*\,Z^*\,d\Lambda_t\,, \quad U_0^* = \mathbb{1}\,, \quad t \in [0,T]\,, \tag{4.6}$$

where $T > 0$ is a fixed finite horizon, the coefficients F, Ψ, Φ, Z are bounded operators on the system space $\mathcal{H}$ and u_t is of the form $-\Pi\,U_t$ for some positive bounded system operator Π.

Then the functional

$$Q_{\xi,T}(u) \;=\; \int_0^T \Big[\langle U_t\,\xi, X^2\,U_t\,\xi\rangle + \langle u_t\,\xi, u_t\,\xi\rangle\Big]\,dt - \langle u_T\,\xi, U_T\,\xi\rangle\,, \tag{4.7}$$

where X is a system space observable, identified with its ampliation $X\otimes I$ to $\mathcal{H}\otimes\Gamma$, is minimized over the set of feedback control processes of the form $u_t = -\Pi\, U_t$, by choosing Π to be a bounded, positive, self-adjoint system operator satisfying

$$\Pi\, F + F^*\Pi + \Phi^*\Pi\Phi - \Pi^2 + X^2 \;=\; 0\,, \tag{4.8}$$

$$\Pi\,\Psi + \Phi^*\,\Pi + \Phi^*\,\Pi\, Z \;=\; 0\,, \tag{4.9}$$

$$\Pi\, Z + Z^*\,\Pi + Z^*\,\Pi\, Z \;=\; 0\,. \tag{4.10}$$

The minimum value is $\langle \xi, \Pi\xi\rangle$. We recognize (4.8) as the algebraic Riccati equation.

Proof. Let

$$\theta_t \;=\; \langle \xi, U_t^*\,\Pi\, U_t\,\xi\rangle\,. \tag{4.11}$$

Using the identity $d(xy) = x\,dy + dx\,y + dx\,dy$ we obtain

$$d\theta_t \;=\; \langle \xi, d(U_t^*\,\Pi\, U_t)\,\xi\rangle \;=\; \langle \xi, (dU_t^*\,\Pi\, U_t + U_t^*\,\Pi\, dU_t + dU_t^*\,\Pi\, dU_t)\,\xi\rangle\,, \tag{4.12}$$

which, after replacing dU_t and dU_t^* by (4.5) and (4.6) respectively and using the Itô table of Proposition 1, becomes

$$\begin{aligned} d\theta_t \;=\; & \langle \xi, U_t^*\,((F^*\,\Pi + \Pi\, F + \Phi^*\,\Pi\,\Phi)\,dt + (\Phi^*\,\Pi + \Pi\,\Psi + \Phi^*\,\Pi\, Z)\,dA_t \\ & + (\Psi\,\Pi^* + \Pi\,\Phi + Z^*\,\Pi\,\Phi)\,dA_t^\dagger + (Z^*\,\Pi + \Pi\, Z + Z^*\,\Pi\, Z)\,d\Lambda_t)\,U_t\,\xi\rangle \\ & + \langle \xi, (u_t^*\,\Pi\, U_t + U_t^*\,\Pi\, u_t)\,dt\,\xi\rangle\,, \end{aligned} \tag{4.13}$$

and by (4.8)–(4.10)

$$d\theta_t \;=\; \langle \xi, U_t^*\,(\Pi^2 - X^2)\,U_t\,dt\,\xi\rangle + \langle \xi, (u_t^*\,\Pi\, U_t + U_t^*\,\Pi\, u_t)\,dt\,\xi\rangle\,. \tag{4.14}$$

By (4.11)

$$\theta_T - \theta_0 \;=\; \langle \xi, U_T^*\,\Pi\, U_T\,\xi\rangle - \langle \xi, \Pi\,\xi\rangle\,, \tag{4.15}$$

while by (4.14)

$$\theta_T - \theta_0 \;=\; \int_0^T \Big(\langle \xi, U_t^*\,(\Pi^2 - X^2)\,U_t\,\xi\rangle + \langle \xi, (u_t^*\,\Pi\, U_t + U_t^*\,\Pi\, u_t)\,\xi\rangle\Big)\,dt\,. \tag{4.16}$$

By (4.15) and (4.16)

$$\begin{aligned} \langle \xi, U_T^*\,\Pi\, U_T\,\xi\rangle - \langle \xi, \Pi\,\xi\rangle \;=\; & \\ & \int_0^T \Big(\langle \xi, U_t^*\,(\Pi^2 - X^2)\,U_t\,\xi\rangle + \langle \xi, (u_t^*\,\Pi\, U_t + U_t^*\,\Pi\, u_t)\,\xi\rangle\Big)\,dt\,. \end{aligned} \tag{4.17}$$

Thus

$$Q_{\xi,T}(u) = (\langle \xi, U_T^*\,\Pi\, U_T\,\xi\rangle - \langle \xi, \Pi\,\xi\rangle) + Q_{\xi,T}(u) - (\langle \xi, U_T^*\,\Pi\, U_T\,\xi\rangle - \langle \xi, \Pi\,\xi\rangle)\,. \tag{4.18}$$

Replacing the first parenthesis on the right hand side of (4.18) by (4.17), and $Q_{\xi,T}(u)$ by (4.7) we obtain after cancellations

$$
\begin{aligned}
Q_{\xi,T}(u) &= \int_0^T \langle \xi, (U_t^* \Pi^2 U_t + u_t^* \Pi U_t + U_t^* \Pi u_t + u_t^* u_t)\xi\rangle \, dt + \langle \xi, \Pi \xi\rangle \\
&= \int_0^T ||(u_t + \Pi U_t)\xi||^2 \, dt + \langle \xi, \Pi \xi\rangle \,, \qquad (4.19)
\end{aligned}
$$

which is clearly minimized by $u_t = -\Pi U_t$ and the minimum is $\langle \xi, \Pi \xi\rangle$. □

DEFINITION 4 The pair $(iH,\, X)$ is called stabilizable if there exists a bounded system operator K such that $iH + KX$ is the generator of an asymptotically stable semigroup $\mathcal{F}_t$ i.e there exist constants $M > 0$ and $\omega < 0$ such that $||\mathcal{F}_t|| \le M\, e^{\omega\, t}$ (see [30]).

THEOREM B *Let X be a bounded self-adjoint system operator such that the pair $(i\,H,\, X)$ is stabilizable. The quadratic performance functional (4.4) associated with the quantum stochastic flow $\{j_t(X) = U_t^*\, X\, U_t \,:\, t \ge 0\}$, where $U = \{U_t \,:\, t \ge 0\}$ is the solution of (4.1), is minimized by*

$$L \;=\; \sqrt{2}\,\Pi^{1/2}\, W_1 \quad (\textit{polar decomposition of } L) \qquad (4.20)$$

and

$$W \;=\; W_2 \,, \qquad (4.21)$$

where Π is a positive self-adjoint solution of the "algebraic Riccati equation"

$$i\,[H, \Pi] + \Pi^2 + X^2 \;=\; 0 \,, \qquad (4.22)$$

and W_1, W_2 are bounded unitary system operators commuting with Π. Moreover

$$\min_{L,W} J_{\xi,T}(L, W) \;=\; \langle \xi, \Pi\, \xi\rangle \qquad (4.23)$$

independently of T.

Remark 1 Eq. (4.22) is a special case of the algebraic Riccati equation (ARE). It is known (see [30]) that if the pair $(iH,\, X)$ is stabilizable, then (4.22) has a positive self-adjoint solution Π.

Proof. Looking at (4.1) as (4.5) with $u_t = -\frac{1}{2} L^*\, L\, U_t$, $F = -i\, H$, $\Psi = -L^*\, W$, $\Phi = L$, and $Z = W - 1$, (4.7) is identical to (4.4). Moreover, equations (4.8)–(4.10) become

$$
\begin{aligned}
i\,[H, \Pi] + L^*\, \Pi\, L - \Pi^2 + X^2 &= 0 \,, & (4.24)\\
L^*\, \Pi - \Pi\, L^*\, W + L^*\, \Pi\, (W - 1) &= 0 \,, & (4.25)\\
(W^* - 1)\, \Pi + \Pi\, (W - 1) + (W^* - 1)\, \Pi\, (W - 1) &= 0 \,. & (4.26)
\end{aligned}
$$

By the self-adjointness of Π, (4.25) implies that

$$[L,\Pi] \;=\; [L^*,\Pi] \;=\; 0\,, \tag{4.27}$$

while (4.26) implies that

$$[W,\Pi] \;=\; [W^*,\Pi] \;=\; 0\,, \tag{4.28}$$

i.e (4.21). By (4.27) and the fact that in this case

$$\Pi \;=\; \frac{1}{2}\,L^*\,L \quad \text{i.e} \quad L^*\,L \;=\; 2\,\Pi\,, \tag{4.29}$$

eq. (4.24) implies (4.22). Eqs (4.27) and (4.29) also imply that

$$[L,L^*] \;=\; 0 \quad \text{(i.e } L \text{ is normal)} \tag{4.30}$$

which implies (4.20). □

5. Control of Square of White Noise Langevin Flows

As shown in [9], eqs. (4.1) and (4.2) are replaced, respectively, by

$$dU_t = \Big(\Big(-\frac{1}{2}\,(D_-^*|D_-^*) + iH\Big)\,dt + d\mathcal{A}_t(D_-) + d\mathcal{A}_t^{\dagger}(-r(W)D_-^*) + d\mathcal{L}_t(W-I)\Big)\,U_t \tag{5.1}$$

and

$$dU_t^* = U_t^*\Big(\Big(-\frac{1}{2}\,(D_-^*|D_-^*) - iH\Big)\,dt + d\mathcal{A}_t^{\dagger}(D_-^*) + d\mathcal{A}_t(-l(W^*)D_-) + d\mathcal{L}_t(W^*-I)\Big) \tag{5.2}$$

with initial conditions

$$U_0 \;=\; U_0^* \;=\; \mathbb{1}\,, \tag{5.3}$$

where H is any bounded self-adjoint system operator, W is a $\circ$-product unitary operator such that $r(W)r(W^*) = r(W^*)r(W) = \mathbb{1}$, I is the $\circ$-product identity, D_- is an arbitrary operator as in (3.30), and $\mathbb{1}$ is the identity operator on $\mathcal{H}\otimes\Gamma$. These conditions guarantee the existence, uniqueness and unitarity of the solutions.

PROPOSITION 3 *In the case of the square of white noise, the quantum Langevin equation (4.3) is replaced by*

$$\begin{aligned} dj_t(X) \;=\; & j_t\Big(i\,[X,H] - \frac{1}{2}((D_-^*|D_-^*)\,X + X\,(D_-^*|D_-^*)) + (r(W)D_-^*|X\,r(W)D_-^*)\Big)\,dt \\ & + j_t\Big(d\mathcal{A}_t^{\dagger}(D_-^*\,X - r(W^*\,X)r(W)D_-^*)\Big) \\ & + j_t\Big(d\mathcal{A}_t(X\,D_- - l(X\,W)l(W^*)D_-)\Big) \\ & + j_t\Big(d\mathcal{L}_t(W^*\,X\circ W - X)\Big)\,, \end{aligned} \tag{5.4}$$

with $j_0(X) = X$, $t \in [0,T]$.

Proof.

$$
\begin{aligned}
dj_t(X) &= (dU_t^*)\,X\,U_t + U_t^*\,X\,(dU_t) + (dU_t^*)\,X\,(dU_t)\\
&= U_t^*\Big\{\Big(-\frac{1}{2}(D_-^*|D_-^*) - iH\Big)\,X\,dt + d\mathcal{A}_t^\dagger(D_-^*\,X) - d\mathcal{A}_t(l(W^*)D_-\,X)\\
&\quad + d\mathcal{L}_t((W^*-I)\,X)) + X\Big(-\frac{1}{2}(D_-^*|D_-^*) + iH\Big)\,dt - d\mathcal{A}_t^\dagger(X\,r(W)D_-^*)\\
&\quad + d\mathcal{A}_t(X\,D_-) + d\mathcal{L}_t(X\,(W-I)) + (r(W)D_-^*|X\,r(W)D_-^*)\,dt\\
&\quad - d\mathcal{A}_t(X\,l((W-I))\,l(W^*)D_-) - d\mathcal{A}_t^\dagger(r((W^*-I)\,X)\,r(W)D_-^*)\\
&\quad + d\mathcal{L}_t((W^*-I)\,X\circ(W-I))\Big\}\,U_t\\
&= U_t^*\Big\{\Big(-\frac{1}{2}((D_-^*|D_-^*)\,X + X\,(D_-^*|D_-^*)) + i\,[X,H]\\
&\quad + (r(W)D_-^*|X\,r(W)D_-^*)\Big)\,dt + d\mathcal{A}_t^\dagger(D_-^*\,X - X\,r(W)D_-^*\\
&\quad - r((W^*-I)\,X)\,r(W)D_-^*) - d\mathcal{A}_t(l(W^*)D_-\,X - X\,D_-\\
&\quad + l(X\,(W-I))\,l(W^*)D_-) + d\mathcal{L}_t((W^*-I)\,X + X\,(W-I)\\
&\quad + (W^*-I)\,X\circ(W-I))\Big\}\,U_t\\
&= U_t^*\Big\{-\frac{1}{2}((D_-^*|D_-^*)\,X + X\,(D_-^*|D_-^*)) + i\,[X,H]\\
&\quad + (r(W)D_-^*|X\,r(W)D_-^*)\Big\}\,U_t\,dt\\
&\quad + U_t^*\{d\mathcal{A}_t^\dagger(D_-^*\,X - X\,r(W)D_-^* - r((W^*-I)\,X)\,r(W)D_-^*)\}\,U_t\\
&\quad + U_t^*\{d\mathcal{A}_t(-l(W^*)D_-\,X + X\,D_- - l(X\,(W-I))\,l(W^*)D_-)\}\,U_t\\
&\quad + U_t^*\{d\mathcal{L}_t(W^*\,X\circ W - X)\}\,U_t\\
&= j_t\Big(i\,[X,H] - \frac{1}{2}((D_-^*|D_-^*)\,X + X\,(D_-^*|D_-^*)) + (r(W)D_-^*|X\,r(W)D_-^*)\Big)\,dt\\
&\quad + j_t\Big(d\mathcal{A}_t^\dagger(D_-^*\,X - X\,r(W)D_-^* - r((W^*-I)\,X)\,r(W)D_-^*)\Big)\\
&\quad + j_t\Big(d\mathcal{A}_t(X\,D_- - l(W^*)D_-\,X - l(X\,(W-I))\,l(W^*)D_-)\Big)\\
&\quad + j_t(d\mathcal{L}_t(W^*\,X\circ W - X))\\
&= j_t\Big(i\,[X,H] - \frac{1}{2}((D_-^*|D_-^*)\,X + X\,(D_-^*|D_-^*))\\
&\quad + (r(W)D_-^*|X\,r(W)D_-^*)\Big)\,dt\\
&\quad + j_t(d\mathcal{A}_t^\dagger(D_-^*\,X - r(W^*\,X)r(W)D_-))\\
&\quad + j_t(d\mathcal{A}_t(X\,D_- - l(X\,W)l(W^*)D_-))\\
&\quad + j_t(d\mathcal{L}_t(W^*\,X\circ W - X))\,.
\end{aligned}
$$

□

DEFINITION 5 On a finite time interval $[0,T]$, the cost functional for the solution of the quantum Langevin equation (5.4) is given by:

$$J_{\xi,T}(D_-,W) \;=\; \int_0^T \Big[\, \|j_t(X)\,\xi\|^2 + \frac{1}{4}\|j_t((D_-^*|D_-^*))\,\xi\|^2 \,\Big]\, dt + \frac{1}{2}\langle \xi, j_T((D_-^*|D_-^*))\,\xi\rangle\,, \tag{5.5}$$

where ξ is an arbitrary vector in the exponential domain of $\mathcal{H}\otimes\Gamma$.

The square of white noise analogues of Theorems 1 and 2 are as follows.

THEOREM C *Let $U=\{U_t\,:\,t\geq 0\}$ be a process satisfying the quantum stochastic differential equation*

$$dU_t = (F\,U_t{+}u_t)\,dt{+}d\mathcal{A}_t(\Psi)\,U_t{+}d\mathcal{A}_t^\dagger(\Phi)\,U_t{+}d\mathcal{L}_t(Z)\,U_t\,,\; U_0=\mathbb{1}\,,\; t\in[0,T], \tag{5.6}$$

with adjoint

$$dU_t^* = (U_t^*\,F^*{+}u_t^*)\,dt{+}U_t^*\,d\mathcal{A}_t^\dagger(\Psi^*){+}U_t^*d\mathcal{A}_t(\Phi^*){+}U_t^*d\mathcal{L}_t(Z^*)\,,\; U_0^*=\mathbb{1},\; t\in[0,T]\,, \tag{5.7}$$

where $T>0$ is a fixed finite horizon, F is a bounded operator on the system space $\mathcal{H}$, Ψ, Φ, and Z are of the same form as D_-, D_+, and D_1 respectively, and u_t is of the form $-\Pi\,U_t$ for some positive bounded system operator Π.

The functional

$$Q_{\xi,T}(u) \;=\; \int_0^T [\langle U_t\,\xi, X^2\,U_t\,\xi\rangle + \langle u_t\,\xi, u_t\,\xi\rangle]\,dt - \langle u_T\,\xi, U_T\,\xi\rangle\,, \tag{5.8}$$

where X is a system space observable, identified with its ampliation $X\otimes I$ to $\mathcal{H}\otimes\Gamma$, is minimized over the set of feedback control processes of the form $u_t=-\Pi\,U_t$ by choosing Π to be a bounded, positive, self-adjoint system operator satisfying

$$\Pi\,F + F^*\Pi + (\Phi|\Pi\,\Phi) - \Pi^2 + X^2 \;=\; 0\,, \tag{5.9}$$

$$\Pi\,\Psi + \Phi^*\,\Pi + l(\Pi\,Z)\Phi^* \;=\; 0\,, \tag{5.10}$$

$$\Pi\,Z + Z^*\,\Pi + (Z^*\,\Pi)\circ Z \;=\; 0\,. \tag{5.11}$$

The minimum value is $\langle\xi,\Pi\xi\rangle$.

Proof. The proof follows in a way similar to that of Theorem 1 with the use of the square of white noise Itô table of Proposition 2. □

THEOREM D *Let X be a bounded self-adjoint system operator such that the pair (iH,X) is stabilizable. The quadratic performance functional (5.5) associated with the quantum stochastic flow $\{j_t(X)=U_t^*\,X\,U_t\,:\,t\geq 0\}$, where $U=\{U_t\,:\,t\geq 0\}$ is the solution of (5.1), is minimized by choosing*

$$D_- \;=\; \sum_n D_{-,n}\otimes e_n \tag{5.12}$$

and

$$W \;=\; \sum_{\alpha,\beta,\gamma} W_{\alpha,\beta,\gamma} \otimes \rho^{+}(B^{+\alpha} M^{\beta} B^{-\gamma}) \tag{5.13}$$

such that

$$\frac{1}{2}\,(D_-^*|D_-^*) \;=\; \Big(\frac{1}{2}\sum_n D_{-,n}\,D_{-,n}^*\Big) \otimes \mathbb{1} \;=\; \Pi\,, \tag{5.14}$$

and

$$[D_{-,n}, D_{-,m}] \;=\; [D_{-,n}, D_{-,m}^*] \;=\; 0\,, \tag{5.15}$$

$$[D_{-,n}, W_{\alpha,\beta,\gamma}] \;=\; [D_{-,n}, W_{\alpha,\beta,\gamma}^*] \;=\; 0\,, \tag{5.16}$$

for all $n, m, \alpha, \beta, \gamma$, *which also implies that* $[D_{-,n}^*, W_{\alpha,\beta,\gamma}] = [D_{-,n}^*, W_{\alpha,\beta,\gamma}^*] = 0$, *where* Π *is a positive self-adjoint solution of the algebraic Riccati equation*

$$i\,[H, \Pi] + \Pi^2 + X^2 \;=\; 0\,. \tag{5.17}$$

Moreover

$$\min_{D_-, W} J_{\xi,T}(D_-, W) \;=\; \langle \xi, \Pi\,\xi \rangle \tag{5.18}$$

independently of T.

Proof. Looking at (5.1) as (5.6) with $u_t = -\frac{1}{2}\,(D_-^*|D_-^*)\,U_t$ i.e

$$\Pi \;=\; \frac{1}{2}\,(D_-^*|D_-^*) \;=\; \Big(\frac{1}{2}\sum_n D_{-,n}\,D_{-,n}^*\Big) \otimes \mathbb{1}\,,$$

$F = iH$, $\Psi = D_-$, $\Phi = -r(W)D_-^*$, and $Z = W - I$, (5.8) is identical to (5.5) and equations (5.9)–(5.11) become

$$i\,[\Pi, H] + (r(W)D_-^*|\Pi\, r(W)D_-^*) - \Pi^2 + X^2 \;=\; 0\,, \tag{5.19}$$

$$\Pi\, D_- - l(W^*)D_-\,\Pi - l(\Pi\,(W - I))\, l(W^*)D_- \;=\; 0\,, \tag{5.20}$$

$$\Pi\,(W - I) + (W^* - I)\,\Pi + ((W^* - I)\,\Pi) \circ (W - I) \;=\; 0\,. \tag{5.21}$$

Equation (5.21) implies $W^*\,\Pi \circ W = \Pi \Rightarrow W^*\,\Pi \circ W \circ W^* = \Pi \circ W^* \Rightarrow W^*\,\Pi \circ I = \Pi \circ W^* \Rightarrow W^*\,\Pi = \Pi\, W^* \Rightarrow [\Pi, W] = [\Pi, W^*] = 0$ and (5.16) follows from (5.14). Similarly, (5.20) implies that $[\Pi, D_-] = 0$ from which (5.15) follows. Finally, using the fact that $(r(W)D_-^*\,|\,r(W)D_-^*) = (D_-^*\,|\,r(W^*)r(W)D_-^*) = (D_-^*\,|\,D_-^*)$, (5.19) implies (5.17). □

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