Quantum Stochastic Linearization of Multi-Linear Interactions

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Abstract. The weak coupling limit for a quantum system, with discrete spectrum, in interaction with a quantum field reservoir is considered. Depending on the nature of the reservoir (i.e. bosonic or fermionic) and the degree of nonlinearity of the interaction, we discover that either a bosonization or fermionization of the collective multi-linear reservoir operators emerges. The stochastic evolution is determined after the weak coupling limit and is shown to be unitary; also we show that our calculations for the system-only dynamics coincide with those previously postulated by physicists.

1. Introduction

There is a well known rule of thumb in quantum mechanics that a pair of fermions may behave in a certain sense like bosonic entity. More generally, we may expect that an even number of fermions should exhibit bosonic properties when considered as a unit, as should any number of bosons; similarly an odd number of fermions should have fermionic properties. In this paper we show how to make this rule of thumb into a rigorous mathematical statement. In all cases, bosonization and fermionization (in the sense that an odd number of fermions can be replaced by one fermionic entity) can be no more than an approximation and specifically we employ a van Hove limit in which the limiting observables are interpreted as quantum stochastic processes: this particular limit we refer to as the *quantum stochastic limit* [1]. The layout of this paper is as follows: in Section 1, we set up notations regarding the quantum field, which we interpret as a reservoir to some quantum mechanical system, and review both the general theory of the noise limit from the point of view of physicists (essentially we discuss the approach of, for instance, Louisell [2]) and the weak coupling limit for linear interactions. In Section 2, we construct mathematically the limit noise. In Section 3, we examine the limiting evolution for multi-linear interactions and deduce the quantum stochastic differential equation that it satisfies. In Section 4, we make some remarks concerning the nature and physical implications of our results.

Our results can be thought of as a generalization of the work of Lu [3], where the quadratic boson interaction was examined, and also of Accardi and Gough [4], where the correct physical interpretation of the weak coupling limit (based on the ideas of [1]) for a general linear interaction was given for the first time. We would also like to mention that there exist other related approaches to stochastic bosonization, especially by Accardi and Mastropietro [6] and also Accardi, Lu and Volovich [7]. Also for other interpretations of bosonization from non-commutative central limits, see Goderis, Verbeure and Vets [11].

1.1. OPEN QUANTUM SYSTEMS

We consider the standard stochastic limit procedure in which a quantum mechanical system, with separable Hilbert space $\mathcal{H}_S$, is coupled to a reservoir which we take to be a quantum field, either bosonic or fermionic, with state space $\mathcal{H}_R$. The reservoir is assumed to be in a quasi-free gauge-invariant state.

The total Hamiltonian for the combined system and reservoir can be formally taken as the operator on $\mathcal{H}_S \otimes \mathcal{H}_R$ given by

$$H^{(\lambda)} = H^{(0)} + \lambda H_I, \tag{1.1}$$

where $H^{(0)} = H_S \otimes \mathbb{1}_R + \mathbb{1}_S \otimes H_R$ is the free Hamiltonian and H_I is the interaction. λ is a real coupling constant. We emphasize that, in our treatment, the reservoir is to be taken as a physical quantum field; so in particular H_R can be taken as the second quantization of a bounded-below operator on the one particle space of the reservoir. That is, the frequency spectrum $\omega = \omega(k)$ for the elementary excitations is bounded below. We set $V_t^{(0)} = \exp\{\frac{t}{i\hbar} H^{(0)}\}$, while the full unitary evolution is given by

$$V_t^{(\lambda)} = \exp\left\{\frac{t}{i\hbar} H^{(\lambda)}\right\} = V_t^{(0)} U_t^{(\lambda)}, \tag{1.2}$$

where $U_t^{(\lambda)}$ is the unitary family given by

$$U_t^{(\lambda)} = \mathcal{T} \exp\left\{\frac{\lambda}{i\hbar} \int_0^t ds\, v_s^{(0)}(H_I)\right\}. \tag{1.3}$$

where $\mathcal{T}$ denotes time ordering and

$$v_s^{(0)}(X) = V_s^{(0)\dagger} X V_s^{(0)}. \tag{1.4}$$

We shall refer to $U_t^{(\lambda)}$ as the *wave operator* at time t. It is responsible for transforming to the interaction picture.

Our objective is to study the transition probabilities and observable expectations under the interaction picture evolution in the limit $\lambda \to 0$ where first we rescale time as t/λ^2 so as to obtain the long-term cumulative behaviour.

The results of previous papers, in which the interaction is linear in the creation/annihilation operators of the reservoir, lead to the same form of approximations purposely made to extract a *Markovian limit* [2]. Moreover, if one considers the situation of quadratic interactions then the stochastic evolution obtained [3] again leads to the Markovian limit situation. The most general interaction considered in [2] is

$$H_I = \hbar \sum_j D_j \otimes F_j, \tag{1.5}$$

where the D_j are operators on the system space with harmonic free evolution of frequency ω_j (cf. (1.22) below) and the F_j are arbitrary operators on the reservoir space: the question we pose in this article is whether the quantum stochastic limit now leads to the same picture as predicted by [2]. Our answer is that, broadly speaking, they do concur however, in the case when the order ν of multi-linearity exceeds 2, we derive this result modulo the proof of the uniform estimate which at present does not seem attainable. Rather we shall show that a regularization of the dynamics (namely retaining in the Dyson series of the wave operator those terms which are technically non-vanishing in the limit and which correspond to the generalization of the non-negligible terms admitted in previous papers concerning the weak coupling limit) does have the correct limiting form consistent with the Markovian approximation theory.

1.2. THE STATE OF THE RESERVOIR

We wish to deal with a quantum field reservoir for which the one-particle space is a separable Hilbert space $\mathcal{H}_R^1$. We take a parameter η to label the statistics of the reservoir; that is to say we set $\eta = 1$ for boson quanta and $\eta = -1$ for Fermi quanta. For $n \geq 1$ we let $\mathcal{H}_R^n = \otimes^n \mathcal{H}_R^1$ and for $\eta = \pm 1$ define the operator P_η, by linear extension, on $\mathcal{H}_R^n$ by

$$P_\eta f_1 \otimes \ldots \otimes f_n = (n!)^{-1} \sum_{\sigma \in S_n} \eta^{\mathrm{par}\,\sigma} f_{\sigma_1} \otimes \ldots \otimes f_{\sigma_n}, \tag{1.6}$$

where S_n is the set of all permutations on the symbols $\{1, \ldots, n\}$ and par σ is the parity of $\sigma \in S_n$ defined to be 0 if σ is even or 1 if σ is odd. P_η is then a projection operator on $\mathcal{H}_R^n$: the space of (anti)-symmetrized n-particle states is then described by

$$\mathcal{H}_R^{n,\eta} = P_\eta \mathcal{H}_R^n. \tag{1.7}$$

Defining as usual $\mathcal{H}_R^0 = \mathbb{C}$; the Bose ($\eta = 1$) and Fermi ($\eta = -1$) Fock spaces are then given by $\Gamma_\eta(\mathcal{H}_R^1) = \bigoplus_{n=0}^\infty \mathcal{H}_R^{n,\eta}$ and this gives the state space of our reservoir.

We shall denote the Fock vacuum vector by Ψ. The canonical (anti)commutation relations for the creation/annihilation operators are

$$[a(g), a(f)]_\eta \;=\; 0 \;=\; [a^\dagger(g), a^\dagger(f)]_\eta$$

and

$$[a(g), a^\dagger(f)]_\eta \;-\; \langle g, f \rangle, \tag{1.8}$$

where $[A, B]_\eta = AB - \eta BA$.

Let $Q \geq 1$ be a self-adjoint operator on $\mathcal{H}_R^1$. Denote by φ_Q the quasi-free gauge-invariant state with covariance Q,

$$\varphi_Q(a^\sharp(g)) \;=\; 0\,,$$
$$\varphi_Q(a(g)a^\dagger(f)) \;=\; \left\langle g, \frac{Q^\eta + 1}{2} f \right\rangle; \tag{1.9}$$

from the canonical relations it follows that

$$\varphi_Q(a^\dagger(f)a(g)) \;=\; \eta \left\langle g, \frac{Q^\eta - 1}{2} f \right\rangle. \tag{1.10}$$

Here $Q^1 = Q$ and Q^{-1} is the inverse of Q. Special cases are

(i) the Fock vacuum, $Q = 1$,

$$\varphi_{\mathrm{vac}}(a(g)a^\dagger(f)) \;=\; \langle g, f \rangle\,, \qquad \varphi_{\mathrm{vac}}(a^\dagger(f)a(g)) \;=\; 0\,; \tag{1.11}$$

(ii) thermal states, $Q = \frac{1 + ze^{-\beta H_R^1}}{1 - ze^{-\beta H_R^1}}$ in which case

$$Q^\eta \;-\; \frac{1 + \eta z e^{-\beta H_R^1}}{1 - \eta z e^{-\beta H_R^1}}\,, \tag{1.12}$$

$$\varphi_{\beta,z}(a(g)a^\dagger(f)) \;=\; \left\langle g, \frac{1}{1 - \eta z e^{-\beta H_R^1}} f \right\rangle,$$

$$\varphi_{\beta,z}(a^\dagger(f)a(g)) \;=\; \left\langle g, \frac{z e^{-\beta H_R^1}}{1 - \eta z e^{-\beta H_R^1}} f \right\rangle, \tag{1.13}$$

where H_R^1 is a bounded-below self-adjoint operator on $\mathcal{H}_R^1$, namely the one-particle Hamiltonian (cf. next subsection). If $\{\mathcal{H}_Q^\eta, \pi_Q, \Psi_Q\}$ denotes the GNS triple associated to the canonical (anti)-commutation relations algebra over $\mathcal{H}_R$ with state φ_Q, then we have the isomorphism [5]

$$\mathcal{H}_Q^\eta \;\cong\; \mathcal{H}_R \otimes \mathcal{H}_R\,, \quad \text{with} \quad \Psi_Q = \Psi \otimes \Psi\,, \tag{1.14}$$

such that if we consider the creation/annihilation operator on $\mathcal{H}_Q^n$ denoted by $a_Q^\natural(g)$, then

$$
\begin{aligned}
a_Q(g) &\equiv a(Q_+ g) \otimes \mathbb{1} + \theta_\eta \otimes a^\dagger(JQ_- g)\,, \\
a_Q^\dagger(g) &\equiv a^\dagger(Q_+ g) \otimes \mathbb{1} + \theta_\eta \otimes a(JQ_- g)\,,
\end{aligned}
\tag{1.15}
$$

where

$$
Q_+ = \sqrt{\frac{Q^\eta + 1}{2}}\,, \qquad Q_- = \sqrt{\eta \frac{Q^\eta - 1}{2}}\,,
\tag{1.16}
$$

$\theta_\eta = \bigoplus_{n=0}^\infty \eta^n$ is the parity operator and J is a complex structure on the one particle Hilbert space $\mathcal{H}_R^1$ (an anti-linear involution satisfying $\langle Jf, Jg \rangle = \langle g, f \rangle$). Note that $\theta_\eta^2 = 1$, while $\theta_\eta = 1$ identically in the boson case. In the following we shall use the notation below for a block of creation or annihilation operators:

$$
a_\nu(g) = a(g_\nu)\ldots a(g_1)\,, \qquad a_\nu^\dagger(g) = a^\dagger(g_1)\ldots a^\dagger(g_\nu)\,.
\tag{1.17}
$$

In the Fock vacuum case we have

$$
\varphi_{\mathrm{vac}}(a_\nu(g) a_\nu^\dagger(f)) = \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\,\sigma} \prod_{k=1}^\nu \langle g_{\sigma_k}, f_k \rangle\,,
\tag{1.18}
$$

where $g = (g_1, \ldots, g_\nu)$ and $f = (f_1, \ldots, f_\nu)$. While in the general situation one has

$$
\varphi_Q(a_\nu(g) a_\nu^\dagger(f)) = \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\,\sigma} \prod_{k=1}^\nu \left\langle g_{\sigma_k}, \frac{Q^\eta + 1}{2} f_k \right\rangle\,,
\tag{1.19a}
$$

$$
\varphi_Q(a_\nu^\dagger(f) a_\nu(g)) = \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\,\sigma} \prod_{k=1}^\nu \left\langle g_{\sigma_k}, \eta \frac{Q^\eta - 1}{2} f_k \right\rangle\,.
\tag{1.19b}
$$

1.3. RESERVOIR AUTO-CORRELATIONS UNDER THE WEAK COUPLING LIMIT

Take H_R^1 to be the operator on $\mathcal{H}_R^1$ corresponding to the one-particle Hamiltonian. The free Hamiltonian H_R is then taken to be its second quantization. The creation/annihilation operators on $\mathcal{H}_R$, under the free evolution, evolve as follows

$$
v_t^{(0)}(1_S \otimes a^\natural(g)) = \mathbb{1}_S \otimes v_t^R(a^\natural(g)) = \mathbb{1}_S \otimes a^\natural(S_t g)\,,
\tag{1.20}
$$

where we have the unitary transformation on the one-particle space

$$
S_t = \exp\left\{ -\frac{t}{i\hbar} H_R^1 \right\} = \exp\{i\Omega t\}\,.
\tag{1.21}
$$

The operator $\Omega = H_R^1/\hbar$ is to be determined from the spectrum $\omega = \omega(k)$ of frequencies of the elementary excitations. v_t^R is the evolute for reservoir observables under the free reservoir evolution. Our basic dynamical assumption shall be that S_t commutes with Q for all t; that is $[\Omega, Q] = 0$. The vacuum and thermal states above clearly satisfy this condition.

We consider an interaction of the type

$$H_I = i\hbar \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} (D_j^\omega \otimes F_j^{\omega\dagger} - h.c.), \tag{1.22}$$

where the $D_j^\omega \in \mathcal{B}(\mathcal{H}_S)$ have harmonic free evolution with frequency ω:

$$v_t^{(0)}(D_j^\omega \otimes \mathbb{1}_R) = e^{-i\omega t} D_j^\omega \otimes \mathbb{1}_R, \tag{1.23}$$

for each $j = 1, \ldots, N(\omega)$. So there is a discrete set F of fundamental frequencies and each frequency ω occurs a total of $N(\omega)$ times in the interaction.

Our starting point is to examine the Dyson series expansion for the wave operator at time t;

$$U_t^{(\lambda)} = 1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{i\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n v_{t_1}^{(0)}(H_I) \ldots v_{t_n}^{(0)}(H_I). \tag{1.24}$$

Now, from our assumptions, we have

$$\frac{1}{i\hbar} v_t^{(0)}(H_I) = \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} (D_j^\omega \otimes v_t^R(F_j^\omega) e^{-i\omega t} - h.c.). \tag{1.25}$$

In a previous paper [4], we have studied the minimal coupling (linear) interactions, where

$$F_j^\omega = a^\dagger(g_j^\omega), \tag{1.26}$$

for certain test-functions $g_j^\omega \subset \mathcal{H}_R^1$.

Here, however, we shall generalize to ν-excitation interactions, where $\nu \in \mathbb{N}$;

$$F_j^\omega = a(g_{j,\nu}^\omega) \ldots a(g_{j,1}^\omega), \tag{1.27}$$

where for each $\omega \in F$ and $j \in \{1, \ldots, N(\omega)\}$ we have ν test-functions $g_{j,k}^\omega \in \mathcal{H}_R^1$. In the notation of (1.17) we have

$$F_j^\omega = a_\nu(g_j^\omega) \tag{1.28}$$

with $g_j^\omega = (g_{j,1}^\omega, \ldots, g_{j,\nu}^\omega) \in \times^\nu \mathcal{H}_R^1$.

In order to obtain an insight into the weak coupling limit we consider the truncated Dyson series

$$U_t^{(\lambda)} = 1 + \frac{\lambda}{i\hbar} \int\limits_0^t du\, v_u^{(0)}(H_I) + O(\lambda^2)$$

$$= 1 + \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} \{D_j^\omega \otimes \mathcal{F}_j^{\omega\dagger}(t,\lambda) - h.c.\} + O(\lambda^2)\,, \tag{1.29}$$

where

$$\mathcal{F}_j^\omega(t,\lambda) = \lambda \int\limits_0^t du\, v_u^R(F_j^\omega) e^{i\omega u}\,. \tag{1.30}$$

We shall examine the first order term in the limit $\lambda \to 0$ with t rescaled as t/λ^2.

$$\left\langle \mathcal{F}_j^\omega(\frac{t}{\lambda^2},\lambda) \mathcal{F}_{j'}^{\omega'}(\frac{s}{\lambda^2},\lambda) \right\rangle_R = \lambda^2 \int\limits_0^t du \int\limits_0^s dv\, \langle v_u^R(F_j^\omega) v_v^R(F_{j'}^{\omega'}) \rangle_R\, e^{i\omega u + i\omega' v}$$

$$= \int\limits_0^{\lambda^2 t} d\sigma \int\limits_{-\sigma/\lambda^2}^{s-\sigma/\lambda^2} d\tau\, \langle F_j^\omega v_\tau^R(F_{j'}^{\omega'}) \rangle_R\, e^{i(\omega+\omega')\sigma/\lambda^2} e^{i\omega\tau}\,, \tag{1.31}$$

where we have undergone a change of variables $\sigma = \lambda^2 u$, $\tau = v - u$ and used the fact that the state $\langle\cdot\rangle_R$ of the reservoir is invariant under the free evolution $v_\tau^R(\cdot)$.

This particular rescaling in time is the only one which allows a non-trivial limit for such expectations as $\lambda \to 0$ for arbitrary choices of the F_j^ω. One finds that one requires $\omega + \omega' = 0$ otherwise the phase $e^{i(\omega+\omega')\sigma/\lambda^2}$ introduces oscillatory behaviour, for an exact treatment of this see [9]. One finds

$$\lim_{\lambda\to 0} \langle \mathcal{F}_j^\omega(t/\lambda^2,\lambda) \mathcal{F}_{j'}^{\omega'}(s/\lambda^2,\lambda) \rangle_R =$$

$$\delta(\omega + \omega') \min\{t,s\} \int\limits_{-\infty}^\infty d\tau\, \langle F_j^\omega v_\tau^R(F_{j'}^{\omega'}) \rangle_R\, e^{-i\omega\tau}\,, \tag{1.32a}$$

likewise

$$\lim_{\lambda\to 0} \langle \mathcal{F}_j^\omega(t/\lambda^2,\lambda) \mathcal{F}_{j'}^{\omega'\dagger}(s/\lambda^2,\lambda) \rangle_R =$$

$$\delta(\omega - \omega') \min\{t,s\} \int\limits_{-\infty}^\infty d\tau\, \langle F_j^\omega v_\tau^R(F_{j'}^{\omega'\dagger}) \rangle_R\, e^{-i\omega\tau}\,, \tag{1.32b}$$

$$\lim_{\lambda \to 0} \langle \mathcal{F}_j^{\omega\dagger}(t/\lambda^2,\lambda)\mathcal{F}_{j'}^{\omega'}(s/\lambda^2,\lambda)\rangle_R =$$

$$\delta(\omega - \omega')\min\{t,s\}\int\limits_{-\infty}^{\infty} d\tau\,\langle F_j^{\omega\dagger}v_\tau^R(F_{j'}^{\omega'})\rangle_R\,e^{-i\omega\tau}\,, \qquad (1.32c)$$

and

$$\lim_{\lambda \to 0} \langle \mathcal{F}_j^{\omega\dagger}(t/\lambda^2,\lambda)\mathcal{F}_{j'}^{\omega'\dagger}(s/\lambda^2,\lambda)\rangle_R =$$

$$\delta(\omega + \omega')\min\{t,s\}\int\limits_{-\infty}^{\infty} d\tau\,\langle F_j^{\omega\dagger}v_\tau^R(F_{j'}^{\omega'\dagger})\rangle_R\,e^{-i\omega\tau}\,. \qquad (1.32d)$$

Specifically in the case where the F_j^ω are given by (1.27) we see that (1.32a) and (1.32d) vanish. On the other hand (1.32b) and (1.32c) are nonzero in general: for example in (1.32) taking a Bose reservoir in the vacuum state we have

$$\int\limits_{-\infty}^{\infty} d\tau\,e^{-i\omega\tau}\langle a_\nu(g_j^\omega)a_\nu^\dagger(S_\tau g_{j'}^{\omega'})\rangle_R = \int\limits_{-\infty}^{\infty} d\tau\,e^{-i\omega\tau}\sum_{\pi\in S_\nu}\prod_{k=1}^{\nu}\langle g_{j,k}^\omega, S_\tau g_{j',\sigma_k}^{\omega'}\rangle\,. \qquad (1.33)$$

For the sake of comparison later on, we give the master equation with arbitrary interaction F_j^ω deduced in [2]; taking s as the reduced system density matrix then

$$\frac{ds}{dt} = \sum_{\omega\in F}\sum_{j,j'=1}^{N(\omega)}\left\{(D_j^\omega D_{j'}^{\omega\dagger}s - D_{j'}^{\omega\dagger}sD_j^\omega)\int\limits_0^\infty\langle v_\tau^R(F_j^{\omega\dagger})F_{j'}^\omega\rangle_R\,e^{i\omega\tau}\,d\tau\right.$$

$$- (D_j^\omega sD_{j'}^{\omega\dagger} - sD_{j'}^\omega D_{j'}^{\omega\dagger})\int\limits_0^\infty\langle F_j^{\omega\dagger}v_\tau^R(F_{j'}^\omega)\rangle_R\,e^{i\omega\tau}\,d\tau \qquad (1.34)$$

$$+ (D_j^{\omega\dagger}D_{j'}^\omega s - D_{j'}^\omega sD_j^{\omega\dagger})\int\limits_0^\infty\langle v_\tau^R(F_j^\omega)F_{j'}^{\omega\dagger}\rangle_R\,e^{i\omega\tau}\,d\tau$$

$$\left. - (D_j^{\omega\dagger}sD_{j'}^\omega - sD_j^{\omega\dagger}D_{j'}^\omega)\int\limits_0^\infty\langle F_j^\omega v_\tau^R(F_{j'}^{\omega\dagger})\rangle_R\,e^{-i\omega\tau}\,d\tau\right\}\,.$$

1.4. DISCUSSION ON THE LINEAR INTERACTION

The linear case has been dealt with fully in [4]; as indicated in (1.12) the interaction takes the form

$$H_I = i\hbar\sum_{\omega\in F}\sum_{j=1}^{N(\omega)}\{D_j^\omega\otimes a(g_j^\omega)^\dagger - h.c.\}\,. \qquad (1.35)$$

It was shown that the wave operator U_t at fixed time t arrived at after a van Hove scaling limit satisfies a quantum stochastic differential equation, in the sense of Hudson and Parthasarathy [10] (boson) or Applebaum and Hudson [11] (fermion), given by

$$dU_t = \Big\{ \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} [D_j^\omega \otimes dB_Q^\omega(g_j^\omega, t)^\dagger - D_j^{\omega\dagger} \otimes dB_Q^\omega(g_j^\omega, t)] - Y \otimes dt \Big\} U_t, \quad (1.36)$$

where the B_Q^ω are quantum Brownian motions of bosonic or fermionic type depending on the choice of the reservoir and Y is an operator on the system space $\mathcal{H}_S$ given by

$$Y = \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} [(g_j^\sim|g_k^\sim)_{Q+}^\sim D_j^{\sim\dagger} D_k^\sim + \overline{(g_j^\omega|g_k^\omega)_{Q-}^\sim} D_j^\sim D_k^{\sim\dagger}] \quad (1.37)$$

with the notations

$$(f|h)_{Q+}^{\omega-} = \int_{-\infty}^0 d\tau \left\langle f, \frac{Q^\eta + 1}{2} S_\tau h \right\rangle e^{-i\omega\tau}, \quad (1.37a)$$

$$(f|h)_{Q-}^{\omega-} = \eta \int_{-\infty}^0 d\tau \left\langle f, \frac{Q^\eta - 1}{2} S_\tau h \right\rangle e^{-i\omega\tau}. \quad (1.37b)$$

The self adjoint part of Y corresponds physically to the damping or dissipative term while the anti-self adjoint part to the energy shift. It was emphasized in [4] that Y is exactly the complex shift of the systems Hamiltonian H_S due to the presence of the reservoir as calculated using standard first order perturbation theory.

Before leaving to study multi-linear interactions, we make the following interesting observations about the linear case: suppose that D operators form an assembly of either bosonic ($\zeta = 1$) or fermionic ($\zeta = -1$) operators; that is to say

$$[D_j^\omega, D_k^{\omega'}]_\zeta = \delta_{\omega,\omega'} \delta_{j,k}. \quad (1.38)$$

This means that the Hamiltonian of the system takes the form

$$\sum_{\omega \in F} \sum_{j=1}^{N(\omega)} \hbar\omega D_j^{\omega\dagger} D_j^\omega.$$

Let Ψ_S denote Fock vacuum for the system, that is $D_j^\omega \Psi_S = 0$ for all ω and j. The typical eigenvectors for H_S are $\{\prod_{\omega \in F} \prod_{j=1}^{N(\omega)} (D_j^{\omega\dagger})^{n(\omega,j)}\} \Psi_S$ with eigenvalues $\sum_{\omega \in F} \sum_{j=1}^{N(\omega)} n(\omega, j)\hbar\omega$. $n(\omega, j)$ gives the number of particles in the jth state with frequency ω and takes the values 0,1 for fermions and $0, 1, 2, 3, \ldots$ for bosons.

Y may be put into normal ordered form yielding

$$Y = \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} [(g_j^\omega | g_k^\omega)_{Q|}^{\omega -} + \zeta \overline{(g_k^\omega | g_j^\omega)_{Q}^{\omega -}}] D_j^{\omega \dagger} D_k^\omega + c \mathbb{1}_S . \tag{1.39}$$

The constant term is $c = \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} \overline{(g_j^\omega | g_j^\omega)_{Q-}^{\omega -}}$. Using the well known identity

$$\int_{-\infty}^{0} d\tau \, e^{ix\tau} = \frac{1}{i(x - i0^+)} = \pi \delta(x) - iP\frac{1}{x}, \tag{1.40}$$

with P denoting the principal part, see [2] for example, we see that Y is readily split into its self and anti-self adjoint parts

$$Y = \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \frac{\pi}{2} \Big\langle g_j^\omega, [Q^\eta + 1 + \eta\zeta(Q^\eta - 1)]\delta(\Omega - \omega)g_k^\omega \Big\rangle D_j^{\omega \dagger} D_k^\omega \tag{1.41}$$

$$- \frac{i}{2} \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} P\Big\langle g_j^\omega, [Q^\eta + 1 - \eta\zeta(Q^\eta - 1)]\frac{1}{\Omega - \omega}g_k^\omega \Big\rangle D_j^{\omega \dagger} D_k^\omega + c \mathbb{1}_S .$$

Two cases arise:

— case (1), $\eta\zeta = 1$

$$Y = \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \Big\{ \pi\Big\langle g_j^\omega, Q^\eta \delta(\Omega - \omega)g_k^\omega \Big\rangle - iP\Big\langle g_j^\omega, \frac{1}{\Omega - \omega}g_k^\omega \Big\rangle \Big\} D_j^{\omega \dagger} D_k^\omega + c \mathbb{1}_S . \tag{1.42a}$$

— case (2), $\eta\zeta = -1$

$$Y = \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \Big\{ \pi\Big\langle g_j^\omega, \delta(\Omega - \omega)g_k^\omega \Big\rangle - iP\Big\langle g_j^\omega, Q^\eta \frac{1}{\Omega - \omega}g_k^\omega \Big\rangle \Big\} D_j^{\omega \dagger} D_k^\omega + c \mathbb{1}_S . \tag{1.42b}$$

In case (1) we have either a Bose system interacting with a Bose reservoir or alternatively a Fermi system and a Fermi reservoir. Here $H_S' = -\hbar \text{Im } Y$ is the energy shift and the Q-dependence is only manifest as a global shift $-\hbar \text{ Im } c = -\hbar \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} P\langle g_j^\omega, \frac{1}{\Omega - \omega}\eta\frac{Q^\eta - 1}{2}g_j^\omega \rangle$; the individual energy levels have then a Q-independent spacing. The spacing of the damping coefficients is however Q-dependent.

In case (2), where the system and the reservoir must be of unequal type (i.e. one is bosonic and the other fermionic), then it is the energy spacing which is Q-dependent but the damping has only a Q-dependence due to the global shift term

$2 \operatorname{Re} c = \pi \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} \langle g_j^\omega, \delta(\Omega - \omega) \eta^{\frac{Q^\eta - 1}{2}} g_j^\omega \rangle$ but the spacing of the damping coefficients is Q-independent.

Finally, it should be pointed out again that U_t has a unitary evolution and examining for instance $X_t = U_t^\dagger (X_0 \otimes 1) U_t$ leads to a Langevin equation whose adjoint is the same master equation (1.34) deduced by Louisell and other authors. The limit from the Heisenberg evolution to the Langevin equation being established rigorously.

2. Limiting Quantum Stochastic Process

2.1. THE QUANTUM NOISE

In this section we outline the basic mathematical structure of the quantum noise processes which shall emerge through our investigations.

For $I \subset \mathbb{R}$, $g = (g_1, \ldots, g_\nu) \in \times^\nu \mathcal{H}_R^1$ and $\lambda > 0$, let

$$B^{(\lambda)\dagger}(g, I) = \lambda \int_{I/\lambda^2} dt a_\nu^\dagger(S_t^{\otimes \nu} g) \equiv \lambda \int_{I/\lambda^2} dt \{ a^\dagger(S_t g_1) \ldots a^\dagger(S_t g_\nu) \}, \quad (2.1)$$

where, for any operator G on $\mathcal{H}_R^1$, we denote its extension on $\otimes^\nu \mathcal{H}_R^1$ by $G^{\otimes \nu}$, that is

$$G^{\otimes \nu} y = (G y_1, \ldots, G y_\nu). \quad (2.2)$$

We wish to study the operator $B^{(\lambda)}(g, I)$ in the limit $\lambda \to 0$. Our claim is that it becomes a quantum Brownian motion. To see what this means we make some rudimentary definitions below.

Let K be the subset of $\mathcal{H}_R^{\nu, \eta}$, the ν-particle space of symmetrized ($\eta = 1$) or anti-symmetrized ($\eta = -1$) states as defined in (1.7), such that

$$\int\limits_{-\infty}^{\infty} dt \sum_{\sigma \in S_\nu} \prod_{j=1}^\nu |\langle f_j, S_t h_{\sigma_j} \rangle| < \infty \quad (2.3)$$

whenever $f = (f_1, \ldots, f_\nu)$ and $h = (h_1, \ldots, h_\nu)$ are elements of K. We define a sesquilinear form $(\cdot|\cdot): K \times K \to \mathbb{C}$ as follows

$$(f|h) = \int\limits_{-\infty}^{\infty} dt e^{-i\omega t} \sum_{\sigma \in S_\nu} \eta^{\operatorname{par}\sigma} \prod_{k=1}^\nu \langle f_{\sigma_j}, S_t h_j \rangle. \quad (2.4)$$

Note that we may write $(f|h) = \int_{-\infty}^{\infty} dt \langle f|S_t^{\otimes \nu} h \rangle e^{-i\omega t}$, where we introduce a sesquilinear form $\langle \cdot|\cdot \rangle$ on $\times^\nu \mathcal{H}_R^1$ defined by

$$\langle f|h \rangle = \sum_{\sigma \in S_\nu} \eta^{\operatorname{par}\sigma} \prod_{j=1}^\nu \langle f_j, h_{\sigma_j} \rangle; \quad (2.5)$$

moreover $(f|h)$ is exactly equal to $\varphi_{\text{vac}}(a_\nu(f)a_\nu^!(h))$ as indicated by (1.18). We now turn K into a Hilbert space as follows: we first of all factor out by the $(.|.)$-null space of K and then take the Hilbert space completion of the remainder again using the $(\cdot|\cdot)$-norm. The resulting Hilbert space we again denote by K and now $(\cdot|\cdot)$ is its inner product.

To construct a quantum Brownian motion over K we proceed as follows. First consider $L^2(\mathbb{R}^+, K)$ the space of square-integrable K-valued functions on $\mathbb{R}^+$. We shall make use of the natural isomorphism $L^2(\mathbb{R}^+, K) \cong L^2(\mathbb{R}^+) \otimes K$. Now let $\Gamma_\xi(L^2(\mathbb{R}^+, K))$ be the bosonic/fermionic Fock space over $L^2(\mathbb{R}^+, K)$ with the convention of Section 1.2 that $\xi = 1$ means Bose statistics while $\xi = -1$ means Fermi statistics. Denote by Φ the vacuum state and by $A^\sharp(\tilde{f})$ the creation/annihilation operator, with test function $\tilde{f} \in L^2(\mathbb{R}^+, K)$, for $\Gamma_\zeta(L^2(\mathbb{R}^+, K))$. In particular, for $f \in K$ we define

$$B(f, t) = A(\chi_{[0,t]} \otimes f), \tag{2.6}$$

then $B(f, t)$ is a quantum Brownian motion on $\Gamma_\xi(L^2(\mathbb{R}^+, K))$ with state given by the vacuum expectation. One notes that

$$[B(f,t), B(h,s)^\dagger]_\xi = (f|h)\min\{t, s\}. \tag{2.7}$$

2.2. CONVERGENCE TO THE NOISE IN THE VACUUM STATE

In the theorem below (taking $\omega = 0$, for clarity) we prove the following assertion: for a reservoir R with statistics η and in the vacuum state, the operator $B^{(\lambda)}(f, [0, t])$ converges weakly in the sense of matrix elements to $B(f, t)$, where the statistics of the quantum Brownian motion is $\xi = \eta^\nu$.

The rule of thumb stated in the introduction is now evident; if $\eta = 1$ then $\xi = 1$ or in other words bosons *stochastically* bosonize, while if $\eta = -1$ then $\xi = (-1)^\nu$ that is to say an even number of fermions also *stochastically* bosonize, whereas an odd number *stochastically* fermionize.

THEOREM 1. *For $N, N' \in \mathbb{N}$, $I_j, J_k \subset \mathbb{R}$ and $g^{(j)}, f^{(k)} \in \mathcal{H}_R^\nu$ for $j = 1, \ldots, N$ and $k = 1, \ldots, N'$, we have*

$$\lim_{\lambda \to 0} \langle B^{(\lambda)\dagger}(g^{(1)}, I_1) \ldots B^{(\lambda)\dagger}(g^{(N)}, I_N)\Psi, B^{(\lambda)\dagger}(f^{(1)}, J_1) \ldots B^{(\lambda)\dagger}(f^{(N')}, J_{N'})\Psi \rangle \tag{2.8}$$

$$= \langle A^\dagger(\chi_{I_1} \otimes g^{(1)}) \ldots A^\dagger(\chi_{I_N} \otimes g^{(N)})\Phi, A^\dagger(\chi_{J_1} \otimes f^{(1)}) \ldots A^\dagger(\chi_{J_{N'}} \otimes f^{(N')})\Phi \rangle. \tag{2.9}$$

Here $A^\sharp$ are the creation/annihilation operators on $\Gamma_\xi(L^2(\mathbb{R}^+, K))$, where we have set $\xi = \eta^\nu$.

Proof. Both (2.8) and (2.9) vanish if $N \neq N'$, so we only need to consider the case $N = N'$. Now, (2.8) for finite λ can be written as

$$\left\langle B^{(\lambda)\dagger}(g^{(1)}, I_1) \ldots B^{(\lambda)\dagger}(g^{(N)}, I_N)\Psi, B^{(\lambda)\dagger}(f^{(1)}, J_1) \ldots B^{(\lambda)\dagger}(f^N, J_N)\Psi \right\rangle =$$

$$\lambda^{2N} \int_{I_1/\lambda^2} du_1 \dots \int_{I_N/\lambda^2} du_N \cdot \int_{J_1/\lambda^2} dv_1 \dots \int_{J_N/\lambda^2} dv_N$$

$$\left\langle \Psi, a(S_{u_N} g_\nu^{(N)}) \dots a(S_{u_N} g_1^{(N)}) \dots a(S_{u_1} g_\nu^{(1)}) \dots a(S_{u_1} g_1^{(1)}) \right. \qquad (2.10)$$

$$\left. a^\dagger(S_{v_1} f_1^{(1)}) \dots a^\dagger(S_{v_1} f_\nu^{(1)}) \dots a^\dagger(S_{v_N} f_1^{(N)}) \dots a^\dagger(S_{v_N} f_\nu^{(N)}) \Psi \right\rangle .$$

For notational convenience we introduce

$$g_{(i,j)} = S_{u_i} g_j^{(i)}, \qquad f_{(i,j)} = S_{v_i} f_j^{(i)}. \qquad (2.11)$$

The expectation above can then be written out as

$$\langle \Psi, a(g_{(N,\nu)}) \dots a(g_{(1,1)}) \cdot a^\dagger(f_{(1,1)}) \dots a^\dagger(f_{(N,\nu)}) \Psi \rangle$$

$$= \sum_{\mu \in S_{N,\nu}} \eta^{\operatorname{par}\mu} \prod_{i=1}^{N} \prod_{j=1}^{\nu} \langle g_{\mu(i,j)}, f_{(i,j)} \rangle , \qquad (2.12)$$

where $S_{N,\nu}$ is the set of all permutations on the $N\nu$ symbols,

$$\{(1,1),(1,2),\dots,(1,\nu),\dots,(N,1),\dots,(N,\nu)\} \qquad (2.13)$$

In particular, we let $S_{N,\nu}^0$ denote the set of permutations of the type $\mu = \sigma \times \epsilon$, where $\sigma \in S_N$ and $\epsilon \in S_\nu$, that is

$$\mu(i,j) = (\sigma_i, \epsilon_j). \qquad (2.14)$$

Then $S_{N,\nu}^0 \cong S_N \times S_\nu$. We let $S_{N,\nu}^1 = S_{N,\nu}/S_{N,\nu}^0$; such permutations give rise to what we call the *crossing terms*.

For $\mu = \sigma \times \epsilon \in S_{N,\nu}^0$, the contribution to (2.10) is

$$\lambda^{2N} \int_{I_1/\lambda^2} du_1 \dots \int_{I_N/\lambda^2} du_N \cdot \int_{J_1/\lambda^2} dv_1 \dots \int_{J_N/\lambda^2} dv_N \, \eta^{\operatorname{par}\mu} \prod_{i=1}^{N} \prod_{j=1}^{\nu} \langle S_{u_{\sigma_i}} g_{\epsilon_j}^{(\sigma_i)}, S_{v_i} f_j^{(i)} \rangle. \qquad (2.15)$$

Now setting $t_i = \lambda^2$ and $\tau_i = u_{\sigma_i} - v_i$ we obtain

$$\int_{J_1} dt_1 \int_{(I_{\sigma_1} - t_1)/\lambda^2} d\tau_1 \dots \int_{J_N} dt_N \int_{(I_{\sigma_N} - t_N)/\lambda^2} d\tau_N \, \eta^{\operatorname{par}\mu} \prod_{i=1}^{N} \prod_{j=1}^{\nu} \langle S_{\tau_i} g_{\epsilon_j}^{(\sigma_i)}, f_j^{(i)} \rangle. \qquad (2.16)$$

But we have the limit

$$\lim_{\lambda \to 0} \int_{J_k} dt_k \int_{(I_{\sigma_k} - t_k)/\lambda^2} d\tau_k \prod_{j=1}^{\nu} \langle S_{\tau_k} g_{\epsilon_j}^{(\sigma_k)}, f_j^{(k)} \rangle$$

$$= |I_{\sigma_k} \cap J_k| \int_{-\infty}^{\infty} d\tau_k \prod_{k=1}^{\nu} \langle S_{\tau_k} y_{\epsilon_j}^{(\sigma_k)}, f_j^{(i)} \rangle. \qquad (2.17)$$

Now note that since $\mu = \sigma \times \epsilon$ we have the relation

$$\eta^{\mathrm{par}\,\mu} = \eta^{\nu\,\mathrm{par}\,\sigma + N\mathrm{par}\,\epsilon}. \tag{2.18}$$

Or, in other words, $\eta^{\mathrm{par}\,\mu} = \xi^{\mathrm{par}\,\sigma} \cdot \eta^{N\mathrm{par}\,\epsilon}$, where we set $\xi = \eta^{\nu}$. Taking all the $S_{N,\nu}^0$ terms together in the limit we therefore obtain

$$\int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_N \sum_{\sigma \subset S_N} \xi^{\mathrm{par}\,\sigma} \prod_{j=1}^{N} \sum_{\epsilon \subset S_\nu} \eta^{\mathrm{par}\,\epsilon} \prod_{k=1}^{\nu} |I_j \cap J_{\sigma_j}| \langle g_k^{(j)}, S_{t_j} f_{\epsilon_k}^{(\sigma_j)} \rangle$$

$$= \sum_{\upsilon \in S_N} \xi^{\mathrm{par}\,\sigma} \prod_{j=1}^{N} \langle \Phi, A(\chi_{I_{\sigma_j}} \otimes g^{(j)}) \cdot A^\dagger(\chi_{J_j} \otimes f^{(j)}) \Phi \rangle, \tag{2.19}$$

however this is equal to

$$\left\langle A^\dagger(\chi_{I_1} \otimes g^{(1)}) \ldots A^\dagger(\chi_{I_N} \otimes g^{(N)}) \Phi, A^\dagger(\chi_{J_1} \otimes f^{(1)}) \ldots A^\dagger(\chi_{J_N} \otimes f^{(N)}) \Phi \right\rangle, \tag{2.20}$$

where $A^\dagger$ is the creation operator on $\Gamma_\zeta(L^2(\mathbb{R}, \mathcal{H}_R^1))$.

So (2.9) clearly comes about from considering all the limit contributions from terms with $\mu \in S_{N,\nu}^0$, that is the non-crossing terms. We must therefore show that the crossing terms do not contribute to the limit (2.8). Essentially, we modify the argument of Lu [3] to show this.

Take $\mu \in S_{N,\nu}^1$, we wish to examine the term

$$\left| \lambda^{2N} \int_{I_1/\lambda^2} du_1 \ldots \int_{I_N/\lambda^2} du_N \cdot \int_{J_1/\lambda^2} dv_1 \ldots \int_{J_N/\lambda^2} dv_N \eta^{\mathrm{par}\,\mu} \prod_{i=1}^{N} \prod_{j=1}^{\nu} \langle g_{\mu(i,j)}, f_{(i,j)} \rangle \right|, \tag{2.21}$$

which is bounded above by

$$\lambda^{2N} \int_{I_1/\lambda^2} du_1 \ldots \int_{I_N/\lambda^2} du_N \cdot \int_{J_1/\lambda^2} dv_1 \ldots \int_{J_N/\lambda^2} dv_N \cdot \prod_{i=1}^{N} \prod_{j=1}^{\nu} |\langle g_{\mu(i,j)}, f_{(i,j)} \rangle|, \tag{2.22}$$

Now, since $\mu \in S_{N,\nu}^1$, there exists an $\alpha \in \{1, \ldots, N\}$ such that

$$\mu(\alpha, a) = (m, l) \quad \text{and} \quad \mu(\alpha, b) = (k, c), \quad \text{where} \quad a \neq b \quad \text{and} \quad m \neq k. \tag{2.23}$$

We can obtain as an upper bound for (2.22)

$$\lambda^{2N} \int_{I_1/\lambda^2} du_1 \ldots \int_{I_N/\lambda^2} du_N \cdot \int_{J_1/\lambda^2} dv_1 \ldots \int_{J_N/\lambda^2} dv_N \cdot c_1$$
$$\times |\langle g_{(m,l)}, f_{(\alpha,a)} \rangle| \cdot |\langle g_{(k,c)}, f_{(\alpha,b)} \rangle| \cdot \prod_{h \neq \alpha} |\langle g_{\mu(h,1)}, f_{(h,1)} \rangle|, \tag{2.24}$$

where $c_1 = \max\{1, \|g_j^{(i)}\|, \|f_j^{(i)}\|\}^{\nu N}$.

Suppose that $\mu(h, 1) = (\mu(h), i(h))$. The above can be rewritten, using the change of variables $t_i' = u_i \lambda^2$, as

$$\int_{I_1} dt_1' \ldots \int_{I_N} dt_N' \cdot \int_{J_1/\lambda^2} dv_1 \ldots \int_{J_N/\lambda^2} dv_N \cdot c_1 \left| \left\langle S_{t_m'/\lambda^2} g_l^{(m)}, S_{v_\alpha} f_a^{(\alpha)} \right\rangle \right| \quad (2.25)$$
$$\times \left| \left\langle S_{t_k'/\lambda^2} g_c^{(k)}, S_{v_\alpha} f_b^{(\alpha)} \right\rangle \right| \cdot \prod_{h \neq \alpha} \left| \left\langle S_{t_{\mu(h)'}/\lambda^2} g_{i(h)}^{\mu(h)}, S_{v_h} f_1^{(h)} \right\rangle \right|.$$

Now, supposing $m < k$, we make the following change of variables

$$\tau_h' = v_h - t_{\mu(h)}'/\lambda^2, \quad \text{for} \quad h \neq \alpha, \quad \text{and} \quad \tau_\alpha' = v_\alpha - t_m'/\lambda^2. \quad (2.26)$$

The above bound can be written as

$$\int_{I_1} dt_1' \ldots \int_{I_N} dt_N' \cdot \prod_{h \neq \alpha} \int_{(J_h - t_{\mu(h)}')/\lambda^2} d\tau_h' \int_{(J_\alpha - t_{\mu(m)}')/\lambda^2} d\tau_\alpha' \cdot c_1$$
$$\times \left| \left\langle g_l^{(m)}, S_{\tau_\alpha'} f_a^{(\alpha)} \right\rangle \right| \cdot \left| \left\langle g_c^{(k)}, S_{\tau_\alpha' + (t_m' - t_k')/\lambda^2} f_b^{(\alpha)} \right\rangle \right| \cdot \prod_{h \neq \alpha} \left| \left\langle g_{i(h)}^{\mu(h)}, S_{\tau_h'} f_1^{(h)} \right\rangle \right|$$
$$= \prod_{r \neq k} \int_{I_r} dt_r' \cdot \left\{ \lambda^2 \int_{(I_k - t_m')/\lambda^2} dt_k'' \right\} \cdot \prod_{h \neq \alpha} \int_{(J_h - t_{\mu(h)}')/\lambda^2} d\tau_h' \int_{(J_\alpha - t_{\mu(m)}')/\lambda^2} d\tau_\alpha'$$
$$\times c_1 \left| \left\langle g_l^{(m)}, S_{\tau_\alpha'} f_a^{(\alpha)} \right\rangle \right| \cdot \left| \left\langle g_c^{(k)}, S_{\tau_\alpha' - t_k''} f_b^{(\alpha)} \right\rangle \right| \cdot \prod_{h \neq \alpha} \left| \left\langle g_{i(h)}^{\mu(h)}, S_{\tau_h'} f_1^{(h)} \right\rangle \right|, \quad (2.27)$$

where we have set $t_k'' = (t_k - t_m)/\lambda^2$.

However, for each $h \neq \alpha$, we have

$$\int_{(J_h - t_{\mu(h)}')/\lambda^2} d\tau_h \left| \left\langle g_{i(h)}^{\mu(h)}, S_{\tau_h} f_1^{(h)} \right\rangle \right| \leq \int_{-\infty}^{\infty} d\tau_h \left| \left\langle g_{i(h)}^{\mu(h)}, S_{\tau_h} f_1^{(h)} \right\rangle \right|. \quad (2.28)$$

Since $\int_{I_h} dt_h = |I_h|$, we use the dominated convergence to obtain an upper bound of the following form:

$$c_2 \lambda^2 \int_{I_1} dt_1' \ldots \int_{I_{k-1}} dt_{k-1}' \int_{(I_k - t_m')/\lambda^2} dt_k'' \int_{(J_\alpha - t_m')/\lambda^2} d\tau_\alpha'$$
$$\times \left| \left\langle g_l^{(m)}, S_{\tau_\alpha'} f_a^{(\alpha)} \right\rangle \right| \cdot \left| \left\langle g_c^{(k)}, S_{\tau_\alpha' - t_k''} f_b^{(\alpha)} \right\rangle \right|$$

for suitable $c_2 > 0$. But now on interchanging the last pair of integrals and changing the variables from t_k'' to $s_k = \tau_\alpha' - t_k''$, this last expression is the same as

$$c_2 \lambda^2 \int_{I_1} dt_1' \ldots \int_{I_{k-1}} dt_{k-1}' \int_{(J_\alpha - t_m')/\lambda^2} d\tau_\alpha' \int_{\tau_\alpha' - (J_\alpha - t_m)/\lambda^2} ds_k \quad (2.29)$$
$$\times \left| \left\langle g_l^{(m)}, S_{\tau_\alpha'} f_a^{(\alpha)} \right\rangle \right| \cdot \left| \left\langle g_c^{(k)}, S_{s_k} f_b^{(\alpha)} \right\rangle \right|,$$

which behaves as $\lambda^2 \cdot O(1)$ as $\lambda \to 0$.

2.3. GENERAL QUASI-FREE STATES

Let K_1 and K_2 be two separable Hilbert spaces, identified with each other as spaces, with inner products $(\cdot|\cdot)_1$ and $(\cdot|\cdot)_2$, respectively. Set $\mathcal{H}_j = \Gamma_\xi(L^2(\mathbb{R}^+, K_j))$ and consider $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. For $\tilde{g} \in L^2(\mathbb{R}^+, K_j)$ we define the following annihilator on $\mathcal{H}$

$$A(\tilde{g}) = A_1(\tilde{g}) \otimes \mathbb{1}_2 + \theta_\xi \otimes A_2^\dagger(\mathbb{1} \otimes J_2\tilde{g}), \tag{2.30}$$

where $A_j(\tilde{g})$ is the annihilator on $\mathcal{H}_j$ and then notations are equivalent to those introduced in Section 1.2; for instance J_2 is an anti-unitary involution on K_2. Note that we have the canonical relations

$$[A_j(\tilde{f}), A_j^\dagger(\tilde{h})]_\xi = (\tilde{f}|\tilde{h})_j \equiv \int_{\mathbb{R}^+} dt(f(t)|h(t))_j. \tag{2.31}$$

It then follows that

$$[A(\tilde{f}), A^\dagger(\tilde{h})]_\xi = (\tilde{f}|\tilde{h})_1 - \xi(\tilde{f}|\tilde{h})_2. \tag{2.32}$$

We denote by Φ_j the vacuum state of $\mathcal{H}_j$ and set $\Phi = \Phi_1 \otimes \Phi_2$. Now for $f \in K_j$ we define

$$B(f, t) = A(\chi_{[0,t]} \otimes f). \tag{2.33}$$

Then $B(f, t)$ in the state Φ on $\mathcal{H}$ is a (doubled up) quantum Brownian motion. Its quantum Ito table is

$$
\begin{aligned}
dB(f,t)\, dB^\dagger(h,t) &= (f|h)_1\, dt & dB^\dagger(f,t)\, dB(h,t) &= (h|f)_2\, dt \\
dB(f,t)\, dB(h,t) &= 0 & dB^\dagger(f,t)\, dB^\dagger(h,t) &= 0.
\end{aligned}
\tag{2.34}
$$

The Hilbert spaces K_1 and K_2 come about as follows; consider the subset K' of $\mathcal{H}_R^{\nu,\eta}$ such that

$$\int_{-\infty}^\infty dt \sum_{\sigma \in S_\nu} \prod_{j=1}^\nu |\langle f_j, Q^\eta S_t h_{\sigma_j} \rangle| < \infty \tag{2.35}$$

whenever $f = (f_1, \ldots, f_\nu)$ and $h = (h_1, \ldots, h_\nu)$ are elements of K'. Note that for $\eta = -1$ we just have $K' = K$. We define a pair of sesquilinear forms on K' as follows

$$(f|h)_1 = \int_{-\infty}^\infty dt e^{-i\omega t} \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\,\sigma} \prod_{j=1}^\nu \left\langle f_{\sigma_j}, \frac{Q^\eta + 1}{2} S_t h_j \right\rangle, \tag{2.36}$$

$$(f|h)_2 = \int_{-\infty}^\infty dt e^{-i\omega t} \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\,\sigma} \prod_{j=1}^\nu \left\langle f_{\sigma_j}, \eta \frac{Q^\eta - 1}{2} S_t h_j \right\rangle. \tag{2.37}$$

That is $(f|h)_j = \int_{-\infty}^{\infty} dt \langle f|S_t^{\otimes\nu}h\rangle_{Q,j}$ where $\langle f|h\rangle_{Q,j}$ is $\varphi_Q(a_\nu(f)a_\nu^\dagger(h))$ for $j = 1$ and $\varphi_Q(a_\nu^\dagger(h)a_\nu(f))$ for $j = 2$. The Hilbert spaces K_1 and K_2 are then just the completions of K' with respect to the sesquilinear forms $(\cdot|\cdot)_1$ and $(\cdot|\cdot)_2$, respectively.

THEOREM 2. *For* $g \in \mathcal{H}_R^\nu$, $I \subset \mathbb{R}$ *and* $\omega \in \mathbb{R}$ *we define*

$$B^{(\lambda)}(g, I) = \lambda \int_{I/\lambda^2} dt\, e^{-i\omega t} a_\nu(S_t^{\otimes\nu}g)\,. \tag{2.38}$$

Then, for $N, N' \in \mathbb{N}$, $I_j, J_k \subset \mathbb{R}$ *and* $g^{(j)}, f^{(k)} \in \mathcal{H}_R^\nu$ *for* $j = 1, \ldots, N$ *and* $k = 1, \ldots, N'$, *we have*

$$\lim_{\lambda\to 0} \varphi_Q(B^{(\lambda)}(g^{(N)}, I_N)\ldots B^{(\lambda)}(g^{(1)}, I_1)B^{(\lambda)\dagger}(f^{(1)}, J_1)\ldots B^{(\lambda)\dagger}(f^{N'}, J_{N'})) \tag{2.39}$$

$$= \left\langle A^\dagger(\chi_{I_1} \otimes g^{(1)})\ldots A^\dagger(\chi_{I_N} \otimes g^{(N)})\Phi, A^\dagger(\chi_{J_1} \otimes f^{(1)})\ldots A^\dagger(\chi_{J_{N'}} \otimes f^{(N')})\Phi\right\rangle, \tag{2.40}$$

where $A^\natural$ *are the creation/annihilation operators on* $\mathcal{H}$ *as defined now by (2.30) with the pair of Hilbert spaces determined from (2.35)–(2.37).*

Proof. Going to the GNS representation $\{\mathcal{H}_Q^\eta, \pi_Q, \Psi_Q\}$ we may write (2.39) as

$$\lambda^{N+N'} \int_{I_1/\lambda^2} du_1 \ldots \int_{I_N/\lambda^2} du_N \cdot \int_{J_1/\lambda^2} dv_1 \ldots \int_{J_{N'}/\lambda^2} dv_{N'} \tag{2.41}$$

$$\left\langle a_Q^\dagger(S_{u_1}g_1^{(1)})\ldots a_Q^\dagger(S_{u_1}g_\nu^{(1)})\ldots a_Q^\dagger(S_{u_N}g_1^{(N)})\ldots a_Q^\dagger(S_{u_N}g_\nu^{(N)})\Psi_Q,\right.$$
$$\left. a_Q^\dagger(S_{v_1}f_1^{(1)})\ldots a_Q^\dagger(S_{v_1}f_\nu^{(1)})\ldots a_Q^\dagger(S_{v_{N'}}f_1^{(N')})\ldots a_Q^\dagger(S_{v_{N'}}f_\nu^{(N')})\Psi_Q\right\rangle.$$

However, using the decomposition (1.14)–(1.16) and noting that

$$a_Q^\dagger(h_1)\ldots a_Q^\dagger(h_m)\Psi_Q \equiv a^\dagger(Q_+h_1)\ldots a^\dagger(Q_+h_m)\Psi \otimes \Psi\,, \tag{2.42}$$

we see that (2.41) is equal to

$$\lambda^{N+N'} \int_{I_1/\lambda^2} du_1 \ldots \int_{I_N/\lambda^2} du_N \cdot \int_{J_1/\lambda^2} dv_1 \ldots \int_{J_{N'}/\lambda^2} dv_{N'} \tag{2.43}$$

$$\left\langle a^\dagger(Q_+S_{u_1}g_1^{(1)})\ldots a^\dagger(Q_+S_{u_1}g_\nu^{(1)})\ldots a^\dagger(Q_+S_{u_N}g_1^{(N)})\ldots a^\dagger(Q_+S_{u_N}g_\nu^{(N)})\Psi,\right.$$
$$\left. a^\dagger(Q_+S_{v_1}f_1^{(1)})\ldots a^\dagger(Q_+S_{v_1}f_\nu^{(1)})\ldots a^\dagger(Q_+S_{v_{N'}}f_1^{(N')})\ldots a^\dagger(Q_+S_{v_{N'}}f_\nu^{(N')})\Psi\right\rangle.$$

So now we just repeat the proof of Theorem 1 with (2.10) replaced by (2.43) above.

2.4. THE LIMIT NOISE

A natural question to ask is whether or not there exists an inner product $\ll \cdot, \cdot \gg$ and an operator $Q \geq 1$ on the space of test functions (completion of K') such that

$$(f|h)_1 = \ll f, \frac{Q^\xi + 1}{2} h \gg, \qquad (f|h)_2 = \xi \ll f, \frac{Q^\xi - 1}{2} h \gg. \qquad (2.44)$$

That is to say, can we recombine the pair of quantum Brownian motions to form a single one by reversing the usual decomposition method? This would be desirable as we could then consider the noise as being essentially Q-quantum Brownian motion on $\Gamma_\xi(L^2(\mathbb{R}^+, K_3))$, where K_3 would be the $\ll \cdot, \cdot \gg$-completion of K'. However, we have that

$$\ll f, h \gg = (f|h)_1 - \xi(f|h)_2, \qquad \ll f, Q^\xi h \gg = (f|h)_1 + \xi(f|h)_2. \qquad (2.45)$$

The expressions (2.46), (2.47) below show that this recombination is not always admissible.

LEMMA 1.

$$(f|h)_1 - \xi(f|h)_2 = 2^{1-\nu} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\sigma} \sum_{\alpha \in \{0,1\}^\nu}^{\alpha_0 = \mathrm{odd}} \prod_{j=1}^{\nu} \langle f_j, [Q^\eta]^{\alpha_j} S_t h_{\sigma_j} \rangle, \qquad (2.46)$$

where $\alpha_0 = \sharp\{j = 1, \ldots, \nu : \alpha_j = 0\}$ and $[Q^\eta]^{\alpha_j} = 1$ $(\alpha_j = 0)$ or $= Q^\eta$ $(\alpha_j = 1)$. Notice that $\ll \cdot, \cdot \gg$ is Q-dependent whenever $\nu > 1$. Furthermore, one has

$$(f|h)_1 + \xi(f|h)_2 = 2^{1-\nu} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\sigma} \sum_{\alpha \in \{0,1\}^\nu}^{\alpha_0 = \mathrm{even}} \prod_{j=1}^{\nu} \langle f_j, [Q^\eta]^{\alpha_j} S_t h_{\sigma_j} \rangle. \qquad (2.47)$$

Proof. Let $x_j^0 = \langle f_{\sigma_j}, S_t h_j \rangle$ and $x_j^1 = \langle f_{\sigma_j}, Q^\eta S_t h_j \rangle$ then

$$(f|h)_1 = 2^{-\nu} \prod_{j=1}^{\nu} (x_j^1 + x_j^0) = 2^{-\nu} \sum_{\alpha \in \{0,1\}^\nu} x_1^{\alpha_1} \cdots x_\nu^{\alpha_\nu}.$$

Similarly, we have that

$$\xi(f|h)_2 = 2^{-\nu} \prod_{j=i}^{\nu} (x_j^1 - x_j^0) = 2^{-\nu} \sum_{\alpha \in \{0,1\}^\nu} (-1)^{\alpha_0} x_1^{\alpha_1} \ldots x_\nu^{\alpha_\nu}.$$

So for example,

$$(f|h)_1 + \xi(f|h)_2 = 2^{1-\nu} \sum_{\alpha \in \{0,1\}^\nu}^{\alpha_0 \text{ even}} \prod_{j=1}^{\nu} x_j^{\alpha_j},$$

which corresponds to the first identity (2.46) above. Identity (2.47) follows similarly.

3. The Normally Ordered Form

We shall restrict our attention first of all to the Fock vacuum state $Q = 1$. The generalization to $Q \geq 1$ being effected using the decomposition $(1.14) - (1.16)$. Writing the interaction H_I as

$$H_I = i\hbar \sum_{\epsilon=0,1} \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} D_j^{\omega,\epsilon} \otimes F_j^{\omega,\epsilon}, \tag{3.1}$$

where we set

$$D_j^{\omega,1} = D_j^{\omega}, \quad D_j^{\omega,0} = -D_j^{\omega\dagger}, \qquad F_j^{\omega,1} = F_j^{\omega\dagger}, \quad F_j^{\omega,0} = F_j^{\omega}, \tag{3.2}$$

with $F_j^{\omega} = a_\nu(g_j^{\omega})$ as in (1.13). From the Dyson series (1.10) we have

$$
\begin{aligned}
U_t^{(\lambda)} = \sum_{n=0}^{\infty} \Big\{ \sum_{\epsilon \in \{0,1\}^n} \sum_{\omega_1,\ldots,\omega_n \in F} \sum_{j_1=1}^{N(\omega_1)} \cdots \sum_{j_n=1}^{N(\omega_n)} \Big\} D_{j_1}^{\omega_1,\epsilon_1} \ldots D_{j_n}^{\omega_n,\epsilon_n} \\
\lambda^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n v_{t_1}^{(0)}(F_{j_1}^{\omega_1,\epsilon_1}) \ldots v_{t_n}^{(0)}(F_{j_n}^{\omega_n,\epsilon_n}) \\
\times e^{\{i[(-1)^{\epsilon_1}\omega_1 t_1 + \ldots + (-1)^{\epsilon_n}\omega_n t_n]\}}.
\end{aligned}
\tag{3.3}
$$

Our objective is to study matrix elements of the type

$$\left\langle \Psi_\lambda, U_{t/\lambda^2}^{(\lambda)} \Psi_\lambda' \right\rangle \tag{3.4}$$

in the limit $\lambda \to 0$, with special choices for the vectors Ψ_λ and Ψ_λ'.

In particular, we define vectors of the following type

$$
\begin{aligned}
\Psi_\lambda(g^{(1)}, \ldots, g^{(N)}; I_1, \ldots, I_N) \\
a_\nu^\dagger\Big(\lambda \int_{I_1/\lambda^2} du_1 S_{u_1} g^{(1)}\Big) \ldots a_\nu^\dagger\Big(\lambda \int_{I_N/\lambda^2} du_N S_{u_N} g^{(N)}\Big) \Psi
\end{aligned}
\tag{3.5}
$$

for $g^{(1)}, \ldots, g^{(N)} \in \times^\nu \mathcal{H}_R^1$ and $I_1, \ldots, I_N \subset \mathbb{R}$. They are called *collective number vectors*. The limit of the operator $U_{t/\lambda^2}^{(\lambda)}$ shall be interpreted as a quantum stochastic process and the convergence is weak in the sense of matrix elements in such collective coherent states. Physically they are multiples of ν-particle states

extended over long time scales so as to pick up the desired algebraic behaviour of the interaction under the weak coupling limit.

Typically, we shall examine matrix terms of the following type

$$
\left\langle \Psi_\lambda(h^{(1)}, \ldots, h^{(N)}; I_1, \ldots, I_N), U^{(\lambda)}_{t/\lambda^2} \Psi_\lambda(f^{(1)}, \ldots, f^{(N')}; J_1, \ldots J_N) \right\rangle
$$

$$
= \sum_{n=0}^\infty \left\{ \sum_{\epsilon \in \{0,1\}^n} \sum_{\omega_1, \ldots, \omega_n \in F} \sum_{j_1=1}^{N(\omega_1)} \cdots \sum_{j_n=1}^{N(\omega_n)} \right\} D^{\omega_1,\epsilon_1}_{j_1} \ldots D^{\omega_n,\epsilon_n}_{j_n} e^{i\{(-1)^{\epsilon_1}\omega_1 t_1 + \ldots + (-1)^{\epsilon_n}\omega_n t_n\}}
$$

$$
\otimes \lambda^{N+n+N'} \int_{I_1/\lambda^2} du_1 \ldots \int_{I_N/\lambda^2} du_N \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n
$$

$$
\int_{J_1/\lambda^2} dv_1 \ldots \int_{J_{N'}/\lambda^2} dv_{N'} \left\langle \Psi, a_\nu(S_{u_1} h^{(1)}) \ldots a_\nu(S_{u_N} h^{(N)}) \right. \tag{3.6}
$$

$$
\left. \times a^{\epsilon_1}_\nu(S_{t_1} g^{\omega_1}_{j_1}) \ldots a^{\epsilon_n}_\nu(S_{t_n} g^{\omega_n}_{j_n}) \, a^\dagger_\nu(S_{v_1} f^{(1)}) \ldots a^\dagger_\nu(S_{v_{N'}} f^{(N')}) \Psi \right\rangle.
$$

Again, for clarity, we shall investigate just one frequency $\omega \in F$ and for simplicity put $\omega = 0$ and set $g^\omega_{i,j} = g^{(i)}_j$.

We now consider the problem of rearranging the expression $a^{\epsilon_1}_\nu(\gamma_1) \ldots a^{\epsilon_n}_\nu(\gamma_n)$, where $\gamma_i = (\gamma_{(i,1)}, \ldots, \gamma_{(i,\nu)})$ with $\gamma_{(i,j)} = S_{t_i} g^{(i)}_j$, into a normally ordered form. We shall rewrite the string of $\nu \cdot n$ operators $a^{\epsilon_1}(\gamma_{(1,1)}) \ldots a^{\epsilon_n}(\gamma_{(n,\nu)})$ as $a^{\mu_1}(\gamma^{(1)}) \ldots a^{\mu_{n\nu}}(\gamma^{(n\nu)})$. Here we have the relabelling $\mu_{m\nu-\nu+1}, \ldots, \mu_{m\nu} = \epsilon_m$ and $\gamma_{(i,j)} = \gamma^{(\nu i - \nu + j)}$.

Suppose $J = \{j_1, \ldots, j_k\}$ is the set of the indices $\epsilon_1, \ldots, \epsilon_n$ which equal 1. We shall assume that $j_1 < \ldots < j_k$. This means that in our string $a^{\epsilon_1}_\nu(\gamma_1) \ldots a^{\epsilon_n}_\nu(\gamma_n)$ there are a total of $k \cdot \nu$ creation operators. We take $i_1 < \quad < i_{k\nu}$ to be the indices in $\{\mu_1, \ldots, \mu_{n\nu}\}$ which equal 1 and set $I = \{i_1, \ldots, i_{k\nu}\}$. The formula for the normally ordered form of the string was derived explicitly by Accardi, Frigerio and Lu [10] and we quote it here below (with the minor modification that there are now ν times as many operators in our present string):

$$
a^{\mu_1}(\gamma^{(1)}) \ldots a^{\mu_{n\nu}}(\gamma^{(n\nu)}) = \sum_{m=0}^{\nu k \wedge (n-k)\nu} \sum_{1 \le r_1 < \ldots < r_m \le \nu k}
$$

$$
\sum_{\substack{h_1=1,\ldots,i_{r_1}-1 \\ \notin I}} \langle \gamma^{(h_1)}, \gamma^{(i_{r_1})} \rangle \eta^{[i_1-1]+[i_2-2]+\ldots+[i_{r_1-1}-(r_1-1)]} \eta^{|\{j:h_1<j<i_{r_1}\}/I|}
$$

$$
\sum_{\substack{h_2=1,\ldots,i_{r_2}-1 \\ \notin I\cup\{h_1\}}} \langle \gamma^{(h_2)}, \gamma^{(i_{r_2})} \rangle
$$

$$
\times \eta^{[i_{r_1+1}-(r_1+1)-1]+[i_{r_1+2}-(r_1+2)-1]+\ldots+[i_{r_2-1}-(r_2-1)-1]} \eta^{N_2(h_2)}
$$

$$\vdots$$

$$\notin I\cup\{h_1,\ldots,h_{m-1}\}$$
$$\sum_{h_m=1,\ldots,i_{r_m}-1} \langle \gamma^{(h_m)}, \gamma^{(i_{r_m})}\rangle \tag{3.7}$$

$$\times \quad \eta^{[i_{r_{m-1}+1}-(r_{m-1}+1)-(m-1)]+[i_{r_{m-1}+2}-(r_{m-2}+2)-(m-1)]+\ldots+[i_{r_{m-1}}-(r_{m-1})-(m-1)]}$$

$$\times \quad \eta^{N_m(h_m)}\eta^{[i_{r_m+1}-(r_m+1)-m]+[i_{r_m+2}-(r_m+2)-m]+\ldots+[i_{\nu k}-\nu k-m]}$$

$$\times \quad \prod_{\alpha\in I/\{i_{r_1},\ldots,i_{r_m}\}} a^\dagger(\gamma^{(\alpha)}) \prod_{\beta\in\{1,\ldots,n\nu\}/(I\cup\{h_1,\ldots,h_m\})} a((\gamma^{(\beta)}),$$

where

$$N_l(h_l) = |\{j : h_l < j < i_{r_l}\}/(I\cup\{h_1,\ldots,h_{l-1}\})|. \tag{3.8}$$

The goal of the weak coupling limit program is to sort out from the above expression those terms which shall have a bearing on the limiting time evolution and the other terms which become negligible as $\lambda \to 0$. In all cases considered so far, it is found that the important terms in (3.7) above are those for which the scalar products appearing involve test functions whose times occur consecutively in the sequence $t_1 \geq t_2 \geq \ldots \geq t_n$. In our case we also have to differentiate between crossing and non-crossing terms.

Suppose that we have a block of ν annihilation operators followed by a block of ν creation operators, say $a_\nu(\gamma_{j-1}) \cdot a_\nu^\dagger(\gamma_j)$ and so occurring consecutively in time. Now we have the following expansion into normal order

$$a_\nu(\gamma_{j-1}) \cdot a_\nu^\dagger(\gamma_j) = \xi a_\nu^\dagger(\gamma_j) \cdot a_\nu(\gamma_{j-1}) + \text{Partial terms}$$
$$+ \sum_{\sigma\in S_\nu} \eta^{\text{par}\sigma} \prod_{l=1}^\nu \langle\gamma_{(j-i,\sigma_l)}, \gamma_{(j,l)}\rangle, \tag{3.9}$$

where the partial terms consist of normal ordered expressions involving m creation operators followed by m annihilation operators with $m = 1, \ldots, \nu-1$. As usual, $\xi = \eta^\nu$. If we replace $a_\nu(\gamma_{j-1}) \cdot a_\nu^\dagger(\gamma_j)$ by (3.9) wherever it occurs in $a_\nu^{\epsilon_1}(\gamma_1)\ldots a_\nu^{\epsilon_n}(\gamma_n)$ and then insert into (3.6) we see that the partial terms always lead to crossings amongst the scalar products in the final expression when the vacuum expectation is taken.

The significant terms in (3.7) are those which come into normal ordered form by reorganizing time consecutive blocks of operators and ignoring the partial terms; we shall denote by $I^\epsilon(n)$ the contribution to (3.7) of exactly these terms: the remainder we shall denote by $II^\epsilon(n)$. Therefore

$$a_\nu^{\epsilon_1}(\gamma_1)\ldots a_\nu^{\epsilon_n}(\gamma_n) = I^\epsilon(n) + II^\epsilon(n) \tag{3.10}$$

with

$$
\begin{aligned}
I^\epsilon(n) \;=\; & \sum_{m=0}^{k\wedge(n-k)} \sum_{\substack{\{j_{r_h}\}_{h=1}^m \cap(\{0\}\cup J)=\emptyset \\ 1\le r_1<\ldots<r_m\le k}} \prod_{h=1}^{m}\Big\{ \sum_{\sigma\in S_\nu} \eta^{\mathrm{par}\,\sigma} \prod_{l=1}^{\nu}\langle \gamma_{(r_h-1,\sigma_l)},\,\gamma_{(r_h,l)}\rangle \Big\} \\
\times\; & \prod_{h=1}^{m} \xi^{[i_{r_h+1}-(r_h+1)-h]+[i_{r_h+2}-(r_h+2)-h]+\ldots+[i_{r_{h+1}-1}-(r_{h+1}-1)-h]}
\end{aligned}
$$

$$
\prod_{\alpha\in J/\{j_{r_h}\}_{h=1}^m} a_\nu^\dagger(g_\alpha) \qquad \prod_{\rho\in\{1,\ldots,n\}/(J\cup\{j_{r_h}-1\}_{h=1}^m)} a_\nu(g_\beta)\,. \tag{3.11}
$$

We are now considering the interaction $H_I = i\hbar(D\otimes a_\nu^\dagger(g) - h.c.)$. We shall write $D_1 := D$, $D_0 := -D^\dagger$. Our objective is to study the limit of the regularized wave operator which we define to be

$$
\widetilde{U}_t^{(\lambda)} := \sum_{n=0}^{\infty}\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \sum_{\epsilon\in\{0,1\}^n} D_{\epsilon_1}\ldots D_{\epsilon_n}\otimes I^\epsilon(n)\,, \tag{3.12}
$$

that is, we exclude the negligible terms. From our long experience $[1,3,4,8,10]$ we can now readily compute the limit of arbitrary matrix elements for collective number operators of $\widetilde{U}_{t/\lambda^2}^{(\lambda)}$. We then have the following theorem.

THEOREM 3. *For* $N,N' \in \mathbb{N}$, $g,f_1,\ldots,f_N,f_1',\ldots,f_{N'}' \in K$, $I_j,J_k \subset \mathbb{R}$ *for* $j = 1,\ldots,N$ *and* $k=1,\ldots,N'$ $t\ge 0$, $u,v\in\mathcal{H}_S,D\in B(\mathcal{H}_S)$, *the limit*

$$
\lim_{\lambda\to 0}\Big\langle u\otimes\Psi_\lambda(f_1,\ldots,f_N;I_1,\ldots,I_N),\,\widetilde{U}_t^{(\lambda)}v\otimes\Psi_\lambda(f_1',\ldots f_{N'}':J_1,\ldots,J_{N'})\Big\rangle \tag{3.13}
$$

exists and equals

$$
\sum_{n=0}^{\infty}\sum_{1\le j_1,\ldots,j_k\le n}\sum_{m=0}^{k\wedge(n-k)}\sum_{\substack{\{j_{r_h}\}_{h+1}^m\cap(\{0\},\{j_1,\ldots,j_r\}+\emptyset \\ 1\le r_1<\ldots<r_m\le k}}
$$

$$
\prod_{\alpha+0}^{m}\xi^{\sum_{\beta=r_\alpha+1}^{r_{\alpha+1}-1}(j_\beta-\beta-\alpha)}\sum_{1\le x_1,\ldots x_{k-m}\le N}^{\prime}\sum_{1\le y_1,\ldots,y_{n-k-m}\le N'}^{\prime}
$$

$$
\int_{t\ge t_1>\ldots>\hat{t}_{j_{r_1}}>\ldots>\hat{t}_{j_{r_m}}>\ldots>t_1\ge 0} dt_1\ldots\widehat{dt_{j_{r_1}}}\ldots\widehat{dt_{j_{r_m}}}\ldots dt_n
$$

$$
\{(g|g)^{-}\}^m \prod_{p=1}^{k-m}\xi^{(x_p,N)}(f_{x_p}|g)\chi_{I_{x_p}}(t_{x_p}) \prod_{q=1}^{n-k-m}\xi^{(y_q,N')}(g|f_{y_q}')\chi_{J_{y_q}}(t_{\beta_q})
$$

$$
\langle u,D_{\epsilon_1}\ldots D_{\epsilon_n}v\rangle\langle\Phi(x),\Phi'(y)\rangle\,. \tag{3.14}
$$

Here we have the convention that a hat on a variable indicates that it is to be omitted and the primes on the summation indicate that the indices summed over must be distinct and furthermore we adopt the notations

$$(g|g)^- = \int_{-\infty}^{0} dt e^{-i\omega t} \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\sigma} \prod_{k=1}^{\nu} \langle g_{\sigma_j}, S_t g_j \rangle , \tag{3.15}$$

$$(x_p, N) = \left| \{1, \ldots, x_p\}/\{x_h\}_{h=1}^{p-1} \right| - 1 \tag{3.16}$$

and

$$\Phi(x) = \prod_{\alpha \in \{1, \ldots, N\}/\{x_p\}_{p=1}^{k-m}} A^\dagger(\chi_{I_\alpha} \otimes f_\alpha)\Phi , \tag{3.17a}$$

$$\Phi'(y) = \prod_{\beta \in \{1, \ldots, N'\}/\{y_q\}_{q+1}^{n-k-m}} A^\dagger(\chi_{J_\beta} \otimes f'_\beta)\Phi , \tag{3.17b}$$

the order of multiplication in (3.17a), (3.17b) being the natural one. The limiting expression (3.14) is then identified with

$$\left\langle u \otimes \prod_{\alpha=1}^{N} A^\dagger(\chi_{I_\alpha} \otimes f_\alpha)\Phi, \ U_t v \otimes \prod_{\beta=1}^{N'} A^\dagger(\chi_{J_\beta} \otimes f'_\beta)\Phi \right\rangle , \tag{3.18}$$

where U_t is a unitary operator process on the $\Gamma_\xi(L^2(\mathbb{R}^+, K))$ which satisfies the quantum stochastic differential equation

$$dU_t = \{D \otimes dB^\dagger(g, t) \quad D^\dagger \otimes dB(g, t) - (g|g)^- D^\dagger D \otimes dt\}U_t . \tag{3.19}$$

With the supplementary condition $U_0 = 1$, the solution to (3.19) is uniquely determined. The proof of this theorem follows so closely arguments of [10] that we omit it.

4. Conclusions

The generalization to several frequencies, including degeneracies, is now easy. The essential ground work was done in [4] and now, even though the interaction is multi-linear, everything follows as before because we have the correct description of the noise space at our disposal. Taking the interaction (1.22) for H_I, we have that the necessary noise space $\mathcal{H}$ is given by

$$\mathcal{H} = \bigotimes_{\omega \in F} \left\{ \bigotimes_{j=1,2} \Gamma_\xi(L^2(\mathbb{R}^+, K_j^\omega)) \right\}, \tag{4.1}$$

where K_j^ω, for fixed ω and $j = 1, 2$, are the pair of Hilbert spaces introduced in Subsection 2.3. Now we make the ω-dependence explicit however. The inner products $(\cdot|\cdot)_j$ defined by $(2.36) - (2.37)$ will be now denoted as follows

$$(f|h)_{Q+}^\omega = \int_{-\infty}^\infty dt\, e^{-i\omega t} \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\sigma} \prod_{j=1}^\nu \left\langle f_{\sigma_j}, \frac{Q^\eta + 1}{2} S_t h_j \right\rangle, \qquad (4.2)$$

$$(f|h)_{Q-}^\omega = \int_\infty^\infty dt\, e^{-i\omega t} \sum_{\sigma \in S_{..}} \eta^{\mathrm{par}\sigma} \prod_{i=1}^\nu \eta \left\langle f_{\sigma_j}, \frac{Q^\eta - 1}{2} S_t h_j \right\rangle. \qquad (4.3)$$

We also introduce the following notations

$$(f|h)_{Q+}^{\omega-} = \int_{-\infty}^\cap dt\, e^{-i\omega t} \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\sigma} \prod_{j=1}^\nu \left\langle f_{\sigma_j}, \frac{Q^\eta + 1}{2} S_t h_j \right\rangle, \qquad (4.4)$$

$$(f|h)_{Q-}^{\omega-} = \int_{-\infty}^\cup dt\, e^{-i\omega t} \sum_{\sigma \in S_\nu} \eta^{\mathrm{par}\sigma} \prod_{j=1}^\nu \eta \left\langle f_{\sigma_j}, \frac{Q^\eta - 1}{2} S_t h_j \right\rangle. \qquad (4.5)$$

The quantum stochastic differential equation on the noise space $\mathcal{H}_S \otimes \mathcal{H}$ satisfied by the regularization of the wave operator in the weak coupling limit is then

$$dU_t = \left\{ \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} [D_j^\omega \otimes dB_Q^\omega(g_j^\omega, t)^\dagger - D_j^{\omega\dagger} \otimes dB_Q^\omega(g_j^\omega, t)] - Y \otimes dt \right\} U_t, \qquad (4.6)$$

where Y is the operator on the system space $\mathcal{H}_S$ given by

$$Y = \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \left[(g_j^\omega | g_k^\omega)_{Q+}^{\omega-} D_j^{\omega\dagger} D_k^\omega + \overline{(g_j^\omega | g_k^\omega)_{Q-}^{\omega-}} D_j^\omega D_k^{\omega\dagger} \right]. \qquad (4.7)$$

One readily notes the similarity of these equations to those of (1.36) and (1.37). The difference now is that the noise terms $B_Q^\omega(g, t)$ are those corresponding to the definitions $(2.30) - (2.33$. To be precise, we shall give below the quantum Ito table for the various integrators arising: (for $\omega, \omega' \in F$)

$$dB_Q^\omega(f, t) dB_Q^{\omega'\dagger}(h, t) = \delta_{\omega,\omega'}(f|h)_{Q+}^\omega dt \qquad dB_Q^{\omega\dagger}(f, t) dB_Q^{\omega'}(h, t) = \delta_{\omega,\omega'}(h|f)_{Q-}^\omega dt$$
$$dB_Q^\omega(f, t) dB_Q^{\omega'}(h, t) = 0 \qquad\qquad dB_Q^{\omega\dagger}(f, t) dB_Q^{\omega'\dagger}(h, t) = 0.$$
$$(4.8)$$

Now the wave operator is not a semigroup but rather is a unitary cocycle with respect to the free evolution. Setting $X_t = U_t^\dagger X U_t$, where X is a bounded observable of the system only; we are interested in computing dX_t/dt at $t = 0$:

$$dX_t = (dU_t)^\dagger X U_t + U_t^\dagger X (dU_t) + (dU_t)^\dagger X (dU_t). \qquad (4.9)$$

Now substituting in for dU_t with the quantum stochastic differential equation (4.6) and using the Ito table (4.8) we see that dX_t equals

$$U_t^\dagger \Big[- XY - Y^\dagger X + \tag{4.10}$$

$$2 \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \{ D_j^\omega X D_k^{\omega\dagger} \mathrm{Re}(g_k^\omega | g_j^\omega)_{Q-}^\omega + D_j^{\omega\dagger} X D_k^\omega \mathrm{Re}(g_j^\omega | g_k^\omega)_{Q+}^\omega \} \Big] \otimes dt\, U_t$$

$$+ \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} U_t^\dagger \Big[(X D_j^\omega - D_j^\omega X) \otimes dB_Q^{\omega\dagger}(g_j^\omega, t) + (D_j^{\omega\dagger} X - X D_j^{\omega\dagger}) \otimes dB_Q^\omega(g_j^\omega, t) \Big] U_t .$$

As we are only interested in the derivative at time $t = 0$, we conclude

$$L_0(X) := \frac{dX_t}{dt}\Big|_{t=0} = -XY - Y^\dagger X \tag{4.11}$$

$$+ 2 \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \Big\{ D_j^\omega X D_k^{\omega\dagger} \mathrm{Re}(g_k^\omega | g_j^\omega)_{Q-}^\omega + D_j^{\omega\dagger} X D_k^\omega \mathrm{Re}\,(g_j^\omega | g_k^\omega)_{Q+}^\omega \Big\},$$

this is the Langevin equation in the interaction picture. To obtain the master equation for the reduced system density matrix s_t we use the adjoint relation

$$\mathrm{tr}\, \frac{ds_t}{dt} X\Big|_{t=0} = \mathrm{Tr} s \frac{dX_t}{dt}\Big|_{t=0} = \mathrm{tr}\, s L_0(X) = \mathrm{tr}\, L_0^*(s) X , \tag{4.12}$$

however, by inspection, the relation

$$\frac{ds_t}{dt}\Big|_{t=0} = L_0^*(s) \tag{4.13}$$

corresponds exactly with master equation (1.34) of Louisell [2].

Bibliography

1. L. Accardi, A. Frigerio, and Y. G. Lu, Comm. Math. Phys. **131**, 537 (1990).
2. W. H. Louisell, *Quantum Statistical Properties of Radiation*, John Wiley and Sons, 1973.
3. Y. G. Lu, Centro Volterra Preprint No. 53 (1990).
4. L. Accardi and J. Gough, Rep. Math. Phys. **36**, 2/3 (1995).
5. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Springer-Verlag, 1979.
6. L. Accardi and V. Mastropietro, Centro Volterra Preprint No. 228 (1995).
7. L. Accardi, Y. G. Lu, and I. Volovich, Centro Volterra Preprint No. 211 (1995).
8. D. Goderis, A. Verbeure, and P. Vets P, Comm. Math. Phys. **122**, 249 (1989); Comm. Math. Phys. **128**, 533 (1990); Probab. Th. Rel. Fields **82**, 527 (1989).
9. L. Accardi, A. Frigerio, and Y. G. Lu, *The Weak Coupling Limit without Rotating Wave Approximation*, Ann. Inst. H. Poincaré, 1990.
10. R. Hudson and K. R. Parthasarathy, Comm. Math. Phys. **1**, 301 (1984).
11. D. B. Applebaum and R. L. Hudson, Comm. Math. Phys. **96**, 473 (1984).
12. L. Accardi, A. Frigerio, and Y. G. Lu, J. Math. Phys. **32**, 1567 (1991).