# PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

*This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.*

*All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to* `hbreiter@email.uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2008. Solutions identified as by students are given preference.*

## Problems for Solution.

**1184.** *Arthur L. Holshouser, Charlotte, NC and Benjamin G. Klein, Davidson College, Davidson, NC*

Suppose that $(S, \odot_1)$ is a group with identity $e_1$. For $s$, arbitrary but fixed in $S$, define a binary operator on $S$ by $a \odot_2 b = a \odot_1 s^{-1} \odot_1 b$ where $a$ and $b$ are elements of $S$ and, for $x$ in $S$, $x^{-1}$ is the inverse of $x$ in the group $(S, \odot_1)$. (a) Show that $(S, \odot_2)$ is a group and that $(S, \odot_2)$ is isomorphic to the group $(S, \odot_1)$. (b) Express $a \odot_1 b$ in terms of operations in the group $(S, \odot_2)$.

**1185.** *Proposed by Matthew McMullen, Otterbein College.*

Find uncountably many functions, $f_r(x)$, that are positive and continuous on $[1, \infty)$ and that satisfy

$$1 = f_r(1) = \int_1^\infty f_r(x)\,\mathrm{d}x.$$

(Note that $g(x) := 1/x^2$ and $h(x) := e^{-(x-1)}$ are two such functions.)

**1186.** *Proposed by H. A. ShahAli, Tehran, IRAN*

Let $a_1, ..., a_n$ be $n \geq 3$ positive reals. Prove that $1 < \sum_{i=1}^{n} \frac{a_i}{a_i + b_i} < n - 1$ for all permutations $(b_1, ..., b_n)$ of $(a_1, ..., a_n)$, and that 1 and $n-1$ are the best possible.

**1187.** *Proposed by Brian Bradie, Christopher Newport University*

Evaluate

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx.$$

**1188.** *Proposed by Javier Gomez-Calderon and David Wells, Penn State University at New Kensington*

Find all real polynomials $P$ having the property that $P(x-1)P(x) = P(x^2)$ for all $x$.

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**1189.** *Proposed by Peter A. Lindstrom, Batavia, NY*

If $F_n$ denotes the $n$th Fibonacci number, show that $F_{4n+2} - (2n+1)$ is divisible by 5.

**1190.** *Proposed by Tom Moore, Bridgewater State College, Bridgewater, MA*

Prove that every even perfect number is both a sum and a difference of two distinct deficient numbers.

**1191.** *Proposed by Fred Weber, Lyndhurst, OH*

For integers $m \geq 2$ and $d \geq 0$, let $\phi(d, m)$ be the number of ordered pairs $< a, b >$ of residues modulo $m$ for which $b = a + d$ and both $a$ and $b$ are relatively prime to $m$. Find a formula for $\phi(d, m)$.

**1192.** *Proposed by Scott D. Kominers, student, Harvard University, Cambridge, MA*

For which positive integers $n$ is it true that for all partitions $P$ of $n$ into more than $\lfloor \frac{n}{2} \rfloor$ parts, there is an undirected graph $G$ with vertex set $V(G)$ such that $\deg(V(G)) = P$?

**1193.** *Proposed by Arthur L. Holshouser, Charlotte, NC*

Suppose $f$ is a reasonably well-behaved real function $f(x)$ that satisfies for all real numbers $a, b$ the condition

$$f(a+b) = \frac{f(a) + f(b)}{1 - f(a)\, f(b)}.$$

Prove that $f(x) = \tan(mx)$ where $m$ is any fixed real number.

**1194.** *Proposed by Herman J. Servatius, Worcester Polytechnic Institute, Worcester, MA*

Given seven points $\{A, \ldots G\}$ in the plane such that $A$, $B$, $C$, and $D$ are colinear, $\overline{AB} = \overline{CD}$, $AB \perp BE$, $CD \perp DF$, $AE \perp AG$, and $CF \perp CG$.

1. Prove that if the four colinear points occur in the order $(A, B, C, D)$ then $\triangle ABE \sim \triangle CDF$.
2. For what other orderings of the four points does that conclusion hold?

**1195.** *Proposed by Mike Pinter, Belmont University, Nashville, TN*

Consider the following alphametic: $M\ C\ C\ A\ I\ N + O\ B\ A\ M\ A\ =\ D\ E\ C\ I\ D\ E$
As we cast our vote, we want to maximize our decision. Find the maximum value of DECIDE for the alphametic.

**Solutions.** We would like to make the following corrections with apologies. In the Spring 2008 issue, we failed to give credit to

**1171.** *Proposed by S.C. Locke, Florida Atlantic University, Boca Raton, FL.*

For any integer $q$, $q \geqslant 2$, let $k(q)$ denote the smallest positive integer $m$ such that there is a monic polynomial $p(n)$ with integer coefficients and which is divisible by $q$ for every integer $n$. For example, $24 \mid n^3\left(n^2 - 1\right)$ for every integer $n$ and, hence, $k(24) \leqslant 5$. Determine, with proof, the value of $k(q)$ for each integer $q$, $q \geqslant 2$.

*Solution by* **Fred Richman**, *Florida Atlantic University, Boca Raton, FL; and the* **Proposer**.

We show that $k(q)$ is the least integer $m$ such that $q \mid m!$. Suppose that $q \mid m!$ . Then, let $p(n) = \prod_{k=0}^{m-1}(n-k) = m!\binom{n}{m} = (-1)^m \binom{-n+m}{m}$. Note that for $n \geqslant 0$, $m!\binom{n}{m}$ is an integer, and for $n < 0$, $\binom{-n+m}{m}$ is an integer. Hence, $k(q) \leqslant m$.

We need to prove that no monic polynomial of smaller degree will suffice. Consider the difference operator $\Delta$, which maps $p(x)$ to $p(x+1) - p(x)$, for any polynomial $p(x)$. Note that $\deg(\Delta p) < \deg(p)$. Also, if $q \mid p(n)$, for every integer $n$, then $q \mid \Delta p(n)$ and, if $p(n) = \sum_{t=0}^{m} a_t n^t$, then $\Delta p(n) = \sum_{t=0}^{m} a_t \Delta n^t = m a_m n^{m-1} + \sum_{t=0}^{m-2} b_t n^t$, for suitable coefficients, $b_t$.

Now, suppose that $p(n)$ is a monic polynomial of degree $m$ with integer coefficients such that $q \mid p(n)$ for every integer $n$. Then, $\Delta^m p(n) = m!$ and $q \mid m!$ .

Also solved by **Paul S. Bruckman**, Sointula, BC.

**1172.** *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.*

Let $a, b, c, d \geq 1$ be natural numbers and let $S = \sum_{m=1}^{\infty} \frac{1}{(am+b)(cm+d)}$. Find the sum

$$S(a,b,c,d) = \sum_{n=1}^{\infty} (-1)^n \left( S - \sum_{m=1}^{n} \frac{1}{(am+b)(cm+d)} \right).$$

*Solution by* **Hongwei Chen**, *Christopher Newport University, Newport News, VA*

First note that

$$\begin{array}{rclllll} S & = & -\frac{1}{(2a+b)(2c+d)} & -\frac{1}{(3a+b)(3c+d)} & -\frac{1}{(4a+b)(4c+d)} & -\cdots & \\ & & & +\frac{1}{(3a+b)(3c+d)} & +\frac{1}{(4a+b)(4c+d)} & +\frac{1}{(5a+b)(5c+d)} & +\cdots \\ & & & & -\frac{1}{(4a+b)(4c+d)} & -\frac{1}{(5a+b)(5c+d)} & -\frac{1}{(6a+b)(6c+d} \quad -\cdots \\ & & & & & +\frac{1}{(5a+b)(5c+d)} & +\frac{1}{(5a+b)(5c+d)} \quad +\frac{1}{(7a+b)(7c+d)} \quad +\cdots \end{array}$$

Thus,

$$S(a,b,c,d) = -\sum_{n=1}^{\infty} \frac{1}{(2an+b)(2cn+d)}.$$

We distinct two cases. First, if $ad - bc = 0$, then there exists a nonzero rational constant $k$ such that $c = ka, d = kb$ and so

$$S(a,b,ka,kb) = -\frac{1}{k}\sum_{n=1}^{\infty} \frac{1}{(2an+b)^2} = -\frac{1}{k}\left(\frac{1}{4a^2}\,\psi'(b/2a) - \frac{1}{b^2}\right),$$

where $\psi(x)$ is the Psi function, defined by

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x).$$

In particular, we have

$$S(1,1,1,1) = 1 - \frac{\pi^2}{8},\ S(2,1,2,1) = 1 - \frac{\pi^2}{16} - \frac{1}{2}G,$$

Where $G$ is the Catalan constant.

Next, if $ad - bc \neq 0$, using partial fractions yields

$$S(a,b,c,d) = -\frac{1}{2(ad-bc)} \sum_{n=1}^{\infty} \left( \frac{2a}{2an+b} - \frac{2c}{2cn+d} \right).$$

Thus,

$$S(a,b,c,d) = -\frac{1}{2(ad-bc)} \; (\psi(c/2d) - \psi(b/2a)) + \frac{1}{bd}.$$

For some special values of $a, b, c, d$, we are able to find close form expressions in terms of well-known constants. For example, since

$$\psi(x) = \psi(1-t) - \pi \, \cot(\pi x),$$

we have

$$S(a,b,a,2a-b) = -\frac{\pi}{4a(a-b)} \; \cot(\pi b/2a) + \frac{1}{b(2a-b)}.$$

In particular,

$$S(2,1,2,3) = \frac{1}{3} - \frac{\pi}{8}, \;\; S(3,2,3,4) = \frac{1}{8} - \frac{\sqrt{3}\,\pi}{36}.$$

Also solved by **Brian Bradie**, Christopher Newport University, Newport News, VA; **Paul S. Bruckman**, Sointula, BC; and the **Proposer**.

**1173.** *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington PA*

$ABC$ is a triangle. A perpendicular $BD$ to $BA$ is constructed such that $BD = r \cdot BA$. Similarly, a perpendicular $CE$ to $CA$ is constructed such that $CE = r \cdot AC$. Find the locus of midpoints of $DE$ for all values of the parameter $r$.

*Solution by*

Also solved by **Mark Evans**, Louisville, KY; **Robert Gebhardt**, Hopatcong, NJ; **Sum People**, Mountain Lakes High School, Mountain Lakes, NJ; **Ricardo Barroso Campos**, Universidad de Sevilla, Spain; and the **Proposer**.

**1174.** *Proposed by Tom Moore, Bridgewater State College, Bridgewater, MA*

The integer 99 has the property that $9 \cdot 9 + (9+9) = 99$. Find all the positive integers N (base 10) with the property that N equals the sum of the product of its digits and the sum of its digits.

*Solution by* **Frank Battles***, Massachusetts Maritime Academy, Buzzards Bay, MA.*

With $a_i$ representing the $i^{\text{th}}$ digit of $N, (a_n \neq 0)$, we have

$$10^n a_n + 10^{n-1} a_{n-1} + \cdots 10 a_1 + a_0 = a_n \cdot a_{n-1} \cdots a_1 \cdot a_0 + a_n + a_{n-1} + \cdots + a_1 + a_0,$$

which we may write as

$$(10^n - 1) a_n + (10^{n-1} - 1) a_{n-1} + \cdots + 9 a_1 = a_n \cdot a_{n-1} \cdots a_1 \cdot a_0.$$

For $n = 1$, we have $9a_1 = a_1 \cdot a_0$ with solution $a_0 = 9$ and $a_1$ arbitrary. For solutions we have $19, 29, 39, 49, 59, 69, 79, 89$, and $99$.

For $n > 1$, we have

$$(10^n - 1)a_n + (10^{n-1} - 1)a_{n-1} + \cdots + 9a_1 \le a_n \cdot 9^n$$

or

$$(10^n - 9^n - 1)a_n + (10^{n-1} - 1)a_{n-1} + \cdots + 9a_1 \le 0.$$

Since all terms after the first are non-negative, we must have

$$10^n - 9^n - 1 < 0$$

but this is not true for $n > 1$. Thus there are no more solutions.

Also solved by **Paul S. Bruckman**, Sointula, BC; **Cal Poly Pomona Problem Solving Group**, Pomona, CA; **Thomas Dence**, Ashland University, Ashland, OH; **Charles R. Diminnie**, Angelo State University, San Angelo, TX; **Mark Evans**, Louisville, KY; **Robert Gebhardt**, Hopatcong, NJ; **Arielle Leitner, student**, California State University, Chico; **James A. Sellers**, Pennsylvania State University, University Park, PA; and the **Proposer**.

**1175.** *Proposed by Zokhrab Mustafaev, Victor Dontsov, Evgeni Maevski, University of Houston-Clear Lake, Houston, TX.*

It is known that numbers 14529, 15197, 20541, 38911, 59619 are multiples of 167. Without actually calculating, prove that the determinant of the $5 \times 5$ matrix $A$ is also a multiple of 167, where $A = \begin{pmatrix} 1 & 4 & 5 & 2 & 9 \\ 1 & 5 & 1 & 9 & 7 \\ 2 & 0 & 5 & 4 & 1 \\ 3 & 8 & 9 & 1 & 1 \\ 5 & 9 & 6 & 1 & 9 \end{pmatrix}$

*Solution by* **Rebecca von Funk, student**, *Elizabethtown College, Elizabethtown, PA.*

Let

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 4 \\ 5 \\ 0 \\ 8 \\ 9 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 5 \\ 1 \\ 5 \\ 9 \\ 6 \end{bmatrix}, \quad \boldsymbol{a}_4 = \begin{bmatrix} 2 \\ 9 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{a}_5 = \begin{bmatrix} 9 \\ 7 \\ 1 \\ 1 \\ 9 \end{bmatrix}$$

be the column vectors of $A$ and let $B = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & (10^4\boldsymbol{a}_1 + 10^3\boldsymbol{a}_2 + 10^2\boldsymbol{a}_3 + 10\boldsymbol{a}_4 + \boldsymbol{a}_5) \end{bmatrix}$. Clearly, $\det A = \det B$. Then

$$B = \begin{pmatrix} 1 & 4 & 5 & 2 & 14529 \\ 1 & 5 & 1 & 9 & 15197 \\ 2 & 0 & 5 & 4 & 20541 \\ 3 & 8 & 9 & 1 & 38911 \\ 5 & 9 & 6 & 1 & 59619 \end{pmatrix} = CD, \quad \text{where } C = \begin{pmatrix} 1 & 4 & 5 & 2 & 87 \\ 1 & 5 & 1 & 9 & 91 \\ 2 & 0 & 5 & 4 & 123 \\ 3 & 8 & 9 & 1 & 233 \\ 5 & 9 & 6 & 1 & 357 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 167 \end{pmatrix},$$

and $\det A = (\det C)(\det D) = (\det C)(167)$. Since $\det C$ is an integer, $\det A$ is an integer multiple of 167.

*Solution by* **Paul M. Kominers and Scott D. Kominers, students**, *Massachusetts Institute of Technology and Harvard University, Cambridge MA.*

We prove the following generalization of the problem:

PROPOSITION: 1. *Let* $\{a_i\}_{i=1}^{n^2}$ *and* $\{b_i\}_{i=1}^{n}$ *be nonnegative integers such that*

$$p \mid \sum_{i=1}^{n} a_{i+kn} \cdot b_i$$

*for some fixed positive integer* $p$ *and for each* $0 \leq k < n$. *Then,*

$$p \mid b_n \cdot \det \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ a_{1+n^2-n} & \cdots & a_{n^2} \end{pmatrix}.$$

*Proof.* It is well-known that $\det M$ is invariant under the operation of adding a multiple of a column of a matrix $M$ to another column of $M$. Likewise, it is well-known that if $M'$ is obtained from $M$ by multiplying a column of $M$ by a scalar $m$ then $\det M' = m \cdot \det M$. Therefore, we have

$$b_n \det \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ a_{1+n^2-n} & \cdots & a_{n^2} \end{pmatrix} = \det \begin{pmatrix} a_1 & \cdots & \sum_{i=1}^{n} a_i \cdot b_i \\ \vdots & & \vdots \\ a_{1+n^2-n} & \cdots & \sum_{i=1}^{n} a_{i+n^2-n} \cdot b_i \end{pmatrix}.$$

Now, from Laplace's formula for the determinant, we see that

$$\det \begin{pmatrix} a_1 & \cdots & \sum_{i=1}^{n} a_i \cdot b_i \\ \vdots & & \vdots \\ a_{1+n^2-n} & \cdots & \sum_{i=1}^{n} a_{i+n^2-n} \cdot b_i \end{pmatrix} = \sum_{k=0}^{n-1} \left( \sum_{i=1}^{n} a_{i+kn} \cdot b_i \right) \cdot C_{k+1,n},$$

where $C_{k+1,n}$ is the $(k+1, n)$ cofactor of the matrix

$$\begin{pmatrix} a_1 & \cdots & \sum_{i=1}^{n} a_i \cdot b_i \\ \vdots & & \vdots \\ a_{1+n^2-n} & \cdots & \sum_{i=1}^{n} a_{i+n^2-n} \cdot b_i \end{pmatrix}.$$

It then immediately follows that

$$p \mid \det \begin{pmatrix} a_1 & \cdots & \sum_{i=1}^{n} a_i \cdot b_i \\ \vdots & & \vdots \\ a_{1+n^2-n} & \cdots & \sum_{i=1}^{n} a_{i+n^2-n} \cdot b_i \end{pmatrix},$$

from which the Proposition follows. □

The problem follows from the Proposition upon taking $b_i = 10^{n-i}$ and

$$\{a_i\}_{i=1}^{n^2} = \{1, 4, 5, 2, 9, 1, 5, 1, 9, 7, 2, 0, 5, 4, 1, 3, 8, 9, 1, 1, 5, 9, 6, 1, 9\}$$

with $n = 5$ and $p = 167$.

Also solved by **Paul S. Bruckman**, Sointula, BC; **Jos Hernndez Santiago, student**, UTM, Oaxaca, Mxico; and the **Proposer**.

**1176.** *Proposed by Jim Jamison, the University of Memphis, Memphis, TN.*

Let $\{a_1, a_2, a_3, \dots\}$ be any sequence of real (or complex) numbers. Define

$$\rho_n := \frac{a_1 + \cdots + a_n}{a_{n+1} + \cdots + a_{2n}}.$$

Observe that if we consider the sequence of odd integers $\{1, 3, 5, \dots\}$ then $\rho_1 = \rho_2 = \rho_3 = \cdots = \frac{1}{3}$. Define $\rho$ to be the constant ratio, i.e. if $\rho_1 = \rho_2 = \rho_3 = \cdots =$ constant, then $\rho_1 = \rho_2 = \cdots = \rho$. Hence we ask for each nonzero $\rho$, does there exist a sequence with the property

$$\rho_1 = \rho_2 = \rho_3 = \cdots = \rho? \tag{1}$$

*Solution by* **Scott D. Kominers and Paul M. Kominers, students** *Harvard University and Massachusetts Institute of Technology, Cambridge MA.*

We show that for any nonzero, real $\rho$ there is a sequence of real numbers $\{a_k\}_{k=1}^{\infty}$ satisfying the condition

$$\rho = \frac{a_1 + \cdots + a_n}{a_{n+1} + \cdots + a_{2n}} \tag{2}$$

for any $n > 0$.

Fix $\mathbb{R} \ni \rho \neq 0$ and let

$$\begin{aligned} a_1 &= \rho, \\ a_2 &= 1. \end{aligned}$$

We will now construct the sequence $\{a_k\}_{k=1}^{\infty}$ inductively. Suppose that we have a sequence $\{a_k\}_{k=1}^{2\hat{k}}$ satisfying (2) for any $0 < n \leq \hat{k}$. Now, take

$$\begin{aligned} a_{2\hat{k}+1} &= 0, \\ a_{2\hat{k}+2} &= \frac{(a_1 + \cdots + a_{\hat{k}+1}) - \rho(a_{\hat{k}+2} + \cdots + a_{2\hat{k}+1})}{\rho}. \end{aligned}$$

Since $\mathbb{R} \ni \rho \neq 0$ and $\mathbb{R}$ is a field (and is therefore closed under addition, multiplication, subtraction, and division by nonzero elements), we see that $a_{2\hat{k}+2} \in \mathbb{R}$. Furthermore, by construction, we have

$$\frac{a_1 + \cdots + a_{\hat{k}+1}}{a_{\hat{k}+2} + \cdots + a_{2\hat{k}+2}} = \frac{a_1 + \cdots + a_{\hat{k}+1}}{a_{\hat{k}+2} + \cdots + a_{2\hat{k}+1} + \frac{(a_1 + \cdots + a_{\hat{k}+1}) - \rho(a_{\hat{k}+2} + \cdots + a_{2\hat{k}+1})}{\rho}} = \rho.$$

Thus, we have obtained a sequence $\{a_k\}_{k=1}^{2(\hat{k}+1)}$ with the desired properties; this completes our induction.

**Remark.** Our construction uses nothing special about $\mathbb{R}$ other than the fact that $\mathbb{R}$ is a field. Thus, the result holds for any field $K$: for any nonzero $\rho \in K$ there is a sequence $\{a_k\}_{k=1}^{\infty}$ ($a_k \in K \forall k$) satisfying the condition (2) for any $n > 0$.

Also solved by **Paul S. Bruckman**, Sointula, BC; **S.C. Locke**, Florida Atlantic University, Boca Raton, FL; and the **Proposer**.

**1177.** *Proposed by Arthur Holshouser, Charlotte, NC.*

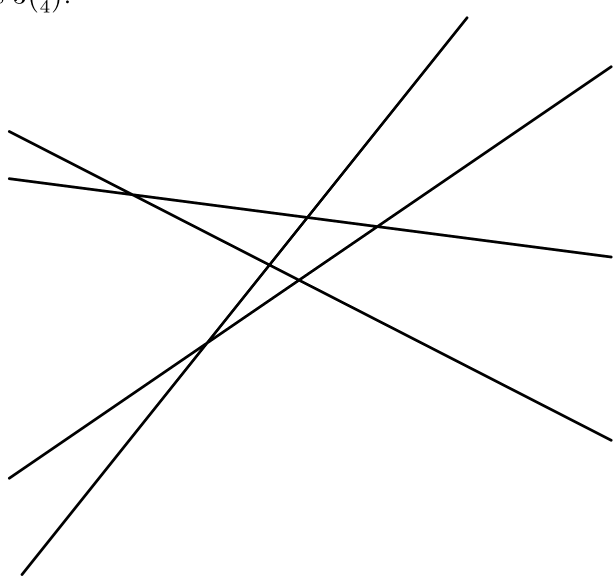
Suppose $n \geq 4$ lines in the plane intersect each other in $\binom{n}{2} = \frac{n(n-1)}{2}$ distinct points. A *quadrilateral set* is a set $S$ having the following properties.

1. $S$ has 4 points as members, and
2. These 4 points can be labeled $\{A, B, C, D\}$ in such a way that $A, B$ are colinear, $B, C$ are colinear, $C, D$ are colinear, $D, A$ are colinear, $A, C$ are not colinear and $B, D$ are not colinear.

How many quadrilateral sets are there?

*Solution by* **Mark Evans**, *Louisville, KY.*

Since the lines intersect in $\binom{n}{2}$ distinct points, only two lines intersect at any point and no two lines are parallel. Consider for the moment the case $n = 4$ as shown below. We have six points of intersection, four of which form the quadrilateral and two that are 'excluded'. By property 2 above, no three points on the quadrilateral set can be on the same line. To achieve this, the two excluded points are not colinear. This means that picking one point to be excluded determines the second point to be excluded. Since there are three possible pairs of excluded points, the number of quadrilaterals in the case $n = 4$ is three. To generalize, since exactly four lines determines three quadrilaterals, we can select $\binom{n}{4}$ unique sets of four lines, so the total number of quadrilaterals is $3\binom{n}{4}$.



Also solved by the **Proposer**.

**1178.** *Proposed by Paolo Perfetti, Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy*

Let $[a]$ the integer part of $a$ and $\{a\} = a - [a]$. Evaluate

$$\int_0^1 \int_0^1 \frac{\left\{\frac{x}{y}\right\}}{\left[\frac{x}{y}\right] + 1} dxdy - \int_{x=0}^1 \int_{y=0}^x \ln\left[\frac{x}{y}\right] dydx.$$

*Solution by* **Brian Bradie**, *Christopher Newport University, Newport News, VA.*
We start with the second integral. Let $n$ be a whole number. If $\frac{x}{n+1} < y \leq \frac{x}{n}$,

then $[x/y] = n$ and

$$\begin{aligned}
\int_{x=0}^{1}\int_{y=0}^{x} \ln\left[\frac{x}{y}\right] dy\,dx &= \sum_{n=1}^{\infty}\int_{x=0}^{1}\int_{y=x/(n+1)}^{x/n} \ln\left[\frac{x}{y}\right] dy\,dx \\
&= \sum_{n=1}^{\infty} \ln n \int_{0}^{1}\int_{y=x/(n+1)}^{x/n} dy\,dx \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)\ln n. \qquad (1)
\end{aligned}$$

For the first integral, we write

$$\int_0^1\int_0^1 \frac{\left\{\frac{x}{y}\right\}}{\left[\frac{x}{y}\right]+1}\,dx\,dy = \int_0^1\int_{y=x}^{y=1} \frac{\left\{\frac{x}{y}\right\}}{\left[\frac{x}{y}\right]+1}\,dy\,dx + \int_0^1\int_{y=0}^{y=x} \frac{\left\{\frac{x}{y}\right\}}{\left[\frac{x}{y}\right]+1}\,dy\,dx.$$

When $y > x$, $\{x/y\} = x/y$, $[x/y] = 0$ and

$$\begin{aligned}
\int_0^1\int_{y=x}^{y=1} \frac{\left\{\frac{x}{y}\right\}}{\left[\frac{x}{y}\right]+1}\,dy\,dx &= \int_0^1\int_x^1 \frac{x}{y}\,dy\,dx \\
&= -\int_0^1 x\ln x\,dx \\
&= \frac{1}{4}. \qquad (2)
\end{aligned}$$

When $y < x$, we have

$$\begin{aligned}
\int_0^1\int_{y=0}^{y=x} \frac{\left\{\frac{x}{y}\right\}}{\left[\frac{x}{y}\right]+1}\,dy\,dx &= \sum_{n=1}^{\infty}\int_0^1\int_{y=x/(n+1)}^{x/n} \frac{\frac{x}{y}-n}{n+1}\,dy\,dx \\
&= \sum_{n=1}^{\infty}\frac{1}{n+1}\int_0^1\int_{y=x/(n+1)}^{x/n}\frac{x}{y}\,dy\,dx - \sum_{n=1}^{\infty}\frac{n}{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)\frac{1}{2} \\
&= \sum_{n=1}^{\infty}\frac{1}{n+1}\ln\frac{n+1}{n}\int_0^1 x\,dx - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(n+1)^2} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n+1}\ln\frac{n+1}{n} - \frac{1}{2}\left(\frac{\pi^2}{6}-1\right) \qquad (3)
\end{aligned}$$

Combining (1), (2) and (3), we find

$$\begin{aligned}
&\int_0^1\int_0^1 \frac{\left\{\frac{x}{y}\right\}}{\left[\frac{x}{y}\right]+1}\,dx\,dy - \int_{x=0}^{1}\int_{y=0}^{x}\ln\left[\frac{x}{y}\right] dy\,dx \\
&\qquad = \frac{3}{4} - \frac{\pi^2}{12} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n+1}\ln\frac{n+1}{n} - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\ln n \\
&\qquad = \frac{3}{4} - \frac{\pi^2}{12} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\ln(n+1) - \frac{1}{n}\ln n\right).
\end{aligned}$$

The remaining summation is a telescoping series whose sum is easily found to be zero. Thus

$$\int_0^1 \int_0^1 \frac{\left\{\frac{x}{y}\right\}}{\left[\frac{x}{y}\right]+1}\,dx\,dy - \int_{x=0}^1 \int_{y=0}^x \ln\left[\frac{x}{y}\right]\,dy\,dx = \frac{3}{4} - \frac{\pi^2}{12}.$$

Also solved by **Paul S. Bruckman**, Sointula, BC; and the **Proposer**.

**1179.** *Proposed by Cecil Rousseau, University of Memphis*

In a MONTHLY problem posed by Freeman J. Dyson[1] , the reader was asked to prove that

$$\sum_{m=1}^{N} \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+1)^2},$$

given $N$ numbers $a_m$ satisfying the $N$ equations

$$\sum_{m=1}^{N} \frac{a_m}{m+n} = \frac{4}{2n+1}, \qquad n = 1, 2, \ldots, N.$$

Now the reader is asked to determine $a_1, a_2, \ldots, a_N$ given the same system.

*Solution by* **Hongwei Chen**, *Christopher Newport University, Newport News, VA.*

Taking $N = 1, 2, 3$ in

$$\sum_{m=1}^{N} \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+1)^2}$$

respectively yields

$$\begin{aligned} \frac{a_1}{3} &= 1 - \frac{1}{3^2} \\ \frac{a_1}{3} + \frac{a_2}{5} &= 1 - \frac{1}{5^2} \\ \frac{a_1}{3} + \frac{a_2}{5} + \frac{a_3}{7} &= 1 - \frac{1}{7^2}. \end{aligned}$$

Solving this system gives

$$a_1 = 3\left(1 - \frac{1}{3^2}\right),\ a_2 = 5\left(\frac{1}{3^2} - \frac{1}{5^2}\right),\ a_3 = 7\left(\frac{1}{5^2} - \frac{1}{7^2}\right).$$

Based on those facts, in general, we guess that

$$a_n = (2n+1)\left(\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}\right),\ \ n = 1, 2, \ldots, N. \tag{1}$$

[1] F. J. Dyson, Problem 4389, *Amer. Math. Monthly* **57** (1950), pp. 188–189. Solutions by E. Trost and G. Szegö, *Amer. Math. Monthly* **58** (1951), pp. 640–641.

Now we confirm (1) by using the mathematical induction. We see that (1) is valid for $N = 1$ already. Assume that (1) is valid for all $1 \le n \le N$. Since

$$\sum_{m=1}^{N+1} \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+3)^2},$$

using the induction hypothesis and telescoping sums, we have

$$\begin{aligned}
\frac{a_{N+1}}{2N+1} &= 1 - \frac{1}{(2N+3)^2} - \sum_{m=1}^{N} \frac{a_m}{2m+1} \\
&= 1 - \frac{1}{(2N+3)^2} - \sum_{m=1}^{N} \left( \frac{1}{(2m-1)^2} - \frac{1}{(2m+1)^2} \right) \\
&= 1 - \frac{1}{(2N+3)^2} - \left( 1 - \frac{1}{(2N+1)^2} \right) \\
&= \left( \frac{1}{(2N+1)^2} - \frac{1}{(2N+3)^2} \right).
\end{aligned}$$

This proves (1) for $n = N + 1$. Thus, by the principle of the mathematical induction, we have found that for $1 \le n \le N$, $a_n$ is given by (1).

Also solved by and the **Proposer**.

**1180.** *Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain*

Let $x$ be a positive real number. Prove that

$$\frac{[x]}{3x + \{x\}} + \frac{\{x\}}{3x + [x]} > \frac{4}{15},$$

where $[x]$ and $\{x\}$ represents the integer and fractional parts of $x$ respectively.

*Solution by* **Victoria Gan, student**, *Perry Hall High School, Baltimore, MD.*

Let $x$ be a positive real number and write $x = [x] + \{x\}$. Then the inequality is equivalent to the inequality

$$(1) \quad \frac{[x]}{3x + (x - [x])} + \frac{x - [x]}{3x + [x]} > \frac{4}{15},$$

or

$$(2) \quad \frac{4x^2 - 2x[x] + 2[x]^2}{12x^2 + x[x] - [x]^2} > \frac{4}{15}.$$

Since $12x^2 + x[x] - [x]^2 \ge 11x^2 + x[x] > 0$ for all positive real numbers $x$, (2) is equivalent to

$$6x^2 - 17x[x] + 17[x]^2 > 0.$$

By completing the square, it is clear that

$$6x^2 - 17x[x] + 17[x]^2 = 6\{(x - \tfrac{17}{12}[x])^2 + \tfrac{17}{6}(1 - \tfrac{17}{24})[x]^2\} = 6(x - \tfrac{17}{12}[x])^2 + \tfrac{119}{24}[x]^2 > 0$$

for all positive $x$. Thus, the inequality is true if $x > 0$.

Also solved by **Brian Bradie**, Christopher Newport University, Newport News, VA; **Paul S. Bruckman**, Sointula, BC; **Cal Poly Pomona Problem Solving Group**, Pomona, CA; **Ricardo Barroso Campos**, Universidad de Sevilla, Spain; **Kenneth B Davenport**, Dallas, PA; **Thomas Dence**, Ashland University, Ashland, OH; **Mark Evans**, Louisville, KY; **FAU Problem Solving Group** Florida Atlantic University, Boca Raton, FL; **Miguel Lerma**, Northwestern University Problem Solving Group, Evanston, IL; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Paolo Perfetti**, Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy; **James A. Sellers**, Pennsylvania State University, University Park, PA; **Sum People**, Mountain Lakes High School, Mountain Lakes, NJ; and the **Proposer**.

**1181.** *Proposed by Brian Bradie, Christopher Newport University, Newport News, VA.*

In *The Problem Department* of the Fall 2007 issue of this journal, readers were challenged to find a closed form expression for the trigonometric sum

$$\cos^{2n} 1^\circ + \cos^{2n} 2^\circ + \cdots + \cos^{2n} 89^\circ,$$

where $n$ is a positive integer. Here, the challenge is to find a closed form expression for the trigonometric sum

$$\cos^{2n+1} 1^\circ + \cos^{2n+1} 2^\circ + \cdots + \cos^{2n+1} 89^\circ,$$

for $n \geq 0$.

*Solution by* **Paolo Perfetti**, *Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy*

Answer: $\displaystyle \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \Big( (-1)^{n-k} \cot \frac{2n+1-2k}{2} - 1 \Big)$

*Proof* We have (the angles in the trigonometric functions are expressed by degrees)

$$\sum_{p=1}^{89} \cos^{2n+1} p = \sum_{p=1}^{89} \frac{(e^{ip} + e^{-ip})^{2n+1}}{2^{2n+1}} = \sum_{p=1}^{89} \frac{1}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} e^{ip(2n+1-2k)}$$

$$\sum_{p=1}^{89} e^{ip(2n+1-2k)} = \frac{1 - e^{i90(2n+1-2k)}}{1 - e^{i(2n+1-2k)}} - 1 = \frac{e^{i(2n+1-2k)} - i(-1)^{n-k}}{1 - e^{i(2n+1-2k)}}$$

Thus

$$\begin{aligned}
&\sum_{p=1}^{89} \cos^{2n+1} p \\
&= \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \Big( \frac{e^{i(2n+1-2k)} - i(-1)^{n-k}}{1 - e^{i(2n+1-2k)}} - \frac{e^{-i(2n+1-2k)} + i(-1)^{n-k}}{1 - e^{-i(2n+1-2k)}} \Big) \\
&= \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{2\cos(2n+1-2k) - 2 + 2(-1)^{n-k} \sin(2n+1-2k)}{4 \sin^2 \frac{2n+1-2k}{2}} \\
&= \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} \Big( (-1)^{n-k} \cot \frac{2n+1-2k}{2} - 1 \Big)
\end{aligned}$$

Also solved by **Paul S. Bruckman**, Sointula, BC; and the **Proposer**.

**1182.** *Proposed by Marcin Kuczma, University of Warsaw, Warsaw, Poland*

Let $a, b, c, d, e$ be decimal digits satisfying $\underline{abc} \cdot a = \underline{bda}$ and $\underline{bda} \cdot a = \underline{cdde}$. What is $\underline{cdde} \cdot a$? Editor's note: this puzzle was sent to friends of the poser in December of a certain year as a gift. This is the eighth of several such problems we plan for this column.

*Solution by* **Robert Gebhardt***, Hopatcong, NJ.*

"I have the solution. It is 2008."

"Really, Holmes, you must have guessed it. It has been but ten minutes since you were presented the problem."

"Eight minutes, Watson, and surely you remember that I almost never guess. Sometimes I do make suppositions which I am certain will prove to be correct."

"But here there is so little information to go on. There are five letters whose values are to be found, but only three equations relating them."

"Two equations, Watson, and a formula for calculating the answer. But three suppositions proved helpful here."

"I would very much like to know what they are."

"First, I assumed that the values of the five letters would be all different. In little problems like this one, the likelihood of that being true is very high. Second, I assumed that there would be only one answer to the problem, which is correct in many such problems."

"And the third, Holmes?"

"It is know that it was used as a numerical greeting in a December, so that it is probably the number of the year that was about to begin. Because the problem is likely to have been created rather recently, I expected the answer to be between 1990 and 2008, inclusive."

"But then how did you solve it?"

"It is quite apparent that $c$ would have to be either 1 or 2. If $c$ is 2, then $a$ would necessarily be 1 and $d$ would have to be 0. Then the first equation becomes $\underline{1b2} \cdot 1 = \underline{b01}$ , or $100+10b+2 = 100b+1$, which cannot be solved with a single integer for $b$. Thus $c$ cannot be 2, so it must be 1.

"You make it appear to be so simple."

" It is simple. If $c$ is 1, then $a$ must be 2. Again $d$ must be 0. Then the first equation is $\underline{2b1} \cdot 2 = \underline{b02}$ or $400 + 20b + 2 = 100b + 2$, so $b$ is 5. Then the second equation is $\underline{502} \cdot 2 = \underline{100e}$, or $1004 = 1000 + e$, so $e$ is 4. Now the question is answered: $\underline{1004} \cdot 2 = 2008.''$

"It is impressive, Holmes. All that in only eight minutes."

"Five, actually. The other three minutes I spent confirming that there could be no other answer."

"If I may say so, Holmes, you have lost none of your skills even now, after so many years."

"I thank you for the compliment, Watson. An occasional problem like this helps me keep my mind active, now that I am so very, very old."

Also solved by **Frank Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA; **Brian Bradie**, Christopher Newport University, Newport News, VA; **Paul S. Bruckman**, Sointula, BC; **Cal Poly Pomona Problem Solving Group**, Pomona, CA; **Mark Evans**, Louisville, KY; and the **Proposer**.

**1183.** *Proposed by Mohammad K. Azarian, University of Evansville*

Evaluate the indefinite integral

$$\int \frac{\sqrt{1-x^2}-x}{x^3-x^2-x+1-\sqrt{1-x^2}+x\sqrt{1-x^2}}dx.$$

*Solution by* **Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie**, *Angelo State University, San Angelo, TX.*

By observation, the numerator can be rewritten as follows

$$\sqrt{1-x^2}-x = (\sqrt{1-x^2}-1)+(1-x),$$

and the denominator can be rewritten as

$$\begin{aligned} x^3-x^2-x+1-\sqrt{1-x^2}+x\sqrt{1-x^2} &= x^2(x-1)-(x-1)+(x-1)\sqrt{1-x^2} \\ &= (x-1)\sqrt{1-x^2}(1-\sqrt{1-x^2}). \end{aligned}$$

Thus, using a trigonometric substitution,

$$\begin{aligned} &\int \frac{\sqrt{1-x^2}-x}{x^3-x^2-x+1-\sqrt{1-x^2}+x\sqrt{1-x^2}}dx \\ &= \int \frac{(\sqrt{1-x^2}-1)+(1-x)}{(x-1)\sqrt{1-x^2}(1-\sqrt{1-x^2})}dx \\ &= -\int \frac{1}{(x-1)\sqrt{1-x^2}}dx - \int \frac{1}{\sqrt{1-x^2}(1-\sqrt{1-x^2})}dx \\ &= -\int \frac{1}{(x-1)\sqrt{1-x^2}}dx - \int \frac{1+\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}dx \\ &= -\int \frac{1}{(x-1)\sqrt{1-x^2}}dx - \int \frac{1}{x^2\sqrt{1-x^2}}dx - \int \frac{1}{x^2}dx \\ &= \frac{\sqrt{1-x^2}}{1-x} + \frac{\sqrt{1-x^2}}{x} + \frac{1}{x} + C \\ &= \frac{\sqrt{1-x^2}-x+1}{x(1-x)} + C. \end{aligned}$$

Also solved by **Frank Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA; **Brian Bradie**, Christopher Newport University, Newport News, VA; **Paul S. Bruckman**, Sointula, BC; **Kenny Davenport**, Dallas, PA; **FAU Problem Solving Group** Florida Atlantic University, Boca Raton, FL; **Robert Gebhardt**, Hopatcong, NJ; **Scott D. Kominers, student** Harvard University, Cambridge MA; **Yoshinobu Murayoshi**, Naha City, Okinawa, Japan; **Paolo Perfetti**, Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy; **James A. Sellers**, Pennsylvania State University, University Park, PA; and the **Proposer**.