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Periodic Solutions for Completely Resonant Nonlinear Wave Equations with Dirichlet Boundary Conditions

Guido Gentile[1], **Vieri Mastropietro**[2], **Michela Procesi**[3]

[1] Dipartimento di Matematica, Università di Roma Tre, 00146 Roma, Italy
[2] Dipartimento di Matematica, Università di Roma "Tor Vergata", 00133 Roma, Italy
[3] SISSA, 34014 Trieste, Italy

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Abstract: We consider the nonlinear string equation with Dirichlet boundary conditions $u_{tt} - u_{xx} = \varphi(u)$, with $\varphi(u) = \Phi u^3 + O(u^5)$ odd and analytic, $\Phi \neq 0$, and we construct small amplitude periodic solutions with frequency ω for a large Lebesgue measure set of ω close to 1. This extends previous results where only a zero-measure set of frequencies could be treated (the ones for which no small divisors appear). The proof is based on combining the Lyapunov-Schmidt decomposition, which leads to two separate sets of equations dealing with the resonant and non-resonant Fourier components, respectively the Q and the P equations, with resummation techniques of divergent powers series, allowing us to control the small divisors problem. The main difficulty with respect to the nonlinear wave equations $u_{tt} - u_{xx} + Mu = \varphi(u)$, $M \neq 0$, is that not only the P equation but also the Q equation is infinite-dimensional.

1. Introduction

We consider the *nonlinear wave equation* in $d = 1$ given by

$$\begin{cases} u_{tt} - u_{xx} = \varphi(u), \\ u(0, t) = u(\pi, t) = 0, \end{cases} \tag{1.1}$$

where Dirichlet boundary conditions allow us to use as a basis in $L^2([0, \pi])$ the set of functions $\{\sin mx, m \in \mathbb{N}\}$, and $\varphi(u)$ is any odd analytic function $\varphi(u) = \Phi u^3 + O(u^5)$ with $\Phi \neq 0$. We shall consider the problem of existence of periodic solutions for (1.1), which represents a completely resonant case for the nonlinear wave equation as in the absence of nonlinearities all the frequencies are resonant.

In the finite dimensional case the problem has its analogue in the study of periodic orbits close to elliptic equilibrium points: results of existence have been obtained in such a case by Lyapunov [31] in the non-resonant case, by Birkhoff and Lewis [6] in the case of resonances of order greater than four, and by Weinstein [37] in the case of any kind of resonances. Systems with infinitely many degrees of freedom (as the nonlinear wave equation, the nonlinear Schrödinger equation and other PDE systems) have been studied much more recently; the problem is much more difficult because of the presence of a small divisors problem, which is absent in the finite dimensional case. For the nonlinear wave equations $u_{tt} - u_{xx} + Mu = \varphi(u)$, with mass M *strictly* positive, existence of periodic solutions has been proved by Craig and Wayne [14], by Pöschel [33] (by adapting the analogous result found by Kuksin and Pöschel [29] for the nonlinear Schrödinger equation) and by Bourgain [8] (see also the review [13]). In order to solve the small divisors problem one has to require that the amplitude and frequency of the solution must belong to a Cantor set, and the main difficulty is to prove that such a set can be chosen with *non-zero* Lebesgue measure. We recall that for such systems also quasi-periodic solutions have been proved to exist in [29, 33, 9] (in many other papers the case in which the coefficient M of the linear term is replaced by a function depending on parameters is considered; see for instance [36, 7] and the reviews [27, 28]).

In all the quoted papers only non-resonant cases are considered. Some cases with some low-order resonances between the frequencies have been studied by Craig and Wayne [15]. The completely resonant case (1.1) has been originally studied with variational methods starting from Rabinowitz [34, 35, 12, 11, 17], where periodic solutions with a period which is a rational multiple of π have been obtained; such solutions correspond to a zero-measure set of values of the amplitudes. The case of irrational periods, which in principle could provide a large measure of values, has been mostly studied only under strong Diophantine conditions (as the ones introduced in [2]) which essentially remove the small divisors problem, leaving in fact again a zero-measure set of values [30, 3, 4]. It is however conjectured that also for $M = 0$ periodic solutions of (1.1) should exist for a large measure set of values of the amplitudes, see for instance [28], and indeed we prove in this paper that this is actually the case: the unperturbed periodic solutions with periods $T_j = 2\pi/j$ can be continued into periodic solutions with periods $T_{\varepsilon,j} = 2\pi/j\sqrt{1-\varepsilon}$, where ε is a small parameter of the order of the squared amplitude of the periodic solution.

In [10] existence of periodic solutions is proved for the equation $u_{tt} - u_{xx} = u^3 + F(x,u)$, with periodic boundary conditions, and with $F(x,u)$ a polynomial in u with coefficients which are trigonometric polynomials in x. Such a problem becomes trivial when F does not depend explicitly on x (in [10] Wayne is credited with such an observation), for instance if $F(x,u) \equiv 0$. On the other hand, when a function $F(x,u)$ depending on x is considered, the perturbation of the exactly solvable problem appears to order higher than 1 (in ε), and this produces a small divisor problem which is solved by imposing a Diophantine condition with an ε-dependent constant (see (5.35) in [10]).

On the contrary in the case of Dirichlet boundary conditions to find a periodic solution just for the cubic equation, $u_{tt} - u_{xx} = u^3$, is non-trivial, and, as will be apparent later on, it is essentially the core of the problem. It already requires the solution of a small divisor problem: one considers the term u^3 as a perturbation and the problem is complicated by the fact that $u_{tt} - u_{xx}$ can be of the same order of u^3; in particular we must impose a Diophantine condition with an ε-independent constant, and this requires careful control of the small divisors.

Of course the techniques used in our and Bourgain's papers are quite different. Bourgain uses the Craig-Wayne approach based on the method of Fröhlich and Spencer [18], while we rely on the Renormalization Group approach proposed in [23], which consists of a Lyapunov-Schmidt decomposition followed by a tree expansion of the solution (with a graphic formalism originally introduced by Gallavotti [19], inspired by Eliasson's work

[16], for investigating the persistence of maximal KAM tori), which allows us to control the small divisors problem. As in [3] and [5] we also consider the problem of finding how many solutions can be obtained with a given period, and we study their minimal period. As a further minor difference between the present paper and [10], we mention that our solutions are analytic in space and time, while the ones found by Bourgain are C^∞.

If $\varphi = 0$ every real solution of (1.1) can be written as

$$u(x,t) = \sum_{n=1}^{\infty} U_n \sin nx \cos(\omega_n t + \theta_n), \tag{1.2}$$

where $\omega_n = n$ and $U_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

For $\varepsilon > 0$ we set $\Phi = \sigma F$, with $\sigma = \operatorname{sgn}\Phi$ and $F > 0$, and rescale $u \to \sqrt{\varepsilon/F} u$ in (1.1), thus obtaining

$$\begin{cases} u_{tt} - u_{xx} = \sigma \varepsilon u^3 + O(\varepsilon^2), \\ u(0,t) = u(\pi,t) = 0, \end{cases} \tag{1.3}$$

where $O(\varepsilon^2)$ denotes an analytic function of u and ε of order at least 2 in ε, and we define $\omega_\varepsilon = \sqrt{1-\lambda\varepsilon}$, with $\lambda \in \mathbb{R}$, so that $\omega_\varepsilon = 1$ for $\varepsilon = 0$.

As the nonlinearity φ is odd the solution of (1.3) can be extended in the x variable to an odd 2π-periodic function (even in the variable t). We shall consider ε small and we shall show that there exists a solution of (1.3), which is $2\pi/\omega_\varepsilon$-periodic in t and ε-close to the function

$$u_0(x, \omega_\varepsilon t) = a_0(\omega_\varepsilon t + x) - a_0(\omega_\varepsilon t - x), \tag{1.4}$$

provided that ε is in an appropriate Cantor set and $a_0(\xi)$ is the odd 2π-periodic solution of the integro-differential equation

$$\sigma\lambda \ddot{a}_0 = -3\left\langle a_0^2 \right\rangle a_0 - a_0^3, \tag{1.5}$$

where the dot denotes the derivative with respect to ξ, and, given any periodic function $F(\xi)$ with period T, we denote by

$$\langle F \rangle = \frac{1}{T}\int_0^T \mathrm{d}\xi \; F(\xi) \tag{1.6}$$

its average. Then a $2\pi/\omega_\varepsilon$-periodic solution of (1.1) is simply obtained by scaling back the solution of (1.3).

Equation (1.5) has odd 2π-periodic solutions, provided that one sets $\sigma\lambda > 0$; we shall choose $\sigma\lambda = 1$ in the following. An explicit computation gives [3]

$$a_0(\xi) = V_{\mathfrak{m}} \operatorname{sn}(\Omega_{\mathfrak{m}}\xi, \mathfrak{m}) \tag{1.7}$$

for $\mathfrak{m}$ a suitable negative constant ($\mathfrak{m} \approx -0.2554$), with $\Omega_{\mathfrak{m}} = 2K(\mathfrak{m})/\pi$ and $V_{\mathfrak{m}} = \sqrt{-2\mathfrak{m}}\Omega_{\mathfrak{m}}$, where $\operatorname{sn}(\Omega_{\mathfrak{m}}\xi, \mathfrak{m})$ is the sine-amplitude function and $K(\mathfrak{m})$ is the elliptic integral of the first kind, with modulus $\sqrt{\mathfrak{m}}$ [25]; see Appendix A1 for further details. Call 2κ the width of the analyticity strip of the function $a_0(\xi)$ and α the maximum value it can assume in such a strip; then one has

$$\left|a_{0,n}\right| \le \alpha e^{-2k|n|}. \tag{1.8}$$

Our result (including also the cases of frequencies which are multiples of ω_ε) can be more precisely stated as follows.

Theorem. *Consider Eq. (1.1), where $\varphi(u) = \Phi u^3 + O(u^5)$ is an odd analytic function, with $F = |\Phi| \neq 0$. Define $u_0(x,t) = a_0(t+x) - a_0(t-x)$, with $a_0(\xi)$ the odd 2π-periodic solution of (1.5). There is a positive constant ε_0 and for all $j \in \mathbb{N}$ a set $\mathcal{E}_j \in [0, \varepsilon_0/j^2]$ satisfying*

$$\lim_{\varepsilon\to 0} \frac{\operatorname{meas}(\mathcal{E}_j \cap [0,\varepsilon])}{\varepsilon} = 1, \tag{1.9}$$

such that for all $\varepsilon \in \mathcal{E}_j$, by setting $\omega_\varepsilon = \sqrt{1-\varepsilon}$ and

$$\|f(x,t)\|_r = \sum_{(n,m)\in\mathbb{Z}^2} f_{n,m} e^{r(|n|+|m|)}, \tag{1.10}$$

for analytic 2π-periodic functions, there exist $2\pi/j\omega_\varepsilon$-periodic solutions $u_{\varepsilon,j}(x,t)$ of (1.1), analytic in (t,x), with

$$\left\| u_{\varepsilon,j}(x,t) - j\sqrt{\varepsilon/F}\, u_0(jx, j\omega_\varepsilon t) \right\|_{\kappa'} \leq C\, j\, \varepsilon\sqrt{\varepsilon}, \tag{1.11}$$

for some constants $C > 0$ and $0 < \kappa' < \kappa$.

Note that such a result provides a solution of the open problem 7.4 in [28], as far as periodic solutions are concerned.

As we shall see for $\varphi(u) = Fu^3$ for all $j \in \mathbb{N}$ one can take the set $\mathcal{E} = [0, \varepsilon_0]$, independently of j, so that for fixed $\varepsilon \in \mathcal{E}$ no restriction on j has to be imposed.

We look for a solution of (1.3) of the form

$$\begin{aligned}
u(x,t) &= \sum_{(n,m)\in\mathbb{Z}^2} e^{inj\omega t + ijmx} u_{n,m} = v(x,t) + w(x,t),\\
v(x,t) &= a(\xi) - a(\xi'), \qquad \xi = \omega t + x, \quad \xi' = \omega t - x,\\
a(\xi) &= \sum_{n\in\mathbb{Z}} e^{in\xi} a_n,\\
w(x,t) &= \sum_{\substack{(n,m)\in\mathbb{Z}^2\\ |n|\neq|m|}} e^{inj\omega t + ijmx} w_{n,m},
\end{aligned} \tag{1.12}$$

with $\omega = \omega_\varepsilon$, such that one has $w(x,t) = 0$ and $a(\xi) = a_0(\xi)$ for $\varepsilon = 0$. Of course by the symmetry of (1.1), hence of (1.4), we can look for solutions (if any) which verify

$$u_{n,m} = -u_{n,-m} = u_{-n,m} \tag{1.13}$$

for all $n, m \in \mathbb{Z}$.

Inserting (1.12) into (1.3) gives two sets of equations, called the Q and P equations [14], which are given, respectively, by

$$\begin{aligned}
&\mathrm{Q} \qquad \begin{cases} n^2 a_n = [\varphi(v+w)]_{n,n}\,, \\ -n^2 a_n = [\varphi(v+w)]_{n,-n}\,, \end{cases}\\
&\mathrm{P} \qquad \left(-\omega^2 n^2 + m^2\right) w_{n,m} = \varepsilon\, [\varphi(v+w)]_{n,m}\,, \qquad |m| \neq |n|,
\end{aligned} \tag{1.14}$$

where we denote by $[F]_{n,m}$ the Fourier component of the function $F(x,t)$ with labels (n,m), so that

$$F(x,t) = \sum_{(n,m)\in\mathbb{Z}^2} e^{in\omega t+mx}[F]_{n,m}. \tag{1.15}$$

In the same way we shall call $[F]_n$ the Fourier component of the function $F(\xi)$ with label n; in particular one has $[F]_0 = \langle F\rangle$. Note also that the two equations Q are in fact the same, by the symmetry property $[\varphi(v+w)]_{n,m} = -[\varphi(v+w)]_{n,-m}$, which follows from (1.13).

We start by considering the case $\varphi(u) = u^3$ and $j = 1$, for simplicity. We shall discuss at the end how the other cases can be dealt with, see Sect. 8.

2. Lindstedt Series Expansion

One could try to write a power series expansion in ε for $u(x,t)$, using (1.14) to get recursive equations for the coefficients. However by proceeding in this way one finds that the coefficient of order k is given by a sum of terms some of which are of order $O(k!^\alpha)$, for some constant α. This is the same phenomenon occurring in the Lindstedt series for invariant KAM tori in the case of quasi-integrable Hamiltonian systems; in such a case however one can show that there are *cancellations* between the terms contributing to the coefficient of order k, which at the end admits a bound C^k, for a suitable constant C. On the contrary such cancellations are absent in the present case and we have to proceed in a different way, equivalent to a resummation (see [23] where such a procedure was applied to the same nonlinear wave equation with a mass term, $u_{tt} - u_{xx} + Mu = \varphi(u)$).

Definition 1. *Given a sequence $\{\nu_m(\varepsilon)\}_{|m|\geq 1}$, such that $\nu_m = \nu_{-m}$, we define the* **renormalized frequencies** *as*

$$\tilde{\omega}_m^2 \equiv \omega_m^2 + \nu_m, \qquad \omega_m = |m|, \tag{2.1}$$

and the quantities ν_m will be called the **counterterms**.

By the above definition and the parity properties (1.13) the P equation in (1.14) can be rewritten as

$$\begin{aligned} w_{n,m}\left(-\omega^2 n^2 + \tilde{\omega}_m^2\right) &= \nu_m w_{n,m} + \varepsilon[\varphi(v+w)]_{n,m} \\ &= \nu_m^{(a)} w_{n,m} + \nu_m^{(b)} w_{n,-m} + \varepsilon[\varphi(v+w)]_{n,m}, \end{aligned} \tag{2.2}$$

where

$$\nu_m^{(a)} - \nu_m^{(b)} = \nu_m. \tag{2.3}$$

With the notations of (1.15), and recalling that we are considering $\varphi(u) = u^3$, we can write

$$\begin{aligned} \left[(v+w)^3\right]_{n,n} &= [v^3]_{n,n} + [w^3]_{n,n} + 3[v^2 w]_{n,n} + 3[w^2 v]_{n,n} \\ &\equiv [v^3]_{n,n} + [g(v,w)]_{n,n}, \end{aligned} \tag{2.4}$$

where, again by using the parity properties (1.13),

$$[v^3]_{n,n} = [a^3]_n + 3\left\langle a^2\right\rangle a_n. \tag{2.5}$$

Then the first Q equation in (1.13) can be rewritten as

$$n^2 a_n = [a^3]_n + 3\left\langle a^2\right\rangle a_n + [g(v,w)]_{n,n}\,, \tag{2.6}$$

so that a_n is the Fourier coefficient of the 2π-periodic solution of the equation

$$\ddot a = -\left(a^3 + 3\left\langle a^2\right\rangle a + G(v,w)\right), \tag{2.7}$$

where we have introduced the function

$$G(v,w) = \sum_{n\in\mathbb{Z}} e^{in\xi}\,[g(v,w)]_{n,n}\,. \tag{2.8}$$

To study Eqs. (2.2) and (2.6) we introduce an auxiliary parameter μ, which at the end will be set equal to 1, by writing (2.2) as

$$w_{n,m}\left(-\omega^2 n^2 + \tilde\omega_m^2\right) = \mu v_m^{(a)} w_{n,m} + \mu v_m^{(b)} w_{n,-m} + \mu\varepsilon[\varphi(v+w)]_{n,m}, \tag{2.9}$$

and we shall look for $u_{n,m}$ in the form of a power series expansion in μ,

$$u_{n,m} = \sum_{k=0}^{\infty} \mu^k u_{n,m}^{(k)}, \tag{2.10}$$

with $u_{n,m}^{(k)}$ depending on ε and on the parameters $v_{m'}^{(c)}$, with $c=a,b$ and $|m'|\ge 1$.

In (2.10) $k=0$ requires $u_{n,\pm n}^{(0)} = \pm a_{0,n}$ and $u_{n,m}^{(0)} = 0$ for $|n|\neq|m|$, for $k\ge 1$, as we shall see later on, the dependence on the parameters $v_{m'}^{(c')}$ will be polynomial, of the form

$$\prod_{m'=2}^{\infty}\prod_{c'=a,b}\left(v_{m'}^{(c')}\right)^{k_{m'}^{(c')}}, \tag{2.11}$$

with $|\underline{k}| = k_1^{(a)} + k_1^{(b)} + k_2^{(a)} + k_2^{(b)} + \cdots \le k-1$. Of course we are using the symmetry property to restrict the dependence only on the positive labels m'.

We derive recursive equations for the coefficients $u_{n,m}^{(k)}$ of the expansion. We start from the coefficients with $|n|=|m|$.

By (1.12) and (2.10) we can write

$$a = a_0 + \sum_{k=1}^{\infty}\mu^k A^{(k)}, \tag{2.12}$$

and inserting this expression into (2.7) we obtain for $A^{(k)}$ the equation

$$\ddot A^{(k)} = -3\left(a_0^2 A^{(k)} + \left\langle a_0^2\right\rangle A^{(k)} + 2\left\langle a_0 A^{(k)}\right\rangle a_0\right) + f^{(k)}, \tag{2.13}$$

with

$$f^{(k)} = - \sum_{\substack{k_1+k_2+k_3=k \\ k_i=k \to |n_i| \neq |m_i|}} \sum_{\substack{n_1+n_2+n_3=n \\ m_1+m_2+m_3=m}} u^{(k_1)}_{n_1,m_1} u^{(k_2)}_{n_2,m_2} u^{(k_3)}_{n_3,m_3}, \tag{2.14}$$

where we have used the notations

$$u^{(k)}_{n,m} = \begin{cases} v^{(k)}_{n,m}, & \text{if } |n| = |m|, \\ w^{(k)}_{n,m}, & \text{if } |n| \neq |m|, \end{cases} \tag{2.15}$$

with

$$v^{(k)}_{n,n} = \begin{cases} A^{(k)}_n, & \text{if } k \neq 0, \\ a_{0,n}, & \text{if } k = 0, \end{cases} \qquad v^{(k)}_{n,-n} = \begin{cases} -A^{(k)}_n, & \text{if } k \neq 0, \\ -a_{0,n}, & if k = 0. \end{cases} \tag{2.16}$$

Before studying how to find the solution of this equation we introduce some preliminary definitions. To shorten notations we write

$$c(\xi) \equiv \text{cn}(\Omega_{\mathfrak{m}}\xi, \mathfrak{m}), \quad s(\xi) \equiv \text{sn}(\Omega_{\mathfrak{m}}\xi, \mathfrak{m}), \quad d(\xi) \equiv \text{dn}(\Omega_{\mathfrak{m}}\xi, \mathfrak{m}), \tag{2.17}$$

and set $cd(\xi) = \text{cn}(\Omega_{\mathfrak{m}}\xi, \mathfrak{m})\,\text{dn}(\Omega_{\mathfrak{m}}\xi, \mathfrak{m})$. Moreover given an analytic periodic function $F(\xi)$ we define

$$\mathbf{P}[F](\xi) = F(\xi) - \langle F \rangle, \tag{2.18}$$

and we introduce a linear operator $\mathbf{I}$ acting on 2π-periodic zero-mean functions and defined by its action on the basis $e_n(\xi) = e^{in\xi}$, $n \in \mathbb{Z} \setminus \{0\}$,

$$\mathbf{I}[e_n](\xi) = \frac{e_n(\xi)}{in}. \tag{2.19}$$

Note that if $\mathbf{P}[F] = F$ then $\mathbf{P}[\mathbf{I}[F]] = \mathbf{I}[F]$ (is simply the zero-mean primitive of F); moreover $\mathbf{I}$ switches parities.

In order to find an odd solution of (2.13) we replace first $\langle a_0 A^{(k)} \rangle$ with a parameter $C^{(k)}$, and we study the modified equation

$$\ddot{A}^{(k)} = -3\left(a_0^2 A^{(k)} + \left\langle a_0^2 \right\rangle A^{(k)} + 2C^{(k)} a_0\right) + f^{(k)}. \tag{2.20}$$

Then we have the following result (proved in Appendix A2).

Lemma 1. *Given an odd analytic 2π-periodic function $h(\xi)$, the equation*

$$\ddot{y} = -3\left(a_0^2 + \left\langle a_0^2 \right\rangle\right) y + h \tag{2.21}$$

admits one and only one odd analytic 2π-periodic solution $y(\xi)$, given by

$$y = \mathbf{L}[h] \equiv B_{\mathfrak{m}}\left(\Omega_{\mathfrak{m}}^{-2} D_{\mathfrak{m}}^2 s \langle s\, h\rangle + \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} \left(s\, \mathbf{I}[cd\, h] - cd\, \mathbf{I}[\mathbf{P}[s\, h]]\right) + cd\, \mathbf{I}[\mathbf{I}[cd\, h]]\right) \tag{2.22}$$

with $B_{\mathfrak{m}} = -\mathfrak{m}/(1-\mathfrak{m})$ and $D_{\mathfrak{m}} = -1/\mathfrak{m}$.

As a_0 is analytic and odd, we find immediately, by induction on k and using Lemma 1, that $f^{(k)}$ is analytic and odd, and that the solution of Eq. (2.20) is odd and given by

$$\tilde{A}^{(k)} = \mathbf{L}[-6C^{(k)}a_0 + f^{(k)}]. \tag{2.23}$$

The function $\tilde{A}^{(k)}$ thus found depends of course on the parameter $C^{(k)}$; in order to obtain $\tilde{A}^{(k)} = A^{(k)}$, we have to impose the constraint

$$C^{(k)} = \left\langle a_0 A^{(k)} \right\rangle, \tag{2.24}$$

and by (2.23) this gives

$$C^{(k)} = -6C^{(k)} \left\langle a_0 \mathbf{L}[a_0] \right\rangle + \left\langle a_0 \mathbf{L}[f^{(k)}] \right\rangle, \tag{2.25}$$

which can be rewritten as

$$\left(1 + 6 \left\langle a_0 \mathbf{L}[a_0] \right\rangle\right) C^{(k)} = \left\langle a_0 \mathbf{L}[f^{(k)}] \right\rangle. \tag{2.26}$$

An explicit computation (see Appendix A3) gives

$$\left\langle a_0 \mathbf{L}[a_0] \right\rangle = \frac{1}{2} V_{\mathfrak{m}}^2 \Omega_{\mathfrak{m}}^{-2} B_{\mathfrak{m}} \left(\left(2D_{\mathfrak{m}} - \frac{1}{2} \right) \left\langle s^4 \right\rangle + \left(2D_{\mathfrak{m}} \left(D_{\mathfrak{m}} - 1 \right) + \frac{1}{2} \right) \left\langle s^2 \right\rangle^2 \right), \tag{2.27}$$

which yields $r_0 = (1 + 6\langle a_0 \mathbf{L}[a_0] \rangle) \neq 0$. At the end we obtain the recursive definition

$$\begin{cases} A^{(k)} = \mathbf{L}[f^{(k)} - 6C^{(k)} a_0], \\ \\ C^{(k)} = r_0^{-1} \left\langle a_0 \mathbf{L}[f^{(k)}] \right\rangle. \end{cases} \tag{2.28}$$

In Fourier space the first of (2.28) becomes

$$\begin{aligned} A_n^{(k)} &= B_{\mathfrak{m}} \Omega_{\mathfrak{m}}^{-2} D_{\mathfrak{m}}^2 s_n \sum_{n_1+n_2=0} s_{n_1} \left(f_{n_2}^{(k)} - 6C^{(k)} a_{0,n_2} \right) \\ &\quad + B_{\mathfrak{m}} \sum_{n_1+n_2+n_3=n}^{*} \frac{1}{i^2 (n_2+n_3)^2} cd_{n_1} cd_{n_2} \left(f_{n_3}^{(k)} - 6C^{(k)} a_{0,n_3} \right) \\ &\quad + B_{\mathfrak{m}} \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} \sum_{n_1+n_2+n_3=n}^{*} \frac{1}{i(n_2+n_3)} s_{n_1} cd_{n_2} \left(f_{n_3}^{(k)} - 6C^{(k)} a_{0,n_3} \right) \\ &\quad - B_{\mathfrak{m}} \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} \sum_{n_1+n_2+n_3=n}^{*} \frac{1}{i(n_2+n_3)} cd_{n_1} s_{n_2} \left(f_{n_3}^{(k)} - 6C^{(k)} a_{0,n_3} \right) \\ &\equiv \sum_{n'} \mathbf{L}_{nn'} \left(f_{n'}^{(k)} - 6C^{(k)} a_{0,n'} \right), \end{aligned} \tag{2.29}$$

where the constants $B_{\mathfrak{m}}$ and $D_{\mathfrak{m}}$ are defined after (2.22), and the $*$ in the sums means that one has the constraint $n_2 + n_3 \neq 0$, while the second of (2.28) can be written as

$$C^{(k)} = r_0^{-1} \sum_{n,n' \in \mathbb{Z}} a_{0,-n} \mathbf{L}_{n,n'} f_{n'}^{(k)}. \tag{2.30}$$

Now we consider the coefficients $u_{n,m}^{(k)}$ with $|n| \neq |m|$. The coefficients $w_{n,m}^{(k)}$ verify the recursive equations

$$w_{n,m}^{(k)} \left[-\omega^2 n^2 + \tilde{\omega}_m^2 \right] = \nu_m^{(a)} w_{n,m}^{(k-1)} + \nu_m^{(b)} w_{n,-m}^{(k-1)} + [(v+w)^3]_{n,m}^{(k-1)}, \tag{2.31}$$

where

$$[(v+w)^3]_{n,m}^{(k)} = \sum_{k_1+k_2+k_3=k} \sum_{\substack{n_1+n_2+n_3=n \\ m_1+m_2+m_3=m}} u_{n_1,m_1}^{(k_1)} u_{n_2,m_2}^{(k_2)} u_{n_3,m_3}^{(k_3)}, \tag{2.32}$$

if we use the same notations (2.15) and (2.16) as in (2.14).

Equations (2.29) and (2.31), together with (2.32), (2.14), (2.30) and (2.32), define recursively the coefficients $u_{n,m}^{(k)}$.

To prove the theorem we shall proceed in two steps. The first step consists in looking for the solution of Eqs. (2.29) and (2.31) by considering $\tilde{\omega} = \{\tilde{\omega}_m\}_{|m|\geq 1}$ as a given set of parameters satisfying the Diophantine conditions (called respectively the first and the second Mel′nikov conditions)

$$\begin{aligned} &|\omega n \pm \tilde{\omega}_m| \geq C_0 |n|^{-\tau} \qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{Z} \setminus \{0\} \text{ such that } |m| \neq |n|, \\ &|\omega n \pm (\tilde{\omega}_m \pm \tilde{\omega}_{m'})| \geq C_0 |n|^{-\tau} \\ &\qquad\qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m, m' \in \mathbb{Z} \setminus \{0\} \text{ such that } |n| \neq |m \pm m'|, \end{aligned} \tag{2.33}$$

with positive constants C_0, τ. We shall prove in Sect. 3 to 5 the following result.

Proposition 1. *Consider a sequence $\tilde{\omega} = \{\tilde{\omega}_m\}_{|m|\geq 1}$ verifying (2.33), with $\omega = \omega_\varepsilon = \sqrt{1-\varepsilon}$ and such that $|\tilde{\omega}_m - |m|| \leq C\varepsilon/|m|$ for some constant C. For all $\mu_0 > 0$ there exists $\varepsilon_0 > 0$ such that for $|\mu| \leq \mu_0$ and $0 < \varepsilon < \varepsilon_0$ there is a sequence $\nu(\tilde{\omega}, \varepsilon; \mu) = \{\nu_m(\tilde{\omega}, \varepsilon; \mu)\}_{|m|\geq 1}$, where each $\nu_m(\tilde{\omega}, \varepsilon; \mu)$ is analytic in μ, such that the coefficients $u_{n,m}^{(k)}$ which solve (2.29) and (2.31) define via (2.10) a function $u(x, t; \tilde{\omega}, \varepsilon; \mu)$ which is analytic in μ, analytic in (x, t) and 2π-periodic in t and solves*

$$\begin{cases} n^2 a_n = \left[a^3\right]_{n,n} + 3\left\langle a^2 \right\rangle a_n + [g(v,w)]_{n,n}, \\ -n^2 a_n = \left[a^3\right]_{n,-n} + 3\left\langle a^2 \right\rangle a_{-n} + [g(v,w)]_{n,-n}, \end{cases} \tag{2.34}$$

$$\left(-\omega^2 n^2 + \tilde{\omega}_m^2\right) w_{n,m} = \mu \nu_m(\tilde{\omega}, \varepsilon; \mu)\, w_{n,m} + \mu\varepsilon \left[\varphi(v+w)\right]_{n,m}, \qquad |m| \neq |n|,$$

with the same notations as in (1.14).

If $\tau \leq 2$ then one can require only the first Mel'nikov conditions in (2.33), as we shall show in Sect. 7.

Then in Proposition 1 one can fix $\mu_0 = 1$, so that one can choose $\mu = 1$ and set $u(x, t; \tilde{\omega}, \varepsilon) = u(x, t; \tilde{\omega}, \varepsilon; 1)$ and $\nu_m(\tilde{\omega}, \varepsilon) = \nu_m(\tilde{\omega}, \varepsilon; 1)$.

The second step, to be proved in Sect. 6, consists in inverting (2.1), with $\nu_m = \nu_m(\tilde{\omega}, \varepsilon)$ and $\tilde{\omega}$ verifying (2.33). This requires some preliminary conditions on ε, given by the Diophantine conditions

$$|\omega n \pm m| \geq C_1 |n|^{-\tau_0} \qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{Z} \setminus \{0\} \text{ such that } |m| \neq |n|, \tag{2.35}$$

with positive constants C_1 and $\tau_0 > 1$.

This allows to solve iteratively (2.1), by imposing further non-resonance conditions besides (2.35), provided that one takes $C_1 = 2C_0$ and $\tau_0 < \tau - 1$, which requires $\tau > 2$. At each iterative step one has to exclude some further values of ε, and at the end the left values fill a Cantor set $\mathcal{E}$ with large relative measure in $[0, \varepsilon_0]$ and $\tilde{\omega}$ verify (2.35).

If $1 < \tau \leq 2$ the first Mel'nikov conditions, which, as we said above, become sufficient to prove Proposition 1, can be obtained by requiring (2.35) with $\tau_0 = \tau$; again this leaves a large measure set of allowed values of ε. This is discussed in Sect. 7.

The result of this second step can be summarized as follows.

Proposition 2. *There are $\delta > 0$ and a set $\mathcal{E} \subset [0, \varepsilon_0]$ with a complement of relative Lebesgue measure of order ε_0^{δ} such that for all $\varepsilon \in \mathcal{E}$ there exists $\tilde{\omega} = \tilde{\omega}(\varepsilon)$ which solves (2.1) and satisfy the Diophantine conditions (2.33) with $|\tilde{\omega}_m - |m|| \leq C\varepsilon/|m|$ for some constant C.*

As we said, our approach is based on constructing the periodic solution of the string equation by a perturbative expansion which is the analogue of the *Lindstedt series* for (maximal) KAM invariant tori in finite-dimensional Hamiltonian systems. Such an approach immediately encounters a difficulty; while the invariant KAM tori are analytic in the perturbative parameter ε, the periodic solutions we are looking for are *not* analytic; hence a power series construction seems at first sight hopeless. Nevertheless it turns out that the Fourier coefficients of the periodic solution have the form $u_{n,m}(\tilde{\omega}(\omega, \varepsilon), \varepsilon; \mu)$; while such functions are not analytic in ε, they turn out to be analytic in μ, provided that $\tilde{\omega}$ satisfies the condition (2.33) and ε is small enough; this is the content of Proposition 1. The smoothness in ε at fixed $\tilde{\omega}$ is what allows us to write as a series expansion $u_{n,m}(\tilde{\omega}, \varepsilon; \mu)$; this strategy was already applied in [23] in the massive case.

3. Tree Expansion: The Diagrammatic Rules

A (connected) graph $\mathbb{G}$ is a collection of points (vertices) and lines connecting all of them. The points of a graph are most commonly known as graph vertices, but may also be called nodes or points. Similarly, the lines connecting the vertices of a graph are most commonly known as graph edges, but may also be called branches or simply lines, as we shall do. We denote with $P(\mathbb{G})$ and $L(\mathbb{G})$ the set of vertices and the set of lines, respectively. A path between two vertices is a subset of $L(\mathbb{G})$ connecting the two vertices. A graph is planar if it can be drawn in a plane without graph lines crossing (i.e. it has graph crossing number 0).

Definition 2. *A* **tree** *is a planar graph $\mathbb{G}$ containing no closed loops (cycles); in other words, it is a connected acyclic graph. One can consider a tree $\mathbb{G}$ with a single special vertex $\mathbb{V}_0$: this introduces a natural partial ordering on the set of lines and vertices, and one can imagine that each line carries an arrow pointing toward the vertex $\mathbb{V}_0$. We can add an extra (oriented) line ℓ_0 connecting the special vertex $\mathbb{V}_0$ to another point which will be called the* **root** *of the tree; the added line will be called the* **root line**. *In this way we obtain a* **rooted tree** *θ defined by $P(\theta) = P(\mathbb{G})$ and $L(\theta) = L(\mathbb{G}) \cup \ell_0$. A* **labeled tree** *is a rooted tree θ together with a label function defined on the sets $L(\theta)$ and $P(\theta)$.*

Note that the definition of rooted tree given above is slightly different from the one which is usually adopted in literature [24, 26] according to which a rooted tree is just a tree with a privileged vertex, without any extra line. However the modified definition that we gave will be more convenient for our purposes. In the following we shall denote with the symbol θ both rooted trees and labeled rooted trees, when no confusion arises.

We shall call equivalent two rooted trees which can be transformed into each other by continuously deforming the lines in the plane in such a way that the latter do not cross each other (i.e. without destroying the graph structure). We can extend the notion of equivalence also to labeled trees, simply by considering equivalent two labeled trees if they can be transformed into each other in such a way that also the labels match.

Given two points $\mathbb{V}, \mathbb{W} \in P(\theta)$, we say that $\mathbb{W} \prec \mathbb{V}$ if $\mathbb{V}$ is on the path connecting $\mathbb{W}$ to the root line. We can identify a line with the points it connects; given a line $\ell = (\mathbb{V}, \mathbb{W})$ we say that ℓ enters $\mathbb{V}$ and comes out of $\mathbb{W}$.

In the following we shall deal mostly with labeled trees: for simplicity, where no confusion can arise, we shall call them just trees. We consider the following *diagrammatic rules* to construct the trees we have to deal with; this will implicitly define also the label function.

(1) We call *nodes* the vertices such that there is at least one line entering them. We call *end-points* the vertices which have no entering line. We denote with $L(\theta)$, $V(\theta)$ and $E(\theta)$ the set of lines, nodes and end-points, respectively. Of course $P(\theta) = V(\theta) \cup E(\theta)$.

(2) There can be two types of lines, w-lines and v-lines, so we can associate with each line $\ell \in L(\theta)$ a *badge* label $\gamma_\ell \in \{v, w\}$ and a *momentum* $(n_\ell, m_\ell) \in \mathbb{Z}^2$, to be defined in item (8) below. If $\gamma_\ell = v$ one has $|n_\ell| = |m_\ell|$, while if $\gamma_\ell = w$ one has $|n_\ell| \neq |m_\ell|$. One can have $(n_\ell, m_\ell) = (0, 0)$ only if ℓ is a v-line. With the v-lines ℓ with $n_\ell \neq 0$ we also associate a label $\delta_\ell \in \{1, 2\}$. All the lines coming out from the end-points are v-lines with $n_\ell \neq 0$.

(3) With each line ℓ coming out from a node we associate a *propagator*

$$g_\ell = g(\omega n_\ell, m_\ell) = \begin{cases} \frac{1}{-\omega^2 n_\ell^2 + \tilde{\omega}_{m_\ell}^2}, & \text{if } \gamma_\ell = w, \\ \frac{1}{(i n_\ell)^{\delta_\ell}}, & \text{if } \gamma_\ell = v, n_\ell \neq 0, \\ 1, & \text{if } \gamma_\ell = v, n_\ell = 0, \end{cases} \tag{3.1}$$

with momentum (n_ℓ, m_ℓ). We can associate also a propagator with the lines ℓ coming out from end-points, simply by setting $g_\ell = 1$.

(4) Given any node $\mathbb{V} \in V(\theta)$ denote with $s_\mathbb{V}$ the number of entering lines (*branching number*): one can have only either $s_\mathbb{V} = 1$ or $s_\mathbb{V} = 3$. Also the nodes $\mathbb{V}$ can be of w-type and v-type: we say that a node is of v-type if the line ℓ coming out from it has label $\gamma_\ell = v$; analogously the nodes of w-type are defined. We can write $V(\theta) = V_v(\theta) \cup V_w(\theta)$, with obvious meaning of the symbols; we also call $V_w^s(\theta)$, $s = 1, 3$, the set of nodes in $V_w(\theta)$ with s entering lines, and analogously we define $V_v^s(\theta)$, $s = 1, 3$. If $\mathbb{V} \in V_v^3(\theta)$ and two entering lines come out of end points then the remaining line entering $\mathbb{V}$ has to be a w-line. If $\mathbb{V} \in V_w^1(\theta)$ then the line entering $\mathbb{V}$ has to be a w-line. If $\mathbb{V} \in V_v^1(\theta)$ then its entering line comes out of an end-node.

(5) With the nodes $\mathbb{V}$ of v-type we associate a label $j_\mathbb{V} \in \{1, 2, 3, 4\}$ and, if $s_\mathbb{V} = 1$, an *order* label $k_\mathbb{V}$, with $k_\mathbb{V} \geq 1$. Moreover we associate with each node $\mathbb{V}$ of v-type two *mode* labels $(n'_\mathbb{V}, m'_\mathbb{V})$, with $m'_\mathbb{V} = \pm n'_\mathbb{V}$, and $(n_\mathbb{V}, m_\mathbb{V})$, with $m_\mathbb{V} = \pm n_\mathbb{V}$, and such that one has

$$\frac{m'_\mathbb{V}}{n'_\mathbb{V}} = \frac{m_\mathbb{V}}{n_\mathbb{V}} = \frac{\displaystyle\sum_{i=1}^{s_\mathbb{V}} m_{\ell_i}}{\displaystyle\sum_{i=1}^{s_\mathbb{V}} n_{\ell_i}}, \tag{3.2}$$

where ℓ_i are the lines entering $\mathbb{V}$. We shall refer to them as the *first mode* label and the *second mode* label, respectively. With a node $\mathbb{V}$ of v-type we associate also a *node factor* $\eta_{\mathbb{V}}$ defined as

$$\eta_{\mathbb{V}} = \begin{cases} -B_{\mathfrak{m}}\Omega_{\mathfrak{m}}^{-2}D_{\mathfrak{m}}^{2} sn_{n'_{\mathbb{V}}} sn_{n_{\mathbb{V}}}, & \text{if } j_{\mathbb{V}}=1 \text{ and } s_{\mathbb{V}}=3,\\ -6B_{\mathfrak{m}}\Omega_{\mathfrak{m}}^{-2}D_{\mathfrak{m}}^{2} sn_{n'_{\mathbb{V}}} sn_{n_{\mathbb{V}}} C^{(k_{\mathbb{V}})}, & \text{if } j_{\mathbb{V}}=1 \text{ and } s_{\mathbb{V}}=1,\\ -B_{\mathfrak{m}} cd_{n'_{\mathbb{V}}} cd_{n_{\mathbb{V}}}, & \text{if } j_{\mathbb{V}}=2 \text{ and } s_{\mathbb{V}}=3,\\ -6B_{\mathfrak{m}} cd_{n'_{\mathbb{V}}} cd_{n_{\mathbb{V}}} C^{(k_{\mathbb{V}})}, & \text{if } j_{\mathbb{V}}=2 \text{ and } s_{\mathbb{V}}=1,\\ -B_{\mathfrak{m}}\Omega_{\mathfrak{m}}^{-1}D_{\mathfrak{m}} sn_{n'_{\mathbb{V}}} cd_{n_{\mathbb{V}}}, & \text{if } j_{\mathbb{V}}=3 \text{ and } s_{\mathbb{V}}=3,\\ -6B_{\mathfrak{m}}\Omega_{\mathfrak{m}}^{-1}D_{\mathfrak{m}} sn_{n'_{\mathbb{V}}} cd_{n_{\mathbb{V}}} C^{(k_{\mathbb{V}})}, & \text{if } j_{\mathbb{V}}=3 \text{ and } s_{\mathbb{V}}=1,\\ B_{\mathfrak{m}}\Omega_{\mathfrak{m}}^{-1}D_{\mathfrak{m}} cd_{n'_{\mathbb{V}}} sn_{n_{\mathbb{V}}}, & \text{if } j_{\mathbb{V}}=4 \text{ and } s_{\mathbb{V}}=3,\\ 6B_{\mathfrak{m}}\Omega_{\mathfrak{m}}^{-1}D_{\mathfrak{m}} cd_{n'_{\mathbb{V}}} s_{n_{\mathbb{V}}} C^{(k_{\mathbb{V}})}, & \text{if } j_{\mathbb{V}}=4 \text{ and } s_{\mathbb{V}}=1. \end{cases} \tag{3.3}$$

Note that the factors $C^{(k_{\mathbb{V}})} = r_0^{-1}\left\langle a_0\mathbf{L}[f^{(k_{\mathbb{V}})}]\right\rangle$ depend on the coefficients $u_{n,m}^{(k')}$, with $k' < k$, so that they have to be defined iteratively. The label δ_ℓ of the line ℓ coming out from a node $\mathbb{V}$ of v-type is related to the label $j_{\mathbb{V}}$ of v: if $j_{\mathbb{V}} = 1$ then $n_\ell = 0$, while if $j_{\mathbb{V}} > 1$ then $n_\ell \neq 0$ and $\delta_\ell = 1 + \delta_{j_{\mathbb{V}},2}$, where $\delta_{i,j}$ denotes the Kronecker delta (so that $\delta_\ell = 2$ if $j_{\mathbb{V}} = 2$ and $\delta_\ell = 1$ otherwise).

(6) With the nodes $\mathbb{V} \in V_w^1(\theta)$, called ν-*vertices*, we associate a label $c_{\mathbb{V}} \in \{a, b\}$. With the nodes $\mathbb{V}$ of w-type we simply associate a *node factor* $\eta_{\mathbb{V}}$ given by

$$\eta_{\mathbb{V}} = \begin{cases} \varepsilon, & \text{if } s_{\mathbb{V}}=3,\\ \nu_{m_\ell}^{(c_{\mathbb{V}})}, & \text{if } s_{\mathbb{V}}=1. \end{cases} \tag{3.4}$$

In the latter case (n_ℓ, m_ℓ) is the momentum of the line coming out from $\mathbb{V}$, and if one has $c_{\mathbb{V}} = a$ the momentum of the entering line is (n_ℓ, m_ℓ) while if $c_{\mathbb{V}} = b$ the momentum of the entering line is $(n_\ell, -m_\ell)$. In order to unify notations we can associate also with the nodes $\mathbb{V}$ of w-type two mode labels, by setting $(n'_{\mathbb{V}}, m'_{\mathbb{V}}) = (0, 0)$ and $(n_{\mathbb{V}}, m_{\mathbb{V}}) = (0, 0)$.

(7) With the end-points $\mathbb{V}$ we associate only a first mode label $(n'_{\mathbb{V}}, m'_{\mathbb{V}})$, with $|m'_{\mathbb{V}}| = |n'_{\mathbb{V}}|$, and an *end-point factor*

$$V_{\mathbb{V}} = (-1)^{1+\delta_{n'_{\mathbb{V}},m'_{\mathbb{V}}}} a_{0,n'_{\mathbb{V}}} = a_{0,m_{\mathbb{V}}}. \tag{3.5}$$

The line coming out from an end-point has to be a v-line.

(8) The momentum (n_ℓ, m_ℓ) of a line ℓ is related to the mode labels of the nodes preceding ℓ; if a line ℓ comes out from a node $\mathbb{V}$ one writes $\ell = \ell_{\mathbb{V}}$ and sets

$$\begin{aligned} n_\ell &= n_{\mathbb{V}} + \sum_{\substack{\mathbb{W}\in V(\theta)\\ \mathbb{W}\prec\mathbb{V}}} \left(n'_{\mathbb{W}} + n_{\mathbb{W}}\right) + \sum_{\substack{\mathbb{W}\in E(\theta)\\ \mathbb{W}\prec\mathbb{V}}} n'_{\mathbb{W}},\\ m_\ell &= m_{\mathbb{V}} + \sum_{\substack{\mathbb{W}\in V(\theta)\\ \mathbb{W}\prec\mathbb{V}}} \left(m'_{\mathbb{W}} + m_{\mathbb{W}}\right) + \sum_{\substack{\mathbb{W}\in E(\theta)\\ \mathbb{W}\prec\mathbb{V}}} m'_{\mathbb{W}} + \sum_{\substack{\mathbb{W}\in V_w^1(\theta)\\ c_{\mathbb{W}}=b}} \left(-2m_{\ell_{\mathbb{W}}}\right), \end{aligned} \tag{3.6}$$

where the sign in m_ℓ is plus if $c_{\mathbb{V}} = a$ and minus if $c_{\mathbb{V}} = b$ and some of the mode labels can be vanishing according to the notations introduced above. If ℓ comes out from an end-point we set $(n_\ell, m_\ell) = (0, 0)$.

We define $\Theta^{*(k)}_{n,m}$ as the *set of inequivalent labeled trees*, formed by following the rules (1) to (8) given above, and with the further following constraints:
(i) if (n_{ℓ_0}, m_{ℓ_0}) denotes the momentum flowing through the root line ℓ_0 and $(n'_{\mathbb{V}_0}, m'_{\mathbb{V}_0})$ is the first mode label associated with the node $\mathbb{V}_0$ which ℓ_0 comes out from (special vertex), then one has $n = n_{\ell_0} + n'_{\mathbb{V}_0}$ and $m = m_{\ell_0} + m'_{\mathbb{V}_0}$;
(ii) one has

$$k = |V_w(\theta)| + \sum_{\mathbb{V} \in V_v^1(\theta)} k_{\mathbb{V}}, \tag{3.7}$$

with k called the *order* of the tree.
An example of tree is given in Fig. 3.1, where only the labels v, w of the nodes have been explicitly written.

Definition 3. *For all $\theta \in \Theta^{*(k)}_{n,m}$, we call*

$$\mathrm{Val}(\theta) = \Big(\prod_{\ell \in L(\theta)} g_\ell \Big) \Big(\prod_{\mathbb{V} \in V(\theta)} \eta_{\mathbb{V}} \Big) \Big(\prod_{\mathbb{V} \in E(\theta)} V_{\mathbb{V}} \Big), \tag{3.8}$$

the **value** *of the tree θ.*

Then the main result about the formal expansion of the solution is provided by the following result.

Lemma 2. *We can write*

$$u^{(k)}_{n,m} = \sum_{\theta \in \Theta^{*(k)}_{n,m}} \mathrm{Val}(\theta), \tag{3.9}$$

and if the root line ℓ_0 is a v-line the tree value is a contribution to $v^{(k)}_{n,\pm n}$, while if ℓ_0 is a w-line the tree value is a contribution to $w^{(k)}_{n,m}$. The factors $C^{(k)}$ are defined as

$$C^{(k)} = r_0^{-1} \sum_{\theta \in \Theta^{*(k)}_{n,n}}{}^{*} a_{0,-n} \mathrm{Val}(\theta), \tag{3.10}$$

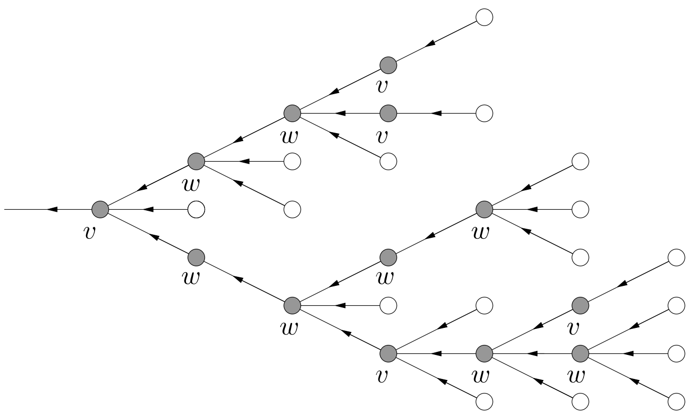

Fig. 3.1.

Fig. 3.2.

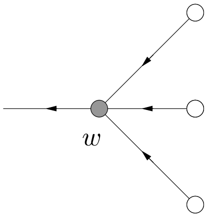

Fig. 3.3.

where the $*$ in the sum means the extra constraint $s_{\mathbb{V}_0} = 3$ for the node immediately preceding the root (which is the special vertex of the rooted tree).

Proof. The proof is done by induction in k. Imagine to represent graphically $a_{0,n}$ as a (small) white bullet with a line coming out from it, as in Fig. 3.2a, and $u^{(k)}_{n,m}$, $k \geq 1$, as a (big) black bullet with a line coming out from it, as in Fig. 3.2b.

One should imagine that labels k, n, m are associated with the black bullet representing $u^{(k)}_{n,m}$, while a white bullet representing $a_{0,n}$ carries the labels $n, m = \pm n$.

For $k = 1$ the proof of (3.9) and (3.10) is just a check from the diagrammatic rules and the recursive definitions (2.27) and (2.29), and it can be performed as follows.

Consider first the case $|n| \neq |m|$, so that $u^{(1)}_{n,m} = w^{(1)}_{n,m}$. By taking into account only the badge labels of the lines, by item (4) there is only one tree whose root line is a w-line, and it has one node $\mathbb{V}_0$ (the special vertex of the tree) with $s_{\mathbb{V}_0} = 3$, hence three end-points $\mathbb{V}_1$, $\mathbb{V}_2$ and $\mathbb{V}_3$. By applying the rules listed above one obtains, for $|n| \neq |m|$,

$$w^{(1)}_{n,m} = \frac{1}{-\omega^2 n^2 + \tilde{\omega}^2_m} \sum_{\substack{n_1+n_2+n_3=n \\ m_1+m_2+m_3=m}} v^{(0)}_{n_1,m_1} v^{(0)}_{n_2,m_2} v^{(0)}_{n_3,m_3} = \sum_{\theta \in \Theta^{*(1)}_{n,m}} \mathrm{Val}(\theta), \tag{3.11}$$

where the sum is over all trees θ which can be obtained from the tree appearing in Fig. 3.3 by summing over all labels which are not explicitly written.

It is easy to realize that (3.11) corresponds to (2.31) for $k = 1$. Each end-point $\mathbb{V}_i$ is graphically a white bullet with first mode labels (n_i, m_i) and second mode labels $(0, 0)$, and has associated an end-point factor $(-1)^{1+\delta_{n_i,m_i}} a_{0,n_i}$ (see 3.5) in item (7)). The node $\mathbb{V}_0$ is represented as a (small) gray bullet, with mode labels $(0, 0)$ and $(0, 0)$, and the factor associated with it is $\eta_{\mathbb{V}_0} = \varepsilon$ (see 3.4) in item (6)). We associate with the line ℓ coming out from the node $\mathbb{V}_0$ a momentum (n_ℓ, n_ℓ), with $n_\ell = n$, and a propagator $g_\ell = 1/(-\omega^2 n^2_\ell + \tilde{\omega}^2_{m_\ell})$ (see (3.1) in item (3)).

Now we consider the case $|n| = |m|$, so that $u^{(1)}_{n,m} = \pm A^{(1)}_n$ (see (2.16)). By taking into account only the badge labels of the lines, there are four trees contributing to $A^{(1)}_n$: they are represented by the four trees in Fig. 3.4 (the tree b and c are simply obtained from the tree by a different choice of the w-line entering the last node).

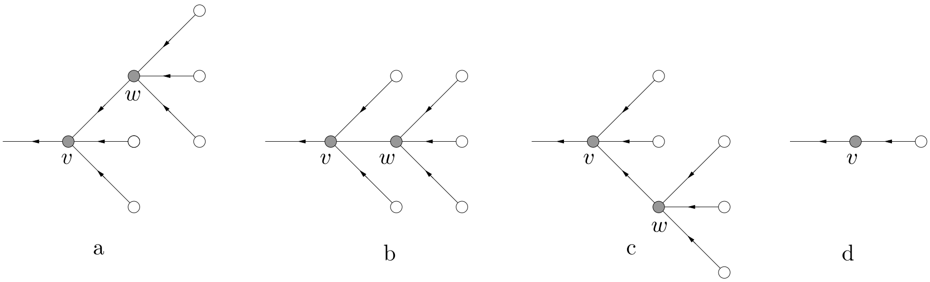

Fig. 3.4.

In the trees of Figs. 3.4a, 3.4b and 3.4c the root line comes out from a node $\mathbb{V}_0$ (the special vertex of the tree) with $s_{\mathbb{V}_0} = 3$, and two of the entering lines come out from end-points: then the other line has to be a w-line (by item (4)), and (3.7) requires that the subtree which has such a line as root line is exactly the tree represented in Fig. 3.2. In the tree of Fig. 4.4d the root line comes out from a node $\mathbb{V}_0$ with $s_{\mathbb{V}_0} = 1$, hence the line entering $\mathbb{V}_0$ is a v-line coming out from an end-point (again see item (4)).

By defining $\Theta^{*(1)}_{n,n}$ as the set of all labeled trees which can be obtained by assigning to the trees in Fig. 3.4 the labels which are not explicitly written, one finds

$$A_n^{(1)} = \sum_{\theta \in \Theta^{*(1)}_{n,n}} \mathrm{Val}(\theta), \tag{3.12}$$

which corresponds to the sum of two contributions. The first one arises from the trees of Figs. 3.4a, 3.4b and 3.4c, and it is given by

$$3 \sum_{n' \in \mathbb{Z}} \mathbf{L}_{n,n'} \sum_{\substack{n'_1+n'_2+n'_3=n' \\ m'_1+m'_2+m'_3=n'}} v^{(0)}_{n'_1,m'_1} v^{(0)}_{n'_2,m'_2} w^{(1)}_{n'_3,m'_3}, \tag{3.13}$$

where one has

$$\begin{aligned} \mathbf{L}_{n,n'} = {} & B_{\mathfrak{m}} \Omega_{\mathfrak{m}}^{-2} D_{\mathfrak{m}}^2 \sum_{\substack{n_1+n_2=n-n' \\ n_2=-n'}}^{*} s_{n_1} s_{n_2} + B_{\mathfrak{m}} \sum_{n_1+n_2=n-n'}^{*} \frac{1}{i^2(n_2+n')^2} cd_{n_1} cd_{n_2} \\ & + B_{\mathfrak{m}} \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} \sum_{n_1+n_2=n-n'}^{*} \frac{1}{i(n_2+n')} s_{n_1} cd_{n_2} \\ & - B_{\mathfrak{m}} \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} \sum_{n_1+n_2=n-n'}^{*} \frac{1}{i(n_2+n')} cd_{n_1} s_{n_2}, \end{aligned} \tag{3.14}$$

with the $*$ denoting the constraint $n_2 + n' \neq 0$. The first and second mode labels associated with the node $\mathbb{V}_0$ are, respectively, $(m'_{\mathbb{V}_0}, n'_{\mathbb{V}_0}) = (n_1, n_1)$ and $(m_{\mathbb{V}_0}, n_{\mathbb{V}_0}) = (n_2, n_2)$, while the momentum flowing through the root line is given by (n_ℓ, m_ℓ), with $|m_\ell| = |n_\ell|$ expressed according to the definition (3.6) in item (8): the corresponding propagator is $(n_\ell)^{\delta_\ell}$ for $n_\ell \neq 0$ and 1 for $n_\ell = 0$, as in (3.1) in item (3).

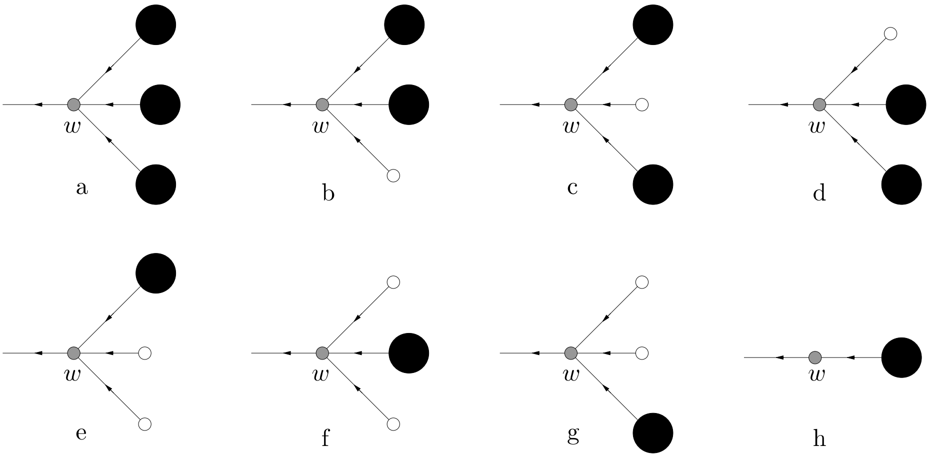

Fig. 3.5.

The second contribution corresponds to the tree of Fig. 3.4d, and it is given by

$$\sum_{n'\in\mathbb{Z}} \mathbf{L}_{n,n'} C^{(1)} a_{0,n'}, \tag{3.15}$$

with the same expression (3.14) for $\mathbf{L}_{n,n'}$ and $C^{(1)}$ still undetermined. The mode labels of the node $\mathbb{V}_0$ and the momentum of the root line are as before.

Then one immediately realizes that the sum of (3.13) and (3.15) corresponds to (2.27) for $k = 1$.

Finally that $C^{(1)}$ is given by (3.9) follows from (2.12). This completes the check of the case $k = 1$.

In general from (2.31) one gets, for $\theta \in \Theta^{*(k)}_{n,m}$ contributing to $w^{(k)}_{n,m}$, that the tree value $\mathrm{Val}(\theta)$ is obtained by summing all contributions either of the form

$$\frac{1}{-\omega^2 n^2 + \tilde{\omega}_m^2}\, \varepsilon \sum_{\substack{n_1+n_2+n_3=n \\ m_1+m_2+m_3=m}} \sum_{k_1+k_2+k_3=k-1} \sum_{\theta_1\in\Theta^{*(k_1)}_{n_1,m_1}} \sum_{\theta_2\in\Theta^{*(k_2)}_{n_2,m_2}} \sum_{\theta_3\in\Theta^{*(k_3)}_{n_3,m_3}} \mathrm{Val}(\theta_1)\,\mathrm{Val}(\theta_2)\,\mathrm{Val}(\theta_3), \tag{3.16}$$

or of the form

$$\frac{1}{-\omega^2 n^2 + \tilde{\omega}_m^2} \sum_{c=a,b} \nu_m^{(c)} \sum_{\theta_1\in\Theta^{*(k,1)}_{n,m^{(c)}}} \mathrm{Val}(\theta_1), \tag{3.17}$$

with $m^{(a)} = m$ and $m^{(b)} = -m$; the corresponding graphical representations are as in Fig. 3.5.

Therefore, by simply applying the diagrammatic rules given above, we see that by summing together the contribution (3.16) and (3.17) we obtain (3.9) for $|n| \neq |m|$.

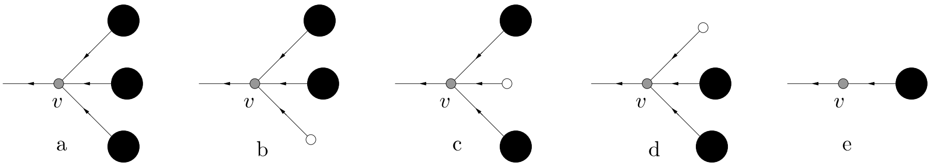

Fig. 3.6.

A similar discussion applies to $A_n^{(k)}$, and one finds that $A_n^{(k)}$ can be written as a sum of contribution either of the form

$$\sum_{n'\in\mathbb{Z}} \mathbf{L}_{n,n'} \sum_{\substack{n_1'+n_2'+n_3'=n' \\ m_1'+m_2'+m_3'=n'}} \sum_{k_1+k_2+k_3=k} \sum_{\theta_1\in\Theta^{*(k_1)}_{n_1,m_1}} \sum_{\theta_2\in\Theta^{*(k_2)}_{n_2,m_2}} \sum_{\theta_3\in\Theta^{*(k_3)}_{n_3,m_3}} \mathrm{Val}(\theta_1)\,\mathrm{Val}(\theta_2)\,\mathrm{Val}(\theta_3), \tag{3.18}$$

or of the form

$$\sum_{n'\in\mathbb{Z}} \mathbf{L}_{n,n'} C^{(k)} a_{0,n'}, \tag{3.19}$$

with $C^{(k)}$ still undetermined. Both (3.18) and (3.19) are of the form $\mathrm{Val}(\theta)$, for $\theta \in \Theta^{*(k)}_{n,m}$. A graphical representation is in Fig. 3.6.

Analogously to the case $k = 1$ the coefficients $C^{(k)}$ are found to be expressed by (3.10). Then the lemma is proved. □

Lemma 3. *For any rooted tree θ one has $|V_v^3(\theta)| \le 2|V_w^3(\theta)| + 2|V_v^1(\theta)|$ and $|E(\theta)| \le 2(|V_v^3(\theta)| + |V_w^3(\theta)|) + 1$.*

Proof. First of all note that $|V_w^3(\theta)| = 0$ requires $|V_v^1(\theta)| \ge 1$, so that one has $|V_w^3(\theta)| + |V_v^1(\theta)| \ge 1$ for all trees θ.

We prove by induction on the number N of nodes the bound

$$\left|V_v^3(\theta)\right| \le \begin{cases} 2|V_w^3(\theta)| + 2|V_v^1(\theta)| - 1, & \text{if the root line is a v-line,} \\ 2|V_w^3(\theta)| + 2|V_v^1(\theta)| - 2 & \text{if the root line is a w-line,} \end{cases} \tag{3.20}$$

which will immediately imply the first assertion.

For $N = 1$ the bound is trivially satisfied, as Figs. 3.3 and 3.4 show.

Then assume that (3.20) holds for the trees with N' nodes, for all $N' < N$, and consider a tree θ with $V(\theta) = N$.

If the special vertex $\mathbb{V}_0$ of θ is not in $V_v^3(\theta)$ (hence it is in $V_w(\theta)$) the bound (3.20) follows trivially by the inductive hypothesis.

If $\mathbb{V}_0 \in V_v^3(\theta)$ then we can write

$$|V_v^3(\theta)| = 1 + \sum_{i=1}^{s} |V_v^3(\theta_i)|, \tag{3.21}$$

where $\theta_1, \ldots, \theta_s$ are the subtrees (not endpoints) whose root line is one of the lines entering $\mathbb{V}_0$. One must have $s \geq 1$, as $s = 0$ would correspond to having all the entering lines of $\mathbb{V}_0$ coming out from end-points, hence to having $N = 1$.

If $s \geq 2$ one has from (3.21) and from the inductive hypothesis

$$|V_v^3(\theta)| \leq 1 + \sum_{i=1}^{s} \Big(2|V_w^3(\theta_i)| + 2|V_v^1(\theta_i)| - 1 \Big) \leq 1 + 2|V_w^3(\theta)| + 2|V_v^1(\theta)| - 2, \tag{3.22}$$

and the bound (3.20) follows.

If $s = 1$ then the root line of θ_1 has to be a w-line by item (4), so that one has

$$|V_v^3(\theta)| \leq 1 + \Big(2|V_w^3(\theta_1)| + 2|V_v^1(\theta)| - 2 \Big) \tag{3.23}$$

which again yields (3.20).

Finally the second assertion follows from the standard (trivial) property of trees

$$\sum_{\mathbb{V} \in V(\theta)} (s_{\mathbb{V}} - 1) = |E(\theta)| - 1, \tag{3.24}$$

and the observation that in our case one has $s_{\mathbb{V}} \leq 3$. $\square$

4. Tree Expansion: The Multiscale Decomposition

We assume the Diophantine conditions (2.33). We introduce a multiscale decomposition of the propagators of the w-lines. Let $\chi(x)$ be a C^∞ non-increasing function such that $\chi(x) = 0$ if $|x| \geq 2C_0$ and $\chi(x) = 1$ if $|x| \leq C_0$ (C_0 is the same constant appearing in (2.33)), and let $\chi_h(x) = \chi(2^h x) - \chi(2^{h+1} x)$ for $h \geq 0$, and $\chi_{-1}(x) = 1 - \chi(x)$; such functions realize a smooth partition of the unity as

$$1 = \chi_{-1}(x) + \sum_{h=0}^{\infty} \chi_h(x) = \sum_{h=-1}^{\infty} \chi_h(x). \tag{4.1}$$

If $\chi_h(x) \neq 0$ for $h \geq 0$ one has $2^{-h-1} C_0 \leq |x| \leq 2^{-h+1} C_0$, while if $\chi_{-1}(x) \neq 0$ one has $|x| \geq C_0$.

We write the propagator of any w-line as the sum of propagators on single scales in the following way:

$$g(\omega n, m) = \sum_{h=-1}^{\infty} \frac{\chi_h(|\omega n| - \tilde{\omega}_m)}{-\omega^2 n^2 + \tilde{\omega}_m^2} = \sum_{h=-1}^{\infty} g^{(h)}(\omega n, m). \tag{4.2}$$

Note that we can bound $|g^{(h)}(\omega n, m)| \leq \frac{2^{h+1}}{C_0}$ (notice that given n, m there are at most two non-zero values of $g^{(h)}(\omega n, m)$).

This means that we can attach to each w-line ℓ in $L(\theta)$ a scale label $h_\ell \geq -1$, which is the scale of the propagator which is associated with ℓ. We can denote with $\Theta_{n,m}^{(k)}$ the set of trees which differ from the previous ones simply because the lines carry also the scale labels. The set $\Theta_{n,m}^{(k)}$ is defined according to the rules (1) to (8) of Sect. 3, by changing item (3) into the following one.

(3′) With each line ℓ coming out from nodes of w-type we associate a *scale label* $h_\ell \geq -1$. For notational convenience we associate a scale label $h = -1$ with the lines coming out from the nodes of v-type and with the lines coming out from the end-points. With each line ℓ we associate a *propagator*

$$g_\ell^{(h_\ell)} \equiv g^{(h_\ell)}(\omega n_\ell, m_\ell) = \begin{cases} \frac{\chi_{h_\ell}(|\omega n_\ell| - \tilde{\omega}_{m_\ell})}{-\omega^2 n_\ell^2 + \tilde{\omega}_{m_\ell}^2}, & \text{if } \gamma_\ell = w, \\ \frac{1}{(i n_\ell)^{\delta_\ell}}, & \text{if } \gamma_\ell = v, n_\ell \neq 0, \\ 1, & \text{if } \gamma_\ell = v, n_\ell = 0, \end{cases} \tag{4.3}$$

with momentum (n_ℓ, m_ℓ).

Definition 4. *For all* $\theta \in \Theta_{n,m}^{(k)}$, *we define*

$$\mathrm{Val}(\theta) = \Big(\prod_{\ell \in L(\theta)} g_\ell^{(h_\ell)} \Big) \Big(\prod_{\mathbb{V} \in V(\theta)} \eta_{\mathbb{V}} \Big) \Big(\prod_{\mathbb{V} \in E(\theta)} V_{\mathbb{V}} \Big), \tag{4.4}$$

the **value of the tree** θ.

Then (3.9) and (3.10) are replaced, respectively, with

$$u_{n,m}^{(k)} = \sum_{\theta \in \Theta_{n,m}^{(k)}} \mathrm{Val}(\theta), \tag{4.5}$$

and

$$C^{(k)} = r_0^{-1} \sum_{\theta \in \Theta_{n,n}^{(k)}}^{*} a_{0,-n} \mathrm{Val}(\theta), \tag{4.6}$$

with the new definition for the tree value $\mathrm{Val}(\theta)$ and with $*$ meaning the same constraint as in (3.10).

Definition 5. *A* **cluster** T *is a connected set of nodes which are linked by a continuous path of lines with the same scale label* h_T *or a lower one and which are maximal; we shall say that the cluster has scale* h_T. *We shall denote with* $V(T)$ *and* $E(T)$ *the set of nodes and the set of end-points, respectively, which are contained inside the cluster* T, *and with* $L(T)$ *the set of lines connecting them. As for trees we call* $V_v(T)$ *and* $V_w(T)$ *the sets of nodes* $\mathbb{V} \in V(T)$ *which are of* v*-type and of* w*-type respectively. Analogously one defines the sets* $V_v^s(T)$ *and* $V_w^s(T)$.

We define the order k_T of a cluster T as the order of a tree (see item (ii) before Definition 3), with the sums restricted to the nodes internal to the cluster.

An inclusion relation is established between clusters, in such a way that the innermost clusters are the clusters with lowest scale, and so on. Each cluster T can have an arbitrary number of lines entering it (incoming lines), but only one or zero line coming from it (outcoming line); we shall denote the latter (when it exists) with ℓ_T^1. We shall call *external lines* of the cluster T the lines which either enter or come out from T, and we shall denote by $h_T^{(e)}$ the minimum among the scales of the external lines of T. Define also

$$\begin{aligned} K(\theta) &= \sum_{\mathbb{V} \in V(\theta)} \big(|n'_{\mathbb{V}}| + |n_{\mathbb{V}}| \big) + \sum_{\mathbb{V} \in E(\theta)} |n'_{\mathbb{V}}|, \\ K(T) &= \sum_{\mathbb{V} \in V(T)} \big(|n'_{\mathbb{V}}| + |n_{\mathbb{V}}| \big) + \sum_{\mathbb{V} \in E(\theta)} |n'_{\mathbb{V}}|, \end{aligned} \tag{4.7}$$

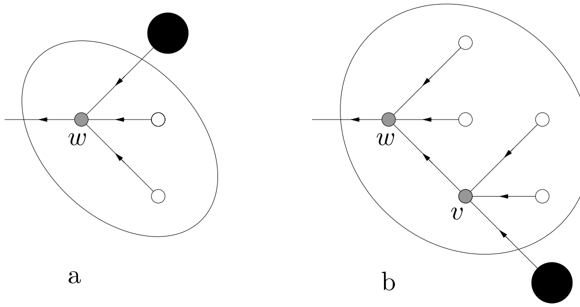

Fig. 4.1.

where we recall that one has $(n'_{\mathbb{V}}, m'_{\mathbb{V}}) = (n_{\mathbb{V}}, m_{\mathbb{V}}) = (0,0)$ if $\mathbb{V} \in V(\theta)$ is of w-type.

If a cluster has only one entering line ℓ_T^2 and (n,m) is the momentum of such a line, for any line $\ell \in L(T)$ one can write $(n_\ell, m_\ell) = (n_\ell^0, m_\ell^0) + \eta_\ell(n,m)$, where $\eta_\ell = 1$ if the line ℓ is along the path connecting the external lines of T and $\eta_\ell = 0$ otherwise.

Definition 6. *A cluster T with only one incoming line ℓ_T^2 such that one has*

$$n_{\ell_T^1} = n_{\ell_T^2} \quad \textit{and} \quad m_{\ell_T^1} = \pm m_{\ell_T^2} \tag{4.8}$$

will be called a **self-energy graph** *or a* **resonance***. In such a case we shall call a* **resonant line** *the line ℓ_T^1, and we shall refer to its momentum as the momentum of the self-energy graph.*

Examples of self-energy graphs T with $k_T = 1$ are represented in Fig. 4.1. The lines crossing the encircling bubbles are the external lines, and they are on scales higher than the lines internal to the bubbles. There are 9 self-energy graphs with $k_T = 1$: they are all obtained by the two which are drawn in Fig. 4.1, simply by considering all possible inequivalent trees.

Definition 7. *The* **value of the self-energy graph** *T with momentum (n,m) associated with the line ℓ_T^2 is defined as*

$$\mathcal{V}_T^h(\omega n, m) = \Big(\prod_{\ell \in T} g_\ell^{(h_\ell)}\Big)\Big(\prod_{\mathbb{V} \in V(T)} \eta_{\mathbb{V}}\Big)\Big(\prod_{\mathbb{V} \in E(T)} V_{\mathbb{V}}\Big), \tag{4.9}$$

where $h = h_T^{(e)}$ is the minimum between the scales of the two external lines of T (they can differ at most by a unit), and one has

$$\begin{aligned} n(T) &\equiv \sum_{\mathbb{V} \in V(T)} \big(n'_{\mathbb{V}} + n_{\mathbb{V}}\big) + \sum_{\mathbb{V} \in E(T)} n'_{\mathbb{V}} = 0, \\ m(T) &\equiv \sum_{\mathbb{V} \in V(T)} \big(m'_{\mathbb{V}} + m_{\mathbb{V}}\big) + \sum_{\mathbb{V} \in E(T)} m'_{\mathbb{V}} + \sum_{\substack{\mathbb{W} \in V_w^1(T) \\ c_{\mathbb{W}} = b}} \big(-2m_{\ell_{\mathbb{W}}}\big) \in \{0, 2m\}, \end{aligned} \tag{4.10}$$

by definition of self-energy graph; one says that T is a resonance of type $c = a$ when $m(T) = 0$ and a resonance of type $c = b$ when $m(T) = 2m$.

Definition 8. *Given a tree θ, we shall denote by $N_h(\theta)$ the number of lines with scale h, and by $C_h(\theta)$ the number of clusters with scale h.*

Then the product of propagators appearing in (4.4) can be bounded as

$$\Big| \prod_{\ell \in L(\theta)} g_\ell^{(h_\ell)} \Big| \le \Big(\prod_{h=0}^{\infty} 2^{h N_h(\theta)} \Big) \Big(\prod_{\substack{\ell \in L(\theta) \\ \gamma_\ell = w}} \frac{\chi_{h_\ell}(|\omega n_\ell| - \tilde{\omega}_{m_\ell})}{|\omega n_\ell| + \tilde{\omega}_{m_\ell}} \Big) \Big(\prod_{\substack{\ell \in L(\theta) \\ \gamma_\ell = v,\, n_\ell \neq 0}} \frac{1}{|\omega n_\ell|} \Big), \tag{4.11}$$

and this will be used later.

Lemma 4. *Assume $0 < C_0 < 1/2$ and that there is a constant C_1 such that one has $|\tilde{\omega}_m - |m|| \le C_1 \varepsilon / |m|$. If ε is small enough for any tree $\theta \in \Theta^{(k)}_{n,m}$ and for any line ℓ on a scale $h_\ell \ge 0$ one has $\min\{m_\ell, n_\ell\} \ge 1/2\varepsilon$.*

Proof. If a line ℓ with momentum (n, m) is on scale $h \ge 0$ then one has

$$\begin{aligned} \frac{1}{2} > C_0 \ge ||\omega n| - \tilde{\omega}_m| &\ge \left| \left(\sqrt{1-\varepsilon} - 1 \right) |n| + (|n| - |m|) - C_1 \varepsilon / |m| \right| \\ &\ge \left| \left| \frac{\varepsilon\, |n|}{1 + \sqrt{1-\varepsilon}} - (|n| - |m|) \right| - C_1 \varepsilon / |m| \right|, \end{aligned} \tag{4.12}$$

with $|n| \neq |m|$, hence $|n - m| \ge 1$, so that $|n| \ge 1/2\varepsilon$. Moreover one has $||\omega n| - \tilde{\omega}_m| \le 1/2$ and $\tilde{\omega}_m - |m| = O(\varepsilon)$, and one obtains also $|m| > 1/2\varepsilon$. □

Lemma 5. *Define h_0 such that $2^{h_0} \le 16 C_0/\varepsilon < 2^{h_0+1}$, and assume that there is a constant C_1 such that one has $|\tilde{\omega}_m - |m|| \le C_1 \varepsilon / |m|$. If ε is small enough for any tree $\theta \in \Theta^{(k)}_{n,m}$ and for all $h \ge h_0$ one has*

$$N_h(\theta) \le 4 K(\theta) 2^{(2-h)/\tau} - C_h(\theta) + S_h(\theta) + M_h^{\nu}(\theta), \tag{4.13}$$

where $K(\theta)$ is defined in (4.7), while $S_h(\theta)$ is the number of self-energy graphs T in θ with $h_T^{(e)} = h$ and $M_h^{\nu}(\theta)$ is the number of ν-vertices in θ such that the maximum scale of the two external lines is h.

Proof. We prove inductively the bound

$$N_h^*(\theta) \le \max\{0, 2K(\theta) 2^{(2-h)/\tau} - 1\}, \tag{4.14}$$

where $N_h^*(\theta)$ is the number of non-resonant lines in $L(\theta)$ on scale $h' \ge h$.

First of all note that for a tree θ to have a line on scale h the condition $K(\theta) > 2^{(h-1)/\tau}$ is necessary, by the first Diophantine conditions in (2.33). This means that one can have $N_h^*(\theta) \ge 1$ only if $K = K(\theta)$ is such that $K > k_0 \equiv 2^{(h-1)/\tau}$: therefore for values $K \le k_0$ the bound (4.14) is satisfied.

If $K = K(\theta) > k_0$, we assume that the bound holds for all trees θ' with $K(\theta') < K$. Define $E_h = 2^{-1}(2^{(2-h)/\tau})^{-1}$: so we have to prove that $N_h^*(\theta) \le \max\{0, K(\theta) E_h^{-1} - 1\}$.

Call ℓ the root line of θ and $\ell_1, \dots, \ell_m$ the $m \ge 0$ lines on scale $\ge h$ which are the closest to ℓ (i.e. such that no other line along the paths connecting the lines $\ell_1, \dots, \ell_m$ to the root line is on scale $\ge h$).

If the root line ℓ of θ is either on scale $< h$ or on scale $\geq h$ and resonant, then

$$N_h^*(\theta) = \sum_{i=1}^{m} N_h^*(\theta_i), \tag{4.15}$$

where θ_i is the subtree with ℓ_i as root line, hence the bound follows by the inductive hypothesis.

If the root line ℓ has scale $\geq h$ and is non-resonant, then $\ell_1, \dots, \ell_m$ are the entering line of a cluster T.

By denoting again with θ_i the subtree having ℓ_i as root line, one has

$$N_h^*(\theta) = 1 + \sum_{i=1}^{m} N_h^*(\theta_i), \tag{4.16}$$

so that the bound becomes trivial if either $m = 0$ or $m \geq 2$.

If $m = 1$ then one has a cluster T with two external lines ℓ and ℓ_1, which are both with scales $\geq h$; then

$$\left| |\omega n_\ell| - \tilde{\omega}_{m_\ell} \right| \leq 2^{-h+1} C_0, \qquad \left| |\omega n_{\ell_1}| - \tilde{\omega}_{m_{\ell_1}} \right| \leq 2^{-h+1} C_0, \tag{4.17}$$

and recall that T is not a self-energy graph.

Note that the validity of both inequalities in (4.17) for $h \geq h_0$ imply that one has $|n_\ell - n_{\ell_1}| \neq |m_\ell \pm m_{\ell_1}|$, as we are going to show.

By Lemma 4 we know that one has $\min\{m_\ell, n_\ell\} \geq 1/2\varepsilon$. Then from (4.17) we have, for some $\eta_\ell, \eta_{\ell_1} \in \{\pm 1\}$,

$$2^{-h+2} C_0 \geq \left| \omega (n_\ell - n_{\ell_1}) + \eta_\ell \tilde{\omega}_{m_\ell} + \eta_{\ell_1} \tilde{\omega}_{m_{\ell_1}} \right|, \tag{4.18}$$

so that if one had $|n_\ell - n_{\ell_1}| = |m_\ell \pm m_{\ell_1}|$ we would obtain for ε small enough

$$2^{-h+2} C_0 \geq \frac{\varepsilon}{1 + \sqrt{1-\varepsilon}} \left| n_\ell - n_{\ell_1} \right| - \frac{C_1 \varepsilon}{|m_\ell|} - \frac{C_1 \varepsilon}{|m_{\ell_1}|} \geq \frac{\varepsilon}{2} - 4 C_1 \varepsilon^2 > \frac{\varepsilon}{4}, \tag{4.19}$$

which is contradictory with $h \leq h_0$; hence one has $|n_\ell - n_{\ell_1}| \neq |m_\ell \pm m_{\ell_1}|$.

Then, by (4.17) and for $|n_\ell - n_{\ell_1}| \neq |m_\ell \pm m_{\ell_1}|$, one has, for suitable $\eta_\ell, \eta_{\ell_1} \in \{+, -\}$,

$$2^{-h+2} C_0 \geq \left| \omega (n_\ell - n_{\ell_1}) + \eta_\ell \tilde{\omega}_{m_\ell} + \eta_{\ell_1} \tilde{\omega}_{m_{\ell_1}} \right| \geq C_0 |n_\ell - n_{\ell_1}|^{-\tau}, \tag{4.20}$$

where the second Diophantine conditions in (2.33) have been used. Hence $K(\theta) - K(\theta_1) > E_h$, which, inserted into (4.16) with $m = 1$, gives, by using the inductive hypothesis,

$$\begin{aligned} N_h^*(\theta) &= 1 + N_h^*(\theta_1) \leq 1 + K(\theta_1) E_h^{-1} - 1 \\ &\leq 1 + \Big(K(\theta) - E_h \Big) E_h^{-1} - 1 \leq K(\theta) E_h^{-1} - 1, \end{aligned} \tag{4.21}$$

hence the bound is proved also if the root line is on scale $\geq h$.

In the same way one proves that, if we denote with $C_h(\theta)$ the number of clusters on scale h, one has

$$C_h(\theta) \leq \max\{0, 2K(\theta) 2^{(2-h)/\tau} - 1\}; \tag{4.22}$$

see [23] for details. $\square$

Note that the argument above is very close to [23]: this is due to the fact that the external lines of any self-energy graph T are both w-lines, so that the only effect of the presence of the v-lines and of the nodes of v-type is in the contribution to $K(T)$.

The following lemma deals with the lines on scale $h < h_0$.

Lemma 6. *Let h_0 be defined as in Lemma 2 and $C_0 < 1/2$, and assume that there is a constant C_1 such that one has $|\tilde{\omega}_m - |m|| \le C_1\varepsilon$. If ε is small enough for $h < h_0$ one has $|g_\ell^{(h)}| \le 32$.*

Proof. Either if $h \neq h_\ell$ or $h = h_\ell = -1$ the bound is trivial. If $h = h_\ell \ge 0$ one has

$$g_\ell^{(h)} = \frac{\chi_h(|\omega n_\ell| - \tilde{\omega}_{m_\ell})}{-|\omega n_\ell| + \tilde{\omega}_m} \, \frac{1}{|\omega n_\ell| + \tilde{\omega}_m}, \tag{4.23}$$

where $|\omega n_\ell| + \tilde{\omega}_m \ge 1/2\varepsilon$ by Lemma 4. Then one has

$$\frac{1}{|\omega n_\ell| + \tilde{\omega}_m} \le 2\varepsilon, \tag{4.24}$$

which, inserted in (4.23), gives $|g_\ell^{(h)}| \le 2^{h+2}\varepsilon/C_0 \le 32$, so that the lemma is proved. $\square$

5. The Renormalized Expansion

It is an immediate consequence of Lemma 5 and Lemma 6 that all the trees θ with no self-energy graphs or ν-vertices admit a bound $O(C^k\varepsilon^k)$, where C is a constant. However the generic tree θ with $S_h(\theta) \neq 0$ admits a much worse bound, namely $O(C^k\varepsilon^k k!^\alpha)$, for some constant α, and the presence of factorials prevent us to prove the convergence of the series; in KAM theory this is called *accumulation of small divisors*. It is convenient then to consider another expansion for $u_{n,m}$, which is essentially a resummation of the one introduced in Sects. 3 and 4.

We define the set $\Theta_{n,m}^{(k)\mathcal{R}}$ of *renormalized trees*, which are defined as $\Theta_{n,m}^{(k)}$ except that the following rules are added.

(9) To each self-energy graph (with $|m| \ge 1$) the $\mathcal{R} = \mathbb{1} - \mathcal{L}$ operation is applied, where $\mathcal{L}$ acts on the self-energy graphs in the following way, for $h \ge 0$ and $|m| \ge 1$,

$$\mathcal{L}\mathcal{V}_T^h(\omega n, m) = \mathcal{V}_T^h(\operatorname{sgn}(n)\,\tilde{\omega}_m, m), \tag{5.1}$$

$\mathcal{R}$ is called a *regularization operator*; its action simply means that each self-energy graph $\mathcal{V}_T^h(\omega n, m)$ must be replaced by $\mathcal{R}\mathcal{V}_T^h(\omega n, m)$.

(10) With the nodes $\mathbb{V}$ of w-type with $s_{\mathbb{V}} = 1$ (which we still call ν-vertices) and with $h \ge 0$ the minimal scale among the lines entering or exiting $\mathbb{V}$, we associate a factor $2^{-h}\nu_{h,m}^{(c)}$, $c = a, b$, where (n, m) and $(n, \pm m)$, with $|m| \ge 1$, are the momenta of the lines and a corresponds to the sign $+$ and b to the sign $-$ in $\pm m$.

(11) The set $\{h_\ell\}$ of the scales associated with the lines $\ell \in L(\theta)$ must satisfy the following constraint (which we call *compatibility*): fixed (n_ℓ, m_ℓ) for any $\ell \in L(\theta)$ and replaced $\mathcal{R}$ with $\mathbb{1}$ at each self-energy graph, one must have $\chi_{h_\ell}(|\omega n_\ell| - \tilde{\omega}_{m_\ell}) \neq 0$.

(12) The factors $C^{(k_{\mathbb{V}})}$ in (3.3) are replaced with Γ, to be considered a parameter.

The set $\Theta^{(k)\mathcal{R}}_{n,m}$ is defined as $\Theta^{(k)}_{n,m}$ with the new rules and with the constraint that the order k is given by $k = |V_w(\theta)| + |V_v^1(\theta)|$.

We consider the following expansion

$$\tilde{u}_{n,m} = \sum_{k=1}^{\infty} \mu^k \sum_{\theta \in \Theta^{(k)\mathcal{R}}_{n,m}} \mathrm{Val}(\theta), \tag{5.2}$$

where, for $|m| \geq 1$ and $h \geq 0$, $\nu^{(c)}_{h,m}$ is given by

$$2^{-h} \mu \nu^{(c)}_{h,m} = \mu \nu^{(c)}_m + \frac{1}{2} \sum_{\sigma=\pm} \sum_{T \in \mathcal{T}^{(c)}_{<h}} \mu^{k_T} \mathcal{V}^h_T(\sigma \tilde{\omega}_m, m), \tag{5.3}$$

with $c = a, b$, and $\mathcal{T}^{(c)}_{<h}$ denoting the set of self-energy graphs T of type c (see item (6) in Sect. 3) with $h_T < h$, and Γ is determined by the self-consistence equation

$$\Gamma = r_0^{-1} \sum_{k=1}^{\infty} \mu^{k-1} \sum_{\theta \in \Theta^{(k)\mathcal{R}}_{n,n}}{}^{*} a_{0,-n} \mathrm{Val}(\theta), \tag{5.4}$$

with $*$ denoting the same constraint as in (3.10).

We shall set also $\nu^{(c)}_m = \nu^{(c)}_{-1,m}$. Note that $\mathcal{V}^h_T(\sigma\tilde{\omega}_m, m)$ is independent of σ.

Calling $L_0(\theta)$, $V_0(\theta)$, $E_0(\theta)$ the set of lines, node and end-points not contained in any self-energy graph, and $S_0(\theta)$ the *maximal self-energy graphs*, *i.e.* the self-energy graphs which are not contained in any self-energy graphs, we can write $\mathrm{Val}(\theta)$ in (5.2) as

$$\mathrm{Val}(\theta) = \Big(\prod_{\ell \in L_0(\theta)} g_\ell^{(h_\ell)} \Big) \Big(\prod_{\mathbb{V} \in V_0(\theta)} \eta_{\mathbb{V}} \Big) \Big(\prod_{\mathbb{V} \in E_0(\theta)} V_{\mathbb{V}} \Big) \Big(\prod_{T \in S_0(\theta)} \mathcal{R}\mathcal{V}^{h^e_T}_T(\omega n_{\ell_T}, m_{\ell_T}) \Big), \tag{5.5}$$

and by definition

$$\mathcal{R}\mathcal{V}^{h^e_T}_T(\omega n_{\ell_T}, m_{\ell_T}) = \mathcal{V}^{h^e_T}_T(\omega n_{\ell_T}, m_{\ell_T}) - \mathcal{V}^{h^e_T}_T(\mathrm{sgn}(n_{\ell_T})\, \tilde{\omega}_{m_{\ell_T}}, m_{\ell_T}), \tag{5.6}$$

and $\mathcal{V}^{h^e_T}_T(\omega n_{\ell_T}, m_{\ell_T})$ is given by

$$\mathcal{V}^{h^e_T}_T(\omega n_{\ell_T}, m_{\ell_T}) = \Big(\prod_{\ell \in L_0(T)} g_\ell^{(h_\ell)} \Big) \Big(\prod_{\mathbb{V} \in V_0(T)} \eta_{\mathbb{V}} \Big) \Big(\prod_{\mathbb{V} \in E_0(T)} V_{\mathbb{V}} \Big) \Big(\prod_{T' \in S_0(T)} \mathcal{R}\mathcal{V}^{h^e_{T'}}_{T'}(\omega n_{\ell_{T'}}, m_{\ell_{T'}}) \Big). \tag{5.7}$$

First (Lemma 7) we will show that the expansion (5.2) is well defined, for $\nu_{h,m}, \Gamma = O(\varepsilon)$; then (Lemmas 8 and 9) we show that under the same conditions also the r.h.s. of (5.3) is well defined; moreover (Lemma 10) we prove by using (5.3) that it is indeed possible to choose $\nu^{(c)}_m$ such that $\nu_{h,m} = O(\varepsilon)$ for any h; then (Lemmas 11 and 12) we show that (5.2) admit a solution $\Gamma = O(\varepsilon)$; finally (Lemma 13) we show that indeed (5.2) solves the last of (1.14) and (2.2); this completes the proof of Proposition 1. In the next section we will solve the implicit function problem of (2.1), thus completing the proof of Theorem 1.

We start from the following lemma stating that, if the $\nu^{(c)}_{h,m}$ and Γ functions are bounded, then the expansion (5.2) is well defined.

Lemma 7. *Assume that there exist a constant C such that one has $|\Gamma| \le C\varepsilon$ and $|\nu_{h,m}^{(c)}| \le C\varepsilon$, with $c = a, b$, for all $|m| \ge 1$ and all $h \ge 0$. Then for all $\mu_0 > 0$ there exists $\varepsilon_0 > 0$ such that for all $|\mu| \le \mu_0$ and for all $0 < \varepsilon < \varepsilon_0$ and for all $(n, m) \in \mathbb{Z}^2$ one has*

$$\left|\tilde{u}_{n,m}\right| \le D_0 \varepsilon \mu \, e^{-\kappa(|n|+|m|)/4}, \tag{5.8}$$

where D_0 is a positive constant. Moreover $u_{n,m}$ is analytic in μ and in the parameters $\nu_{m',h'}^{(c)}$, with $c = a, b$ and $|m'| \ge 1$.

Proof. In order to take into account the $\mathcal{R}$ operation we write (5.6) as

$$\mathcal{R}\mathcal{V}_T^{h_T^e}(\omega n_{\ell_T}, m_{\ell_T}) = \left(\omega n_{\ell_T} - \tilde{\omega}_{m_{\ell_T}}\right) \int_0^1 \mathrm{d}t \, \partial \mathcal{V}_T^{h_T^e}(\omega n_{\ell_T} + t(\omega n_{\ell_T} - \tilde{\omega}_{m_{\ell_T}}), m_{\ell_T}), \tag{5.9}$$

where ∂ denotes the derivative with respect to the argument $\omega n_{\ell_T} + t(\omega n_{\ell_T} - \tilde{\omega}_{m_{\ell_T}})$.

By (5.7) we see that the derivatives can be applied either on the propagators in $L_0(T)$, or on the $\mathcal{R}\mathcal{V}_{T'}^{h_{T'}^e}$. In the first case there is an extra factor $2^{-h_T^{(e)}+h_T}$ with respect to the bound (4.11): $2^{-h_T^{(e)}}$ is obtained from $\omega n_{\ell_T} - \tilde{\omega}_{m_{\ell_T}}$ while $\partial g^{(h_T)}$ is bounded proportionally to 2^{2h_T}; in the second case note that $\partial_t \mathcal{R}\mathcal{V}_{T'}^{h_{T'}^e} = \partial_t \mathcal{V}_{T'}^{h_{T'}^e}$ as $\mathcal{L}\mathcal{V}_{T'}^{h_{T'}^{(e)}}$ is independent of t; if the derivative acts on the propagator of a line $\ell \in L(T)$, we get a gain factor

$$2^{-h_T^{(e)}+h_{T'}} \le 2^{-h_T^{(e)}+h_T} 2^{-h_{T'}^{(e)}+h_{T'}}, \tag{5.10}$$

as $h_{T'}^{(e)} \le h_T$. We can iterate this procedure until all the $\mathcal{R}$ operations are applied on propagators; at the end (i) the propagators are derived at most one time; (ii) the number of terms so generated is $\le k$; (iii) with each self-energy graph T a factor $2^{-h_T^{(e)}+h_T}$ is associated.

Assuming that $|\nu_{h,m}^{(c)}| \le C\varepsilon$ and $|\Gamma| \le C\varepsilon$, for any θ one obtains, for a suitable constant $\overline{D}$,

$$\begin{aligned}
&|\mathrm{Val}(\theta)| \le \varepsilon^{|V_w(\theta)|+|V_v^{(1)}(\theta)|} \overline{D}^{|V(\theta)|} \\
&\Big(\prod_{h=h_0}^{\infty} \exp\Big[h \log 2 \Big(4K(\theta) 2^{-(h-2)/\tau} - C_h(\theta) + S_h(\theta) + M_h^{\nu}(\theta) \Big) \Big] \Big) \\
&\Big(\prod_{\substack{T \in \mathcal{S}(\theta) \\ h_T^{(e)} \ge h_0}} 2^{-h_T^{(e)}+h_T} \Big) \Big(\prod_{h=h_0}^{\infty} 2^{-h M_h^{\nu}(\theta)} \Big) \\
&\Big(\prod_{\mathbb{V} \in V(\theta) \cup E(\theta)} e^{-\kappa(|n_{\mathbb{V}}|+|n'_{\mathbb{V}}|)} \Big) \Big(\prod_{\mathbb{V} \in V(\theta) \cup E(\theta)} e^{-\kappa(|m_{\mathbb{V}}|+|m'_{\mathbb{V}}|)} \Big),
\end{aligned} \tag{5.11}$$

where the second line is a bound for $\prod_{h \ge h_0} 2^{hN_h(\theta)}$ and we have used that by item (12) $N_h(\theta)$ can be bounded through Lemma 5, and Lemma 4 has been used for the lines on scales $h < h_0$; moreover $\prod_{h=h_0}^{\infty} 2^{-hM_\nu^h(\theta)}$ takes into account the factors 2^{-h} arising from the running coupling constants $\nu_{h,m}^{(c)}$ and the action of $\mathcal{R}$ produces, as discussed above, the factor $\prod_{T \in \mathcal{S}(\theta)} 2^{-h_T^{(e)}+h_T}$. Then one has

$$\Big(\prod_{h=h_0}^{\infty} 2^{hS_h(\theta)} \Big) \Big(\prod_{T\in\mathcal{S}(\theta)} 2^{-h_T^{(e)}} \Big) = 1,$$
$$\Big(\prod_{h=h_0}^{\infty} 2^{-hC_h(\theta)} \Big) \Big(\prod_{T\in\mathcal{S}(\theta)} 2^{h_T} \Big) \le 1. \tag{5.12}$$

We have to sum the values of all trees, so we have to worry about the sum of the labels. Recall that a labeled tree is obtained from an unlabeled tree by assigning all the labels to the points and the lines: so the sum over all possible labeled trees can be written as sum over all unlabeled trees and of labels. For a fixed unlabeled tree θ with a given number of nodes, say N, we can assign first the mode labels $\{(n'_{\mathbb{V}}, m'_{\mathbb{V}}), (n_{\mathbb{V}}, m_{\mathbb{V}})\}_{v\in V(\theta)\cup E(\theta)}$, and we sum over all the other labels, which gives $4^{|V_v(\theta)|}$ (for the labels $j_{\mathbb{V}}$) times $2^{|L(\theta)|}$ (for the scale labels): then all the other labels are uniquely fixed. Then we can perform the sum over the mode labels by using the exponential decay arising from the node factors (3.3) and end-point factors (3.4). Finally we have to sum over the unlabeled trees, and this gives a factor 4^N [26]. By Lemma 3, one has $|V(\theta)| = |V_w(\theta)| + |V_v^{(1)}(\theta)| + |V_v^{(3)}(\theta)| \le 3(|V_w^{(3)}(\theta)| + |V_v^{(1)}(\theta)|)$, hence $N \le 3k$, so that $\sum_{\theta\in\Theta_{n,m}^{(k)\mathcal{R}}} |\mathrm{Val}(\theta)| \le D^k \varepsilon^k$, for some positive constant D.

Therefore, for fixed (n, m) one has

$$\sum_{k=1}^{\infty} \sum_{\theta\in\Theta_{n,m}^{(k)}} \mu^k |\mathrm{Val}(\theta)| \le D_0 \mu\varepsilon\, e^{-\kappa(|n|+|m|)/4}, \tag{5.13}$$

for some positive constant D_0, so that (5.8) is proved. □

From (5.3) we know that the quantities $\nu_{h,m}^{(c)}$, for $h \ge 0$ and $|m| \ge 1$, verify the recursive relations

$$\mu\nu_{h+1,m}^{(c)} = 2\mu\nu_{h,m}^{(c)} + \beta_{h,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{h',m'}^{(c')}\}), \tag{5.14}$$

where, by defining $\mathcal{T}_h^{(c)}$ as the set of self-energy graphs in $\mathcal{T}_{<h+1}^{(c)}$ which are on scale h, the *beta function*

$$\beta_{h,m}^{(c)} \equiv \beta_{h,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{h',m'}^{(c')}\}) = 2^{h+1}\frac{1}{2}\sum_{\sigma=\pm}\sum_{T\in\mathcal{T}_h^{(c)}} \mu^{k_T}\mathcal{V}_T^{h+1}(\sigma\tilde{\omega}_m, m), \tag{5.15}$$

depends only on the scales $h' \le h$.

In order to obtain a bound on the beta function, hence on the running coupling constants, we need to bound $\mathcal{V}_T^{h+1}(\pm\tilde{\omega}_m, m)$ for $T \in \mathcal{T}_h^{(c)}$. We define $\widetilde{\Theta}_{n,m}^{(k)\mathcal{R}}$ as the set $\Theta_{n,m}^{(k)\mathcal{R}}$ introduced before, but by changing item (7) into the following one:

(7′) We divide the set $\widetilde{E}(\theta)$ of end-points into two sets $E(\theta)$ and $E_0(\theta)$. With each end-point $\mathbb{V} \in E(\theta)$ we associate a first mode label $(n'_{\mathbb{V}}, m'_{\mathbb{V}})$, with $|m'_{\mathbb{V}}| = |n'_{\mathbb{V}}|$, a second mode label $(0, 0)$ and an *end-point factor* $V_{\mathbb{V}} = (-1)^{1+\delta_{n'_{\mathbb{V}},m'_{\mathbb{V}}}} a_{0,n'_{\mathbb{V}}}$, while $E_0(\theta)$ is either the empty set or a single end-point $\mathbb{v}_0$, and, in the latter case, with the end-point $\mathbb{V} \in E_0(\theta)$ we associate a first mode label $(\overline{\omega}_{m_{\mathbb{V}}}, n_{\mathbb{V}})$, where $\overline{\omega}_{m_{\mathbb{V}}} = \tilde{\omega}_{m_{\mathbb{V}}}/\omega$, a second mode label $(0, 0)$ and an *end-point factor* $V_{\mathbb{V}} = 1$.

Then we have the following generalization of Lemma 4.

Lemma 8. *If ε is small enough for any tree $\theta \in \widetilde{\Theta}^{(k)\mathcal{R}}_{n,m}$ one has*

$$N_h(\theta) \le 4K(\theta)2^{(2-h)/\tau} - C_h(\theta) + S_h(\theta) + M^{\nu}_h(\theta), \tag{5.16}$$

where the notations are as in Lemma 4.

Proof. Lemma 4 holds for $E_0(\theta) = 0$; we mimic the proof of Lemma 4 proving that

$$N^*_h(\theta) \le \max\{0, 2K(\theta)2^{(2-h)/\tau}\}, \tag{5.17}$$

for all trees θ with $E_0(\theta) \ne \emptyset$, again by induction on $K(\theta)$.

For any line $\ell \in L(\theta)$ set $\eta_\ell = 1$ if the line is along the path connecting $\mathtt{v}_0$ to the root and $\eta_\ell = 0$ otherwise, and write

$$n_\ell = n^0_\ell + \eta_\ell \overline{\omega}_m, \qquad m_\ell = m^0_\ell + \eta_\ell m, \tag{5.18}$$

which implicitly defines n^0_ℓ and m^0_ℓ.

Define $k_0 = 2^{(h-1)/\tau}$. One has $N^*_h(\theta) = 0$ for $K(\theta) < k_0$, because if a line $\bar\ell \in L(\theta)$ is indeed on scale h then $|\omega n_{\bar\ell} - \tilde\omega_{m_{\bar\ell}}| < C_0 2^{1-h}$, so that (5.18) and the Diophantine conditions imply

$$K(\theta) \ge \left| n^0_{\bar\ell} \right| > 2^{(h-1)/\tau} \equiv k_0. \tag{5.19}$$

Then, for $K \ge k_0$, we assume that the bound (5.17) holds for all $K(\theta) = K' < K$, and we show that it follows also for $K(\theta) = K$.

If the root line ℓ of θ is either on scale $< h$ or on scale $\ge h$ and resonant, the bound (5.17) follows immediately from the bound (4.13) and from the inductive hypothesis.

The same occurs if the root line is on scale $\ge h$ and non-resonant, and, by calling $\ell_1, \dots, \ell_m$ the lines on scale $\ge h$ which are the closest to ℓ, one has $m \ge 2$: in fact in such a case at least $m - 1$ among the subtrees $\theta_1, \dots, \theta_m$ having $\ell_1, \dots, \ell_m$, respectively, as root lines have $E_0(\theta_i) = \emptyset$, so that we can write, by (4.13) and by the inductive hypothesis,

$$N^*_h(\theta) = 1 + \sum_{i=1}^{m} N^*_h(\theta_i) \le 1 + E^{-1}_h \sum_{i=1}^{m} K(\theta_i) - (m-1) \le E_h K(\theta), \tag{5.20}$$

so that (5.17) follows.

If $m = 0$ then $N^*_h(\theta) = 1$ and $K(\theta)2^{(2-h)/\tau} \ge 1$ because one must have $K(\theta) \ge k_0$.

So the only non-trivial case is when one has $m = 1$. If this happens ℓ_1 is, by construction, the root line of a tree θ_1 such that $K(\theta) = K(T) + K(\theta_1)$, where T is the cluster which has ℓ and ℓ_1 as external lines and $K(T)$, defined in (4.7), satisfies the bound $K(T) \ge |n_{\ell_1} - n_\ell|$.

Moreover, if $E_0(\theta_1) \ne \emptyset$, one has

$$\begin{aligned} \left| |\omega n^0_\ell + \tilde\omega_m| - \tilde\omega_{m_\ell} \right| &\le 2^{-h+1} C_0, \\ \left| |\omega n^0_{\ell_1} + \tilde\omega_m| - \tilde\omega_{m_{\ell_1}} \right| &\le 2^{-h+1} C_0, \end{aligned} \tag{5.21}$$

so that, for suitable $\eta_\ell, \eta_{\ell_1} \in \{-, +\}$, we obtain

$$2^{-h+2}C_0 \geq \left|\omega(n_\ell^0 - n_{\ell_1}^0) + \eta_\ell \tilde{\omega}_{m_\ell} + \eta_{\ell_1}\tilde{\omega}_{m_{\ell_1}}\right| \geq C_0|n_\ell^0 - n_{\ell_1}^0|^{-\tau} \equiv C_0|n_\ell - n_{\ell_1}|^{-\tau}, \tag{5.22}$$

by the second Diophantine conditions in (2.33), as the quantities $\tilde{\omega}_m$ appearing in (5.21) cancel out. Therefore one obtains by the inductive hypothesis

$$N_h^*(\theta) \leq 1 + K(\theta_1)E_h^{-1} \leq 1 + K(\theta)E_h^{-1} - K(T)E_h^{-1} \leq K(\theta)E_h^{-1}, \tag{5.23}$$

hence the first bound in (5.17) is proved.

If $E_0(\theta_1) = \emptyset$, one has

$$N_h^*(\theta) \leq 1 + K(\theta_1)E_h^{-1} - 1 \leq 1 + K(\theta)E_h^{-1} - 1 \leq K(\theta)E_h^{-1}, \tag{5.24}$$

and (5.17) follows also in such a case. □

The following bound for $\mathcal{V}_T^{h+1}(\pm\tilde{\omega}_m, m)$, $h \geq h_0$, can then be obtained.

Lemma 9. *Assume that there exists a constant C such that one has $|\Gamma| \leq C\varepsilon$ and $|\nu_{h,m}^{(c)}| \leq C\varepsilon$, with $c = a, b$, for all $|m| \geq 1$ and all $h \geq 0$. Then if ε is small enough for all $h \geq 0$ and for all $T \in \mathcal{T}_h^{(c)}$ one has*

$$|\mathcal{V}_T^{h+1}(\pm\tilde{\omega}_m, m)| \leq B^{|V(T)|}e^{-\kappa 2^{(h-1)/\tau}/4}e^{-\kappa K(T)/4}\varepsilon^{|V_v^{(1)}(T)|+|V_w(T)|}, \tag{5.25}$$

where B is a constant and $K(T)$ is defined in (4.7).

Proof. By using Lemma 7 we obtain for all $T \in \mathcal{T}_h^{(c)}$ and assuming $h \geq h_0$ we get the bound

$$\begin{aligned}&\left|\mathcal{V}_T^{h+1}(\pm\tilde{\omega}_m, m)\right| \leq \overline{B}^{|V(T)|}\varepsilon^{|V_v^{(1)}(T)|+|V_w(T)|}\\ &\prod_{h'=h_0}^{h} \exp\left[4K(T)\log 2h'2^{(2-h')/\tau} - C_{h'}(T) + S_{h'}(T) + M_{h'}^{\nu}(T)\right]\\ &\Big(\prod_{\substack{T'\subset T\\ h_T^{(\varepsilon)}\geq h_0}} 2^{-h_T^{(e)}+h_T}\Big)\Big(\prod_{h'=h_0}^{h} 2^{-h'M_{h'}^{\nu}(T)}\Big)e^{-\kappa|K(T)|/2},\end{aligned} \tag{5.26}$$

where $\overline{B}$ is a suitable constant. If $h < h_0$ the bound trivializes as the r.h.s. reduces simply to $C^{|V(T)|}|\varepsilon|^{|V_v^{(1)}(T)|+|V_w(T)|}e^{-\frac{\kappa}{2}|K(T)|}$. The main difference with respect to Lemma 6 is that, given a self-energy graph $T \in \mathcal{T}_h^{(c)}$, there is at least a line $\ell \in L(T)$ on scale $h_\ell = h$ and with propagator

$$\frac{1}{-\omega^2(n_\ell^0 + \eta_\ell\overline{\omega}_m)^2 + \tilde{\omega}^2_{m_\ell^0+\eta_\ell m}}, \tag{5.27}$$

where $\eta_\ell = 1$ if the line ℓ belongs to the path of lines connecting the entering line (carrying a momentum (n, m)) of T with the line coming out of T, and $\eta_\ell = 0$ otherwise. Then one has by the Mel'nikov conditions

$$C_0|n_\ell^0|^{-\tau} \le \left|\omega n_\ell^0 + \eta_\ell \tilde{\omega}_m \pm \tilde{\omega}_{m_\ell^0+\eta_\ell m}\right| \le C_0 2^{-h+1}, \tag{5.28}$$

so that $|n_\ell^0| \ge 2^{(h-1)/\tau}$. On the other hand one has $|n_\ell^0| \le K(T)$, hence $K(T) \ge 2^{(h-1)/\tau}$; so we get the bound (5.25). $\square$

It is an immediate consequence of the above lemma that for all $\mu_0 > 0$ there exists $\varepsilon_0 > 0$ such that for all $|\mu| \le \mu_0$ and $0 < \varepsilon < \varepsilon_0$ one has $|\beta_{h,m}^{(c)}| \le B_1\varepsilon|\mu|$, with B_1 a suitable constant.

We have then proved convergence *assuming that* the parameters $\nu_{h,m}$ and Γ are bounded; we have to show that this is actually the case, if the $\nu_m^{(c)}$ in (2.34) are chosen in a proper way.

We start proving that it is possible to choose $\nu^{(c)} = \{\nu_m^{(c)}\}_{|m|\ge 1}$ such that, for a suitable positive constant C, one has $|\nu_{h,m}^{(c)}| \le C\varepsilon$ for all $h \ge 0$ and for all $|m| \ge 1$.

For any sequence $a \equiv \{a_m\}_{|m|\ge 1}$ we introduce the norm

$$\|a\|_\infty = \sup_{|m|\ge 1} |a_m|. \tag{5.29}$$

Then we have the following result.

Lemma 10. *Assume that there exists a constant C such that $|\Gamma| \le C\varepsilon$. Then for all $\mu_0 > 0$ there exists $\varepsilon_0 >$ such that for all $|\mu| \le \mu_0$ and for all $0 < \varepsilon < \varepsilon_0$ there is a family of intervals $I_{c,m}^{(\bar h)}$, $\bar h \ge 0$, $|m| \ge 1$, $c = a, b$, such that $I_{c,m}^{(\bar h+1)} \subset I_{c,m}^{(\bar h)}$, $|I_{c,m}^{\bar h}| \le 2\varepsilon(\sqrt{2})^{-(\bar h+1)}$ and, if $\nu_m^{(c)} \in I_{c,m}^{(\bar h)}$, then*

$$\|\nu_h^{(c)}\|_\infty \le D\varepsilon, \qquad \bar h \ge h \ge 0, \tag{5.30}$$

for some positive constant D. Finally one has $\nu_{h,-m}^{(c)} = \nu_{h,m}^{(c)}$, $c = a, b$, for all $\bar h \ge h \ge 0$ and for all $|m| \ge 1$. Therefore one has $\|\nu_h\|_\infty \le C\varepsilon$ for all $h \ge 0$, for some positive constant D; in particular $|\nu_m| \le D\varepsilon$ for all $m \ge 1$.

Proof. The proof is done by induction on $\bar h$. Let us define $J_{c,m}^{(h)} = [-\varepsilon, \varepsilon]$ and call $J^{(h)} = \times_{|m|\ge 1, c=a,b} J_{c,m}^{(h)}$ and $I^{(h)} = \times_{|m|\ge 1, c=a,b} I_{c,m}^{(h)}$.

We suppose that there exists $I^{(\bar h)}$ such that, if ν spans $I^{(\bar h)}$ then $\nu_{\bar h}$ spans $J^{(\bar h)}$ and $|\nu_{h,m}^{(c)}| \le D\varepsilon$ for $\bar h \ge h \ge 0$; we want to show that the same holds for $\bar h + 1$. Let us call $\tilde J^{(\bar h+1)}$ the interval spanned by $\{\nu_{\bar h+1,m}^{(c)}\}_{|m|\ge 1, c=a,b}$ when $\{\nu_m^{(c)}\}_{|m|\ge 1, c=a,b}$ span $I^{(\bar h)}$. For any $\{\nu_m^{(c)}\}_{|m|\ge 1, c=a,b} \in I^{(\bar h)}$ one has $\{\nu_{\bar h+1,m}\}_{|m|\ge 1, c=a,b} \in [-2\varepsilon - D\varepsilon^2, 2\varepsilon + D\varepsilon^2]$, where the bound (5.25) has been used. This means that $J^{(\bar h+1)}$ is strictly contained in $\tilde J^{(\bar h+1)}$. On the other hand it is obvious that there is a one-to-one correspondence between $\{\nu_m^{(c)}\}_{|m|>1, c=a,b}$ and the sequence $\{\nu_{h,m}^{(c)}\}_{|m|\ge 1, c=a,b}$, $\bar h + 1 \ge h \ge 0$. Hence there is a set $I^{(\bar h+1)} \subset I^{(\bar h)}$ such that, if $\{\nu_m^{(c)}\}_{|m|\ge 1, c=a,b}$ spans $I^{(\bar h+1)}$, then $\{\nu_{\bar h+1,m}^{(c)}\}_{|m|\ge 1, c=a,b}$ spans the interval $J^{(\bar h)}$ and, for ε small enough, $|\nu_h|_\infty \le C\varepsilon$ for $\bar h + 1 \ge h \ge 0$.

The previous computations also show that the inductive hypothesis is verified also for $\bar{h} = 0$ so that we have proved that there exists a decreasing sets of intervals $I^{(\bar{h})}$ such that if $\{\nu_m^{(c)}\}_{|m|>1,c=a,b} \in I^{(\bar{h})}$ then the sequence $\{\nu_{h,m}^{(c)}\}_{|m|\geq 1,c=a,b}$ is well defined for $h \leq \bar{h}$ and it verifies $|\nu_{h,m}^{(c)}| \leq C|\varepsilon|$. In order to prove the bound on the size of $I_{c,m}^{(\bar{h})}$ let us denote by $\{\nu_{h,m}^{(c)}\}_{|m|\geq 1,c=a,b}$ and $\{\nu_{h,m}^{\prime(c)}\}_{|m|\geq 1,c=a,b}$, $0 \leq h \leq \bar{h}$, the sequences corresponding to $\{\nu_m^{(c)}\}_{|m|\geq 1,c=a,b}$ and $\{\nu_m^{\prime(c)}\}_{|m|\geq 1,c=a,b}$ in $I^{(\bar{h})}$. We have

$$\mu\nu_{h+1,m}^{(c)} - \mu\nu_{h+1,m}^{\prime(c)} = 2\left(\mu\nu_{h,m}^{(c)} - \mu\nu_{h,m}^{\prime(c)}\right) + \beta_{h,m}^{(c)} - \beta_{h,m}^{\prime(c)}, \tag{5.31}$$

where $\beta_{h,m}^{(c)}$ and $\beta_{h,m}^{\prime(c)}$ are shorthands for the beta functions. Then, as $|\nu_k - \nu_k'|_\infty \leq |\nu_h - \nu_h'|_\infty$ for all $k \leq h$, we have

$$|\nu_h - \nu_h'|_\infty \leq \frac{1}{2}|\nu_{h+1} - \nu_{h+1}'|_\infty + D\varepsilon^2|\nu_h - \nu_h'|_\infty. \tag{5.32}$$

Hence if ε is small enough then one has

$$\|\nu - \nu'\|_\infty \leq (\sqrt{2})^{-(\bar{h}+1)}\|\nu_{\bar{h}} - \nu_{\bar{h}}'\|_\infty. \tag{5.33}$$

Since, by definition, if ν spans $I^{(\bar{h})}$, then $\nu_{\bar{h}}$ spans the interval $J^{(\bar{h})}$, of size $2|\varepsilon|$, the size of $I^{(\bar{h})}$ is bounded by $2|\varepsilon|(\sqrt{2})^{(-\bar{h}-1)}$.

Finally note that one can choose $\nu_m^{(c)} = \nu_{-m}^{(c)}$ and then $\nu_{h,m}^{(c)} = \nu_{h,-m}^{(c)}$ for any $|m| \geq 1$ and any $\bar{h} \geq h \geq 0$; this follows from the fact that the function $\beta_{k,m}^{(c)}$ in (5.15) is even under the exchange $m \to -m$; it depends on m through $\tilde{\omega}_m$ (which is an even function of m), through the end-points $v \in E(\theta)$ (which are odd under the exchange $m \to -m$; but their number must be even) and finally through $\nu_{k,m}^{(q-1)}$ which are assumed inductively to be even. $\square$

It will be useful to explicitly construct the $\nu_{h,m}^{(c)}$ by a contraction method. By iterating (5.14) we find, for $|m| \geq 1$,

$$\mu\nu_{h,m}^{(c)} = 2^{h+1}\left(\mu\nu_m^{(c)} + \sum_{k=-1}^{h-1} 2^{-k-2}\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\})\right), \tag{5.34}$$

where $\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\})$ depends on $\nu_{k',m'}$ with $k' \leq k-1$. If we put $h = \bar{h}$ in (5.34) we get

$$\mu\nu_m^{(c)} = -\sum_{k=-1}^{\bar{h}-1} 2^{-k-2}\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\}) + 2^{-\bar{h}-1}\mu\nu_{\bar{h},m}^{(c)} \tag{5.35}$$

and, combining (5.34) with (5.35), we find, for $\bar{h} > h \geq 0$,

$$\mu\nu_{h,m}^{(c)} = -2^{h+1}\left(\sum_{k=h}^{\bar{h}-1} 2^{-k-2}\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\})\right) + 2^{h-\bar{h}}\mu\nu_{\bar{h},m}^{(c)}. \tag{5.36}$$

The sequences $\{\nu_{h,m}^{(c)}\}_{|m|>1}$, $\bar{h} > h \geq h_0$, parameterized by $\{\nu_{\bar{h},m}^{(c)}\}_{|m|\geq 2}$ such that $\|\nu_{\bar{h}}^{(c)}\|_\infty \leq C\varepsilon$, can be obtained as the limit as $q \to \infty$ of the sequences $\{\nu_{h,m}^{(c)(q)}\}$, $q \geq 0$, defined recursively as

$$\begin{aligned}\mu\nu_{h,m}^{(c)(0)} &= 0,\\ \mu\nu_{h,m}^{(c)(q)} &= -2^{h+1}\left(\sum_{k=h}^{\bar{h}-1} 2^{-k-2}\beta_{k,m}^{(c)}(\tilde{\omega},\varepsilon,\{\nu_{k',m'}^{(c')(q-1)}\})\right) + 2^{h-\bar{h}}\mu\nu_{\bar{h},m}^{(c)}.\end{aligned} \tag{5.37}$$

In fact, it is easy to show inductively that, if ε is small enough, $\|\nu_h^{(q)}\|_\infty \leq C\varepsilon$, so that (5.25) is meaningful, and

$$\max_{0\leq h\leq\bar{h}} \|\nu_h^{(q)} - \nu_h^{(q-1)}\|_\infty \leq (C\varepsilon)^q. \tag{5.38}$$

For $q = 1$ this is true as $\nu_h^{(c)(0)} = 0$; for $q > 1$ it follows by the fact that $\beta_k^{(c)}(\tilde{\omega},\varepsilon,\{\nu_{k',m'}^{(c')(q-1)}\}) - \beta_k^{(c)}(\tilde{\omega},\varepsilon,\{\nu_{k',m'}^{(c')(q-2)}\})$ can be written as a sum of terms in which there are at least one ν-vertex, with a difference $\nu_{h'}^{(c')(q-1)} - \nu_{h'}^{(c')(q-2)}$, with $h' \geq k$, in place of the corresponding $\nu_{h'}^{(c')}$, and one node carrying an ε. Then $\nu_h^{(q)}$ converges as $q \to \infty$, for $\bar{h} < h \leq 1$, to a limit ν_h, satisfying the bound $\|\nu_h\|_\infty \leq C\varepsilon$. Since the solution is unique, it must coincide with one in Lemma 10.

We have then constructed a sequence of $\nu_{h,m}^{(c)}$ solving (5.36) for any $\bar{h} > 1$ and any $\nu_{\bar{h},m}^{(c)}$; we shall call $\nu_{h,m}^{(c)}(\Gamma)$ the solution of (5.36) with $\bar{h} = \infty$ and $\nu_{\infty,m}^{(c)} = 0$, to stress the dependence on Γ.

We will prove the following lemma.

Lemma 11. *Under the the same conditions of Lemma 10 it holds that for any $h \geq 0$,*

$$\|\nu_h(\Gamma^1) - \nu_h(\Gamma^2)\|_\infty \leq D\varepsilon|\Gamma^1 - \Gamma^2|, \tag{5.39}$$

for a suitable constant D.

Proof. Calling $\nu_h^{(q)}(\Gamma)$ the l.h.s. of (5.25) with $\bar{h} = \infty$ and $\nu_{\infty,m} = 0$, we can show by induction on q that

$$\|\nu_h^{(q)}(\Gamma^1) - \nu_h^{(q)}(\Gamma^2)\|_\infty \leq D\varepsilon|\Gamma^1 - \Gamma^2|. \tag{5.40}$$

We find convenient to write explicitly the dependence of the function $\beta_{h,m}^{(c)}$ from the parameter Γ, so that we rewrite $\beta_{k,m}^{(c)}(\tilde{\omega},\varepsilon,\{\nu_{k',m'}^{(c')(q-1)}\}$ in the r.h.s. of (5.37) as $\beta_{k,m}^{(c)}(\tilde{\omega},\varepsilon,\Gamma,\{\nu_{k',m'}^{(c')(q-1)}(\Gamma)\}$. Then from (5.37) we get

$$\begin{aligned}\mu\nu_m^{(c)(q)}(\Gamma^1) - \mu\nu_m^{(c)(q)}(\Gamma^2) = \sum_{k=h}^{\infty} 2^{h-k-1}[&\beta_{k,m}^{(c)}(\tilde{\omega},\varepsilon,\Gamma^1,\{\nu_{k',m'}^{(c')(q-1)}(\Gamma^1)\})\\ &-\beta_{k,m}^{(c)}(\tilde{\omega},\varepsilon,\Gamma^2,\{\nu_{k',m'}^{(c')(q-1)}(\Gamma^2)\})].\end{aligned} \tag{5.41}$$

When $q = 1$ we have that $\beta^{(c)}_{k,m}(\tilde{\omega}, \varepsilon, \Gamma^1, \{\nu^{(c')(q-1)}_{k',m'}(\Gamma^1)\}) - \beta^{(c)}_{k,m}(\tilde{\omega}, \varepsilon, \Gamma^2, \{\nu^{(c')(q-1)}_{k',m'}(\Gamma^2)\})$ is given by a sum of self energy graphs with one node $\mathbb{V}$ with a factor $\eta_{\mathbb{V}}$ with Γ replaced by $\Gamma^1 - \Gamma^2$; as there is at least a vertex $\mathbb{V}$ of w-type by the definition of the self energy graphs we obtain

$$\|\nu^{(1)}_h(\Gamma^1) - \nu^{(1)}_h(\Gamma^2)\|_\infty \le \left(D_1\varepsilon + \tilde{D}_1\varepsilon^2\right)|\Gamma^1 - \Gamma^2|, \tag{5.42}$$

for positive constants $D_1 < 2D$ and $\tilde{D}_2$, where $D_1\varepsilon|\Gamma^1 - \Gamma^2|$ is a bound for the self-energy first order contribution.

For $q > 1$ we can write the difference in (5.41) as

$$\begin{aligned}&\left(\beta^{(c)}_{k,m}(\tilde{\omega}, \varepsilon, \Gamma^1, \{\nu^{(c')(q-1)}_{k',m'}(\Gamma^1)\}) - \beta^{(c)}_{k,m}(\tilde{\omega}, \varepsilon, \Gamma^2, \{\nu^{(c)(q-1)}_{k',m'}(\Gamma^1)\})\right)\\ &+\left(\beta^{(c)}_{k,m}(\tilde{\omega}, \varepsilon, \Gamma^2, \{\nu^{(c)(q-1)}_{k',m'}(\Gamma^1)\}) - \beta^{(c)}_{k,m}(\tilde{\omega}, \varepsilon, \Gamma^2, \{\nu^{(c)(q-1)}_{k',m'}(\Gamma^2)\})\right).\end{aligned} \tag{5.43}$$

The first factor is given by a sum over self-energy graphs with one node $\mathbb{V}$ with a factor $\eta_{\mathbb{V}}$ with Γ replaced by $\Gamma^1 - \Gamma^2$; the other difference is given by a sum over self energy graphs with a ν-vertex with which is associated a factor $\nu^{(c)(q-1)}_{k',m'}(\Gamma^1) - \nu^{(c)(q-1)}_{k',m'}(\Gamma^2)$; hence we find

$$\begin{aligned}\|\nu^{(q)}_h(\Gamma^1) - \nu^{(q)}_h(\Gamma^2)\|_\infty \le{}& \left(D_1\varepsilon + D_3\varepsilon^2\right)|\Gamma^1 - \Gamma^2|\\ &+\varepsilon D_2 \sup_{h\ge 0}\|\nu^{(q-1)}_h(\Gamma^1) - \nu^{(q-1)}_h(\Gamma^2)\|_\infty,\end{aligned} \tag{5.44}$$

where $D_1\varepsilon|\Gamma^1 - \Gamma^2|$ is a bound for the first order contribution coming from the first line in (5.43), while the last summand in (5.44) is a bound from the terms from the last line of (5.43). Then (5.40) follows with $D = 2D_1$, for ε small enough. □

By using Lemma 11 we can show that the self consistence equation for Γ (5.4) has a unique solution $\Gamma = O(\varepsilon)$.

Lemma 12. *For all $\mu_0 > 0$ there exists $\varepsilon_0 > 0$ such that, for all $|\mu| \le \mu_0$ and for all $0 < \varepsilon < \varepsilon_0$, given $\nu^{(c)}_m(\Gamma)$ chosen as in Lemma 9 (with $\bar{h} = \infty$ and $\nu^{(c)}_{\infty,m} = 0$) it holds that (5.4) has a solution $|\Gamma| \le C\varepsilon$ where C is a suitable constant.*

Proof. The solution of (5.4) can be obtained as the limit as $q \to \infty$ of the sequence $\Gamma^{(q)}$, $q \ge 0$, defined recursively as

$$\begin{aligned}&\Gamma^{(0)} = 0,\\ &\Gamma^{(q)} = r_0^{-1}\sum_{k=1}^{\infty}\ \sum_{\theta\in\Theta^{(k)(q-1))\mathcal{R}}_{n,n}}{}^{\!*}\, a_{0,-n}\mathrm{Val}(\theta),\end{aligned} \tag{5.45}$$

where we define $\Theta^{(k)(q)\mathcal{R}}_{n,m}$ as the set of trees identical to $\Theta^{(k)\mathcal{R}}_{n,m}$ except that Γ in $\eta_{\mathbb{V}}$ is replaced by $\Gamma^{(q)}$ and each $\nu_{h,m}$ is replaced by $\nu_{h,m}(\Gamma^{(q)})$, for all $h \ge 0$, $|m| \ge 1$.Equation (5.45) is a contraction defined on the set $|\Gamma| \le C\varepsilon$, for ε small. In fact if $|\Gamma^{(q-1)}| \le C\varepsilon$, then by (5.45) $|\Gamma^{(q)}| \le C_1\varepsilon + C_2C\varepsilon^2$, where we have used that the first order contribution to the r.h.s. of (5.45) is Γ-independent (see Sect. 3), and $C_1\varepsilon < \varepsilon C/$ is a bound for it; hence for ε small enough (5.45) send the interval $|\Gamma| \le C\varepsilon$ to itself.

Moreover we can show inductively that

$$\|\Gamma^{(q)} - \Gamma^{(q-1)}\|_\infty \le (C\varepsilon)^q. \tag{5.46}$$

For $q = 1$ this is true; for $q > 1$ $\Gamma^{(q)} - \Gamma^{(q-1)}$ can be written as sum of trees in which a) either with a node $\mathbb{V}'$ is associated a factor proportional to $\Gamma^{(q-1)} - \Gamma^{(q-2)}$; or b) with a ν vertex is associated $\nu_{h',m'}(\Gamma^{(q-1)}) - \nu_{h',m'}(\Gamma^{(q-2)})$ for some h', m'. In the first case we note that the constraint in the sum in the r.h.s. of (5.45) implies that $s_{v_0} = 3$ for the special vertex of θ; hence, item (4) in Sect. 3, says that $\mathbb{V}' \neq v_0$ so that such terms are bounded by $D_1\varepsilon|\Gamma^{(1)} - \Gamma^{(2)}|$ (a term order $O(\Gamma^1 - \Gamma^2)$ should have three v lines entering v_0 and two of them coming from end points, which is impossible). In the second case we use (5.39), and we bound such terms by $D_2\varepsilon|\Gamma^{(1)} - \Gamma^{(2)}|$. Hence by induction (5.46) is found, if $C \ge D_1/4, D_2/4, C_1/4)$. □

We have finally to prove that $\tilde{u}_{n,m}$ solves the last of (1.14) and (2.2).

Lemma 13. *For all $\mu_0 > 0$ there exists $\varepsilon_0 >$ such that, for all $|\mu| \le \mu_0$ and for all $0 < \varepsilon < \varepsilon_0$, given $\nu_m^{(c)}(\Gamma)$ chosen as in Lemma 9 and Γ chosen as in Lemma 12 then $\tilde{u}_{n,m}$ solves the last of (1.14) and (2.2).*

Proof. Let us consider first the case in which $|n| \neq |m|$ and we call $\Theta^{\mathcal{R}}_{n,m} = \bigcup_k \Theta^{(k)\mathcal{R}}_{n,m}$; assume also (what of course is not restrictive) that n, m is such that $\chi_{h_0}(|\omega n| - \tilde{\omega}_m) + \chi_{h_0+1}(|\omega n| - \tilde{\omega}_m) = 1$. We call $\Theta^{\mathcal{R}}_{n,m,}$ the set of trees $\theta \in \Theta^{\mathcal{R}}_{n,m}$ with root line at scale h_0, so that

$$\tilde{u}_{n,m} = \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m}} \mathrm{Val}(\theta) = \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m,h_0}} \mathrm{Val}(\theta) + \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m,h_0+1}} \mathrm{Val}(\theta), \tag{5.47}$$

and we write $\Theta^{\mathcal{R}}_{n,m,h_0} = \Theta^{\alpha,\mathcal{R}}_{n,m,h_0} \bigcup \Theta^{\beta,\mathcal{R}}_{n,m,h_0}$, where $\Theta^{\alpha,\mathcal{R}}_{n,m,h_0}$ are the trees with $s_{\mathbb{V}_0} = 1$, while $\Theta^{\beta,\mathcal{R}}_{n,m,\bar{h}}$ are the trees with $s_{\mathbb{V}_0} = 3$ and $\mathbb{V}_0$ is the special vertex (see Definition 2). Then

$$\begin{aligned}
\sum_{\theta\in\Theta^{\alpha,\mathcal{R}}_{n,m,h_0}} \mathrm{Val}(\theta) = \sum_{c=a,b} \Big(& g^{(\bar{h})}(n,m)2^{-h_0}\nu^{(c)}_{h_0,m} \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0}} \mathrm{Val}(\theta) \\
& + g^{(\bar{h})}(n,m)2^{-\bar{h}}\nu^{(c)}_{h_0,m} \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0+1}} \mathrm{Val}(\theta) \\
& + g^{(h_0+1)}(n,m)2^{-h_0}\nu^{(c)}_{h_0,m} \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m,h_0}} \mathrm{Val}(\theta) \\
& + g^{(h_0+1)}(n,m)2^{-h_0-1}\nu^{(c)}_{h_0+1,m} \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0+1}} \mathrm{Val}(\theta) \Big),
\end{aligned} \tag{5.48}$$

where m_c is such that $m_a = m$ and $m_b = -m$. On the other hand we can write $\Theta^{\beta,\mathcal{R}}_{n,m,h_0} = \Theta^{\beta 1,\mathcal{R}}_{n,m,h_0} \bigcup \Theta^{\beta 2,\mathcal{R}}_{n,m,h_0}$, where $\Theta^{\beta 1,\mathcal{R}}_{n,m,h_0}$ are the trees such that the root line ℓ_0 is the external line of a self-energy graph, and $\Theta^{\beta 2,\mathcal{R}}_{n,m,h_0}$ is the complement. Then we can write

$$\begin{aligned}\sum_{\theta\in\Theta^{\beta 1,\mathcal{R}}_{n,m,h_0}} \mathrm{Val}(\theta) = \sum_{c=a,b} & g^{(h_0)}(n,m) \sum_{T\in\tilde{\mathcal{T}}^{(c)}_{<h_0}} \mathcal{R}\mathcal{V}^{\bar{h}}_T(\omega n,m) \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0}} \mathrm{Val}(\theta) \\ & + g^{(h_0)}(n,m) \sum_{T\in\tilde{\mathcal{T}}^{(c)}_{<h_0}} \mathcal{R}\mathcal{V}^{h_0}_T(\omega n,m) \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0+1}} \mathrm{Val}(\theta) \\ & + g^{(h_0+1)}(n,m) \sum_{T\in\tilde{\mathcal{T}}^{(c)}_{<h_0}} \mathcal{R}\mathcal{V}^{h_0}_T(\omega n,m) \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0}} \mathrm{Val}(\theta) \\ & + g^{(h_0+1)}(n,m) \sum_{T\in\tilde{\mathcal{T}}^{(c)}_{<h_0+1}} \mathcal{R}\mathcal{V}^{h_0+1}_T(\omega n,m) \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0+1}} \mathrm{Val}(\theta),\end{aligned} \tag{5.49}$$

where $\tilde{\mathcal{T}}^{(c)}_{<h_0}$ is the set of self-energy graphs such that if $\mathbb{V}$ is the vertex to which the external line ℓ_0 is attached, then $s_{\mathbb{V}} = 3$. Note that if T belongs to the complementary of $\tilde{\mathcal{T}}^{(c)}_{<\bar{h}}$, then $\mathcal{L}\mathcal{V}^h_T = 0$ by the compact support properties of $g^{(h)}$.

Summing (5.6) and (5.49) and using (5.1), (5.3) we get

$$\begin{aligned}\sum_{\theta\in\Theta^{\beta 1,\mathcal{R}}_{n,m,h_0}} \mathrm{Val}(\theta) = \sum_{c=a,b} & g^{(h_0)}(n,m) \sum_{T\in\mathcal{T}^{(c)}_{<h_0}} \mathcal{V}^{h_0}_T(\omega n,m) \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0}} \mathrm{Val}(\theta) \\ & + g^{(h_0)}(n,m) \sum_{T\in\mathcal{T}^{(c)}_{<h_0}} \mathcal{V}^{h_0}_T(\omega n,m) \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m,h_0+1}} \mathrm{Val}(\theta) \\ & + g^{(h_0+1)}(n,m) \sum_{T\in\mathcal{T}^{(c)}_{<h_0}} \mathcal{V}^{h_0}_T(\omega n,m) \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0}} \mathrm{Val}(\theta) \\ & + g^{(h_0+1)}(n,m) \sum_{T\in\mathcal{T}^{(c)}_{<h_0+1}} \mathcal{V}^{h_0+1}_T(\omega n,m) \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0+1}} \mathrm{Val}(\theta) \\ & + \nu^{(c)}_m \left(g^{(h_0)}(n,m) + g^{(h_0+1)}(n,m)\right) \Bigg(\sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0}} \mathrm{Val}(\theta) + \sum_{\theta\in\Theta^{\mathcal{R}}_{n,m_c,h_0+1}} \mathrm{Val}(\theta) \Bigg).\end{aligned} \tag{5.50}$$

The last line is equal to

$$\frac{1}{-\omega^2 n^2 + \tilde{\omega}_m^2} [\nu^{(a)}_m \tilde{u}_{n,m} + \nu^{(b)}_m \tilde{u}_{n,-m}], \tag{5.51}$$

while adding the first three lines in (5.49) to $\sum_{\theta\in\Theta^{\beta 1,\mathcal{R}}_{n,m,h_0}} \mathrm{Val}(\theta)$ we get

$$\varepsilon \sum_{\substack{n_1+n_2+n_3=n \\ m_1+m_2+m_3=m}} \tilde{u}_{n_1,m_1} \tilde{u}_{n_2,m_2} \tilde{u}_{n_3,m_3}, \tag{5.52}$$

from which we get that $\sum_{\theta\in\Theta^{\mathcal{R}}_{n,m}} \mathrm{Val}(\theta)$, for $\mu = 1$, is a formal solution of (2.2). A similar result holds for $|n| = |m|$. $\square$

6. Construction of the Perturbed Frequencies

In the following it will be convenient to set $\tilde{\omega} = \{\omega_m\}_{|m|\geq 2}$. By the analysis of the previous sections we have found the counterterms $\{\nu_m(\tilde{\omega}, \varepsilon)\}_{|m|\geq 2}$ as functions of ε and $\tilde{\omega}$. We have now to invert the relations

$$\tilde{\omega}_m^2 - \nu_m(\tilde{\omega}, \varepsilon) = m^2, \tag{6.1}$$

in order to prove Proposition 2.

We shall show that there exists a sequence of sets $\{\mathcal{E}^{(p)}\}_{p=0}^{\infty}$ in $[0, \varepsilon_0]$, such that $\mathcal{E}^{(p+1)} \subset \mathcal{E}^{(p)}$, and a sequence of functions $\{\tilde{\omega}^{(p)}(\varepsilon)\}_{p=0}^{\infty}$, with each $\tilde{\omega}^{(p)} \equiv \tilde{\omega}^{(p)}(\varepsilon)$ defined for $\varepsilon \in \mathcal{E}^{(p)}$, such that for all $\varepsilon \in \mathcal{E}$, with

$$\mathcal{E} = \bigcap_{p=0}^{\infty} \mathcal{E}^{(p)} = \lim_{p\to\infty} \mathcal{E}^{(p)}, \tag{6.2}$$

there exists the limit

$$\tilde{\omega}^{(\infty)}(\varepsilon) = \lim_{p\to\infty} \tilde{\omega}^{(p)}(\varepsilon), \tag{6.3}$$

and it solves (6.1).

To fulfill the program above we shall define since the beginning $\omega = \omega_\varepsilon = \sqrt{1-\varepsilon}$, and we shall follow an iterative scheme by setting, for $|m| \geq 1$,

$$\begin{aligned} \tilde{\omega}_m^{(0)2} &= m^2, \\ \tilde{\omega}_m^{(p)2} &= \tilde{\omega}_m^{(p)2}(\varepsilon) = m^2 + \nu_m(\tilde{\omega}^{(p-1)}, \varepsilon), \qquad p \geq 1, \end{aligned} \tag{6.4}$$

and reducing recursively the set of admissible values of ε.

We start by imposing on ε the Diophantine conditions

$$|\omega n \pm m| \geq 2C_0 |n|^{-\tau_0} \qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{Z} \setminus \{0\} \text{ such that } |m| \neq |n|, \tag{6.5}$$

where C_0 and τ_0 are two positive constants.

This will imply some restrictions on the admissible values of ε, as the following result shows.

Lemma 14. *For all $0 < C_0 \leq 1/2$ there exist $\varepsilon_0 > 0$ and $\gamma_0, \delta_0 > 0$ such that the set $\mathcal{E}^{(0)}$ of values $\varepsilon \in [0, \varepsilon_0]$ for which (6.5) are satisfied has Lebesgue measure $\mathrm{meas}(\mathcal{E}^{(0)}) \geq \varepsilon_0(1 - \gamma_0 \varepsilon_0^{\delta_0})$ provided that one has $\tau_0 > 1$.*

Proof. For (n, m) such that $|\omega n \pm m| \geq 2C_0$ the Diophantine conditions in (6.5) are trivially satisfied. We consider then (n, m) such that $|\omega n \pm m| < 2C_0$ and we write, if $0 < C_0 \leq 1/2$,

$$1 > 2C_0 > |\omega n \pm m| \geq \left| \frac{\varepsilon n}{1 + \sqrt{1-\varepsilon}} - n \pm m \right|, \tag{6.6}$$

and as $|n \pm m| \geq 1$ one gets $|n| \geq 1/2\varepsilon \geq 1/2\varepsilon_0$. Moreover, for fixed n, the set $\mathcal{M}$ of m's such that $|\omega n \pm m| < 1$ contains at most $2 + \varepsilon_0|n|$ values. By writing

$$f(\varepsilon(t)) = n\sqrt{1-\varepsilon(t)} \pm m = t\frac{2C_0}{|n|^{\tau_0}}, \qquad t \in [0, 1], \tag{6.7}$$

and, calling $\mathcal{I}^{(0)}$ the set of ε such that $|\omega n \pm m| < 2C_0|n|^{-\tau_0}$ is verified for some (n, m), one finds for the Lebesgue measure of $\mathcal{I}^{(0)}$,

$$\operatorname{meas}(\mathcal{I}^{(0)}) = \int_{\mathcal{I}^{(0)}} \mathrm{d}\varepsilon = \sum_{|n|\geq 1/2\varepsilon_0} \sum_{|m|\in\mathcal{M}} \int_{-1}^{1} \mathrm{d}t \left| \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} \right|. \tag{6.8}$$

We have from (6.7)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{2C_0}{|n|^{\tau_0}}, \tag{6.9}$$

so that, noting that one has $|\partial f/\partial \varepsilon| \geq |n|/4$, we have to exclude a set of measure

$$\sum_{|n|\geq N} \frac{8C_0}{|n|^{\tau_0+1}} (2 + \varepsilon_0|n|) \leq \text{const.}\, \varepsilon_0^{\tau_0}, \tag{6.10}$$

and one has to impose $\tau_0 = 1 + \delta_0$, with $\delta_0 > 0$. □

For $p \geq 1$ the sets $\mathcal{E}^{(p)}$ will be defined recursively as

$$\begin{aligned} \mathcal{E}^{(p)} = \Big\{ \varepsilon \in \mathcal{E}^{(p-1)} \ : \ & |\omega n \pm \tilde{\omega}_m^{(p)}| > C_0|n|^{-\tau} \qquad \forall |m| \neq |n|, \\ & |\omega n \pm (\tilde{\omega}_m^{(p)} \pm \tilde{\omega}_{m'}^{(p)})| > C_0|n|^{-\tau} \qquad |n| \neq |m \pm m'| \Big\}, \qquad p \geq 1, \end{aligned} \tag{6.11}$$

for $\tau > \tau_0$ to be fixed.

In Appendix A4 we prove the following result.

Lemma 15. *For all $p \geq 1$ one has*

$$\left\| \tilde{\omega}^{(p)}(\varepsilon) - \tilde{\omega}^{(p-1)}(\varepsilon) \right\|_\infty \leq C\varepsilon_0^p \qquad \forall \varepsilon \in \mathcal{E}^{(p)}, \tag{6.12}$$

for some constant C.

Therefore we can conclude that there exists a sequence $\{\tilde{\omega}^{(p)}(\varepsilon)\}_{p=0}^{\infty}$ converging to $\tilde{\omega}^{(\infty)}(\varepsilon)$ for $\varepsilon \in \mathcal{E}$. We now have to show that the set $\mathcal{E}$ has positive (large) measure.

It is convenient to introduce a set of variables $\mu(\tilde{\omega}, \varepsilon)$ such that

$$\tilde{\omega}_m + \mu_m(\tilde{\omega}, \varepsilon) = \omega_m \equiv |m|; \tag{6.13}$$

the variables $\mu(\tilde{\omega}, \varepsilon)$ and the counterterms are trivially related by

$$-\nu_m(\tilde{\omega}, \varepsilon) = \mu_m^2(\tilde{\omega}, \varepsilon) + 2\tilde{\omega}_m \mu_m(\tilde{\omega}, \varepsilon). \tag{6.14}$$

One can write $\tilde{\omega}_m^{(p)} = \omega_m - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)$, according to (6.4). We shall impose the Diophantine conditions

$$\begin{aligned} &|\omega n \pm (\omega_m - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon))| > C_0|n|^{-\tau}, \\ &|\omega n \pm ((\omega_m \pm \omega_{m'}) - (\mu_m(\tilde{\omega}^{(p-1)}, \varepsilon) \pm \mu_{m'}(\tilde{\omega}^{(p-1)}, \varepsilon)))| > C_0|n|^{-\tau}. \end{aligned} \tag{6.15}$$

Suppose that for $\varepsilon \in \mathcal{E}^{(p-1)}$ the functions $\mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)$ are well defined; then define $\mathcal{I}^{(p)} = \mathcal{I}_1^{(p)} \cup \mathcal{I}_2^{(p)} \cup \mathcal{I}_3^{(p)}$, where $\mathcal{I}_1^{(p)}$ is the set of values $\varepsilon \in \mathcal{E}^{(p-1)}$ verifying the conditions

$$\left|\omega n \pm \left(\omega_m - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)\right)\right| \leq C_0 \, |n|^{-\tau} \,, \tag{6.16}$$

$\mathcal{I}_2^{(p)}$ is the set of values ε verifying the conditions

$$\left|\omega n \pm \left((\omega_m - \omega_{m'}) - (\mu_m(\tilde{\omega}^{(p-1)}, \varepsilon) - \mu_{m'}(\tilde{\omega}^{(p-1)}, \varepsilon))\right)\right| \leq C_0 |n|^{-\tau}, \tag{6.17}$$

and $\mathcal{I}_3^{(p)}$ is the set of values ε verifying the conditions

$$\left|\omega n \pm \left((\omega_m + \omega_{m'}) - (\mu_m(\tilde{\omega}^{(p-1)}, \varepsilon) + \mu_{m'}(\tilde{\omega}^{(p-1)}, \varepsilon))\right)\right| \leq C_0 |n|^{-\tau}. \tag{6.18}$$

For future convenience we shall call, for $i = 1, 2, 3$, $\mathcal{I}_i^{(p)}(n)$ the subsets of $\mathcal{I}_i^{(p)}$ which verify the Diophantine conditions (6.16), (6.17) and (6.18), respectively, for fixed n.

We want to bound the measure of the set $\mathcal{I}^{(p)}$. First we need to know a little better the dependence on ε and $\tilde{\omega}$ of the counterterms: this is provided by the following result.

Lemma 16. *For all $p \geq 1$ and for all $\varepsilon \in \mathcal{E}^{(p)}$ there exists a positive constant C such that*

$$\begin{aligned}
&\left|\nu_m(\tilde{\omega}^{(p)}, \varepsilon)\right| \leq C\varepsilon, \qquad \left|\partial_{\tilde{\omega}_{m'}^{(p)}} \nu_m(\tilde{\omega}^{(p)}, \varepsilon)\right| \leq C\varepsilon, \qquad m, m' \geq 1, \\
&\left|\mu_m(\tilde{\omega}^{(p)}, \varepsilon)\right| \leq \frac{C\varepsilon}{m}, \qquad \left|\partial_{\tilde{\omega}_{m'}^{(p)}} \mu_m(\tilde{\omega}^{(p)}, \varepsilon)\right| \leq \frac{C\varepsilon}{m}, \qquad m, m' \geq 1, \\
&\left|\partial_\varepsilon \mu_m^{(p)}(\omega, \varepsilon)\right| \leq C, \qquad m \geq 1,
\end{aligned} \tag{6.19}$$

where the derivatives are in the sense of Whitney [38].

Proof. The bound for $\nu_m(\tilde{\omega}^{(p)}, \varepsilon)$ is obvious by construction.

In order to prove the bound for $\partial_{\omega_{m'}^{(p)}} \nu_m(\tilde{\omega}^{(p)}, \varepsilon)$ note that one has

$$\left| g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) - (\tilde{\omega} - \tilde{\omega}')\, \partial_{\tilde{\omega}} g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) \right| \leq C\, n_{\ell_j}^2 2^{-3h_{\ell_j}} \left\| \tilde{\omega}' - \tilde{\omega} \right\|_\infty^2, \tag{6.20}$$

from the compact support properties of the propagator.

Let us consider the quantity $\nu(\tilde{\omega}', \varepsilon) - \nu(\tilde{\omega}, \varepsilon) - (\omega - \omega')\, \partial_{\tilde{\omega}} \nu(\tilde{\omega}, \varepsilon)$, where $\partial_{\tilde{\omega}} \nu(\tilde{\omega}, \varepsilon)$ denotes the derivative in the sense of Whitney, and note that it can be expressed as a sum over trees each one containing a line with propagator $g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) - (\tilde{\omega} - \tilde{\omega}')\, \partial_{\tilde{\omega}} g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega})$, by proceeding as in the proof of Lemma 9 of [23]. Then we find

$$\left\| \nu(\tilde{\omega}', \varepsilon) - \nu(\tilde{\omega}, \varepsilon) - (\tilde{\omega} - \tilde{\omega}')\, \partial_{\tilde{\omega}} \nu(\tilde{\omega}, \varepsilon) \right\|_\infty \leq C\varepsilon \left\| \tilde{\omega}' - \tilde{\omega} \right\|_\infty^2, \tag{6.21}$$

and the second bound in (6.19) follows.

The bounds for $\mu_m(\tilde{\omega}^{(p)}, \varepsilon)$ and $\partial_{\omega^{(p)}_{m'}} \mu_m(\tilde{\omega}^{(p)}, \varepsilon)$ simply follow from (6.14) which gives

$$\mu_m(\tilde{\omega}^{(p)}, \varepsilon) = \frac{\nu_m(\tilde{\omega}^{(p)}, \varepsilon)}{|m|} \, \frac{-1}{1 + \sqrt{1 - \nu_m(\tilde{\omega}^{(p)}, \varepsilon)/m}}. \tag{6.22}$$

In order to prove the last bound in (6.19) we prove by induction the bound

$$\left| \partial_\varepsilon \tilde{\omega}^{(p)}_m(\omega, \varepsilon) \right| \le C, \qquad m \ge 1, \tag{6.23}$$

by assuming that it holds for $\tilde{\omega}^{(p-1)}$; then from (6.4) we have

$$\begin{aligned} 2\tilde{\omega}^{(p)} \partial_\varepsilon \tilde{\omega}^{(p)}_m(\varepsilon) &= -\partial_\varepsilon \nu_m(\tilde{\omega}^{(p-1)}(\varepsilon), \varepsilon) \\ &= -\sum_{m' \in \mathbb{Z}} \partial_{\tilde{\omega}^{(p-1)}_{m'}} \nu_m(\tilde{\omega}^{(p-1)}(\varepsilon), \varepsilon)\, \partial_\varepsilon \tilde{\omega}^{(p-1)}_{m'}(\varepsilon) - \left. \partial_\eta \nu_m(\tilde{\omega}^{(p-1)}_m(\varepsilon), \eta) \right|_{\eta=\varepsilon}, \end{aligned} \tag{6.24}$$

as it is easy to realize by noting that μ_m can depend on $\tilde{\omega}_{m'}$ when $|m'| > |m|$ only if the sum of the absolute values of the mode labels m_v is greater than $|m' - m|$, while we can bound the sum of the contributions with $|m'| < |m|$ by a constant times ε, simply by using the second line in (6.19).

Hence from the inductive hypothesis and the proved bounds in (6.19), we obtain

$$\left\| \tilde{\omega}^{(p)}(\varepsilon') - \tilde{\omega}^{(p)}(\varepsilon) - (\varepsilon - \varepsilon')\, \partial_\varepsilon \tilde{\omega}^{(p)}(\varepsilon) \right\|_\infty \le C \left| \varepsilon' - \varepsilon \right|, \tag{6.25}$$

so that also the bound (6.23) and hence the last bound in (6.16) follows. □

Now we can bound the measure of the set we have to exclude: this will conclude the proof of Proposition 2.

We start with the estimate of the measure of the set $\mathcal{I}^{(p)}_1$. When (6.16) is satisfied one must have

$$\begin{aligned} C_2|m| &\le |\omega_m - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \le \omega|n| + C_0|n|^{-\tau} \le C_2'|n|, \\ C_1|m| &\ge |\omega_m - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \ge \omega|n| - C_0|n|^{-\tau} \ge C_1'|n|, \end{aligned} \tag{6.26}$$

which implies

$$M_1|n| \le |m| \le M_2|n|, \qquad M_1 = \frac{C_1}{C_1'}, \qquad M_2 = \frac{C_2'}{C_2}, \tag{6.27}$$

and it is easy to see that, for fixed n, the set $\mathcal{M}_0(n)$ of m's such that (6.16) are satisfied contains at most $2 + \varepsilon_0|n|$ values.

Furthermore (by using also (6.5)) from (6.16) one obtains also

$$\begin{aligned} 2C_0|n|^{-\tau_0} &\le |\omega|n| - \omega_m| \le |\omega|n| - \omega_m + \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| + |\mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \\ &\le C_0|n|^{-\tau} + C\varepsilon_0/|m|, \end{aligned} \tag{6.28}$$

which implies, together with (6.36), for $\tau \ge \tau_0$,

$$|n| \ge N_0 \equiv \left(\frac{C_0 M_1}{C\varepsilon_0} \right)^{1/(\tau_0 - 1)}. \tag{6.29}$$

Let us write $\omega(\varepsilon) = \omega_\varepsilon = \sqrt{1-\varepsilon}$ and consider the function $\mu(\tilde{\omega}^{(p-1)}, \varepsilon)$: we can define a map $t \to \varepsilon(t)$ such that

$$f(\varepsilon(t)) \equiv \omega(\varepsilon(t))|n| - \omega_m + \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon(t)) = t\,\frac{C_0}{|n|^\tau}, \qquad t \in [-1, 1], \tag{6.30}$$

describes the interval defined by (6.16); then the Lebesgue measure of $\mathcal{I}_1^{(p)}$ is

$$\operatorname{meas}(\mathcal{I}_1^{(p)}) = \int_{\mathcal{I}_1^{(p)}} \mathrm{d}\varepsilon = \sum_{|n|\geq N_0} \sum_{m\in\mathcal{M}_0(n)} \int_{-1}^{1} \mathrm{d}t \left|\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}\right|. \tag{6.31}$$

We have from (6.30),

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{C_0}{|n|^\tau}, \tag{6.32}$$

hence

$$\operatorname{meas}(\mathcal{I}_1^{(p)}) = \sum_{|n|\geq N_0} \sum_{m\in\mathcal{M}_0(n)} \frac{C_0}{|n|^\tau} \int_{-1}^{1} \mathrm{d}t \left|\frac{\mathrm{d}f(\varepsilon(t))}{\mathrm{d}\varepsilon(t)}\right|^{-1}. \tag{6.33}$$

In order to perform the derivative in (6.30) we write

$$\frac{\mathrm{d}\mu_m}{\mathrm{d}\varepsilon} = \partial_\varepsilon \mu_m + \sum_{|m'|\geq 1} \partial_{\omega_{m'}^{(p-1)}} \mu_m \partial_\varepsilon \tilde{\omega}_{m'}^{(p-1)}, \tag{6.34}$$

where $\partial_\varepsilon \mu_m$ and $\partial_\varepsilon \tilde{\omega}_{m'}^{(p-1)}$ are bounded through Lemma 11. Moreover one has

$$\left|\partial_{\tilde{\omega}_{m'}^{(p-1)}} \mu_m\right| \leq C\varepsilon\, e^{-\kappa|m'-m|/2}, \qquad |m'| > |m|, \tag{6.35}$$

and the sum over m' can be dealt with as in (6.24).

At the end we get that the sum in (6.34) is $O(\varepsilon)$, and from (6.29) and (6.30) we obtain, if ε_0 is small enough,

$$\left|\frac{\partial f(\varepsilon(t))}{\partial\varepsilon(t)}\right| \geq \frac{|n|}{4}, \tag{6.36}$$

so that one has

$$\begin{aligned}\int_{\mathcal{I}_1^{(p)}} \mathrm{d}\varepsilon &\leq \text{const.} \sum_{|n|\geq N_0} \frac{C_0}{|n|^{\tau+1}}\,(2+\varepsilon_0|n|) \\ &\leq \text{const.}\ \varepsilon_0 \left(\varepsilon_0^{(\tau-1)/(\tau_0-1)} + \varepsilon_0^{(\tau-\tau_0+1)/(\tau_0-1)}\right).\end{aligned} \tag{6.37}$$

Therefore for $\tau > \max\{1, \tau_0 - 1\}$ the Lebesgue measure of the set $\mathcal{I}_1^{(p)}$ is bounded by $\varepsilon_0^{1+\delta_1}$, with $\delta_1 > 0$.

Now we discuss how to bound the measure of the set $\mathcal{I}_2^{(p)}$. We start by noting that from (6.30) we obtain, if $m, \ell > 0$,

$$\left|\tilde{\omega}_{m+\ell}^{(p)} - \tilde{\omega}_m^{(p)} - \ell\right| \leq \frac{C\varepsilon}{m} + \frac{C\varepsilon}{m+\ell}, \tag{6.38}$$

for all $p \geq 1$.

By the parity properties of $\tilde{\omega}_m$ without loss of generality we can confine ourselves to the case $n > 0$, $m' > m \geq 2$, and $|\omega n - (\tilde{\omega}_{m'}^{(p)} - \tilde{\omega}_m^{(p)})| < 1$. Then the discussion proceeds as follows.

When the conditions (6.17) are satisfied, one has

$$\begin{aligned} 2C_0|n|^{-\tau_0} &\leq |\omega n - (\omega_{m'} - \omega_m)| \\ &\leq |\omega n - (\omega_{m'} - \mu_{m'}^{(p-1)}(\tilde{\omega}^{(p-1)}, \varepsilon)) \\ &\quad + (\omega_m - \mu_m^{(p-1)}(\tilde{\omega}^{(p-1)}, \varepsilon))| \\ &\quad + |\mu_{m'}(\tilde{\omega}^{(p-1)}, \varepsilon) - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \\ &\leq C_0|n|^{-\tau} + \frac{C\varepsilon}{m} + \frac{C\varepsilon}{m'} \leq C_0|n|^{-\tau} + \frac{2C\varepsilon_0}{m}, \end{aligned} \tag{6.39}$$

which implies for $\tau \geq \tau_0$,

$$|n| \geq N_1 \equiv \left(\frac{C_0}{2C\varepsilon_0}\right)^{1/\tau_0}. \tag{6.40}$$

We can bound the Lebesgue measure of the set $\mathcal{I}_2^{(p)}$ by distinguishing, for fixed (n, ℓ), with $\ell = m' - m > 0$, the values $m \leq m_0$ and $m > m_0$, where m_0 is determined by the request that one has for $m > m_0$,

$$\frac{2C\varepsilon_0}{m} \leq \frac{C_0}{|n|^{\tau_0}}, \tag{6.41}$$

which gives

$$m_0 = m_0(n) = \left(\frac{2C|n|^{\tau_0}\varepsilon_0}{C_0}\right). \tag{6.42}$$

Therefore for $m > m_0$ and $\tau \geq \tau_0$ one has, from (6.5), (6.38) and (6.41),

$$\left|\omega n - (\tilde{\omega}_{m'}^{(p)} - \tilde{\omega}_m^{(p)})\right| \geq |\omega n - \ell| - \frac{2C\varepsilon}{m} \geq \frac{2C_0}{|n|^{\tau_0}} - \frac{C_0}{|n|^{\tau_0}} \geq \frac{C_0}{|n|^{\tau_0}} \geq \frac{C_0}{|n|^{\tau}}, \tag{6.43}$$

so that one has to exclude no further value from $\mathcal{E}^{(p-1)}$, provided one takes $\tau \geq \tau_0$.

For $m < m_0$ define L_0 such that

$$C_3\ell \leq \left|\tilde{\omega}_{m+\ell}^{(p)} - \tilde{\omega}_m^{(p)}\right| < \omega n + 1 < C_3'|n|, \qquad L_0 = \frac{C_3'}{C_3}, \tag{6.44}$$

where (6.26) has been used. Again, for fixed n, the set $\mathcal{L}_0(n)$ of ℓ's such that (6.26) is satisfied with $m' - m = \ell$ contains at most $2 + \varepsilon_0|n|$ values.

Therefore, by reasoning as in obtaining (6.37), one finds that for $m < m_0$ one has to exclude from $\mathcal{E}^{(p-1)}$ a set of measure bounded by a constant times

$$\sum_{|n|\geq N_1} \sum_{\ell\in\mathcal{L}_0(n)} \sum_{m<m_0(n)} \frac{C_0}{|n|^{\tau+1}} \leq \text{const.}\, \varepsilon_0 \left(\varepsilon_0^{(\tau-\tau_0)/\tau_0} + \varepsilon_0^{1+(\tau-\tau_0-1)/\tau_0}\right), \tag{6.45}$$

provided that one has $\tau > \tau_0 + 1 > 2$ the Lebesgue measure of the set $\mathcal{I}_2^{(p)}$ is bounded by $\varepsilon_0^{1+\delta_2}$, with $\delta_2 > 0$.

Finally we study the measure of the set $\mathcal{I}_3^{(p)}$. If $n > 0$, $|m'| > |m|$ and $|(\tilde{\omega}_m^{(p)} + \tilde{\omega}_{m'}) - \omega n| < 1$ (which again is the only case we can confine ourselves to study), then one has to sum over $|n| \le N_1$, with N_1 given by (6.40). For such values of n one has

$$\begin{aligned} \left|\omega n - (\tilde{\omega}_{m'}^{(p)} + \tilde{\omega}_m^{(p)})\right| &\ge \left|\omega n - (|m| + |m'|)\right| - \frac{2C\varepsilon}{|m|} \\ &\ge \frac{2C_0}{|n|^{\tau_0}} - \frac{C_0}{|n|^{\tau_0}} \ge \frac{C_0}{|n|^{\tau_0}} \ge \frac{C_0}{|n|^{\tau}}, \end{aligned} \tag{6.46}$$

as soon as $|m| > m_0$, with m_0 given by (6.42). Therefore we have to take into account only the values of m such that $|m| < m_0$, and we can also note that $|m'|$ is uniquely determined by the values of n and m. Then one can proceed as in the previous case and in the end one excludes a further subset of $\mathcal{E}^{(p-1)}$ whose Lebesgue measure is bounded by a constant times

$$\sum_{|n|\ge N_1} \sum_{m<m_0(n)} \frac{C_0}{|n|^{\tau+1}} \le \text{const.}\, \varepsilon_0^{1+(\tau-\tau_0)/\tau_0}, \tag{6.47}$$

so that the Lebesgue measure of the set $\mathcal{I}_3^{(p)}$ is bounded by $\varepsilon_0^{1+\delta_3}$, with $\delta_3 > 0$, provided that one takes $\tau > \tau_0$.

By summing together the bounds for $\mathcal{I}_1^{(p)}$, for $\mathcal{I}_2^{(p)}$ and for $\mathcal{I}_3^{(p)}$, then the bound

$$\text{meas}(\mathcal{I}^{(p)}) \le b\, \varepsilon_0^{\delta+1} \tag{6.48}$$

with $\delta > 0$, follows for all $p \ge 1$, if τ is chosen to be $\tau > \tau_0 + 1 > 2$.

We can conclude the proof of Proposition 2 through the following result, which shows that the bound (6.42) essentially extends to the union of all $\mathcal{I}^{(p)}$ (at the cost of taking a larger constant B instead of b).

Lemma 17. *Define $\mathcal{I}^{(p)}$ as the set of values in $\mathcal{E}^{(p)}$ verifying (6.26) to (6.28) for $\tau > 1$. Then one has, for two suitable positive constants B and δ,*

$$\text{meas}\left(\cup_{p=0}^{\infty} \mathcal{I}^{(p)}\right) \le B\varepsilon_0^{\delta+1}, \tag{6.49}$$

where meas *denotes the Lebesgue measure.*

Proof. First of all we check that, if we call $\varepsilon_j^{(p)}(n)$ the centers of the intervals $\mathcal{I}_j^{(p)}(n)$, with $j = 1, 2, 3$, then one has

$$\left|\varepsilon_j^{(p+1)}(n) - \varepsilon_j^{(p)}(n)\right| \le D\varepsilon_0^{p}, \tag{6.50}$$

for a suitable constant D.

The center $\varepsilon_1^{(p)}(n)$ is defined by the condition

$$\omega n \pm \left(\omega_m - \mu_m(\tilde{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)), \varepsilon_1^{(p)}(n))\right) = 0, \tag{6.51}$$

where Whitney extensions are considered outside $\mathcal{E}^{(p-1)}$; then, by subtracting (6.51) from the analogous expression for $p + 1$, we have

$$\left(\mu_m(\tilde{\omega}^{(p)}(\varepsilon_1^{(p+1)}(n)), \varepsilon_1^{(p+1)}(n)) - \mu_m(\tilde{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)), \varepsilon_1^{(p)}(n))\right) = 0. \tag{6.52}$$

In (6.48) one has

$$\begin{aligned}\mu_m(\tilde{\omega}'(\varepsilon'),\varepsilon')-\mu_m(\tilde{\omega}(\varepsilon),\varepsilon)&=\mu_m(\tilde{\omega}'(\varepsilon'),\varepsilon')-\mu_m(\tilde{\omega}(\varepsilon'),\varepsilon')\\+\mu_m(\tilde{\omega}(\varepsilon'),\varepsilon')-\mu_m(\tilde{\omega}(\varepsilon),\varepsilon')&+(\mu_m(\tilde{\omega}(\varepsilon),\varepsilon')-\mu_m(\tilde{\omega}(\varepsilon),\varepsilon)),\end{aligned}\tag{6.53}$$

and, from Lemma 13,

$$\begin{aligned}&\left|\mu_m(\tilde{\omega}'(\varepsilon'),\varepsilon')-\mu_m(\tilde{\omega}(\varepsilon'),\varepsilon')\right|\le C\varepsilon\left\|\omega'-\omega\right\|_\infty,\\&\left|\mu_m(\tilde{\omega}(\varepsilon'),\varepsilon')-\mu_m(\tilde{\omega}(\varepsilon),\varepsilon')\right|\le C\varepsilon\left|\varepsilon'-\varepsilon\right|,\end{aligned}\tag{6.54}$$

so that we get, by Lemma 10,

$$\left|\varepsilon_1^{(p+1)}(n)-\varepsilon_1^{(p)}(n)\right|\le C\varepsilon_0^p,\tag{6.55}$$

for a suitable positive constant C. This proves the bound (6.50) for $j=1$. Analogously one can consider the cases $j=2$ and $j=3$, and a similar result is found.

By (6.10), (6.51), (6.54) and (6.12) it follows for $p>p_0$,

$$|\varepsilon_j^{(p)}(n)-\varepsilon_j^{(p_0)}(n)|\le C\sum_{k=p_0}^{\infty}\varepsilon_0^k=C\frac{\varepsilon_0^{p_0}}{1-\varepsilon_0}\tag{6.56}$$

so that one can ensure that $|\varepsilon_j^{(p)}(n)-\varepsilon_j^{(p_0)}(n)|\le C_0|n|^{-\tau}$ for $p\ge p_0$ by choosing

$$p_0=p_0(n,j)\le\text{const.}\log|n|.\tag{6.57}$$

For all $p\le p_0$ define $\mathcal{J}_j^{(p)}(n)$ as the set of values ε such that (6.16), (6.17) and (6.18) are satisfied with C_0 replaced with $2C_0$. By the definition of p_0 all the intervals $\mathcal{I}_j^{(p)}(n)$ fall inside the union of the intervals $\mathcal{J}_j^{(0)}(n),\ldots,\mathcal{J}_j^{(p_0)}(n)$ as soon as $p>p_0$. This means that, by (6.31) to (6.37),

$$\begin{aligned}\text{meas}\left(\cup_{p=0}^{\infty}\mathcal{I}_1^{(p)}\right)&\le\text{meas}\left(\cup_{p=0}^{p_0}\mathcal{J}_1^{(p)}\right)\le\sum_{|n|\ge N_0}\text{meas}(\mathcal{I}_1^{(p)}(n))\\&\le\text{const.}\sum_{|n|\ge N_0}\sum_{p=0}^{p_0(n,1)}\frac{2C_0}{|n|^{\tau+1}}\le\text{const.}\sum_{n=N_0}^{\infty}n^{-(\tau+1)}\log n\le B\varepsilon_0^{1+\delta},\end{aligned}\tag{6.58}$$

with suitable B and δ, in order to take into account the logarithmic corrections due to (6.55). Analogously one obtains the bounds $\text{meas}(\mathcal{I}_2)\le B\varepsilon_0^{1+\delta}$ and $\text{meas}(\mathcal{I}_3)\le B\varepsilon_0^{1+\delta}$ for the Lebesgue measures of the sets $\mathcal{I}_2$ and $\mathcal{I}_3$ (possibly redefining the constant B). This completes the proof of the bound (6.48). □

7. The case $\tau \leq 2$

Proposition 1 was proved assuming that $\tilde{\omega}$ verifies the first and the second Diophantine conditions (2.33) with $\tau > 2$. Here we want to prove that it is possible to obtain a result similar to Proposition 1 assuming only the first Diophantine condition and $1 < \tau \leq 2$, and that also in such a case the set of allowed values of ε have large Lebesgue measure, so that a result analogous to Proposition 2 holds.

The proof of the analogue of Proposition 1 is an immediate adaptation of the analysis in Sect. 4 and 5. First we consider a slightly different multiscale decomposition of the propagator; instead of (4.2) we write

$$\frac{1}{-\omega^2 n_\ell^2+\tilde{\omega}_{m_\ell}^2}=\frac{\chi(6\sqrt{|\omega^2 n_\ell^2-\tilde{\omega}_{m_\ell}^2|})}{-\omega^2 n_\ell^2+\tilde{\omega}_{m_\ell}^2}+\frac{\chi_{-1}(6\sqrt{|\omega^2 n_\ell^2-\tilde{\omega}_{m_\ell}^2|})}{-\omega^2 n_\ell^2+\tilde{\omega}_{m_\ell}^2}, \tag{7.1}$$

and the denominator in the first addend of the r.h.s. of (7.1aa) is smaller than $C_0/6$; as in Sect. 4 we assume $C_0 \leq 1/2$ without loss of generality. If $|n| \neq |m|$ and $|\omega^2 n^2 - \tilde{\omega}_m^2| < C_0/6$, then, by reasoning as in (4.12), we obtain

$$|n| > |m| > \frac{3}{4}|n|, \qquad \min\{|m|,|n|\} > \frac{1}{2\varepsilon}, \tag{7.2}$$

and

$$|\omega|n| - \tilde{\omega}_m| < \frac{C_0}{6(\omega|n|+\tilde{\omega}_m)} < \frac{C_0}{6|n|}. \tag{7.3}$$

We can decompose the first summand in (7.1), obtaining

$$\chi(6\sqrt{|\omega^2 n_\ell^2-\tilde{\omega}_{m_\ell}^2|})\sum_{h=-1}^{\infty}\frac{\chi_h(|\omega n|-\tilde{\omega}_m)}{-\omega^2 n^2+\tilde{\omega}_m^2} \equiv \sum_{h=-1}^{\infty} g_a^{(h)}(\omega n, m), \tag{7.4}$$

and the scales from -1 to h_0, with h_0 given in the statement of Lemma 5, can be bounded as in Lemma 6. We shall call the line of type a a line with which a propagator $g_a^{(h)}(\omega n, m)$ is associated and the line of type b a line with which the second summand in (7.1) is associated; the scale of the b lines is set equal to -1. If $N_h(\theta)$ is the number of lines on scale h, the following result holds.

Lemma 18. *Assume that there is a constant C_1 such that one has $|\tilde{\omega}_m - |m|| \leq C_1\varepsilon/|m|$ and that $\tilde{\omega}$ verifies the first Melnikov condition in (2.33) with $\tau > 1$. If ε is small enough for any tree $\theta \in \Theta_{n,m}^{(k)}$ and for all $h \geq h_0$ one has*

$$N_h(\theta) \leq 4K(\theta)2^{(2-h)/\tau^2} - C_h(\theta) + M_h^{\nu}(\theta) + S_h(\theta), \tag{7.5}$$

where $M_h^{\nu}(\theta)$ and $S_h(\theta)$ are defined as the number of ν-vertices in θ such that the maximum scale of the two external lines is h and, respectively, the number of self-energy graphs in θ with $h_T^e = h$.

Proof. We prove inductively the bound

$$N_h^*(\theta) \le \max\{0, 2K(\theta)2^{(1-h)/\tau^2} - 1\}, \tag{7.6}$$

where $N_h^*(\theta)$ is the number of non-resonant lines in $L(\theta)$ on scale $h' \ge h$. We proceed exactly as in the proof of Lemma 5. First of all note that for a tree θ to have a line on scale h the condition $K(\theta) > 2^{(h-1)/\tau} > 2^{(h-1)/\tau^2}$ is necessary, by the first Diophantine conditions in (2.33). This means that one can have $N_h^*(\theta) \ge 1$ only if $K = K(\theta)$ is such that $K > k_0 \equiv 2^{(h-1)/\tau}$: therefore for values $K \le k_0$ the bound (7.6) is satisfied. If $K = K(\theta) > k_0$, we assume that the bound holds for all trees θ' with $K(\theta') < K$. Define $E_h = 2^{-1}(2^{(1-h)/\tau^2})^{-1}$, we have to prove that $N_h^*(\theta) \le \max\{0, K(\theta)E_h^{-1} - 1\}$.

The dangerous case is if $m = 1$ then one has a cluster T with two external lines ℓ and ℓ_1, which are both with scales $\ge h$; then

$$\big||\omega n_\ell| - \tilde{\omega}_{m_\ell}\big| \le 2^{-h+1}C_0, \qquad \Big||\omega n_{\ell_1}| - \tilde{\omega}_{m_{\ell_1}}\Big| \le 2^{-h+1}C_0, \tag{7.7}$$

and recall that T is not a self-energy graph, so that $n_\ell \ne n_{\ell_1}$. Note that the validity of both inequalities in (7.7) for $h \ge h_0$ imply, by Lemma 5, that one has

$$|n_\ell - n_{\ell_1}| \ne |m_\ell \pm m_{\ell_1}|. \tag{7.8}$$

Moreover from (7.3), (7.8) and (7.2) one obtains

$$\frac{C_0}{|n_\ell - n_{\ell_1}|^\tau} \le |\omega|n_\ell - n_{\ell_1}| - |m_{\ell_1} \pm m_{\ell_1}|| < \frac{C_0}{3\min\{|n_\ell|, |n_{\ell_1}|\}}, \tag{7.9}$$

which implies

$$\left|n_\ell - n_{\ell_1}\right| \ge 3^{1/\tau}\min\{|n_\ell|, |n_{\ell_1}|\}^{1/\tau}, \tag{7.10}$$

Finally if $C_0|n|^{-\tau} \le ||\omega n| - \tilde{\omega}_m| < C_0 2^{-h+1}$ we have $|n| \ge 2^{(h-1)/\tau}$, so that from (7.8) we obtain $|n_\ell - n_{\ell_1}| \ge 3^{1/\tau}2^{(h-1)/\tau^2}$. Then $K(\theta) - K(\theta_1) > E_h$, which gives (7.6), by using the inductive hypothesis. □

The analysis in Sect. 5 can be repeated with the following modifications. The *renormalized trees* are defined as in Sect. 5, but the rule (9) is replaced with

(9′) To each self-energy graph T the $\mathcal{L}'$ operation is applied, where

$$\mathcal{L}'\mathcal{V}_T^h(\omega n, m) = \mathcal{V}_T^h(\omega n, m), \tag{7.11}$$

if T is such that its two external lines are attached to the same node $\mathbb{V}_0$ of T (see as an example the first graph of Fig. 4.1), and $\mathcal{L}'\mathcal{V}_T^h(\omega n, m) = 0$ otherwise.

With this definition we have the expansion (5.2) to (5.4), with $\nu_{h,m}$ replaced with ν_h (as $\mathcal{L}'\mathcal{V}_T^h(\omega n, m)$ is independent of n, m).

We have that the analogue of Lemma 7 still holds also with $\mathcal{L}', \mathcal{R}'$ replacing $\mathcal{L}, \mathcal{R}$; indeed by construction the dependence on (n, m) in $\mathcal{R}'\mathcal{V}_T^{h_T}(\omega n, m)$ is due to the propagators of lines ℓ along the path connecting the external lines of the self-energy graph. If a line ℓ in the path is a line of type a, the propagator has the form

$$\chi(6\sqrt{|\omega^2 n_\ell^2 - \tilde{\omega}_{m_\ell}^2|})\frac{\chi_h(|\omega n_\ell| - \tilde{\omega}_{m_\ell})}{-\left(|\omega n_\ell| + \tilde{\omega}_{m_\ell}\right)\left(|\omega n_\ell| - \tilde{\omega}_{m_\ell}\right)}, \tag{7.12}$$

and the second factor in the denominator is bounded proportionally to 2^{-h_ℓ}, while the first is bounded proportionally to $|m_\ell^0 + m_{\ell_T^e}|$. Then we get for $h_T^e \le h_0$,

$$|\mathcal{R}'\mathcal{V}_T^{h_T}(\omega n, m) g^{(h_{\ell_T^e})}(\omega n, m)| \le C\varepsilon^2 \frac{e^{-\kappa|m_\ell^0|}}{|m_\ell^0 + m_{\ell_T^e}|} |n_{\ell_T^e}|^{\tau-1}, \tag{7.13}$$

which means that propagator of the external line of the resonance T is compensated, if $1 < \tau \le 2$, by the extra factor $(m_{\ell_T^e})^{-1}$. Here we used that there are at least two nodes, carrying a node factor proportional to ε, not contained in any inner internal self-energy graphs, as otherwise the points to which the external lines are attached would coincide.

The same happens if ℓ is of type b, as we are going to show. In the following with C we denote any constant. We can assume that $|m_{\ell_T^e}|/2 < |m_\ell| < 2|m_{\ell_T^e}|$, otherwise one has $|m_\ell^0| > |m_{\ell_T^e}|/2$ and we can use the factor $e^{-\kappa|m_\ell^0|/2} < C|m_{\ell_T^e}|^{-1}$, to compensate the propagator of one of the external lines of T. We can decompose the propagator of the line ℓ as

$$\chi_{-1}(6\sqrt{|\omega^2 n_\ell^2 - \tilde{\omega}_{m_\ell}^2|}) \sum_{h=-1}^{\infty} \frac{\chi_h(|\omega n| - \tilde{\omega}_m)}{-\omega^2 n^2 + \tilde{\omega}_m^2}. \tag{7.14}$$

If the line has scale $h \le h_0$ then we can bound the propagator with $O(|\varepsilon m_{\ell_T^e}|^{-1})$, and we get

$$|\mathcal{R}'\mathcal{V}_T^{h_T}(\omega n, m) g^{(h_{\ell_T^e})}(\omega n, m)| \le C\varepsilon \frac{1}{|m_{\ell_T^e}|} |n_{\ell_T^e}|^{\tau-1}. \tag{7.15}$$

Moreover if $|m_{\ell_T^e}| < 6\varepsilon^{-1}$ still (7.15) holds. We have then to consider the case in which the line ℓ is a line of type b with scale $h > h_0$ and $|m_{\ell_T^e}| \ge 6\varepsilon^{-1}$. We have still two cases: either ℓ is such that $|\omega n_\ell - \tilde{\omega}_{m_\ell}| \le 2C_0/|m_{\ell_T^e}|$ or not. In the first case, remembering that $|\omega n_{\ell_T^e} - \tilde{\omega}_{m_{\ell_T^e}}| \le 2C_0/|m_{\ell_T^e}|$ because ℓ_T^e is a line of type a, we find

$$\frac{C_0}{|n_{\ell^1} - n_{\ell_T^e}|^\tau} \le |\omega(n_{\ell^1} - n_{\ell_T^e}) \pm m_{\ell^1} \pm m_{\ell_T^e}| < 5C_0|m_{\ell_T^e}|^{-1}, \tag{7.16}$$

where we have used that $\tilde{\omega}_m = |m| + O(\varepsilon m^{-1})$. This implies $|n_{\ell^1} - n_{\ell_T^e}| = |n_{\ell^1}^0| > 5^{1/\tau}|m_{\ell_T^e}|^{1/\tau}$, so that one has

$$|\mathcal{R}'\mathcal{V}_T^{h_T}(\omega n, m) g^{h_{\ell_T^e}}(\omega n, m)| \le C\varepsilon^2 e^{-\kappa C|m_{\ell_T^e}|^{\frac{1}{\tau}}} |n_{\ell_T^e}|^{\tau-1}. \tag{7.17}$$

In the second case one has $|\omega n_\ell - \tilde{\omega}_{m_\ell}| > 2C_0/|m_{\ell_T^e}|$. If h_1 is the scale of ℓ we can distinguish two subcases: either $h_1 < h_{\ell_T^e}$ or $h_1 \ge h_{\ell_T^e}$. If $h_1 < h_{\ell_T^e}$ we sum find

$$\frac{C_0}{|n_{\ell^1} - n_{\ell_T^e}|^\tau} \le |\omega(n_{\ell^1} - n_{\ell_T^e}) \pm m_{\ell^1} \pm m_{\ell_T^e}| < 5C_0 2^{-h_1}, \tag{7.18}$$

as $\tilde{\omega}_{m_{\ell_T^e}} = |m_{\ell_T^e}| + O(\varepsilon|m_{\ell_T^e}|^{-1}) = |m_{\ell_T^e}| + O(\varepsilon 2^{-h_1})$ and $\tilde{\omega}_{m_\ell} = |m_\ell| + O(\varepsilon 2^{-h_1})$. Hence we find $|n_\ell^0| > 2^{(h_1-2)/\tau}$ and by bounding the propagator corresponding to ℓ by

$O(|m_\ell|^{-1})2^{h_1})$ and using the factor $e^{-\kappa 2^{(h_1-2)/\tau}}$ we get also in this case $\mathcal{R}'\mathcal{V}_T^{h_T}(\omega n, m)$ is $O(|m_{\ell_T^e}^{-1}|)$.

Finally if $h_1 \geq h_{\ell_T^e}$ we sum again the denominators of ℓ and ℓ_T^e and we find

$$\frac{C_0}{|n_{\ell^1} - n_{\ell_T^e}|^\tau} \leq |\omega(n_{\ell^1} - n_{\ell_T^e}) \pm m_{\ell^1} \pm m_{\ell_T^e}| < 5C_0 2^{-h_{\ell_T^e}}, \tag{7.19}$$

as $\tilde{\omega}_{m_{\ell_T^e}} = |m_{\ell_T^e}| + O(\varepsilon |m_{\ell_T^e}|^{-1}) = |m_{\ell_T^e}| + O(\varepsilon 2^{-h_1})$ and $\tilde{\omega}_{m_{\ell_1}} = |m_{\ell^1}| + O(\varepsilon 2^{-h_1})$. Hence we find $|n^0_{\ell_T^e}| > 2^{(h_{\ell_T^e}-2)/\tau}$ and this factor compensates the extra factor $2^{h_{\ell_T^e}}$ arising from the entering line of T.

Then the bound (5.11) is replaced, using also Lemma 18, by

$$\begin{aligned}
|\mathrm{Val}(\theta)| \leq \varepsilon^{k/2}\overline{D}^k & \Bigg(\prod_{h=h_0}^{\infty} \exp\Big[h \log 2 \Big(4K(\theta) 2^{-(h-2)/\tau^2} - C_h(\theta) + M_h^\nu(\theta) \Big) \Big] \Bigg) \\
& \Bigg(\prod_{h=h_0}^{\infty} 2^{-hM_h^\nu(\theta)} \Bigg) \\
& \Bigg(\prod_{\mathbb{V} \in V(\theta) \cup E(\theta)} e^{-\kappa(|n_v| + |n'_v|)} \Bigg) \Bigg(\prod_{\mathbb{V} \in V(\theta) \cup E(\theta)} e^{-\kappa(|m_v| + |m'_v|)} \Bigg).
\end{aligned} \tag{7.20}$$

There is no need of Lemma 8, while the analogues of Lemma 9 and Lemma 10 can be proved with some obvious changes. For instance in (5.28) one has $\eta_\ell = 0$, which shows that the first Mel'nikov is required, and ν_m is replaced with ν, i.e. one needs only one counterterm.

Finally also the analogue of Proposition 2 can be proved by reasoning as in Sect. 6, simply by observing that in order to impose the first Mel'nikov conditions (6.16) one can take $\tau_0 = \tau$; note indeed that the condition $\tau > 2$ was made necessary to obtain the second Mel'nikov conditions (6.17) and (6.18).

8. Generalizations of the Results

So far we have considered only the case $\varphi(u) = u^3$ in (1.1). Now we consider the case in which the function $\varphi(u)$ in (1.1) is replaced with any odd analytic function

$$\varphi(u) = \Phi u^3 + O(u^5), \qquad \Phi \neq 0. \tag{8.1}$$

Define $\omega = \sqrt{1 - \lambda\varepsilon}$ as in Sect. 1. Then by choosing $\lambda = \sigma$ we obtain, instead of (1.14),

$$\begin{aligned}
&\mathrm{Q} \qquad \begin{cases} n^2 a_n = \big[(v+w)^3 + O(\varepsilon)\big]_{n,n}, \\ -n^2 a_n = \big[(v+w)^3 + O(\varepsilon)\big]_{n,-n}, \end{cases} \\
&\mathrm{P} \qquad \Big(-\omega^2 n^2 + m^2\Big) w_{n,m} = \varepsilon \Big[(v+w)^3 + O(\varepsilon)\Big]_{n,m}, \qquad |m| \neq |n|,
\end{aligned} \tag{8.2}$$

where $O(\varepsilon)$ denotes analytic functions in u and ε of order at least one in ε. Then we can introduce an auxiliary parameter μ, by replacing ε with $\varepsilon\mu$ in (8.2) (recall that at the end one has to set $\mu = 1$). Then we can proceed as in the previous sections. The equation for a_0 is the same as before, and only the diagrammatic rules for the coefficients $u_{n,m}^{(k)}$ have to be slightly modified. The nodes can have any odd number of entering lines, that is $s_{\mathbb{V}} = 1, 3, 5, \ldots$. For nodes $\mathbb{V}$ of w-type with $s_{\mathbb{V}} \geq 3$ the node factor is given by $\eta_{\mathbb{V}} = \varepsilon^{(s_{\mathbb{V}}-1)/2}$, which has to be added to the list of node factors (3.3), while for nodes $\mathbb{V}$ of v-type one has to add to the list in (3.2) other (obvious) contributions, arising from the fact that the function $f_n^{(k)}$ in (2.4) has to be replaced with

$$f_n^{(k)} = \sum_{k'=1}^{k} f_n^{(k,k')}, \tag{8.3}$$

where the function $f_n^{(k,1)}$ is given by (2.12), while for $k \geq 2$ and $1 < k' \leq k$ one has

$$f_n^{(k,k')} = - \sum_{k_1+\ldots+k_{2k'+1}=k-k'} \sum_{\substack{n_1+\ldots+n_{2k'+1}=n \\ m_1+\ldots+m_{2k'+1}=m}} u_{n_1,m_1}^{(k_1)} \ldots u_{n_{2k'+1},m_{2k'+1}}^{(k_{2k'+1})}, \tag{8.4}$$

where the symbols have to be interpreted according to (2.13) and (2.14); note that $s = 2k' + 1$ is the number of lines entering the node in the corresponding graphical representation.

In the same way one has to modify (2.31) by replacing the last term in the r.h.s. with

$$[\varphi(v + w)]_{n,m}^{(k)} = \sum_{k'=1}^{k} [\varphi(v + w)]_{n,m}^{(k,k')}, \tag{8.5}$$

where $[\varphi(v + w)]_{n,m}^{(k,1)}$ is given by the old (2.32), while all the other terms are given by expressions analogous to (8.4).

The discussion then proceeds exactly as in the previous cases. Of course we have to use that, by writing

$$\varphi(u) = \sum_{k=1}^{\infty} \Phi_k u^{2k+1}, \tag{8.6}$$

with $\Phi_1 = \Phi$, the constants Φ_k can be bounded for all $k \geq 1$ by a constant to the power k (which follows from the analyticity assumption).

By taking into account the new diagrammatic rules, one change in the proper way the definition of the tree value, and a result analogous to Lemma 2 is easily obtained. The second statement of Lemma 3 has to be changed into $|E(\theta)| \leq \sum_{\mathbb{V} \in V(\theta)} (s_{\mathbb{V}} - 1) + 1$ (see (3.24)), while the bound on $|V_v^3(\theta)|$ still holds. For $V_v^s(\theta)$, with $s \geq 5$, one has to use the factors $\varepsilon^{(s-1)/2}$.

Nothing changes in the following sections, except that the bound (5.12) has to be suitably modified in order to take into account the presence of the new kinds of nodes (that is the nodes with branching number more than three).

At the end we obtain the proof of Theorem 1 for general odd nonlinearities starting from the third order. Until now we are still confining ourselves to the case $j = 1$.

If we choose $j > 1$ we have to perform a preliminary rescaling $u(t, x) \to ju(jt, jx)$, and we write down the equation for $U(t, x) = u(jt, jx)$. If $\varphi(u) = Fu^3$ we see immediately that the function U solves the same equation as before, so that the same conditions on ε has to be imposed in order to find a solution. In the general case (8.2) holds with $j^2\varepsilon$ replacing ε. This completes the proof of Theorem 1 in all cases.

Appendix A1. The Solution of (1.5)

The odd 2π-periodic solutions of (1.5) can be found in the following way [30, 32]. First we consider (1.5), with $\langle a_0^2 \rangle$ replaced by a parameter c_0, and we see that

$$a_0(\xi) = V \,\mathrm{sn}(\Omega\xi, \mathfrak{m}), \tag{A1.1}$$

where $\mathrm{sn}(\Omega\xi, \mathfrak{m})$ is the sine-amplitude function with modulus $\sqrt{\mathfrak{m}}$ [25, 1], is an odd solution of

$$\ddot{a}_0 = -3c_0 a_0 - a_0^3, \tag{A1.2}$$

if the following relations are verified by $V, \Omega, c_0, \mathfrak{m}$

$$V = \sqrt{-2\mathfrak{m}}\Omega, \qquad \frac{V^2}{6c_0 + V^2} = -\mathfrak{m}. \tag{A1.3}$$

Of course both V and $\mathfrak{m}$ can be written as a function of c_0 and Ω as

$$\mathfrak{m} = \frac{3c_0}{\Omega^2} - 1, \qquad V = \sqrt{2 - \frac{6c_0}{\Omega^2}}\,\Omega. \tag{A1.4}$$

In particular one finds

$$\partial_\Omega V = \sqrt{-2\mathfrak{m}} + \frac{1}{\sqrt{-2\mathfrak{m}}}\frac{6c_0}{\Omega^2} = \frac{\sqrt{-2\mathfrak{m}}}{\mathfrak{m}}\left(\mathfrak{m} - \frac{3c_0}{\Omega^2}\right) = \sqrt{\frac{2}{-\mathfrak{m}}}, \tag{A1.5}$$

so that one has

$$\frac{\Omega\partial_\Omega V}{V} = \frac{1}{-\mathfrak{m}}. \tag{A1.6}$$

If we impose also that the solution (A1.1) is 2π-periodic (and we recall that $4K(\mathfrak{m})$ is the natural period of the sine-amplitude $\mathrm{sn}(\xi, \mathfrak{m})$, [1]), we obtain $\Omega = \Omega_{\mathfrak{m}} = 2K(\mathfrak{m})/\pi$, and V is fixed to the value $V_{\mathfrak{m}} = \sqrt{-2\mathfrak{m}}\Omega_{\mathfrak{m}}$.

Finally imposing that c_0 equals the average of a_0^2 fixes $\mathfrak{m}$ to be the solution of [32]

$$E(\mathfrak{m}) = K(\mathfrak{m})\,\frac{7 + \mathfrak{m}}{6}, \tag{A1.7}$$

where

$$K(\mathfrak{m}) = \int_0^{\pi/2} \mathrm{d}\theta\, \frac{1}{\sqrt{1 - \mathfrak{m}\sin^2\theta}}, \qquad E(\mathfrak{m}) = \int_0^{\pi/2} \mathrm{d}\theta\, \sqrt{1 - \mathfrak{m}\sin^2\theta} \tag{A1.8}$$

are, respectively, the complete elliptic integral of the first kind and the complete elliptic integral of the second kind [1], and we have used that the average of $\mathrm{sn}^2(\Omega\xi, \mathfrak{m})$ is $(K(\mathfrak{m}) - E(\mathfrak{m}))/K(\mathfrak{m})$. This gives $\mathfrak{m} \approx -0.2554$.

One can find $2\pi/j$-periodic solutions by noticing that Eqs. (1.5) are invariant by the simmetry $a_0(\xi) \to \alpha\, a_0(\alpha\xi)$, so that a complete set of odd 2π-periodic solutions is provided by $a_0(\xi, j) = j\, a_0(j\xi)$.

Appendix A2. Proof of Lemma 1

The solution of (2.21) is found by variation of constants. We write $\langle a_0^2 \rangle = c_0$, and consider c_0 as a parameter. Let $a_0(\xi)$ be the solution (A1.1) of Eq. (A1.2), and define the Wronskian matrix

$$W(\xi) = \begin{pmatrix} w_{11}(\xi) & w_{12}(\xi) \\ w_{21}(\xi) & w_{22}(\xi) \end{pmatrix}, \tag{A2.1}$$

which solves the linearized equation

$$\dot{W} = M(\xi) W, \qquad M(\xi) = \begin{pmatrix} 0 & 1 \\ -3a_0^2(\xi) - 3c_0 & 0 \end{pmatrix}, \tag{A2.2}$$

and is such that $W(0) = \mathbb{1}$ and $\det W(\xi) = 1 \ \forall \xi$. We need two independent solutions of the linearized equation. We can take one as $\dot{a}_0$, the other as $\partial_\Omega a_0$ (we recall that $\mathfrak{m}$ and V are functions of Ω and c_0 through (A1.3)); then one has

$$\begin{aligned}
w_{11}(\xi) &= \frac{1}{\Omega V} \dot{a}_0(\xi) = \mathrm{cn}(\Omega\xi, \mathfrak{m})\, \mathrm{dn}(\Omega\xi, \mathfrak{m}), \\
w_{21}(\xi) &= \dot{w}_{11}(\xi) = -\mathrm{sn}(\Omega\xi, \mathfrak{m}) \left(\mathrm{dn}^2(\Omega\xi, m) + \mathfrak{m}\, \mathrm{cn}^2(\Omega\xi, \mathfrak{m}) \right), \\
w_{12}(\xi) &= \frac{1}{V(1 + D_\mathfrak{m})} \partial_\Omega a_0(\xi) \\
&= B_\mathfrak{m} \left(\xi \, \mathrm{cn}(\Omega\xi, \mathfrak{m})\, \mathrm{dn}(\Omega\xi, \mathfrak{m}) + \Omega^{-1} D_\mathfrak{m} \mathrm{sn}(\Omega\xi, \mathfrak{m}) \right), \\
w_{22}(\xi) &= \dot{w}_{12}(\xi) \\
&= \mathrm{cn}(\Omega\xi, \mathfrak{m})\, \mathrm{dn}(\Omega\xi, \mathfrak{m}) - \Omega B_\mathfrak{m} \xi \, \mathrm{sn}(\Omega\xi, \mathfrak{m}) \left(\mathrm{dn}^2(\Omega\xi, \mathfrak{m}) + \mathfrak{m}\, \mathrm{cn}^2(\Omega\xi, \mathfrak{m}) \right),
\end{aligned} \tag{A2.3}$$

where we have used (A1.3) to define the dimensionless constants

$$D_\mathfrak{m} = \frac{\Omega \partial_\Omega V}{V} = \frac{1}{-\mathfrak{m}}, \qquad B_\mathfrak{m} = \frac{1}{(1 + D_\mathfrak{m})} = \frac{-\mathfrak{m}}{1 - \mathfrak{m}}. \tag{A2.4}$$

As we are interested in the case $c_0 = \langle a_0^2 \rangle$ we set $\Omega = \Omega_\mathfrak{m} = 2K(\mathfrak{m})/\pi$ (in order to have the period equal to 2π) and we fix $\mathfrak{m}$ as in Appendix A1.

Then, by defining $X = (y, \dot{y})$ and $F = (0, h)$, we can write the solution of (2.21) as the first component of

$$X(\xi) = W(\xi)\, \overline{X} + W(\xi) \int_0^\xi \mathrm{d}\xi' \, W^{-1}(\xi')\, F(\xi'), \tag{A2.5}$$

where $\overline{X} = (0, \dot{y}(0))$ denote the corrections to the initial conditions $(a_0(0), \dot{a}_0(0))$, and $y(0) = 0$ as we are looking for an odd solution.

Shorten $c(\xi) \equiv \mathrm{cn}(\Omega_\mathfrak{m}\xi, \mathfrak{m})$, $s(\xi) \equiv \mathrm{sn}(\Omega_\mathfrak{m}\xi, \mathfrak{m})$, and $d(\xi) \equiv \mathrm{dn}(\Omega_\mathfrak{m}\xi, \mathfrak{m})$, and define $cd(\xi) = \mathrm{cn}(\Omega_\mathfrak{m}\xi, \mathfrak{m})\, \mathrm{dn}(\Omega_\mathfrak{m}\xi, \mathfrak{m})$. One can write the first component of (A2.5) as

$$\begin{aligned} y(\xi) &= w_{12}(\xi)\,\dot y(0) + \int_0^{\xi} \mathrm{d}\xi' \left(w_{12}(\xi)\,w_{11}(\xi') - w_{11}(\xi)\,w_{12}(\xi') \right) h(\xi') \\ &= B_{\mathfrak{m}} \Big(\xi\, cd(\xi) + \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} s(\xi) \Big)\, \dot y(0) \\ &\quad + B_{\mathfrak{m}} \Bigg(cd(\xi) \int_0^{\xi} \mathrm{d}\xi' \int_0^{\xi'} \mathrm{d}\xi''\, cd(\xi'')\, h(\xi'') \\ &\quad + \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} \Big(s(\xi) \int_0^{\xi} \mathrm{d}\xi'\, cd(\xi')\, h(\xi') - cd(\xi) \int_0^{\xi} \mathrm{d}\xi'\, s(\xi')\, h(\xi') \Big) \Bigg), \end{aligned} \tag{A2.6}$$

as we have explictly written

$$\begin{aligned} & w_{12}(\xi)\, w_{11}(\xi') - w_{11}(\xi)\, w_{12}(\xi') \\ &\quad = B_{\mathfrak{m}} \Big(\xi\, cd(\xi)\, cd(\xi') + \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} s(\xi)\, cd(\xi') \\ &\qquad\qquad - cd(\xi)\, \xi'\, cd(\xi') - \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} cd(\xi)\, s(\xi') \Big), \end{aligned} \tag{A2.7}$$

and integrated by parts

$$\begin{aligned} & \int_0^{\xi} \mathrm{d}\xi' \left(\xi\, cd(\xi)\, cd(\xi') - cd(\xi)\, \xi'\, cd(\xi') \right) h(\xi') \\ &\quad = cd(\xi) \int_0^{\xi} \mathrm{d}\xi' \int_0^{\xi'} \mathrm{d}\xi'' cd(\xi'')\, h(\xi''). \end{aligned} \tag{A2.8}$$

By using that if $\mathbf{P}[F] = F$ then $\mathbf{P}[\mathbf{I}[F]] = \mathbf{I}[F]$ and that $\mathbf{I}$ switches parity we can rewrite (A2.6) as

$$\begin{aligned} y(\xi) = B_{\mathfrak{m}} \Bigg(& \xi\, cd(\xi) \Big(\dot y(0) - \mathbf{I}[cd\, h](0) - \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} \langle s\, h \rangle \Big) \\ & + \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} s(\xi) \Big(\dot y(0) - \mathbf{I}[cd\, h](0) \Big) \\ & + \Omega_{\mathfrak{m}}^{-1} D_{m} \big(s(\xi)\, \mathbf{I}[cd\, h](\xi) - cd(\xi)\, \mathbf{I}[\mathbf{P}[s\, h]](\xi) \big) + cd(\xi)\, \mathbf{I}[\mathbf{I}[cd\, h]](\xi) \Bigg). \end{aligned} \tag{A2.9}$$

This is an odd 2π-periodic analytic function provided that we choose

$$\dot y(0) - \mathbf{I}[cd\, h](0) - \Omega_{\mathfrak{m}}^{-1} D_{\mathfrak{m}} \langle s\, h \rangle = 0, \tag{A2.10}$$

which fixes the parameter $\dot y(0)$; hence (2.22) is found.

Appendix A3. Proof of (2.27)

We have to compute $\langle a_0 \mathbf{L}[a_0]\rangle$, which is given by (recall that one has $a_0(\xi) = V_{\mathfrak{m}}\, s(\xi)$ and $cd(\xi) = \Omega_{\mathfrak{m}}^{-1} \dot{s}(\xi)$)

$$\langle a_0 \mathbf{L}[a_0]\rangle = B_{\mathfrak{m}} \Omega_{\mathfrak{m}}^{-2} V_{\mathfrak{m}}^2 \Big(D_{\mathfrak{m}}^2 \left\langle s^2 \right\rangle^2 + D_{\mathfrak{m}} \left\langle s^2\, \mathbf{I}[s\dot{s}] \right\rangle - D_{\mathfrak{m}} \left\langle s\dot{s}\, \mathbf{I}[\mathbf{P}[s^2]] \right\rangle + \langle s\dot{s}\mathbf{I}[\mathbf{I}[s\dot{s}]]\rangle \Big). \tag{A3.1}$$

Using that one has $\mathbf{I}[s\dot{s}] = (s^2 - \langle s^2\rangle)/2$, we obtain

$$\left\langle s^2 \mathbf{I}[s\dot{s}] \right\rangle = \frac{1}{2}\left(\left\langle s^4 \right\rangle - \left\langle s^2 \right\rangle^2 \right); \tag{A3.2}$$

moreover, integrating by parts, we find

$$\begin{aligned} \left\langle s\dot{s}\, \mathbf{I}[\mathbf{P}[s^2]] \right\rangle &= -\frac{1}{2}\left\langle s^4 \right\rangle + \frac{1}{2}\left\langle s^2 \right\rangle^2 , \\ \left\langle s\dot{s}\mathbf{I}[\mathbf{I}[s\dot{s}]] \right\rangle &= -\frac{1}{4}\left\langle s^4 \right\rangle + \frac{1}{4}\left\langle s^2 \right\rangle^2 , \end{aligned} \tag{A3.3}$$

so that we finally get

$$\langle a_0 \mathbf{L}[a_0]\rangle = \frac{1}{2} V_{\mathfrak{m}}^2 \Omega_{\mathfrak{m}}^{-2} B_{\mathfrak{m}} \Big(\Big(2D_{\mathfrak{m}} - \frac{1}{2} \Big) \langle s^4\rangle + \Big(2D_{\mathfrak{m}} \left(D_{\mathfrak{m}} - 1 \right) + \frac{1}{2} \Big) \langle s^2\rangle^2 \Big), \tag{A3.4}$$

which is strictly positive by (A2.4) and by the choice of $\mathfrak{m}$ according to Appendix A1; then (2.27) follows.

Appendix A4. Proof of Lemma 15

We shall prove inductively on p the bounds (6.12). From (6.4) we have

$$|\tilde{\omega}_m^{(p)}(\varepsilon) - \tilde{\omega}_m^{(p-1)}(\varepsilon)| \le C|\nu_m(\tilde{\omega}^{(p-1)}(\varepsilon), \varepsilon) - \nu_m(\tilde{\omega}^{(p-2)}(\varepsilon), \varepsilon)|, \tag{A4.1}$$

as we can bound $|\tilde{\omega}_m^{(p)}(\varepsilon) + \tilde{\omega}_m^{(p-1)}(\varepsilon)| \ge 1$ for $\varepsilon \in \mathcal{E}^{(p)}$.

We set, for $|m| \ge 1$,

$$\Delta\nu_{h,m} \equiv \nu_{h,m}(\tilde{\omega}^{(p-1)}(\varepsilon), \varepsilon) - \nu_{h,m}(\tilde{\omega}^{(p-2)}(\varepsilon), \varepsilon) = \lim_{q\to\infty} \Delta\nu_{h,m}^{(q)}, \tag{A4.2}$$

where we have used the notations (5.37) to define

$$\Delta\nu_{h,m}^{(q)} = \nu_{h,m}^{(q)}(\tilde{\omega}^{(p-1)}(\varepsilon), \varepsilon) - \nu_{h,m}^{(q)}(\tilde{\omega}^{(p-2)}(\varepsilon), \varepsilon). \tag{A4.3}$$

We want to prove inductively on q the bound

$$\left| \Delta\nu_{h,m}^{(q)} \right| \le C\varepsilon \|\tilde{\omega}^{(p-1)}(\varepsilon) - \tilde{\omega}^{(p-2)}(\varepsilon)\|_{\infty}, \tag{A4.4}$$

for some constant C, uniformly in q, h and m.

For $q = 0$ the bound (A4.4) is trivially satisfied. Then assume that (A4.4) hold for all $q' < q$.

For simplicity we set $\tilde{\omega} = \tilde{\omega}^{(p-1)}(\varepsilon)$ and $\tilde{\omega}' = \tilde{\omega}^{(p-2)}(\varepsilon)$. We can write, from (6.25), for $|m| \geq 1$ and for $h \geq 0$,

$$\Delta\nu_{h,m}^{(q)} = -\sum_{k=h}^{\bar{h}-1} 2^{-k-2}\Big(\beta_{k,m}^{(q)}(\tilde{\omega}, \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde{\omega}', \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}', \varepsilon)\})\Big), \tag{A4.5}$$

where we recall that $\beta_{k,m}^{(q)}(\tilde{\omega}, \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\})$ depend only on $\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)$ with $k' \leq k-1$, and we can set

$$\begin{aligned} &\beta_{k,m}^{(q)}(\tilde{\omega}, \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\}) \\ &= \beta_{k,m}^{(a)(q)}(\tilde{\omega}, \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\}) - \beta_{k,m}^{(b)(q)}(\tilde{\omega}, \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\}), \end{aligned} \tag{A4.6}$$

according to the settings in (2.3). Then we can split the differences in (A4.6) into

$$\begin{aligned} &\beta_{k,m}^{(q)}(\tilde{\omega}, \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde{\omega}', \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}', \varepsilon)\}) \\ &= \Big(\beta_{k,m}^{(q)}(\tilde{\omega}, \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde{\omega}', \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\})\Big) \\ &\quad + \Big(\beta_{k,m}^{(q)}(\tilde{\omega}', \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde{\omega}', \varepsilon, \{\nu_{k'}^{(q-1)}(\tilde{\omega}', \varepsilon)\})\Big), \end{aligned} \tag{A4.7}$$

and we bound separately the two terms.

The second term can be expressed as the sum of trees θ which differ from the previously considered ones as, among the nodes v of w-type with only one entering line, there are some with $\nu_{k_{\mathbb{V}}}^{(c_{\mathbb{V}})(q-1)}(\tilde{\omega}, \varepsilon)$, some with $\nu_{k_{\mathbb{V}}}^{(c_{\mathbb{V}})(q-1)}(\tilde{\omega}', \varepsilon)$ and one with $\nu_{k_{\mathbb{V}}}^{(c_{\mathbb{V}})(q-1)}(\tilde{\omega}, \varepsilon) - \nu_{k_{\mathbb{V}}}^{(c_{\mathbb{V}})(q-1)}(\tilde{\omega}', \varepsilon)$. Then we can bound

$$\begin{aligned} &\left|\beta_{k,m}^{(q)}(\tilde{\omega}', \{\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde{\omega}', \{\nu_{k'}^{(q-1)}(\tilde{\omega}', \varepsilon)\})\right| \\ &\leq D_1\varepsilon \sup_{h'\geq 0}\sup_{|m'|\geq 2} \Delta\nu_{h',m'}^{(q-1)} \leq D_1 C\varepsilon^2 \left\|\tilde{\omega}' - \tilde{\omega}\right\|_\infty, \end{aligned} \tag{A4.8}$$

by the inductive hypothesis.

We are left with the first term in (A4.8). We can reason as in [23] (which we refer to for details), and at the end, instead of (A9.14) of [23], we obtain

$$\begin{aligned} &\left|\mathrm{Val}'(\theta_0)\left(g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega})\right)\mathrm{Val}'(\theta_1)\Big(\prod_{i=2}^{s}\mathrm{Val}(\theta_i)\Big)\right| \\ &\leq C^{|V(T)|}e^{-\kappa K(T)/2}\varepsilon^{|V_w(T)|+|V_v^1(T)|}\left\|\tilde{\omega}' - \tilde{\omega}\right\|_\infty \\ &\leq C^{|V(T)|}\varepsilon^{|V_w(T)|+|V_v^1(T)|}e^{-\kappa K(T)/4}e^{-\kappa 2^{(k-1)/\tau}/4}\left\|\tilde{\omega}' - \tilde{\omega}\right\|_\infty, \end{aligned} \tag{A4.9}$$

where $K(T)$ is defined in (4.7).

Therefore we can bound $\left\|\tilde{\omega}^{(p)}(\varepsilon) - \tilde{\omega}^{(p-1)}(\varepsilon)\right\|_\infty$ with a constant times ε times the same expression with p replaced with $p-1$, *i.e.* $\left\|\tilde{\omega}^{(p-1)}(\varepsilon) - \tilde{\omega}^{(p-2)}(\varepsilon)\right\|_\infty$, so that, by the inductive hypothesis, the bound (6.12) follows.

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