Infinite Dimensional Analysis, Quantum Probability and Related Topics

A STOCHASTIC GOLDEN RULE AND QUANTUM LANGEVIN EQUATION FOR THE LOW DENSITY LIMIT

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A rigorous derivation of quantum Langevin equation from microscopic dynamics in the low density limit is given. We consider a quantum model of a microscopic system (test particle) coupled with a reservoir (gas of light Bose particles) via interaction of scattering type. We formulate a mathematical procedure (the so-called stochastic golden rule) which allows us to determine the quantum Langevin equation in the limit of large time and small density of particles of the reservoir. The quantum Langevin equation describes not only dynamics of the system but also the reservoir. We show that the generator of the corresponding master equation has the Lindblad form of most general generators of completely positive semigroups.

Keywords: low density, quantum stochastic equations, quantum Langevin equation

1. Introduction

In the past years many important physical models have been investigated by using the stochastic limit method (see [1, 2] for a survey). In particular Fermi golden rule has been generalized to a *stochastic golden rule* which allows to solve the following problem: give a quantum Hamiltonian system, determine the associated Langevin equation (including the structure of the so-called master fields or quantum noises) and obtain as a corollary the master equation associated to this system.

The Langevin equation is an asymptotic equation, valid in a regime of large times and small parameters (e.f. weak coupling) and its interest for the applications

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comes from the fact that the relevant physical phenomena in this regime are much more transparent than in the original Hamiltonian equation and in this sense an approximate equation gives a better description of the physical phenomenon than the original exact equation.

The stochastic golden rule plays a fundamental role in the stochastic limit because it allows to separate the problem of finding the correct solution from the problem of giving a rigorous proof of the convergence of the method.

Although well developed in the weak coupling case, in the low density case, the stochastic golden rule was formulated in [1] only for Fock fields and this limitation exclude of the most interesting cases for the applications.

The development of a stochastic golden rule for the low density limit required a totally different approach from that developed for the weak coupling case and the first step in this direction was done in the paper [3] where the problem was solved under the (very strong) assumption of rotating wave approximation (RWA). In the present paper we bring to a completion the programme begun in [3] and we establish the stochastic golden rule for general discrete spectrum system interacting with a Boson reservoir without RWA. This also gives, when combined with the analytical estimates of part III of [1], a simplified proof of the deduction of the stochastic Schrödinger equation for the low density limit. The new proof is much simpler and allows a deeper understanding of the emergence of the scattering operator, more precisely the T-operator, in this equation.

To describe a quantum physical model to which we apply the low density limit let us consider an N-level atom immersed in a free gas whose molecules can collide with the atom; the gas is supposed to be very dilute. Then the reduced time evolution for the atom will be Markovian, since the characteristic time t_S for appreciable action of the surroundings on the atom (time between collisions) is much larger than the characteristic time t_R for relaxation of correlations in the surroundings. Rigorous results substantiating this idea have been obtained in [4].

The dynamics of the N-level atom interacting with the free gas converges, in the low density limit, to the solution of a quantum stochastic differential equation driven by quantum Poisson noise [5]. Indeed, from a semiclassical point of view, collision times, being times of occurrence of rare events, will tend to become Poisson distributed, whereas the effect of each collision will be described by the (quantum-mechanical) scattering operator of the atom with one gas particle (see the description of the quantum Poisson process in [6]).

We consider a microscopic system (test particle) interacting with a gas of Bose particles (reservoir) via interaction of scattering type so that the particles of the reservoir are only scattered and not created or destroyed. The reservoir is supposed to be a very dilute gas at equilibrium state. We obtain starting from microphysical dynamics, in the limit of long time and small density of Bose particles, the quantum Langevin equation which describes the evolution of any system observable. The Langevin equation includes not only the system but also the reservoir dynamics. After averaging over the reservoir equilibrium state we obtain a master equation which describes the evolution of any system observable, i.e. the reduced evolution of the test particle. The generator of this master equation has the standard Lindblad form [7].

Starting from the quantum stochastic equation (7.29) for the evolution operator one can easy deduce the corresponding quantum linear Boltzmann equation for the density matrix of the system. This is a linear dissipative equation describing an irreversible evolution of the test particle. A general structure of a master equation driving a completely positive time evolution for a test particle in a quantum (Rayleigh) gas has been investigated in [8]. It turns out that general requirements of the underlying symmetry are not sufficient to fix the form of the master equation and one needs a microscopic derivation of the master equation [9]. A linear dissipative equation describing the long time dynamics of a particle interacting with a reservoir (a quantum Fokker-Planck equation) has been studied in [10] where the problem of rigorous derivation of such equations was emphasized.

Using the white noise approach developed in [1] and the energy representation [3] we derive a new algebra for the quadratic master fields in the low density limit. An advantage of this method is the simplicity of derivation of the white noise equations and in the computation of correlation functions. We show that the drift term in the QSDE is simply related with T-operator describing the scattering of the system on the one particle sector of the reservoir.

The main results of the paper are:

(i) the explicit determination of the algebra of the master field in energy representation (section 3)

(ii) the determination of the white noise Hamiltonian equation (6.20) for the low density limit

(iii) the determination of the (causally) ordered form (7.29) of this equation which, as known from the general theory developed in [1], is equivalent to a quantum stochastic differential equation in the sense of [11]

(iv) formula (8.37), which gives an explicit expression for the drift in terms of the 1-particle T-operator

(v) the determination of the quantum Langevin equation (9.39) and the quantum Markovian generator (9.40) of the corresponding master equation.

Our strategy is the following. In section 2 we describe the model which we consider; and for this model, we introduce the Fock-anti-Fock representation. In section 3 we introduce the rescalled fields whose stochastic limits are the master fields in the LDL. In section 4 we derive the commutation relations of the master fields (Theorem 1). Then in section 5 we construct a concrete representation of this algebra. Using this representation we write in section 6 the stochastic Schrödinger equation (6.20) for the limiting evolution operator. In section 7 we rewrite the stochastic Schrödinger equation in normally ordered form, i.e. in the form of a quantum stochastic differential equation. In section 8 we prove formula (8.37) which expresses the drift, in the equation for the evolution operator in the LDL in terms of: the 1-particle T-operator, the 1-particle gas Hamiltonian and the spectral projections of the system Hamiltonian. In section 9 we obtain the quantum Langevin equation and the quantum Markovian generator of the corresponding master equation.

2. The Model

Now let us explain our notations. Let $\mathcal{H}_{\rm S}$ be a Hilbert space of the system with the Hamiltonian H_S. We will suppose that the system Hamiltonian H_S has a discrete spectrum

$$H_S = \sum_k \varepsilon_k P_k$$

where ε_k are the eigenvalues and P_k the spectral projections.

We will consider the case of a Boson reservoir in this paper. Therefore the reservoir is described by the Boson Fock space $\Gamma(\mathcal{H}_1)$ over the one particle Hilbert space $\mathcal{H}_1 = L^2(\mathbb{R}^d)$ (the scalar product in $\mathcal{H}_1$ we denote as $< \cdot, \cdot >$), where $d = 3$ in physical case. Moreover, the free Hamiltonian of the reservoir is given by $H_R := \Gamma(H_1)$ (the second quantization of the one particle Hamiltonian H_1) and the total Hamiltonian of the compound system is given by a self–adjoint operator on the total Hilbert space $\mathcal{H}_S \otimes \Gamma(\mathcal{H}_1)$, which has the form

$$H_{\rm tot} := H_S \otimes 1 + 1 \otimes H_R + V =: H_0 + V$$

Here V is the interaction Hamiltonian between the system and the reservoir. The evolution operator at time t is given by:

$$U(t) := e^{itH_0} \cdot e^{-itH_{\rm tot}} \tag{2.1}$$

and it satisfies the differential equation

$$\partial_t U(t) = -iV(t)U(t)$$

where the quantity the evolved interaction $V(t)$ is defined by

$$V(t) = e^{itH_0} V e^{-itH_0}.$$

The interaction Hamiltonian will be assumed to have the following form:

$$V = D \otimes A^+(g_0)A(g_1) + D^+ \otimes A^+(g_1)A(g_0)$$

where D is a bounded operator in $\mathcal{H}_S$, $D \in \mathbf{B}(\mathcal{H}_S)$, A and A^+ are annihilation and creation operators and $g_0, g_1 \in \mathcal{H}_1$ are form-factors describing the interaction of the system with the reservoir. Therefore, with the notations

$$S_t := e^{itH_1} \; ; \qquad D(t) := e^{itH_S} D e^{-itH_S}$$

the evolved interaction can be written in the form

$$V(t) = D(t) \otimes A^+(S_t g_0)A(S_t g_1) + D^+(t) \otimes A^+(S_t g_1)A(S_t g_0) \tag{2.2}$$

The initial state of the compound system is supposed to be of the form

$$\rho = \rho_S \otimes \varphi^{(\xi)}.$$

Here ρ_S is arbitrary density matrix of the system and the initial state of the reservoir $\varphi^{(\xi)}$ is the Gibbs state, at inverse temperature β, of the free evolution, i.e. the gauge invariant quasi-free state characterized by

$$\varphi^{(\xi)}(W(f)) = \exp\Big(-\frac{1}{2} < f, (1 + \xi e^{-\beta H_1})(1 - \xi e^{-\beta H_1})^{-1} f >\Big) \tag{2.3}$$

for each $f \in \mathcal{H}_1$. Here $W(f)$ is the Weyl operator, β the inverse temperature of the reservoir, $\xi = e^{\beta\mu}$ the fugacity, μ the chemical potential. We suppose that the temperature $\beta^{-1} > 0$. Therefore for sufficiently low density one is above the transition temperature, and no condensate is present.

We will study the dynamics, generated by the Hamiltonian (2.2) and the initial state of the reservoir (2.3) in the low density regime: $n \to 0$, $t \sim 1/n$ (n is the density of particles of the reservoir). We do not fix the initial state of the system so our results can be applied to an arbitrary initial density matrix of the system. In the low density limit the fugacity ξ and the density of particles of the reservoir n have the same asymptotic, i.e.

$$\lim_{n\to 0} \frac{\xi(n)}{n} = 1$$

Therefore the limit $n \to 0$ is equivalent to the limit $\xi \to 0$.

Throughout this paper, for simplicity, the following technical condition is assumed: the two test functions in the interaction Hamiltonian have disjoint supports in the energy representation. This condition is invariant under the action of any function of H_1 and implies that the two test function g_0, g_1 in the interaction Hamiltonian satisfy:

$$< g_0, S_t e^{-\beta H_1} g_1 >= 0 \qquad \forall t \in \mathbb{R}.$$

In the paper [3] we obtained for the model described above a quantum stochastic differential equation under additional rotating wave approximation condition:

$$D(t) = e^{itH_S} D e^{-itH_S} = e^{-it\omega_0} D \tag{2.4}$$

for some real number ω_0.

In the present article we will derive the white noise Schrödinger equation without assuming any relation between D and H_{S}.

Let us rewrite the free evolution of D in a form which is convenient for derivation of the white noise equation. For this we introduce the set of all Bohr frequencies

$$\mathrm{B(H_S)} =: \mathrm{B} := \{\omega \,|\, \exists\, \varepsilon_k, \varepsilon_m \in \mathrm{spec}\, \mathrm{H_S} \text{ s.t. } \omega = \varepsilon_k - \varepsilon_m\}$$

Using this notion and the properties of the spectral projections one can rewrite the free evolution of an arbitrary system operator D in the form

$$D(t) = \sum_{k,m} e^{it(\varepsilon_k - \varepsilon_m)} P_k D P_m = \sum_{\omega \in \mathrm{B}} e^{-it\omega} D_\omega$$

where we denote

$$D_\omega := \sum_{k,m:\varepsilon_m - \varepsilon_k = \omega} P_k D P_m$$

We realize the representation space as the tensor product of a Fock and anti-Fock representations. Then the expectation values with respect to the state $\varphi^{(\xi)}$ for the model with the interaction Hamiltonian (2.2) can be conveniently represented as the vacuum expectation values in the Fock-anti-Fock representation for the modified Hamiltonian.

Denote by $\mathcal{H}_1^\iota$ the conjugate of $\mathcal{H}_1$, i.e. $\mathcal{H}_1^\iota$ is identified to $\mathcal{H}_1$ as a set and the identity operator $\iota : \ \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is antilinear

$$\iota(\lambda f) := \bar{\lambda}\iota(f)$$

$$< \iota(f), \iota(g) >_\iota := < g, f >$$

Then, $\mathcal{H}_1^\iota$ is a Hilbert space and, if the vectors of $\mathcal{H}_1$ are thought as ket-vectors $|\xi >$, then the vectors of $\mathcal{H}_1^\iota$ can be thought as bra-vectors $< \xi|$. The corresponding Fock space $\Gamma(\mathcal{H}_1^\iota)$ is called the anti-Fock space.

It was shown in [5] that, with the notations $D_{\omega,0} = D_\omega$, $D_{\omega,1} = D_\omega^+$ and $\lambda = +\sqrt{\xi}$, the part of the modified Hamiltonian which gives a nontrivial contribution in the LDL, acts in $\mathcal{H}_S \otimes \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_1^\iota)$ as

$$H_\lambda(t) = \sum_{\varepsilon=0,1} \sum_{\omega\in B} D_{\omega,\varepsilon} e^{-(-1)^\varepsilon it\omega} \otimes \Big(A^+(S_t g_\varepsilon) A(S_t g_{1-\varepsilon}) \otimes 1$$

$$+ \lambda \big(A(S_t g_{1-\varepsilon}) \otimes A(S_t L g_\varepsilon) + A^+(S_t g_\varepsilon) \otimes A^+(S_t L g_{1-\varepsilon}) \big) \Big).$$

Here A and A^+ are Bose annihilation and creation operators acting in the Fock spaces $\Gamma(\mathcal{H}_1)$ and $\Gamma(\mathcal{H}_1^\iota)$ and $L := e^{-\beta H_1/2}$.

The interaction Hamiltonian $H_\lambda(t)$ determines the evolution operator $U_t^{(\lambda)}$ which is the solution of the Schrödinger equation in interaction representation:

$$\partial_t U_t^{(\lambda)} = -i H_\lambda(t) U_t^{(\lambda)}$$

with initial condition

$$U_0^{(\lambda)} = 1.$$

This is equivalent to the following integral equation for the evolution operator

$$U_t^{(\lambda)} = 1 - i \int_0^t dt' H_\lambda(t') U_{t'}^{(\lambda)}.$$

The convenience of the Fock-anti-Fock representation, as it was mentioned above, is in the fact that the expectation value, for example, of any time evolved system observable $X_t = U^*(t)(X \otimes 1)U(t)$ with respect to the state (2.3) is equal to the vacuum expectation value in the Fock-anti-Fock representation, i.e.

$$\varphi^{(\xi)}(U^*(t)(X \otimes 1)U(t)) = < U_t^{*(\lambda)} (X \otimes 1 \otimes 1) U_t^{(\lambda)} >_{\text{vac}}$$

3. Energy Representation

We will investigate the limit of the evolution operator $U_{t/\xi}^{(\lambda)}$ when $\xi \to +0$ after the time rescaling $t \to t/\xi$, where $\xi = \lambda^2$. After this time rescaling the equation for the evolution operator becomes

$$\partial_t U_{t/\lambda^2}^{(\lambda)} = -i \sum_{\omega\in B} \sum_{\varepsilon=0,1} D_{\omega,\varepsilon} \otimes \big[N_{\varepsilon,1-\varepsilon,\lambda}((-1)^\varepsilon \omega, t)$$

$$+B_{1-\varepsilon,\varepsilon,\lambda}((-1)^{1-\varepsilon}\omega,t)+B^{+}_{\varepsilon,1-\varepsilon,\lambda}((-1)^{\varepsilon}\omega,t)\big]U^{(\lambda)}_{t/\lambda^2}$$

where we introduced for each $\varepsilon_1,\varepsilon_2=0,1$ and $\omega\in B$ the rescaled fields:

$$N_{\varepsilon_1,\varepsilon_2,\lambda}(\omega,t):=\frac{1}{\lambda^2}e^{-it\omega/\lambda^2}A^{+}(S_{t/\lambda^2}g_{\varepsilon_1})A(S_{t/\lambda^2}g_{\varepsilon_2})\otimes 1 \tag{3.5}$$

$$B_{\varepsilon_1,\varepsilon_2,\lambda}(\omega,t):=\frac{1}{\lambda}e^{it\omega/\lambda^2}A(S_{t/\lambda^2}g_{\varepsilon_1})\otimes A(S_{t/\lambda^2}Lg_{\varepsilon_2}). \tag{3.6}$$

and $B^{+}_{\varepsilon_1,\varepsilon_2,\lambda}(\omega,t)$ is the adjoint of $B_{\varepsilon_1,\varepsilon_2,\lambda}(\omega,t)$.

The energy representation for the creation and annihilation operators is defined by the formulae

$$A^{+}_{E}(g):=A^{+}(P_Eg),\qquad A_E(g):=A(P_Eg) \tag{3.7}$$

with

$$P_E:=\frac{1}{2\pi}\int_{-\infty}^{\infty}dtS_te^{-itE}=\delta(H_1-E) \tag{3.7a}$$

It has the properties

$$S_t=\int dEP_Ee^{itE} \tag{3.7b}$$

$$P_EP_{E'}=\delta(E-E')P_E,\qquad P^{*}_{E}=P_E$$

The meaning of the δ-function in (3.7a) is explained in [1], section (1.2). In our case for $\mathcal{H}_1=L^2(\mathbb{R}^d)$ the one-particle Hamiltonian is the multiplication operator by the function $\omega(k)$ and acts on an element $f\in L^2(\mathbb{R}^d)$ as $(H_1f)(k)=\omega(k)f(k)$ so that $P_E=\delta(\omega(k)-E)$.

It is easy to check that

$$[A_E(f),A^{+}_{E'}(g)]=\delta(E-E')<f,P_Eg>.$$

Using the energy representation (3.7b), (3.5) and (3.6) become respectively

$$N_{\varepsilon_1,\varepsilon_2,\lambda}(\omega,t)=\int\int dE_1dE_2N_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t)$$

$$B_{\varepsilon_1,\varepsilon_2,\lambda}(\omega,t)=\int\int dE_1dE_2B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t)$$

where the rescaled fields in energy representation are given by

$$N_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t):=\frac{e^{it(E_1-E_2-\omega)/\lambda^2}}{\lambda^2}A^{+}_{E_1}(g_{\varepsilon_1})A_{E_2}(g_{\varepsilon_2})\otimes 1 \tag{3.8}$$

$$B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t):=\frac{e^{it(E_2-E_1+\omega)/\lambda^2}}{\lambda}A_{E_1}(g_{\varepsilon_1})\otimes A_{E_2}(Lg_{\varepsilon_2}). \tag{3.9}$$

and $B^{+}_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t)$ is the adjoint of $B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t)$. Let us also denote

$$\gamma_\varepsilon(E) := \int_{-\infty}^{0} < g_\varepsilon, S_t g_\varepsilon > e^{-itE} dt.$$

4. The Algebra of the Master Fields

In this section we derive the algebra of master fields which is the limit as $\lambda \to 0$ of rescaled fields (3.8), (3.9) in the sense of convergence of correlators.

Denote Ω the lattice of all linear combinations of the Bohr frequencies with integer coefficients:

$$\Omega = \{\omega \,|\, \omega = \sum_{k=1}^{N} n_k \omega_k \text{ with } N \in \mathbb{N},\, n_k \in \mathbb{Z},\, \omega_k \in \mathrm{B}\}$$

and extend the definitions (3.8) and (3.9), of the rescaled fields, by allowing in them an arbitrary $\omega \in \Omega$. The following theorem describes the algebra of commutation relations for the master field in the LDL.

Theorem 4.1 *The limits of the rescaled fields*

$$X_{\varepsilon_1,\varepsilon_2}(E_1,E_2,\omega,t) := \lim_{\lambda\to 0} X_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t), \qquad X = B, B^+, N$$

exist in the sense of convergence of correlators and satisfy the commutation relations

$$[B_{\varepsilon_1,\varepsilon_2}(E_1,E_2,\omega,t), B^{+}_{\varepsilon_3,\varepsilon_4}(E_3,E_4,\omega',t')] = 2\pi\delta_{\omega,\omega'}\delta_{\varepsilon_1,\varepsilon_3}\delta_{\varepsilon_2,\varepsilon_4}\delta(t'-t)$$

$$\times\, \delta(E_1-E_3)\delta(E_2-E_4)\delta(E_1-E_2-\omega) < g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} >< g_{\varepsilon_2}, P_{E_2} L^2 g_{\varepsilon_2} > \quad (4.10)$$

$$[B_{\varepsilon_1,\varepsilon_2}(E_1,E_2,\omega,t), N_{\varepsilon_3,\varepsilon_4}(E_3,E_4,\omega',t')] = 2\pi\delta_{\varepsilon_1,\varepsilon_3}\delta(t'-t)$$

$$\times\, \delta(E_1-E_3)\delta(E_1-E_2-\omega) < g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} > B_{\varepsilon_4,\varepsilon_2}(E_4,E_2,\omega-\omega',t') \qquad (4.11)$$

$$[N_{\varepsilon_1,\varepsilon_2}(E_1,E_2,\omega,t), N_{\varepsilon_3,\varepsilon_4}(E_3,E_4,\omega',t')] = 2\pi\delta(t'-t)$$

$$\times \Big\{\delta_{\varepsilon_2,\varepsilon_3}\delta(E_2-E_3)\delta(E_3-E_1+\omega) < g_{\varepsilon_2}, P_{E_2} g_{\varepsilon_2} > N_{\varepsilon_1,\varepsilon_4}(E_1,E_4,\omega+\omega',t')$$

$$-\,\delta_{\varepsilon_1,\varepsilon_4}\delta(E_1-E_4)\delta(E_3-E_1-\omega') < g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} > N_{\varepsilon_3,\varepsilon_2}(E_3,E_2,\omega+\omega',t)\Big\} \quad (4.12)$$

The causal commutation relations of the master field are obtained replacing in (4.10),(4.11),(4.12) the factor $\delta(t'-t)$ by $\delta_+(t'-t)$ where the causal δ-function $\delta_+(t'-t)$ is defined in section (8.4) of [1], $2\pi\delta(E_1-E_2-\omega)$ by $(i(E_1-E_2-\omega-i0))^{-1}$ and $2\pi\delta(E_3-E_1\pm\omega)$ by $(i(E_3-E_1\pm\omega-i0))^{-1}$.

Proof. Introduce the operators:

$$N^i_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t) := \frac{e^{it(E_2-E_1-\omega)/\lambda^2}}{\lambda^2}\ 1\otimes A^+_{E_1}(Lg_{\varepsilon_1})A_{E_2}(Lg_{\varepsilon_2})$$

with $A_E(Lg_\varepsilon)$ and $A^+_E(Lg_\varepsilon)$ defined by (3.7), the commutators of the rescalled fields are:

$$[B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t), B^+_{\varepsilon_3,\varepsilon_4,\lambda}(E_3,E_4,\omega',t')] = \frac{e^{i(t'-t)(E_1-E_2-\omega)/\lambda^2}}{\lambda^2}$$

$$\times \Big(\delta_{\varepsilon_1,\varepsilon_3}\delta_{\varepsilon_2,\varepsilon_4}e^{it'(\omega-\omega')/\lambda^2}\delta(E_1-E_3)\delta(E_2-E_4) < g_{\varepsilon_1}, P_{E_1}g_{\varepsilon_1} >< g_{\varepsilon_2}, P_{E_2}L^2 g_{\varepsilon_2} >$$

$$+ \lambda^2\delta_{\varepsilon_1,\varepsilon_3}\delta(E_1-E_3) < g_{\varepsilon_1}, P_{E_1}g_{\varepsilon_1} > N^i_{\varepsilon_4,\varepsilon_2,\lambda}(E_4,E_2,\omega'-\omega,t')$$

$$+ \lambda^2\delta_{\varepsilon_2,\varepsilon_4}\delta(E_2-E_4) < g_{\varepsilon_2}, P_{E_2}L^2 g_{\varepsilon_2} > N_{\varepsilon_3,\varepsilon_1,\lambda}(E_3,E_1,\omega'-\omega,t')\Big) \tag{4.13}$$

$$[B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t), N_{\varepsilon_3,\varepsilon_4,\lambda}(E_3,E_4,\omega',t')] = \frac{e^{i(t'-t)(E_1-E_2-\omega)/\lambda^2}}{\lambda^2}$$

$$\times\ \delta_{\varepsilon_1,\varepsilon_3}\delta(E_1-E_3) < g_{\varepsilon_1}, P_{E_1}g_{\varepsilon_1} > B_{\varepsilon_4,\varepsilon_2,\lambda}(E_4,E_2,\omega-\omega',t') \tag{4.14}$$

$$[N_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,\omega,t), N_{\varepsilon_3,\varepsilon_4,\lambda}(E_3,E_4,\omega',t')]$$

$$= \frac{e^{i(t'-t)(E_3-E_1+\omega)/\lambda^2}}{\lambda^2}\delta_{\varepsilon_2,\varepsilon_3}\delta(E_2-E_3) < g_{\varepsilon_2}, P_{E_2}g_{\varepsilon_2} > N_{\varepsilon_1,\varepsilon_4,\lambda}(E_1,E_4,\omega+\omega',t')$$

$$-\frac{e^{i(t'-t)(E_3-E_1-\omega')/\lambda^2}}{\lambda^2}\delta_{\varepsilon_1,\varepsilon_4}\delta(E_1-E_4) < g_{\varepsilon_1}, P_{E_1}g_{\varepsilon_1} > N_{\varepsilon_3,\varepsilon_2,\lambda}(E_3,E_2,\omega+\omega',t). \tag{4.15}$$

Notice that in the sense of distributions one has the limit

$$\lim_{\lambda\to 0}\frac{e^{i(t'-t)(E_1-E_2-\omega)/\lambda^2}}{\lambda^2}e^{it'(\omega-\omega')/\lambda^2} = 2\pi\delta_{\omega,\omega'}\delta(t'-t)\delta(E_1-E_2-\omega) \tag{4.16}$$

and, in the sense of distributions over the standard simplex (see [1] for details) one has the limit

$$\lim_{\lambda\to 0}\frac{e^{i(t'-t)(E_1-E_2-\omega)/\lambda^2}}{\lambda^2}e^{it'(\omega-\omega')/\lambda^2} = \delta_{\omega,\omega'}\delta_+(t'-t)\frac{1}{i(E_1-E_2-\omega-i0)}. \tag{4.17}$$

The proof of the theorem follows by induction from the commutation relations (4.13)-(4.15) using the limits (4.16) and (4.17) and standard methods of the stochastic limit. □

5. The Master Space and the Associated White Noise

In this section we construct a representation of the limiting algebra (4.10)-(4.12).

Let K be a vector space of finite rank operators acting on the one particle Hilbert space $\mathcal{H}_1$ introduced in section 2, with the property that, for any $\omega \in \Omega$ and for any $X,\ Y \in K$

$$< X, Y >_\omega := \int dt Tr\left(e^{-\beta H_1} X^* S_t Y S_t^*\right) e^{-i\omega t}$$

$$= 2\pi \int dE Tr\left(e^{-\beta H_1} X^* P_E Y P_{E-\omega}\right) < \infty$$

Because of our assumptions on $\mathcal{H}_1$, the space K is non empty and $< \cdot, \cdot >_\omega$ defines a prescalar product on K. We denote $\{K, < \cdot, \cdot >_\omega\}$ or simply K_ω the Hilbert space with inner product $< \cdot, \cdot >_\omega$ obtained as completion of the quotient of K by the zero $< \cdot, \cdot >_\omega$-norm elements. Denoting $Y_t = S_t Y S_t^*$ the free evolution of Y, one can rewrite the inner product as

$$< X, Y >_\omega = \int dt Tr\left(e^{-\beta H_1} X^* Y_t\right) e^{-i\omega t}$$

With these notations the representation space of the algebra (4.10)-(4.12) is the Fock space

$$\Gamma(L^2(\mathbb{R}_+) \otimes \bigoplus_{\omega\in\Omega} K_\omega) \equiv \Gamma(\bigotimes_{\omega\in\Omega} L^2(\mathbb{R}_+, K_\omega)) \equiv \bigotimes_{\omega\in\Omega} \Gamma(L^2(\mathbb{R}_+, K_\omega))$$

where the infinite tensor product is referred to the vacuum vectors. With these notations the master fields are realized as a family of white noise operators $b_{t,\omega}(\cdot)$ which act on $\Gamma(L^2(\mathbb{R}_+) \otimes \bigoplus_{\omega\in\Omega} K_\omega)$ and satisfy the commutation relations

$$[b_{t,\omega}(X), b^+_{t',\omega'}(Y)] = \delta(t'-t)\delta_{\omega,\omega'} < X, Y >_\omega .$$

Moreover each white noise operator $b_{t,\omega}(\cdot)$ acts as an usual annihilation operator in $\Gamma(L^2(\mathbb{R}_+) \otimes K_\omega)$ and as identity operator in other subspaces.

We will construct a representation of the algebra (4.10)-(4.12) in the Fock space defined above by the identification of the operators $B_{\varepsilon_1,\varepsilon_2}(E_1, E_2, \omega, t)$ with the white noise operators

$$B_{\varepsilon_1,\varepsilon_2}(E_1, E_2, \omega, t) = b_{t,\omega}(|P_{E_1} g_{\varepsilon_1} >< P_{E_2} g_{\varepsilon_2}|)$$

The number operators will then have the form

$$N_{\varepsilon_1,\varepsilon_2}(E_1, E_2, \omega, t) =$$

$$\sum_{\varepsilon\in\{0,1\}} \sum_{\omega_1\in\Omega} \mu_\varepsilon(E_1-\omega_1) b^+_{t,\omega_1}(|g_{\varepsilon_1} >< P_{E_1-\omega_1} g_\varepsilon|) b_{t,\omega_1-\omega}(|P_{E_2} g_{\varepsilon_2} >< P_{E_1-\omega_1} g_\varepsilon|)$$

with

$$\mu_\varepsilon(E) := \frac{1}{< g_\varepsilon, P_E L^2 g_\varepsilon >}.$$

One easily checks that these operators satisfy the algebra (4.10)-(4.12).

Let us introduce, for simplicity of calculations, the operators

$$B_{\varepsilon_1,\varepsilon_2}(E,\omega,t) := \int dE' B_{\varepsilon_1,\varepsilon_2}(E',E,\omega,t)$$

$$N_{\varepsilon_1,\varepsilon_2}(\omega,t) := \int dE_1 dE_2 N_{\varepsilon_1,\varepsilon_2}(E_1,E_2,\omega,t) = \int dE N_{\varepsilon_1,\varepsilon_2}(E,\omega,t)$$

with

$$N_{\varepsilon_1,\varepsilon_2}(E,\omega,t) = \sum_{\varepsilon\in\{0,1\}} \sum_{\omega_1\in\Omega} \mu_\varepsilon(E) B^+_{\varepsilon_1,\varepsilon}(E,\omega_1,t) B_{\varepsilon_2,\varepsilon}(E,\omega_1-\omega,t)$$

The operators $B_{\varepsilon_1,\varepsilon_2}(E,\omega,t)$, $B^+_{\varepsilon_1,\varepsilon_2}(E,\omega,t)$ and $N_{\varepsilon_1,\varepsilon_2}(\omega,t)$ satisfy the (causal) commutation relations:

$$[B_{\varepsilon_1,\varepsilon_2}(E,\omega,t), B^+_{\varepsilon_3,\varepsilon_4}(E',\omega',t')]$$

$$= \delta_+(t'-t)\delta_{\varepsilon_1,\varepsilon_3}\delta_{\varepsilon_2,\varepsilon_4}\delta_{\omega,\omega'}\delta(E-E')\gamma_{\varepsilon_1}(E+\omega)\mu^{-1}_{\varepsilon_2}(E) \tag{5.18}$$

$$[B_{\varepsilon_1,\varepsilon_2}(E,\omega,t), N_{\varepsilon_3,\varepsilon_4}(\omega',t')] = \delta_+(t'-t)\delta_{\varepsilon_1,\varepsilon_3}\gamma_{\varepsilon_1}(E+\omega)B_{\varepsilon_4,\varepsilon_2}(E,\omega-\omega',t'). \tag{5.19}$$

In these notations the limiting white noise Hamiltonian acts on $\Gamma(L^2(\mathbb{R}_+)\otimes \bigoplus_{\omega\in\Omega} K_\omega)$ as

$$H(t) = \sum_{\omega\in B} D_\omega \otimes \Big\{ N_{0,1}(\omega,t) + \int dE \Big[B_{1,0}(E,-\omega,t) + B^+_{0,1}(E,\omega,t) \Big] \Big\}$$

$$+ \sum_{\omega\in B} D^+_{-\omega} \otimes \Big\{ N_{1,0}(\omega,t) + \int dE \Big[B_{0,1}(E,-\omega,t) + B^+_{1,0}(E,\omega,t) \Big] \Big\}.$$

6. White Noise Stochastic Schrödinger Equation

The results of the preceding section allow us to write the white noise Schrödinger equation for the evolution operator in the stochastic limit

$$\partial_t U_t = -iH(t)U_t = -i \sum_{\omega\in B} \Big\{ D_\omega \otimes \Big[N_{0,1}(\omega,t) + \int dE \Big(B_{1,0}(E,-\omega,t) + B^+_{0,1}(E,\omega,t) \Big) \Big]$$

$$+ D^+_{-\omega} \otimes \Big[N_{1,0}(\omega,t) + \int dE \Big(B_{0,1}(E,-\omega,t) + B^+_{1,0}(E,\omega,t) \Big) \Big] \Big\} U_t \tag{6.20}$$

Following the general theory of white noise equations, in order to give a precise meaning to this equation we will rewrite it in the causally normally ordered form in which all annihilators are on the right hand side of the evolution operator and all creators on the left hand side. After this procedure we obtain a quantum stochastic differential equation (QSDE) in the sense of Hudson and Parthasarathy [11].

For each $\omega \in \Omega\backslash\mathrm{B}$ define $D_\omega = 0$. Then define for any $\omega, \omega' \in \Omega$ the operators $T^0_{\omega,\omega'}(E)$ and $T^1_{\omega,\omega'}(E)$ by

$$T^0_{\omega,\omega'}(E) := \sum_{\omega''} \gamma_0(E+\omega)\gamma_1(E-\omega'')D_{\omega''+\omega}D^+_{\omega''+\omega'}$$

$$T^1_{\omega,\omega'}(E) := \sum_{\omega''} \gamma_1(E+\omega)\gamma_0(E+\omega'')D^+_{\omega''-\omega}D_{\omega''-\omega'}$$

We will also denote for each $\varepsilon = 0, 1$

$$(1+T_\varepsilon)_{\omega,\omega'}(E) = \delta_{\omega,\omega'} + T^\varepsilon_{\omega,\omega'}(E)$$

The following lemma plays an important role in the derivation of the normally ordered white noise equation.

Lemma 6.1 *In the above notations one has*

$$\sum_{\omega'}(1+T_0)_{\omega,\omega'}(E)B_{0,0}(E,\omega',t)U_t = U_tB_{0,0}(E,\omega,t) - i\sum_{\omega'} D_{\omega-\omega'}\gamma_0(E+\omega)$$
$$\times U_tB_{1,0}(E,\omega',t) - \sum_{\omega'} D_{\omega'+\omega}D^+_{\omega'}\mu_0^{-1}(E)\gamma_0(E+\omega)\gamma_1(E-\omega')U_t \tag{6.21}$$

$$\sum_{\omega'}(1+T_0)_{\omega,\omega'}(E)B_{0,1}(E,\omega',t)U_t = U_tB_{0,1}(E,\omega,t) - i\sum_{\omega'} D_{\omega-\omega'}\gamma_0(E+\omega)$$
$$\times U_tB_{1,1}(E,\omega',t) - iD_\omega\mu_1^{-1}(E)\gamma_0(E+\omega)U_t \tag{6.22}$$

$$\sum_{\omega'}(1+T_1)_{\omega,\omega'}(E)B_{1,1}(E,\omega',t)U_t = U_tB_{1,1}(E,\omega,t) - i\sum_{\omega'} D^+_{\omega'-\omega}\gamma_1(E+\omega)$$
$$\times U_tB_{0,1}(E,\omega',t) - \sum_{\omega'} D^+_{\omega'-\omega}D_{\omega'}\mu_1^{-1}(E)\gamma_1(E+\omega)\gamma_0(E+\omega')U_t \tag{6.23}$$

$$\sum_{\omega'}(1+T_1)_{\omega,\omega'}(E)B_{1,0}(E,\omega',t)U_t = U_tB_{1,0}(E,\omega,t) - i\sum_{\omega'} D^+_{\omega'-\omega}\gamma_1(E+\omega)$$
$$\times U_tB_{0,0}(E,\omega',t) - iD^+_{-\omega}\mu_0^{-1}(E)\gamma_1(E+\omega)U_t \tag{6.24}$$

Proof. It is clear that

$$B_{\varepsilon_1,\varepsilon_2}(E,\omega,t)U_t = [B_{\varepsilon_1,\varepsilon_2}(E,\omega,t), U_t] + U_tB_{\varepsilon_1,\varepsilon_2}(E,\omega,t)$$

Using the integral form of equation (6.20) for the evolution operator one gets

$$[B_{\varepsilon_1,\varepsilon_2}(E,\omega,t), U_t] = -i\int_0^t dt_1\Big([B_{\varepsilon_1,\varepsilon_2}(E,\omega,t), H(t_1)]U_{t_1}$$

$$+H(t_1)[B_{\varepsilon_1,\varepsilon_2}(E,\omega,t),U_{t_1}]\Big) \tag{6.25}$$

Notice that due to the time consecutive principle (see [1] for details) one has for $t > t_1$

$$[B_{\varepsilon_1,\varepsilon_2}(E,\omega,t),U_{t_1}] = 0.$$

Using the causal commutation relations (5.18), (5.19) one can compute the causal commutator $[B_{\varepsilon_1,\varepsilon_2}(E,\omega,t),H(t_1)]$. After substitution of this commutator in (6.25) one gets

$$[B_{\varepsilon_1,\varepsilon_2}(E,\omega,t),U_t] =$$

$$-i\delta_{0,\varepsilon_1}\gamma_0(E+\omega)\Big(\sum_{\omega'} D_{\omega'}B_{1,\varepsilon_2}(E,\omega-\omega',t)U_t + \delta_{1,\varepsilon_2}D_\omega\mu_1^{-1}(E)U_t\Big)$$

$$-i\delta_{1,\varepsilon_1}\gamma_1(E+\omega)\Big(\sum_{\omega'} D^+_{-\omega'}B_{0,\varepsilon_2}(E,\omega-\omega',t)U_t + \delta_{0,\varepsilon_2}D^+_{-\omega}\mu_0^{-1}(E)U_t\Big)$$

From this it follows that

$$B_{0,0}(E,\omega,t)U_t = U_tB_{0,0}(E,\omega,t) - i\gamma_0(E+\omega)\sum_{\omega'} D_{\omega'}B_{1,0}(E,\omega-\omega',t)U_t \tag{6.26}$$

$$B_{1,0}(E,\omega,t)U_t = U_tB_{1,0}(E,\omega,t) - i\gamma_1(E+\omega)$$

$$\times\Big(\sum_{\omega'} D^+_{-\omega'}B_{0,0}(E,\omega-\omega',t)U_t + D^+_{-\omega}\mu_0^{-1}(E)U_t\Big) \tag{6.27}$$

After substitution of (6.27) in (6.26) one gets

$$B_{0,0}(E,\omega,t)U_t = -\sum_{\omega'}\sum_{\omega''} D_{\omega'}D^+_{-\omega''}\gamma_0(E+\omega)\gamma_1(E+\omega-\omega')B_{0,0}(E,\omega-\omega'-\omega'',t)U_t$$

$$+U_tB_{0,0}(E,\omega,t) - i\gamma_0(E+\omega)\sum_{\omega'} D_{\omega'}U_tB_{1,0}(E,\omega-\omega',t)$$

$$-\sum_{\omega'} D_{\omega'}D^+_{\omega'-\omega}\gamma_0(E+\omega)\gamma_1(E+\omega-\omega')\mu_0^{-1}(E)U_t$$

Now changing the summation index in the double sum, $\tilde{\omega}'' = \omega - \omega' - \omega''$, using the definition of $T^0_{\omega,\omega'}(E)$, and bringing the double sum to the left hand side of the equality, one obtains (6.21). The derivation of (6.22)-(6.24) can be done in a similar way. □

7. The Normally Ordered Equation

In this section we will bring equation (6.20) to the normally ordered form. For this goal we will express terms like $B_{\varepsilon_1,\varepsilon_2}(E,\omega,t)U_t$ in the RHS of equation (6.20) in a form in which the annihilation operators are on the RHS of the evolution operator. This form is based on equations (6.21)-(6.24) of Lemma 1, which we will solve with respect to $B_{\varepsilon_1,\varepsilon_2}(E,\omega,t)U_t$.

Suppose that for each $\varepsilon = 0, 1$ the operators $(1 + T_\varepsilon)_{\omega,\omega'}(E)$ are invertible so that there exist the operators $(1 + T_\varepsilon)^{-1}_{\omega,\omega''}(E)$ with the propertie

$$\sum_{\omega''}(1 + T_\varepsilon)^{-1}_{\omega,\omega''}(E)(1 + T_\varepsilon)_{\omega'',\omega'}(E) = \delta_{\omega,\omega'}$$

and that these operators are given by the convergent series

$$(1 + T_\varepsilon)^{-1}_{\omega,\omega'}(E) = \delta_{\omega,\omega'} + \sum_{n=1}^{\infty}(-1)^n$$

$$\times \sum_{\omega_1,\dots,\omega_{n-1}} T^{\varepsilon}_{\omega,\omega_1}(E)T^{\varepsilon}_{\omega_1,\omega_2}(E)\dots T^{\varepsilon}_{\omega_{n-1},\omega'}(E) \tag{7.28}$$

Detailed investigation of conditions under which this series converges will be done in a future paper.

Then let us define

$$R^{0,1}_{\omega,\omega'}(E) = -i\sum_{\omega_1} D_{\omega-\omega_1}(1 + T_1)^{-1}_{\omega_1,\omega'}(E)$$

$$R^{1,0}_{\omega,\omega'}(E) = -i\sum_{\omega_1} D^{+}_{\omega_1-\omega}(1 + T_0)^{-1}_{\omega_1,\omega'}(E)$$

$$R^{0,0}_{\omega,\omega'}(E) = -\sum_{\omega_1,\omega_2} D_{\omega-\omega_1}(1 + T_1)^{-1}_{\omega_1,\omega_2}(E)D^{+}_{\omega'-\omega_2}\gamma_1(E + \omega_2)$$

$$R^{1,1}_{\omega,\omega'}(E) = -\sum_{\omega_1,\omega_2} D^{+}_{\omega_1-\omega}(1 + T_0)^{-1}_{\omega_1,\omega_2}(E)D_{\omega_2-\omega'}\gamma_0(E + \omega_2)$$

With these notations the normally ordered form of the equation (6.20) is given by the following theorem

Theorem 7.2 *The normally ordered form of equation (6.20) is*

$$\partial_t U_t = \sum_{\varepsilon_1,\varepsilon_2}\int dE\Big[\sum_{\omega,\omega'} R^{\varepsilon_1,\varepsilon_2}_{\omega,\omega'}(E)\sum_{\varepsilon}\mu_\varepsilon(E)B^{+}_{\varepsilon_1,\varepsilon}(E,\omega,t)U_t B_{\varepsilon_2,\varepsilon}(E,\omega',t)$$

$$+\sum_{\omega}\Big(R^{\varepsilon_1,\varepsilon_2}_{\omega,0}(E)B^{+}_{\varepsilon_1,\varepsilon_2}(E,\omega,t)U_t + R^{\varepsilon_2,\varepsilon_1}_{0,\omega}(E)U_t B_{\varepsilon_1,\varepsilon_2}(E,\omega,t)\Big)$$

$$+R^{\varepsilon_1,\varepsilon_2}_{0,0}(E) < g_{\varepsilon_1}, P_E e^{-\beta H_1} g_{\varepsilon_2} > U_t\Big] \tag{7.29}$$

Proof. Using the inverse operators $(1 + T_\varepsilon)^{-1}_{\omega,\omega'}(E)$ in equations (6.21)-(6.24) one can express the products $B_{\varepsilon_1,\varepsilon_2}(E,\omega,t)U_t$ in terms of the products $U_t B_{\varepsilon'_1,\varepsilon'_2}(E,\omega',t)$. For example:

$$B_{0,0}(E,\omega,t)U_t = \sum_{\omega'}(1 + T_0)^{-1}_{\omega,\omega'}(E)\Big[U_t B_{0,0}(E,\omega',t) - i\sum_{\omega''} D_{\omega'-\omega''}\gamma_0(E + \omega')$$

$$\times U_t B_{1,0}(E,\omega'',t) - \sum_{\omega''} D_{\omega''+\omega'} D^{+}_{\omega''} < g_0, P_E L^2 g_0 > \gamma_0(E+\omega')\gamma_1(E-\omega'')U_t\Big]$$

and similarly for the other terms. Then after substitution of these expressions in (6.20) one obtains (7.29). □

It is known [1] that any normally ordered white noise equation is equivalent to a quantum stochastic differential equation. In particular equation (7.29) is equivalent to the quantum stochastic differential equation for the evolution operator:

$$dU_t = \sum_{\varepsilon_1,\varepsilon_2} \int dE \Bigg[\sum_{\omega,\omega'} R^{\varepsilon_1,\varepsilon_2}_{\omega,\omega'}(E) dN_t(Z^{\varepsilon_1,\varepsilon_2}_{\omega,\omega'}(E))$$

$$+ \sum_{\omega} \Big(R^{\varepsilon_1,\varepsilon_2}_{\omega,0}(E) dB^{+}_t((|g_{\varepsilon_1} >< P_E g_{\varepsilon_2}|)_\omega) + R^{\varepsilon_2,\varepsilon_1}_{0,\omega}(E) dB_t((|g_{\varepsilon_1} >< P_E g_{\varepsilon_2}|)_\omega) \Big)$$

$$+ R^{\varepsilon_1,\varepsilon_2}_{0,0}(E) < g_{\varepsilon_1}, P_E e^{-\beta H_1} g_{\varepsilon_2} > dt \Bigg] U_t \tag{7.30}$$

Here we denote by $(|f >< g|)_\omega$ the element of $\bigoplus_{\omega\in\Omega} K_\omega$ which belongs to K_ω subspace. Moreover,

$$Z^{\varepsilon_1,\varepsilon_2}_{\omega,\omega'}(E) : \bigoplus_{\omega\in\Omega} K_\omega \to \bigoplus_{\omega\in\Omega} K_\omega$$

and acts on an element $X \in K_{\omega''}$ as

$$Z^{\varepsilon_1,\varepsilon_2}_{\omega,\omega'}(E) X = \delta_{\omega',\omega''} \sum_{\varepsilon} \mu_\varepsilon(E) < (|g_{\varepsilon_2} >< P_E g_\varepsilon|)_{\omega'}, X >_{\omega'} (|g_{\varepsilon_1} >< P_E g_\varepsilon|)_\omega \in K_\omega$$

8. Connection with Scattering Theory

Let us show that the evolution operator in the LDL is directly related with the (1-particle) T-operator describing the scattering of the system on one particle of the reservoir. Notice that the mean value of the evolution operator (2.1) with respect to the state (2.3) in the low density limit is equal to the vacuum mean value of the solution of the QSDE (7.30)

$$\lim_{\xi\to 0} \varphi^\xi \Big(U(t/\xi) \Big) = < U_t >_{vac} = e^{-\Gamma t} \tag{8.31}$$

where, since we average only over the reservoir degrees of freedom, the drift term Γ is an operator acting in the system Hilbert space $\mathcal{H}_S$ as

$$\Gamma = - \sum_{\varepsilon=0,1} \int dE R^{\varepsilon,\varepsilon}_{0,0}(E) < g_\varepsilon, P_E e^{-\beta H_1} g_\varepsilon >$$

with $R^{\varepsilon,\varepsilon}_{0,0}(E)$ given at the beginning of section 7.

Let us remind the definition of T-operator. For the interaction of scattering type the closed subspace of $\mathcal{H}_S \otimes \Gamma(\mathcal{H}_1)$ generated by vectors of the form $u \otimes A^{+}(f)\Phi$

($u \in \mathcal{H}_S$, $f \in \mathcal{H}_1$ and $\Phi \in \Gamma(\mathcal{H}_1)$ is the vacuum vector) which is naturally isomorphic to $\mathcal{H}_S \otimes \mathcal{H}_1$, is globally invariant under the time evolution operator $\exp[i(H_S \otimes 1 + 1 \otimes H_R + V)t]$. Explicitly the restriction of the time evolution operator to this subspace is given by

$$\exp[i(H_S \otimes 1 + 1 \otimes H_1 + V_1)t] \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_1)$$

with

$$V_1 = \sum_{\varepsilon=0,1} D_\varepsilon \otimes |g_\varepsilon >< g_{1-\varepsilon}| \tag{8.32}$$

The 1-particle Møller wave operators are defined by:

$$\Omega_\pm = s - \lim_{t \to \pm\infty} \exp[-i(H_S \otimes 1 + 1 \otimes H_1 + V_1)t] \exp[i(H_S \otimes 1 + 1 \otimes H_1)t] \tag{8.33}$$

and the 1-particle T-operator is defined by:

$$T = V_1 \Omega_+ \tag{8.34}$$

From (8.32) and (8.33) it follows that

$$\Omega_\pm = s - \lim_{t \to \pm\infty} U_t^{(1)}$$

where $U_t^{(1)}$ is the solution of

$$\partial_t U_t^{(1)} = -i U_t^{(1)} V_1(t), \quad U_0^{(1)} = 1.$$

Here

$$V_1(t) = \sum_\omega D_\omega e^{it\omega} \otimes |S_{-t} g_0 >< S_{-t} g_1| + \sum_\omega D_\omega^+ e^{-it\omega} \otimes |S_{-t} g_1 >< S_{-t} g_0| \tag{8.35}$$

We will show in Appendix that

$$T = i \sum_{\varepsilon,\varepsilon'=0,1} \int dE \sum_\omega R_{\omega,0}^{\varepsilon,\varepsilon'}(E) \otimes |g_\varepsilon >< g_{\varepsilon'}| P_E. \tag{8.36}$$

From this formula it follows (see Appendix) that the drift term Γ is connected with the T-operator by the equality

$$\Gamma = i \sum_k P_k \left(Tr_{\mathcal{H}_1} e^{-\beta H_1} T \right) P_k \tag{8.37}$$

This formula means that the drift term is the diagonal part (in the sense of the spectral projections of the system Hamiltonian) of the partial expectation of the T-operator in the 1-particle reservoir Gibbs state. In particular, under the RWA assumption (2.4) with $\omega_0 = 0$ the expression for the drift term has the form

$$\Gamma = i Tr_{\mathcal{H}_1} e^{-\beta H_1} T$$

Formula (8.37) has important implications for the master equation which will be discussed in a future paper.

9. The Langevin Equation

Using the stochastic golden rule we can find the Langevin equation, which is the limit of the Heisenberg evolution in interaction representation, of any observable $X = X_S \otimes 1_R$ of the system. The Langevin equation is the equation satisfied by the stochastic flow j_t, defined by

$$j_t(X) \equiv X_t := U_t^* X U_t$$

where U_t satisfies equation (7.30). To derive the Langevin equation we may apply the stochastic golden rule, described in previous sections. As a result we get in terms of the white noise operators

$$\dot{X}_t = \sum_{\varepsilon_1,\varepsilon_2} \int dE \Bigg[\sum_{\omega_1,\omega_2} \sum_{\varepsilon} \mu_\varepsilon(E) B^+_{\varepsilon_1,\varepsilon}(E,\omega_1,t) U_t^* \Theta^{\varepsilon_1,\varepsilon_2}_{\omega_1,\omega_2}(X) U_t B_{\varepsilon_2,\varepsilon}(E,\omega_2,t)$$

$$+ \sum_{\omega} \Big(B^+_{\varepsilon_1,\varepsilon_2}(E,\omega,t) U_t^* \Theta^{\varepsilon_1,\varepsilon_2}_{\omega,0}(X) U_t + U_t^* \Theta^{\varepsilon_2,\varepsilon_1}_{0,\omega}(X) U_t B_{\varepsilon_1,\varepsilon_2}(E,\omega,t) \Big)$$

$$+ < g_{\varepsilon_1}, P_E e^{-\beta H_1} g_{\varepsilon_2} > U_t^* \Theta^{\varepsilon_1,\varepsilon_2}_{0,0}(X) U_t \Bigg] \tag{9.38}$$

with the maps

$$\Theta^{\varepsilon_1,\varepsilon_2}_{\omega_1,\omega_2}(X) := X R^{\varepsilon_1,\varepsilon_2}_{\omega_1,\omega_2}(E) + R^{+\varepsilon_2,\varepsilon_1}_{\omega_2,\omega_1}(E) X + 2 \sum_{\varepsilon,\omega} \mathrm{Re}\gamma_\varepsilon(E+\omega) R^{+\varepsilon,\varepsilon_1}_{\omega,\omega_1}(E) X R^{\varepsilon,\varepsilon_2}_{\omega,\omega_2}(E)$$

The equation (9.38) can be rewritten in terms of the stochastic differentials as

$$dj_t(X) = j_t \circ \sum_{\varepsilon_1,\varepsilon_2} \int dE \Bigg[\sum_{\omega_1,\omega_2} \Theta^{\varepsilon_1,\varepsilon_2}_{\omega_1,\omega_2}(X) dN_t(Z^{\varepsilon_1,\varepsilon_2}_{\omega_1,\omega_2}(E))$$

$$+ \sum_{\omega} \Big(\Theta^{\varepsilon_1,\varepsilon_2}_{\omega,0}(E) dB_t^+((|g_{\varepsilon_1} >< P_E g_{\varepsilon_2}|)_\omega) + \Theta^{\varepsilon_2,\varepsilon_1}_{0,\omega}(E) dB_t((|g_{\varepsilon_1} >< P_E g_{\varepsilon_2}|)_\omega) \Big) \Bigg]$$

$$+ j_t \circ \Theta_0(X) dt \tag{9.39}$$

Here

$$\Theta_0(X) := \sum_{\varepsilon} \int dE \mu_\varepsilon^{-1}(E) \Big[X R^{\varepsilon,\varepsilon}_{0,0}(E) + R^{+\varepsilon,\varepsilon}_{0,0}(E) X$$

$$+ 2 \sum_{\varepsilon',\omega} \mathrm{Re}\gamma_{\varepsilon'}(E+\omega) R^{+\varepsilon',\varepsilon}_{\omega,0}(E) X R^{\varepsilon',\varepsilon}_{\omega,0}(E) \Big] \tag{9.40}$$

is a quantum Markovian generator. The structure map $\Theta_0(X)$ has the standard form of the generator of a master equation [7]

$$\Theta_0(X) = \Psi(X) - \frac{1}{2}\{\Psi(1), X\} + i[H, X]$$

where

$$\Psi(X) := 2\sum_{\varepsilon}\int dE\mu_{\varepsilon}^{-1}(E)\sum_{\varepsilon',\omega}\mathrm{Re}\gamma_{\varepsilon'}(E+\omega)R_{\omega,0}^{+\varepsilon',\varepsilon}(E)XR_{\omega,0}^{\varepsilon',\varepsilon}(E)$$

is a completely positive map and

$$H := \sum_{\varepsilon}\int dE\mu_{\varepsilon}^{-1}(E)\frac{R_{0,0}^{+\varepsilon,\varepsilon}(E)-R_{0,0}^{\varepsilon,\varepsilon}(E)}{2i}$$

is selfadjoint.

10. Conclusions

An important problem in theory of open quantum systems is a rigorous derivation of a quantum Boltzmann equation. In the present paper the quantum model of the test particle interacting with the Bose gas has been considered. For this model we have presented a rigorous derivation of the quantum Langevin and the quantum master equation.

We developed the stochastic limit method for the low density case. The procedure of the deduction of a (unitary) evolution of the compound system in the limit of long time and small density of particles of the gas was developed. This procedure is being called the stochastic golden rule for the low density limit. The limiting evolution is directly expressed in terms of the 1-particle T–operator describing scattering of the test particle on one particle of the reservoir. After that we obtain the quantum Langevin equation. This equation includes not only the system but also the reservoir dynamics. The master equation can be obtained after averaging of this equation over the equilibrium state of the reservoir. We find the generator of this master equation which describes the reduced evolution of the test particle. We show that this generator has the Lindblad form of most general generators of completely positive semigroups.

We considered the situation when the temperature of the reservoir is high enough so that no condensate is present. It is an interesting important problem to generalize the stochastic limit method to the case that Bose condensate is present.

Appendix

Let us first derive an explicit formula for the T-operator (8.36).

The perturbation expansion for Ω_+ is

$$\Omega_+ = \sum_{n=0}^{\infty}(-i)^n\int_0^{\infty}dt_1\ldots\int_0^{t_{n-1}}dt_nV_1(t_n)\ldots V_1(t_1)$$

This expansion induces the following expansion for the T-operator:

$$T = \sum_{n=1}^{\infty}T_n$$

with

$$T_{n+1} = (-i)^n \int_0^{\infty} dt_1 \ldots \int_0^{t_{n-1}} dt_n V_1 V_1(t_n) \ldots V_1(t_1)$$

and $V_1(t)$ is given by (8.35).

By direct calculations one can prove that

$$T_{2n} = i \int dE \left(T^{00}_{2n}(E)|g_0 >< P_E g_0| + T^{11}_{2n}(E)|g_1 >< P_E g_1| \right)$$

$$T_{2n+1} = i \int dE \left(T^{01}_{2n+1}(E)|g_0 >< P_E g_1| + T^{10}_{2n+1}(E)|g_1 >< P_E g_0| \right)$$

with

$$T^{00}_{2n}(E) = (-1)^n \sum_{\omega,\omega_1 \ldots \omega_{2n-1}} D_\omega D^+_{\omega_1} \ldots D_{\omega_{2n-2}} D^+_{\omega_{2n-1}} \gamma_1(E - \omega_{2n-1})$$

$$\times \gamma_0(E - \omega_{2n-1} + \omega_{2n-2}) \ldots \gamma_0(E - \omega_{2n-1} + \ldots + \omega_2) \gamma_1(E - \omega_{2n-1} + \ldots + \omega_2 - \omega_1)$$

$$T^{11}_{2n}(E) = (-1)^n \sum_{\omega,\omega_1 \ldots \omega_{2n-1}} D^+_\omega D_{\omega_1} \ldots D^+_{\omega_{2n-2}} D_{\omega_{2n-1}} \gamma_0(E + \omega_{2n-1})$$

$$\times \gamma_1(E + \omega_{2n-1} - \omega_{2n-2}) \ldots \gamma_1(E + \omega_{2n-1} - \ldots - \omega_2) \gamma_0(E + \omega_{2n-1} - \ldots - \omega_2 + \omega_1)$$

$$T^{01}_{2n+1}(E) = -i(-1)^n \sum_{\omega,\omega_1 \ldots \omega_{2n}} D_\omega D^+_{\omega_1} \ldots D^+_{\omega_{2n-1}} D_{\omega_{2n}} \gamma_0(E + \omega_{2n})$$

$$\times \gamma_1(E + \omega_{2n} - \omega_{2n-1}) \ldots \gamma_0(E + \omega_{2n} - \ldots + \omega_2) \gamma_1(E + \omega_{2n} - \ldots + \omega_2 - \omega_1)$$

$$T^{10}_{2n+1}(E) = -i(-1)^n \sum_{\omega,\omega_1 \ldots \omega_{2n}} D^+_\omega D_{\omega_1} \ldots D_{\omega_{2n-1}} D^+_{\omega_{2n}} \gamma_1(E - \omega_{2n})$$

$$\times \gamma_0(E - \omega_{2n} + \omega_{2n-1}) \ldots \gamma_1(E - \omega_{2n} + \ldots - \omega_2) \gamma_0(E - \omega_{2n} + \ldots - \omega_2 + \omega_1)$$

Let us show, for example, that

$$\sum_\omega R^{1,0}_{\omega,0}(E) = \sum_{n=0}^{\infty} T^{10}_{2n+1}(E) \tag{A.1}$$

In fact,

$$\sum_\omega R^{1,0}_{\omega,0}(E) = -i \sum_{\omega,\omega_1} D^+_\omega (1 + T_0)^{-1}_{\omega_1,0}(E) =$$

$$-i \sum_{\omega,\omega_1} D^+_\omega \delta_{\omega_1,0} - i \sum_{n=1}^{\infty} (-1)^n \sum_{\omega,\omega_1 \ldots \omega_n} D^+_\omega T^0_{\omega_1,\omega_2}(E) T^0_{\omega_2,\omega_3}(E) \ldots T^0_{\omega_n,0}(E)$$

One has

$$(-1)^n \sum_{\omega,\omega_1\ldots\omega_n} D^+_\omega T^0_{\omega_1,\omega_2}(E)T^0_{\omega_2,\omega_3}(E)\ldots T^0_{\omega_n,0}(E) =$$

$$(-1)^n \sum_{\omega,\omega_1,\omega_3\ldots\omega_{2n-1}} D^+_\omega T^0_{\omega_1,\omega_3}(E)T^0_{\omega_3,\omega_5}(E)\ldots T^0_{\omega_{2p-1},\omega_{2p+1}}(E)\ldots T^0_{\omega_{2n-1},0}(E) =$$

$$(-1)^n \sum_{\omega,\omega_1,\omega_2\ldots\omega_{2n}} D^+_\omega D_{\omega_2+\omega_1} D^+_{\omega_2+\omega_3}\ldots D_{\omega_{2p}+\omega_{2p-1}} D^+_{\omega_{2p}+\omega_{2p+1}}\ldots D_{\omega_{2n}+\omega_{2n-1}} D^+_{\omega_{2n}}$$

$$\times \gamma_0(E+\omega_1)\gamma_1(E-\omega_2)\ldots\gamma_0(E+\omega_{2p-1})\gamma_1(E-\omega_{2p})\ldots\gamma_0(E+\omega_{2n-1})\gamma_1(E-\omega_{2n})$$

Now let us make a change of summation index $\omega_{2n-1} \to \omega_{2n-1} - \omega_{2n}$, then $\omega_{2n-2} \to \omega_{2n-2} - \omega_{2n-1} + \omega_{2n}$, ... and finally $\omega_1 \to \omega_1 - \omega_2 + \ldots + \omega_{2n-1} - \omega_{2n}$. After this change of summation indices we get for the RHS of the previous equality

$$(-1)^n \sum_{\omega,\omega_1\ldots\omega_{2n}} D^+_\omega D_{\omega_1} D^+_{\omega_2}\ldots D_{\omega_{2p-1}} D^+_{\omega_{2p}}\ldots D_{\omega_{2n-1}} D^+_{\omega_{2n}}\gamma_0(E+\omega_1-\omega_2+\ldots-\omega_{2n})$$

$$\times\gamma_1(E-\omega_2+\ldots-\omega_{2n})\ldots\gamma_0(E+\omega_{2n-1}-\omega_{2n})\gamma_1(E-\omega_{2n}) \equiv T^{01}_{2n+1}(E)$$

and from this (A.1) follows. One can prove in the similar way that

$$\sum_\omega R^{0,1}_{\omega,0}(E) = \sum_{n=0}^{\infty} T^{01}_{2n+1}(E), \quad \sum_\omega R^{\varepsilon,\varepsilon}_{\omega,0}(E) = \sum_{n=1}^{\infty} T^{\varepsilon\,\varepsilon}_{2n}(E), \quad \varepsilon = 0,1$$

Therefore the T-operator is given by formula (8.36).

Now let us prove the relation (8.37) beetwen the drift term and the T-operator. Since g_0 and g_1 are mutually ortogonal one has

$$iTr_{\mathcal{H}_1} e^{-\beta H_1} T = -\sum_{\varepsilon=0,1}\int dE \sum_\omega R^{\varepsilon,\varepsilon}_{\omega,0}(E) < g_\varepsilon, P_E e^{-\beta H_1} g_\varepsilon >$$

Then by expanding the operators $(1+T_\varepsilon)^{-1}_{\omega,\omega'}$ in the series (7.28) one can show that for $\omega \neq 0$

$$\sum_k P_k R^{\varepsilon,\varepsilon}_{\omega,0} P_k = 0$$

and

$$\sum_k P_k R^{\varepsilon,\varepsilon}_{0,0} P_k = R^{\varepsilon,\varepsilon}_{0,0}$$

From this (8.37) follows.

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