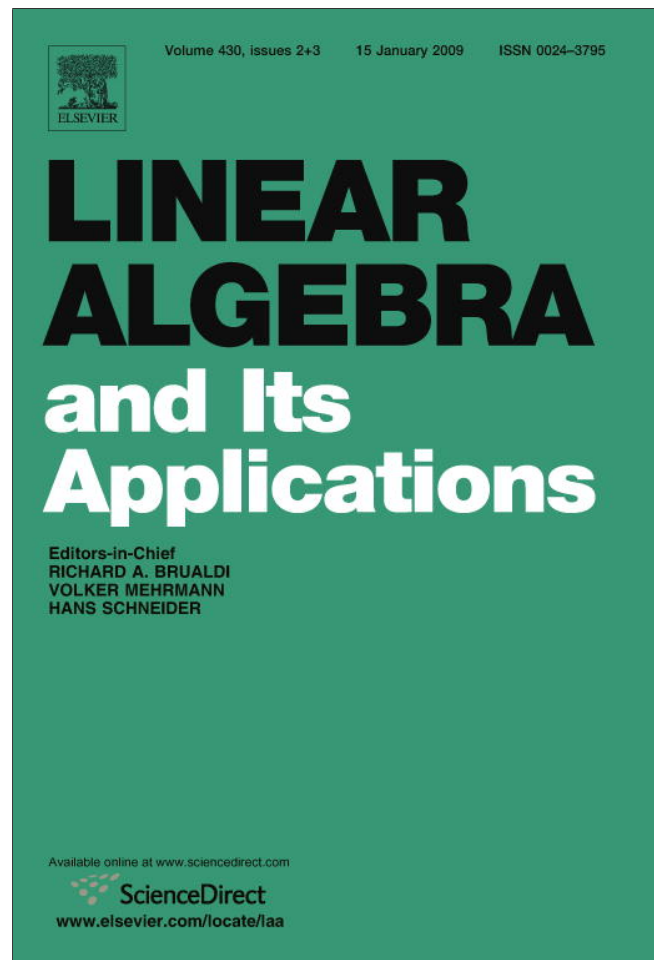


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## A note on noncommutative unique ergodicity and weighted means

Luigi Accardi <sup>a</sup>, Farrukh Mukhamedov <sup>b,\*</sup>,<sup>1</sup><sup>a</sup> Centro Interdisciplinare Vito Volterra II, Università di Roma "Tor Vergata", Via Columbia 2, 00133 Roma, Italy<sup>b</sup> Department of Computational and Theoretical Sciences, Faculty of Sciences, International Islamic University Malaysia, P.O. Box 141, 25710 Kuantan, Pahang, Malaysia

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## ABSTRACT

In this paper we study unique ergodicity of  $C^*$ -dynamical system  $(\mathfrak{A}, T)$ , consisting of a unital  $C^*$ -algebra  $\mathfrak{A}$  and a Markov operator  $T : \mathfrak{A} \mapsto \mathfrak{A}$ , relative to its fixed point subspace, in terms of Riesz summation which is weaker than Cesaro one. Namely, it is proven that  $(\mathfrak{A}, T)$  is uniquely ergodic relative to its fixed point subspace if and only if its Riesz means

$$\frac{1}{p_1 + \dots + p_n} \sum_{k=1}^n p_k T^k x$$

converge to  $E_T(x)$  in  $\mathfrak{A}$  for any  $x \in \mathfrak{A}$ , as  $n \rightarrow \infty$ , here  $E_T$  is a projection of  $\mathfrak{A}$  to the fixed point subspace of  $T$ . It is also constructed a uniquely ergodic entangled Markov operator relative to its fixed point subspace, which is not ergodic.

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### 1. Introduction

It is known [16,22] that one of the important notions in ergodic theory is unique ergodicity of a homeomorphism  $T$  of a compact Hausdorff space  $\Omega$ . Recall that  $T$  is *uniquely ergodic* if there is a unique  $T$ -invariant Borel probability measure  $\mu$  on  $\Omega$ . The well known Krylov–Bogolyubov theorem [16] states that  $T$  is uniquely ergodic if and only if for every  $f \in C(\Omega)$  the averages

\* Corresponding author.

E-mail addresses: [accardi@volterra.uniroma2.it](mailto:accardi@volterra.uniroma2.it) (L. Accardi), [far75m@yandex.ru](mailto:far75m@yandex.ru), [farrukh\\_m@iiu.edu.my](mailto:farrukh_m@iiu.edu.my) (F. Mukhamedov).<sup>1</sup> The author (F.M.) is on leave from National University of Uzbekistan, Department of Mechanics and Mathematics, Vuzgorodok, 100174 Tashkent, Uzbekistan.

$$\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

converge uniformly to the constant  $\int f d\mu$ , as  $n \rightarrow \infty$ .

The study of ergodic theorems in recent years showed that the ordinary Cesaro means have been replaced by weighted averages

$$\sum_{k=0}^{n-1} a_k f(T^k x). \tag{1.1}$$

Therefore, it is natural to ask: is there a weaker summation than Cesaro, ensuring the unique ergodicity. In [15] it has been established that unique ergodicity implies uniform convergence of (1.1), when  $\{a_k\}$  is Riesz weight (see also [14] for similar results). In [4] similar problems were considered for transformations of Hilbert spaces.

On the other hand, since the theory of quantum dynamical systems provides a convenient mathematical description of irreversible dynamics of an open quantum system (see [1,5]) investigation of ergodic properties of such dynamical systems have had a considerable growth. In a quantum setting, the matter is more complicated than in the classical case. Some differences between classical and quantum situations are pointed out in [1,19]. This motivates an interest to study dynamics of quantum systems (see [8,9,12]). Therefore, it is then natural to address the study of the possible generalizations to quantum case of various ergodic properties known for classical dynamical systems. In [17,18] a non-commutative notion of unique ergodicity was defined, and certain properties were studied. Recently in [2] a general notion of unique ergodicity for automorphisms of a  $C^*$ -algebra with respect to its fixed point subalgebra has been introduced. The present paper is devoted to a generalization of such a notion for positive mappings of  $C^*$ -algebras, and its characterization in term of Riesz means.

The paper is organized as follows: Section 2 is devoted to preliminaries, where we recall some facts about  $C^*$ -dynamical systems and the Riesz summation of a sequence on  $C^*$ -algebras. Here we define a notion of unique ergodicity of  $C^*$ -dynamical system relative to its fixed point subspace. In Section 3 we prove that a  $C^*$ -dynamical system  $(\mathfrak{A}, T)$  is uniquely ergodic relative to its fixed point subspace if and only if its Riesz means (see below)

$$\frac{1}{p_1 + \dots + p_n} \sum_{k=1}^n p_k T^k x$$

converge to  $E_T(x)$  in  $\mathfrak{A}$  for any  $x \in \mathfrak{A}$ , here  $E_T$  is a projection of  $\mathfrak{A}$  onto the fixed point subspace of  $T$ . Note however that if  $T$  is completely positive then  $E_T$  is a conditional expectation (see [6,20]). On the other hand it is known [18] that unique ergodicity implies ergodicity. Therefore, one can ask: can a  $C^*$ -dynamical system which is uniquely ergodic relative to its fixed point subspace be ergodic? It turns out that this question has a negative answer. More precisely, in Section 4 we construct entangled Markov operator which is uniquely ergodic relative to its fixed point subspace, but which is not ergodic.

## 2. Preliminaries

In this section we recall some preliminaries concerning  $C^*$ -dynamical systems.

Let  $\mathfrak{A}$  be a  $C^*$ -algebra with unit  $\mathbf{1}$ . An element  $x \in \mathfrak{A}$  is called *positive* if there is an element  $y \in \mathfrak{A}$  such that  $x = y^*y$ . The set of all positive elements will be denoted by  $\mathfrak{A}_+$ . By  $\mathfrak{A}^*$  we denote the conjugate space to  $\mathfrak{A}$ . A linear functional  $\varphi \in \mathfrak{A}^*$  is called *Hermitian* if  $\varphi(x^*) = \overline{\varphi(x)}$  for every  $x \in \mathfrak{A}$ . A Hermitian functional  $\varphi$  is called *state* if  $\varphi(x^*x) \geq 0$  for every  $x \in \mathfrak{A}$  and  $\varphi(\mathbf{1}) = 1$ . By  $S_{\mathfrak{A}}$  (resp.  $\mathfrak{A}_n^*$ ) we denote the set of all states (resp. Hermitian functionals) on  $\mathfrak{A}$ . By  $M_n(\mathfrak{A})$  we denote the set of all  $n \times n$ -matrices  $a = (a_{ij})$  with entries  $a_{ij}$  in  $\mathfrak{A}$ .

**Definition 2.1** A linear operator  $T : \mathfrak{A} \mapsto \mathfrak{A}$  is called:

- (i) *positive*, if  $Tx \geq 0$  whenever  $x \geq 0$ ;

- (ii)  $n$ -positive if the linear mapping  $T_n : M_n(\mathfrak{A}) \mapsto M_n(\mathfrak{A})$  given by  $T_n(a_{ij}) = (T(a_{ij}))$  is positive;
- (iii) completely positive if it is  $n$ -positive for all  $n \in \mathbb{N}$ .

A positive mapping  $T$  with  $T\mathbb{1} = \mathbb{1}$  is called *Markov operator*. A pair  $(\mathfrak{A}, T)$  consisting of a  $C^*$ -algebra  $\mathfrak{A}$  and a Markov operator  $T : \mathfrak{A} \mapsto \mathfrak{A}$  is called a  $C^*$ -dynamical system. The  $C^*$ -dynamical system  $(\mathfrak{A}, \varphi, T)$  is called *uniquely ergodic* if there is a unique invariant state  $\varphi$  (i.e.  $\varphi(Tx) = \varphi(x)$  for all  $x \in \mathfrak{A}$ ) with respect to  $T$ . Denote

$$\mathfrak{A}^T = \{x \in \mathfrak{A} : Tx = x\}. \tag{2.1}$$

It is clear that  $\mathfrak{A}^T$  is a closed linear subspace of  $\mathfrak{A}$ , but in general it is not a subalgebra of  $\mathfrak{A}$  (see Section 3). We say that  $(\mathfrak{A}, T)$  is *uniquely ergodic relative to  $\mathfrak{A}^T$*  if every state of  $\mathfrak{A}^T$  has a unique  $T$ -invariant state extension to  $\mathfrak{A}$ . In the case when  $\mathfrak{A}^T$  consists only of scalar multiples of the identity element, this reduces to the usual notion of unique ergodicity. Note that for an automorphism such a notion has been introduced in [2].

Now suppose we are given a sequence of numbers  $\{p_n\}$  such that  $p_1 > 0, p_k \geq 0$  with  $\sum_{k=1}^\infty p_k = \infty$ . We say that a sequence  $\{s_n\} \subset \mathfrak{A}$  is *Riesz convergent* to an element  $s \in \mathfrak{A}$  if the sequence

$$\frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k s_k$$

converges to  $s$  in  $\mathfrak{A}$ , and it is denoted by  $s_n \rightarrow s(R, p_n)$ . The numbers  $p_n$  are called *weights*. If  $s_n \rightarrow s$  implies  $s_n \rightarrow s(R, p_n)$  then Riesz-convergence is said to be *regular*. The regularity condition (see [13, Theorem 14]) is equivalent to

$$\frac{p_n}{p_1 + p_2 + \dots + p_n} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{2.2}$$

Basics about  $(R, p_n)$  convergence can be found in [13].

Recall the following lemma which shows that Riesz convergence is weaker than Cesaro convergence (see [13, 15]).

**Lemma 2.2** [13, Theorem 16]. *Assume that  $p_{n+1} \leq p_n$  and*

$$\frac{np_n}{p_1 + \dots + p_n} \leq C \quad \forall n \in \mathbb{N} \tag{2.3}$$

for some constant  $C > 0$ . Then Cesaro convergence implies  $(R, p_n)$  convergence.

### 3. Unique ergodicity

In this section we are going to characterize unique ergodicity relative to  $\mathfrak{A}^T$  of  $C^*$ -dynamical systems. To do it we need the following.

**Lemma 3.1** (cf. [18, 2]). *Let  $(\mathfrak{A}, T)$  be uniquely ergodic relative to  $\mathfrak{A}^T$ . If  $h \in \mathfrak{A}^*$  is invariant with respect to  $T$  and  $h|_{\mathfrak{A}^T} = 0$ , then  $h = 0$ .*

**Proof.** Let us first assume that  $h$  is Hermitian. Then there is a unique Jordan decomposition [21] of  $h$  such that

$$h = h_+ - h_-, \quad \|h\|_1 = \|h_+\|_1 + \|h_-\|_1, \tag{3.1}$$

where  $\|\cdot\|_1$  is the dual norm on  $\mathfrak{A}^*$ . The invariance of  $h$  implies that

$$h \circ T = h_+ \circ T - h_- \circ T = h_+ - h_-.$$

Using  $\|h_+ \circ T\|_1 = h_+(\mathbb{1}) = \|h_+\|_1$ , similarly  $\|h_- \circ T\|_1 = \|h_-\|_1$ , from uniqueness of the decomposition we find  $h_+ \circ T = h_+$  and  $h_- \circ T = h_-$ . From  $h|_{\mathfrak{A}^T} = 0$  one gets  $h(\mathbb{1}) = 0$ , which implies that  $\|h_+\|_1 =$

$\|h_-\|_1$ . On the other hand, we also have  $\frac{h_+}{\|h_+\|_1} = \frac{h_-}{\|h_-\|_1}$  on  $\mathfrak{A}^T$ . So, according to the unique ergodicity relative to  $\mathfrak{A}^T$  we obtain  $h_+ = h_-$  on  $\mathfrak{A}$ . Consequently,  $h = 0$ . Now let  $h$  be an arbitrary bounded, linear functional. Then it can be written as  $h = h_1 + ih_2$ , where  $h_1$  and  $h_2$  are Hermitian. Again invariance of  $h$  implies that  $h_i \circ T = h_i, i = 1, 2$ . From  $h \upharpoonright \mathfrak{A}^T = 0$  one gets  $h_k \upharpoonright \mathfrak{A}^T = 0, k = 1, 2$ . Consequently, according to the above argument, we obtain  $h = 0$ .  $\square$

Now we are ready to formulate a criterion for unique ergodicity of  $C^*$ -dynamical system in terms of  $(R, p_n)$  convergence. In the proof we will follow some ideas used in [2,15,18].

**Theorem 3.2.** Let  $(\mathfrak{A}, T)$  be a  $C^*$ -dynamical system. Assume that the weight  $\{p_n\}$  satisfies

$$P(n) := \frac{p_1 + |p_2 - p_1| + \dots + |p_n - p_{n-1}| + p_n}{p_1 + p_2 + \dots + p_n} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.2}$$

Then the following conditions are equivalent:

- (i)  $(\mathfrak{A}, T)$  is uniquely ergodic relative to  $\mathfrak{A}^T$ .
- (ii) The set  $\mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}$  is dense in  $\mathfrak{A}$ .
- (iii) For all  $x \in \mathfrak{A}$ ,

$$T^n x \rightarrow E_T(x) \text{ (R, } p_n),$$

where  $E_T(x)$  is a positive norm one projection onto  $\mathfrak{A}^T$  such that  $E_T T = T E_T = E_T$ . Moreover, the following estimation holds:

$$\left\| \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x) - E_T(x) \right\| \leq P(n) \|x\|, \quad n \in \mathbb{N} \tag{3.3}$$

for every  $x \in \mathfrak{A}$ .

- (iv) For every  $x \in \mathfrak{A}$  and  $\psi \in S_{\mathfrak{A}}$

$$\psi(T^k(x)) \rightarrow \psi(E_T(x)) \text{ (R, } p_n).$$

**Proof.** Consider the implication (i) $\Rightarrow$ (ii). Assume that  $\overline{\mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}} \neq \mathfrak{A}$ ; then there is an element  $x_0 \in \mathfrak{A}$  such that  $x_0 \notin \overline{\mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}}$ . Then according to the Hahn–Banach theorem there is a functional  $h \in \mathfrak{A}^*$  such that  $h(x_0) = 1$  and  $h \upharpoonright \overline{\mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}} = 0$ . The last condition implies that  $h \upharpoonright \mathfrak{A}^T = 0$  and  $h \circ T = h$ . Hence, Lemma 3.1 yields that  $h = 0$ , which contradicts to  $h(x_0) = 1$ .

(ii) $\Rightarrow$ (iii): It is clear that for every element of the form  $y = x - T(x), x \in \mathfrak{A}$  by (3.2) we have

$$\begin{aligned} \frac{1}{\sum_{k=1}^n p_k} \left\| \sum_{k=1}^n p_k T^k(y) \right\| &= \frac{1}{\sum_{k=1}^n p_k} \left\| \sum_{k=1}^n p_k (T^{k+1}(x) - T^k(x)) \right\| \\ &= \frac{1}{\sum_{k=1}^n p_k} \|p_1 T x + (p_2 - p_1) T^2 x + \dots \\ &\quad + (p_n - p_{n-1}) T^n x - p_n T^{n+1} x\| \\ &\leq P(n) \|x\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.4}$$

Now let  $x \in \mathfrak{A}^T$ , then

$$\lim_{n \rightarrow \infty} \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x) = x. \tag{3.5}$$

Hence, for every  $x \in \mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}$  the limit

$$\lim_{n \rightarrow \infty} \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k x$$

exists, which is denoted by  $E_T(x)$ . It is clear that  $E_T$  is a positive linear operator from  $\mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}$  onto  $\mathfrak{A}^T$ . Positivity and  $E_T \mathbf{1} = \mathbf{1}$  imply that  $E_T$  is bounded. From (3.4) one obviously gets that  $E_T T = T E_T = E_T$ . According to (ii) the operator  $E_T$  can be uniquely extended to  $\mathfrak{A}$ , this extension is denoted by the same symbol  $E_T$ . It is evident that  $E_T$  is a positive projection with  $\|E_T\| = 1$ .

Now take an arbitrary  $x \in \mathfrak{A}$ . Then again using (ii), for any  $\epsilon > 0$  we can find  $x_\epsilon \in \mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}$  such that  $\|x - x_\epsilon\| \leq \epsilon$ . By means of (3.4), (3.5) we conclude that

$$\left\| \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x_\epsilon) - E_T(x_\epsilon) \right\| \leq P(n) \|x_\epsilon\|.$$

Hence, one has

$$\begin{aligned} \left\| \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x) - E_T(x) \right\| &\leq \left\| \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x - x_\epsilon) \right\| \\ &\quad + \left\| \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x_\epsilon) - E_T(x_\epsilon) \right\| \\ &\quad + \|E_T(x - x_\epsilon)\| \\ &\leq 2\|x - x_\epsilon\| + P(n)\|x_\epsilon\| \\ &\leq P(n)\|x\| + (2 + P(n))\epsilon, \end{aligned}$$

which with the arbitrariness of  $\epsilon$  implies (3.3).

Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k x = E_T(x)$$

is valid for every  $x \in \mathfrak{A}$ .

The mapping  $E_T$  is a unique  $T$ -invariant positive projection. Indeed, if  $\tilde{E} : \mathfrak{A} \rightarrow \mathfrak{A}^T$  is any  $T$ -invariant positive projection onto  $\mathfrak{A}^T$ , then

$$\tilde{E}(x) = \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k \tilde{E}(T^k(x)) = \tilde{E} \left( \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x) \right).$$

Taking the limit as  $n \rightarrow \infty$  gives

$$\tilde{E}(x) = \tilde{E}(E_T(x)) = E_T(x).$$

The implication (iii) $\Rightarrow$ (iv) is obvious. Let us consider (iv) $\Rightarrow$ (i). Let  $\psi$  be any state on  $\mathfrak{A}^T$ , then  $\psi \circ E_T$  is a  $T$ -invariant extension of  $\psi$  to  $\mathfrak{A}$ . Assume that  $\phi$  is any  $T$ -invariant, linear extension of  $\psi$ . Then

$$\phi(x) = \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k \phi(T^k(x)) = \phi \left( \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x) \right).$$

Now taking the limit from both sides of the last equality as  $n \rightarrow \infty$  one gives

$$\phi(x) = \phi(E_T(x)) = \psi(E_T(x)),$$

so  $\phi = \psi \circ E_T$ .  $\square$

**Remark 3.3.** If we choose  $p_n = 1$  for all  $n \in \mathbb{N}$  then it is clear that the condition (3.2) is satisfied, hence we infer that unique ergodicity relative to  $\mathfrak{A}^T$  is equivalent to the norm convergence of the mean averages, i.e.

$$\frac{1}{n} \sum_{k=1}^n T^k(x),$$

which recovers the result of [2].

**Remark 3.4.** If the condition (2.3) is satisfied then condition (3.2) is valid as well. This means that unique ergodicity would remain true if Cesaro summation is replaced by a weaker. Theorem 3.2 extends a result of Mukhamedov and Temir [18].

**Example.** If we define  $p_n = n^\alpha$  with  $\alpha > 0$ , then one can see that  $\{p_n\}$  is an increasing sequence and condition (3.2) is also satisfied. This provides a concrete example of weights.

**Remark 3.5.** Note that some nontrivial examples of uniquely ergodic quantum dynamical systems based on automorphisms, has been given in [2]. Namely, it was proved that free shifts based on reduced  $C^*$ -algebras of RD-groups (including the free group on infinitely many generators), and amalgamated free product  $C^*$ -algebras, are uniquely ergodic relative to the fixed-point subalgebra. In [11] it has been proved that such shifts possess a stronger property called  $F$ -strict weak mixing (see also [18]).

**Observation.** We note that, in general, the projection  $E_T$  is not a conditional expectation, but when  $T$  is an automorphism then it is so. Now we are going to provide an example of Markov operator which is uniquely ergodic relative to its fixed point subspace for which the projector  $E_T$  is not a conditional expectation.

Consider the algebra  $M_d(\mathbb{C})$  –  $d \times d$  matrices over  $\mathbb{C}$ . For a matrix  $\mathbf{x} = (x_{ij})$  by  $\mathbf{x}^t$  we denote its transpose matrix, i.e.  $\mathbf{x}^t = (x_{ji})$ . Define a mapping  $\phi : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C})$  by  $\phi(\mathbf{x}) = \frac{\mathbf{x} + \mathbf{x}^t}{2}$ . Then it is known [20] that such a mapping is positive, but not completely positive. One can see that  $\phi$  is a Markov operator. Due to the equality

$$\mathbf{x} = \frac{\mathbf{x} + \mathbf{x}^t}{2} + \frac{\mathbf{x} - \mathbf{x}^t}{2}$$

condition (ii) of Theorem 3.2 is satisfied, so  $\phi$  is uniquely ergodic with respect to  $M_d(\mathbb{C})^\phi$ . Hence, the corresponding projection  $E_\phi$  is given by  $E_\phi(\mathbf{x}) = \frac{\mathbf{x} + \mathbf{x}^t}{2}$ , which is not completely positive. Moreover,  $M_d(\mathbb{C})^\phi$  is the set of all symmetric matrices, which do not form an algebra. So,  $E_\phi$  is not a conditional expectation.

#### 4. A uniquely ergodic entangled Markov operator

In recent developments of quantum information many people have discussed the problem of finding a satisfactory quantum generalization of classical random walks. Motivating this in [3,10] a new class of quantum Markov chains was constructed which are at the same time purely generated and uniquely determined by a corresponding classical Markov chain. Such a class of Markov chains was constructed by means of entangled Markov operators. In one's turn they were associated with Schur multiplication. In that paper, ergodicity and weak clustering properties of such chains were established. In this section we are going to provide entangled Markov operator which is uniquely ergodic relative to its fixed point subspace, but which is not ergodic.

Let us recall some notations. To define Schur multiplication, we choose an orthonormal basis  $\{e_j\}$ ,  $j = 1, \dots, d$  in a  $d$ -dimensional Hilbert space  $H_d$  which is kept fixed during the analysis. In such a way, we have the natural identification  $H_d$  with  $C^d$ . The corresponding system of matrix units  $e_{ij} = e_i \otimes e_j$  identifies  $B(H_d)$  with  $M_d(\mathbb{C})$ . Then, for  $\mathbf{x} = \sum_{i,j=1}^d x_{ij} e_{ij}$ ,  $\mathbf{y} = \sum_{i,j=1}^d y_{ij} e_{ij}$  elements of  $M_d(\mathbb{C})$ , we define Schur multiplication in  $M_d(\mathbb{C})$  as usual,

$$\mathbf{x} \diamond \mathbf{y} = \sum_{i,j=1}^d (x_{ij} y_{ij}) e_{ij}, \tag{4.1}$$

that is, componentwise,  $(\mathbf{x} \diamond \mathbf{y})_{ij} := x_{ij} y_{ij}$ .

A linear map  $P : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C})$  is said to be *Schur identity-preserving* if its diagonal projection is the identity, i.e.  $\mathbf{1} \diamond P(\mathbf{1}) = \mathbf{1}$ . It is called an *entangled Markov operator* if, in addition,  $P(\mathbf{1}) \neq \mathbf{1}$ .

The entangled Markov operator (see [3]) associated to a stochastic matrix  $\Pi = (p_{ij})_{i,j=1}^d$  and to the canonical systems of matrix units  $\{e_{ij}\}_{i,j=1}^d$  of  $M_d(\mathbb{C})$  is defined by

$$P(\mathbf{x})_{ij} := \sum_{k,l=1}^d \sqrt{p_{ik}p_{jl}}x_{kl}, \tag{4.2}$$

where as before  $\mathbf{x} = \sum_{i,j=1}^d x_{ij}e_{ij}$ .

Define a Markov operator  $\Psi : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C})$  by

$$\Psi(\mathbf{x}) = \mathbf{1} \diamond P(\mathbf{x}), \quad \mathbf{x} \in M_d(\mathbb{C}). \tag{4.3}$$

Given a stochastic matrix  $\Pi = (p_{ij})$  put

$$\text{Fix}(\Pi) = \{\psi \in \mathbb{C}^d : \Pi\psi = \psi\}.$$

To every vector  $a = (a_1, \dots, a_d) \in \mathbb{C}^d$  corresponds a diagonal matrix  $\mathbf{x}_a$  in  $M_d(\mathbb{C})$  defined by

$$\mathbf{x}_a = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_d \end{pmatrix}. \tag{4.4}$$

**Lemma 4.1.** For a Markov operator given by (4.3) one has

$$M_d(\mathbb{C})^\Psi = \{\mathbf{x}_\psi : \psi \in \text{Fix}(\Pi)\}$$

**Proof.** Let  $\mathbf{x} = (x_{ij}) \in M_d(\mathbb{C})^\Psi$ , i.e.  $\Psi(\mathbf{x}) = \mathbf{x}$ . From (4.1) and (4.3) we conclude that  $x_{ij} = 0$  if  $i \neq j$ . Therefore, due to (4.2) one finds

$$\sum_{j=1}^d \sqrt{p_{ij}p_{ij}}x_{jj} = x_{ii}$$

which implies that  $(x_{11}, \dots, x_{dd}) \in \text{Fix}(\Pi)$ .  $\square$

Furthermore, we assume that the dimension of  $\text{Fix}(\Pi)$  is greater or equal than 2, i.e.  $\dim(\text{Fix}(\Pi)) \geq 2$ . Hence, according to Lemma 4.1 we conclude that  $M_d(\mathbb{C})^\Psi$  is a nontrivial commutative subalgebra of  $M_d(\mathbb{C})$ .

**Theorem 4.2.** Let  $\Pi$  be a stochastic matrix such that  $\dim(\text{Fix}(\Pi)) \geq 2$ . Then the corresponding Markov operator  $\Psi$  given by (4.3) is uniquely ergodic w.r.t.  $M_d(\mathbb{C})^\Psi$ .

**Proof.** To prove the statement, it is enough to establish condition (ii) of Theorem 3.2. Take any  $\mathbf{x} = (x_{ij}) \in M_d(\mathbb{C})$ . Now we are going to show that it can be represented as follows:

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2, \tag{4.5}$$

where  $\mathbf{x}_1 \in M_d(\mathbb{C})^\Psi$  and  $\mathbf{x}_2 \in \{\mathbf{y} - \Psi(\mathbf{y}) : \mathbf{y} \in M_d(\mathbb{C})\}$ .

Due to Lemma 4.1 there is a vector  $\psi \in \text{Fix}(\Pi)$  such that  $\mathbf{x}_1 = \mathbf{x}_\psi$ , and hence, from (4.3), (4.5) one finds that

$$\mathbf{x}_2 = \begin{pmatrix} \varphi_{11} & x_{12} & \dots & x_{1d} \\ x_{21} & \varphi_{22} & \dots & x_{2d} \\ \dots & \dots & \dots & \dots \\ x_{d1} & x_{d2} & \dots & \varphi_{dd} \end{pmatrix}, \tag{4.6}$$

where



$$\varphi_{ii} = \xi_i - \sum_{j=1}^d p_{ij}\xi_j - \sum_{\substack{k,l=1 \\ k \neq j}}^d \sqrt{p_{ik}p_{il}}x_{kl}. \tag{4.7}$$

The existence of the vectors  $\psi = (\psi_1, \dots, \psi_d)$  and  $(\xi_1, \dots, \xi_d)$  follows immediately from the following relations:

$$\psi_i + \xi_i - \sum_{j=1}^d p_{ij}\xi_j = x_{ii} + \sum_{\substack{k,l=1 \\ k \neq j}}^d \sqrt{p_{ik}p_{il}}x_{kl}, \quad i = 1, \dots, d, \tag{4.8}$$

since the number of unknowns is greater than the number of equations. Note that the equality (4.8) comes from (4.3)–(4.7). Hence, one concludes that the equality

$$M_d(\mathbb{C})^\Psi + \{\mathbf{x} - \Psi(\mathbf{x}) : \mathbf{x} \in M_d(\mathbb{C})\} = M_d(\mathbb{C}),$$

which completes the proof.  $\square$

Let us provide a more concrete example.

**Example.** Consider on  $M_3(\mathbb{C})$  the following stochastic matrix  $\Pi_0$  defined by:

$$\Pi_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & u & v \end{pmatrix}, \tag{4.9}$$

here  $u, v \geq 0, u + v = 1$ .

One can immediately find that

$$\text{Fix}(\Pi_0) = \{(x, y, y) : x, y \in \mathbb{C}\}. \tag{4.10}$$

Then for the corresponding Markov operator  $\Psi_0$ , given by (4.3), (4.2), due to Lemma 4.1 one has

$$M_3(\mathbb{C})^{\Psi_0} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{pmatrix} : x, y \in \mathbb{C} \right\}. \tag{4.11}$$

So,  $M_3(\mathbb{C})^{\Psi_0}$  is a nontrivial commutative subalgebra of  $M_3(\mathbb{C})$  having dimension 2.

So, according to Theorem 4.2 we see that  $\Psi_0$  is uniquely ergodic relative to  $M_3(\mathbb{C})^{\Psi_0}$ . But (4.11) implies that  $\Psi_0$  is not ergodic. Note that ergodicity of entangled Markov chains has been studied in [3].

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