Mathproblems
ISSN: 2217-446X, url: http://www.mathproblems-ks.com
Volume 1, Issue 4 (2011), Pages 32–47

PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before 2 January 2012

Problems

22. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.*
Let x, y, z be positive real numbers. Prove that

$$\sum_{cyc} \frac{4x(x+2y+2z)}{(x+3y+3z)^2} \geq \sum_{cyc} \frac{(x+y)(3x+3y+4z)}{(2x+2y+3z)^2}$$

23. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let f be a real, continuous integrable function defined on $[0,1]$ such that $\int_0^1 f(x)dx = 0$, and let $m = \min\limits_{0\leq x\leq 1} f(x)$ and $M = \max\limits_{0\leq x\leq 1} f(x)$. Let us define $F(x) = \int_0^x f(y)dy$. Prove that

$$\frac{1}{6}\,\frac{Mm^2(5M-3m)}{(M-m)^2} \leq \int_0^1 F^2(x)dx \leq \frac{1}{6}\,\frac{mM^2(5m-3M)}{(M-m)^2}$$

24. *Proposed by D.M. Bătinetu - Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.* Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be sequences of positive real numbers such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{n^2 \cdot a_n} = \lim\limits_{n\to\infty} \frac{b_{n+1}}{n^3 \cdot b_n} = a > 0$. Compute

$$\lim_{n\to\infty} \left(\sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} - \sqrt[n]{\frac{b_n}{a_n}} \right)$$

25. *Proposed by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.* Let D, E, F be three points lying on the sides BC, AB, CA of $\triangle ABC$. Let M be a point lying on cevian AD. If E, M, F are collinear then show that

$$\left(\frac{BC \cdot MD}{MA} \right) \left(\frac{EA}{DC \cdot EB} + \frac{FA}{BD \cdot FC} \right) \geq 4$$

26. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.* Determine all functions $f : \mathbb{R} - \{0, 1\} \to \mathbb{R}$, which satisfy the relation

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$.

27. *Proposed by David R. Stone, Georgia Southern University, Statesboro, GA USA.* With $\pi(x) =$ the number of primes $\leq x$, show that there exist constants a and b such that

$$e^{ax} < x^{\pi(x)} < e^{bx}$$

for x sufficiently large.

28. *Proposed by Florin Stanescu, School Cioculescu Serban, Gaesti, jud. Dambovita, Romania.* Let ABC be a triangle with semi-perimeter p. Prove that

$$\frac{a}{\sqrt{p-a}} + \frac{b}{\sqrt{p-b}} + \frac{c}{\sqrt{p-c}} \geq 2\sqrt{3p},$$

where $[AB] = c, [AC] = b, [BC] = a$.

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

15. *Proposed by Valmir Bucaj, Texas Lutheran University, Seguin, TX.* Consider the set given by

$$L_{a,r} = \{ar^n | n \in \mathbb{Z}_+\}, \text{ where } r = \frac{1}{q}, q \in \mathbb{Z}_+, \text{ and } \gcd(a,q) = 1.$$

Show that

$$B_q = \{L_{a,r} | r = \frac{1}{q}, q \in \mathbb{Z}_+, q \geq 2 \text{ and } \gcd(a,q) = 1\},$$

forms a basis for a topology on $\mathbb{Q}_+$.

Solution by the proposer. Hereafter, we present some results more or less well-known that will be used later on. We begin with

Lemma 1. *If $gcd(a,b) = 1$ and $a|bc$ then $a|c$.*

Proof. Since $gcd(a,b) = 1$ then $ah + bk = 1$ with $h, k \in \mathbb{Z}$. From the hypothesis $a|bc$ it follows that $bc = na$, with $n \in \mathbb{Z}_+$. Then

$$c = c(ah + bk) = a(ch) + (bc)k = a(ch) + a(nk) = a(ch + nk).$$

This shows that $a|c$, as we wanted to show. □

Lemma 2. *If $a_1 \neq a_2$ and $gcd(a_1, q_1) = gcd(a_2, q_2) = 1$ then $\frac{a_1}{q_1} \neq \frac{a_2}{q_2}$.*

Proof. The case when $q_1 = q_2$ is trivial. So, assume that $q_1 \neq q_2$. We suppose the contrary. That is, assume that $\frac{a_1}{q_1} = \frac{a_2}{q_2}$. Then $a_1 q_2 = a_2 q_1$. This shows that $a_1 | a_2 q_1$ and also $a_2 | a_1 q_2$. Now, since $gcd(a_1, q_1) = 1$, from Lemma 1 we have $a_1 | a_2$. Similarly, since $gcd(a_2, q_2) = 1$ from Lemma 1 follows that $a_2 | a_1$. Therefore, $a_1 | a_2$ and $a_2 | a_1$ imply $a_1 = a_2$, contrary to the hypothesis of the lemma. □

From this lemma, immediately follows

Corollary 1. *If $a_1 \neq a_2$ and $gcd(a_1, q_1) = gcd(a_2, q_2) = 1$ then $\frac{a_1}{q_1^n} \neq \frac{a_2}{q_2^m}$, for any $n, m \in \mathbb{Z}_+$.*

Proof. The proof is similar to that of Lemma 2, therefore I will omit it here. □

Lemma 3. *Let L_{a_1,r_1} and L_{a_2,r_2} be two sets as defined in the statement.*
(i) If $a_1 \neq a_2$ then $L_{a_1,r_1} \bigcap L_{a_2,r_2} = \emptyset$
(ii) If $q_1 \neq q_2^k, \forall k \in Z_+$, then $L_{a_1,r_1} \bigcap L_{a_2,r_2} = \emptyset$

Proof. (i) Let L_{a_1,r_1} and L_{a_2,r_2} be two sets as defined in the statement with $a_1 \neq a_2$. Let x be any element from L_{a_1,r_1} and y be any element from L_{a_2,r_2}. Then they are of the form

$$x = \frac{a_1}{q_1^n}, \text{ and } y = \frac{a_2}{q_2^m},$$

where $n, m \in Z_+$ and $gcd(a_1, q_1) = gcd(a_2, q_2) = 1$. Then from Corollary 1 follows that $x \neq y, \forall m, n \in \mathbb{Z}_+$. Therefore $L_{a_1,r_1} \bigcap L_{a_2,r_2} = \emptyset$.

(ii) Let $x \in L_{a_1,r_1}$ and $y \in L_{a_2,r_2}$. Then each of them is of the form

$$x = \frac{a_1}{q_1^n}, \text{ and } y = \frac{a_2}{q_2^m},$$

where $n, m \in \mathbb{Z}_+$. First consider the case when $a_1 = a_2 = a$. Then, since $q_1 \neq q_2^k$ for any $k \in \mathbb{Z}_+$, it follows that $q_1^n \neq q_2^m$ for any $n, m \in \mathbb{Z}_+$. In turn, this implies that $x \neq y$, as well. The case where $a_1 \neq a_2$ was already considered in part (i). This concludes the proof. □

Theorem 1. *If $a_1 = a_2$ and $q_1 = q_2^k$ for some $k \in \mathbb{Z}_+$, then*

$$L_{a_1,r_1} \bigcap L_{a_2,r_2} = L_{a_1,r_1}$$

Proof. Assume that $a_1 = a_2 = a$ and $q_1 = q_2^k$ for some $k \in \mathbb{Z}_+$. Let $x \in L_{a_1,r_1}$, then $x = \frac{a}{q_1^n}, n \in \mathbb{Z}_+$. Since, $q_1 = q_2^k$, then $x = \frac{a}{q_2^{kn}} \in L_{a_2,r_2}, kn \in \mathbb{Z}_+ \Rightarrow L_{a_1,r_1} \subset L_{a_2,r_2}$ Therefore $L_{a_1,r_1} \bigcap L_{a_2,r_2} = L_{a_1,r_1}$. This concludes the proof. □

Corollary 2. *Let L_{a_1,r_1} and L_{a_2,r_2} be two sets as defined in the statement. Then either $L_{a_1,r_1} \bigcap L_{a_2,r_2} = \emptyset$ or $L_{a_1,r_1} \bigcap L_{a_2,r_2} = L_{a_1,r_1}$.*

Proof. We will show that they cannot have any element in common. Let $x \in L_{a_1,r_1} \bigcap L_{a_2,r_2}$. Then, clearly the intersection is not empty. By (i) and (ii) of Lemma 3 then $a_1 = a_2$ and $q_1 = q_2^k$, for some $k \in \mathbb{Z}_+$, otherwise the intersection would be empty. Then, from Theorem 1 follows that $L_{a_1,r_1} \bigcap L_{a_2,r_2} = L_{a_1,r_1}$. Which is what we wished to show. □

Now we are ready to construct a topology T_q on the set $\mathbb{Q}_+$ of positive rational numbers. Let

$$B_q = \{L_{a,r} \mid r = \frac{1}{q}, q \in Z_+, q \geq 2 \text{ and } gcd(a, q) = 1\},$$

where $L_{a,r}$ is given in the statement, be a collection of subsets of $\mathbb{Q}_+$. In what follows we show that B_q forms a basis for a topology on $\mathbb{Q}_+$. Indeed,
1. Let x be any element in $\mathbb{Q}_+$. Then $x = \dfrac{u}{v}$ where $u, v \in \mathbb{Z}_+$. For simplicity assume that $gcd(u, v) = 1$. Then, it is clear that $x \in L_{u,r}$, where $r = \dfrac{1}{v}$. So, B_q satisfies the first condition for a basis.

2. If x belongs to the intersection of two elements of B_q, namely L_{a_1,r_1} and L_{a_2,r_2}, then from Corollary 2 follows that the intersection itself is an element of B_q. Therefore the collection B_q satisfies the second condition for a basis.

We conclude that the collection B_q of subsets of $\mathbb{Q}_+$ generates a topology on the set $\mathbb{Q}_+$ of positive rational numbers.

16. *Proposed by José Luis Díaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain.* Find all triplets (x, y, z) of real numbers which are solutions of the following system of equations

$$\left.\begin{array}{l} x^2(y+z)^2 = (3x^2 - x + 1)y^2z^2 \\ y^2(z+x)^2 = (4y^2 - y + 1)z^2x^2 \\ z^2(x+y)^2 = (5z^2 - z + 1)x^2y^2 \end{array}\right\}$$

Solution by the proposer. We observe that triplets $(t, 0, 0), (0, t, 0), (0, 0, t)$, $t \in \mathbb{R}$ are solutions of the given system. To find other solutions we assume that $xyz \neq 0$. Dividing each of the preceding equations by $x^2y^2z^2$ yields

$$\left.\begin{array}{l} \left(\dfrac{1}{y} + \dfrac{1}{z}\right)^2 = 3 - \dfrac{1}{x} + \dfrac{1}{x^2} \\ \left(\dfrac{1}{z} + \dfrac{1}{x}\right)^2 = 4 - \dfrac{1}{y} + \dfrac{1}{y^2} \\ \left(\dfrac{1}{x} + \dfrac{1}{y}\right)^2 = 5 - \dfrac{1}{z} + \dfrac{1}{z^2} \end{array}\right\}$$

Setting $a = 1/x, b = 1/y$ and $c = 1/z$ in the above system of equations, we obtain

$$\left.\begin{array}{l} (b+c)^2 = 5 - a + a^2 \\ (c+a)^2 = 7 - b + b^2 \\ (a+b)^2 = 8 - c + c^2 \end{array}\right\}$$

Adding up the preceding equations yields $(a+b+c)^2 = 12 - (a+b+c)$. Putting $a+b+c = t$, we have $t^2+t-12 = (t-3)(t+4) = 0$ from which follows $a+b+c = 3$ and $a+b+c = -4$. If $a+b+c = 3$, then $b+c = 3-a$. Substituting in the equation $(b+c)^2 = 5 - a + a^2$, yields $a = 6/5$. Likewise, we get $b = 1$ and $c = 4/5$. If $a+b+c = -4$, then we obtain $a = -13/9$, $b = -4/3$ and $c = -11/9$. Hence, $(5/6, 1, 5/4)$ and $(-9/13, -3/4, -9/11)$ are also solutions and we are done.

17. *Proposed by José Luis Díaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain.* Let $A(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \neq 0)$ be a nonconstant polynomial with complex coefficients. Prove that all its zeros lie in the ring shaped region $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{4^k \binom{n}{k}}{5^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{1/k} \quad \text{and} \quad r_2 = \max_{1 \leq k \leq n} \left\{ \frac{5^n - 1}{4^k \binom{n}{k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}$$

Solution by the proposer. If we assume $|z| < r_1$ then from $A(z) = \displaystyle\sum_{k=0}^{n} a_k z^k$, we have

$$\begin{aligned} |A(z)| &= \left| \sum_{k=0}^{n} a_k z^k \right| \geq |a_0| - \sum_{k=1}^{n} |a_k||z|^k > |a_0| - \sum_{k=1}^{n} |a_k| r_1^k \\ &= |a_0| \left(1 - \sum_{k=1}^{n} \left| \frac{a_k}{a_0} \right| r_1^k \right) \qquad (1) \end{aligned}$$

From the expression of r_1 and taking into account the identity

$$\sum_{k=0}^{n} 4^k \binom{n}{k} = 5^n,$$

immediately follows

$$\left|\frac{a_k}{a_0}\right| r_1^k \leq \frac{4^k \binom{n}{k}}{5^n - 1}, \quad 1 \leq k \leq n \tag{2}$$

Substituting (2) into (1), we have

$$|A(z)| > |a_0| \left(1 - \sum_{k=1}^{n} \left|\frac{a_k}{a_0}\right| r_1^k\right) \geq |a_0| \left(1 - \sum_{k=1}^{n} \frac{4^k \binom{n}{k}}{5^n - 1}\right) = 0$$

Consequently, $A(z)$ does not have zeros in $\{z \in \mathcal{C} \,:\, |z| < r_1\}$.

To prove the second inequality we will use the well-known fact that all the zeros of $A(z)$ have modulus less than or equal to the unique positive root of the equation

$$B(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \ldots - |a_1|z - |a_0| = 0$$

Therefore, the second part of our statement will be proved if we show that $B(r_2) \geq 0$. In fact, from the expression of r_2 immediately follows

$$\left|\frac{a_{n-k}}{a_n}\right| \leq \frac{4^k \binom{n}{k}}{5^n - 1} r_2^k, \quad 1 \leq k \leq n$$

and

$$B(r_2) = |a_n| \left[r_2^n - \sum_{k=1}^{n} \left|\frac{a_{n-k}}{a_n}\right| r_2^{n-k}\right] \geq |a_n| \left[r_2^n - \sum_{k=1}^{n} \left(\frac{4^k \binom{n}{k}}{5^n - 1} r_2^k\right) r_2^{n-k}\right]$$

$$= |a_n| r_2^n \left(1 - \sum_{k=1}^{n} \frac{4^k \binom{n}{k}}{5^n - 1}\right) = 0,$$

as desired. This completes the proof.

18. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.* If, $f(x) = \frac{1-\sqrt{1-2x}}{2}$, and $f_n^{-1} = \underbrace{f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}}_{n}$, then evaluate:

$$\lim_{n\to\infty} \int_0^{1/2} f_n^{-1}(x)\, dx$$

Solution by Paolo Perfetti, Tor Vergata University, Rome, Italy. The answer is 1/4. Indeed, let $f : (-\infty, 1/2] \to (-\infty, 1/2]$ defined by $f(x) = \frac{1-\sqrt{1-2x}}{2}$, then the inverse function $f^{-1} : (-\infty, 1/2] \to (-\infty, 1/2]$ is defined by $f^{-1}(x) = \frac{1-(1-2x)^2}{2}$, where $x \leq 1/2$. Putting $f_n^{-1}(x) = (f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1})(x) = y$ then follows $x = (f \circ f \circ \ldots \circ f)(y)$ and the integral becomes

$$\int_0^{1/2} y\Big(f \circ f \circ \ldots \circ f)(y)\Big)' dy$$

Integrating by parts, yields

$$y \cdot \Big(f \circ f \circ \ldots \circ f \Big)(y) \Big|_0^{1/2} - \int_0^{1/2} (f \circ f \circ \ldots \circ f)(y)\, dy$$

To compute the last integral we will use the following Lemma.

Lemma. *Let* $\{f_n(x)\}_{n\geq 1}$ *be the sequence defined by* $f_n(x) = \underbrace{(f \circ f \circ \ldots \circ f)(x)}_{n}$ *Then* $\lim_{n\to\infty} f_n(x) = 0$ *for any* $0 \leq x < 1/2$.

Proof. Observe that $\{f_n(x)\}_{n\geq 1}$ is decreasing because for all $x \in [0, 1/2]$ is $1-2x \leq \sqrt{1-2x}$ which implies $f_n(x) < f_{n-1}(x) \Leftrightarrow f(x) < x$, as can be easily proven. Now observe that $\{0, 1\}$ are fixed points of the sequence $\{f_n(x)\}_{n\geq 1}$ yielding $0 \leq f_n(x) \leq 1/2$ whenever $0 \leq x \leq 1/2$. Since $\{f_n(x)\}_{n\geq 1}$ is decreasing, then

$$\lim_{n\to\infty} f_n(x) = \inf_{0\leq x\leq 1/2} f_n(x) = L = 0$$

□

Since f is continuous, then exchanging the limit with the integral yields

$$\lim_{n\to\infty} \int_0^{1/2} (f \circ f \circ \ldots \circ f)(y)\, dy = \int_0^{1/2} \lim_{n\to\infty} (f \circ f \circ \ldots \circ f)(y)\, dy = 0$$

and

$$y \cdot \Big(f \circ f \circ \ldots \circ f \Big)(y) \Big|_0^{1/2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

as claimed.

Also solved by Moubinool Omarjee, Paris France; and the proposer.

19. *Proposed by Paolo Perfetti, Tor Vergata University, Rome, Italy.* Let $\{a_k\}_{k\geq 1}$ be a monotonic sequence of real positive numbers such that $\sum_{n=1}^{\infty} a_n < +\infty$. Moreover, $\{a_k\}_{k\geq 1}$ fulfills the condition

$$a_k - a_{k+1} \geq 2^{-n} a_{2^{n+1}}, \quad \text{for all} \quad k \quad \text{with} \quad 2^n \leq k \leq 2^{n+1} - 1.$$

Let α be a quadratic irrational. Prove that the following series converges for any $\delta > 0$:

$$\sum_{n=1}^{\infty} \frac{a_n}{n(\ln n)^{\delta}} \sum_{k=n+1}^{2n} \frac{1}{|\sin(k\pi\alpha)|^{ka_k \ln k}}$$

Solution by the proposer. Let C and C_1 be two positive real numbers. A well-known property of all quadratic irrational α states that $|q\alpha - p| \geq C/q$ for some $C > 0$ and for any integers p, q. Let us denote by $||x||$ the distance from x to the nearest integer. Then $|\sin(\pi k\alpha)| \geq 2||k\alpha|| \geq 2C/k$, and

$$\sum_{k=n+1}^{2n} \frac{a_k}{\sin(k\pi\alpha)} \leq \sum_{k=n+1}^{2n} \frac{a_k}{2||k\alpha||}$$

Since $\Big| \, ||x|| - ||y|| \, \Big| = \min\Big\{ ||x-y||, ||x+y|| \Big\}$, then

$$n+1 \leq k < k' \leq 2n \quad \Longrightarrow \quad \Big| \, ||k\alpha|| - ||k'\alpha|| \, \Big| \in \Big\{ ||\alpha||, ||(n+1)\alpha||, \ldots, ||(4n-1)\alpha|| \Big\}$$

from which follows

$$\Big| ||k\alpha|| - ||k'\alpha|| \Big| \geq \frac{2C}{4n-1} \geq \frac{C}{2n} \quad \forall\ n \leq k < k' \leq 2n$$

Thus, if we write $\Big\{ ||n\alpha||, ||(n+1)\alpha||, \ldots, ||2n\alpha|| \Big\}$ according to the increasing order, the k–th member of the list verifies

$$||k\alpha|| \geq \frac{Ck}{2n}$$

Likewise for the sequence $\{a_k\}$ we write

$$a_{2^k} - a_{2^{k+1}} = \sum_{j=2^k}^{2^{k+1}-1} (a_j - a_{j+1}) \geq 2^{-k} a_{2^{k+1}} (2^{k+1} - 2^k) = a_{2^{k+1}}$$

That is,

$$a_{2^k} \geq 2a_{2^{k+1}} \quad \text{or} \quad 2^{k+1} a_{2^{k+1}} \leq 2^k a_{2^k} \quad \Longrightarrow \quad \{2^k a_{2^k}\} \quad \text{is monotonic.}$$

Applying Cauchy's condensation criteria to the sum $\sum_{k=1}^{\infty} a_k$ immediately follows $na_n \ln n \to 0$ when $n \to +\infty$ or $na_n \ln n < \varepsilon$ for all $n > n'$. Now for $n > n'$ we can write

$$\sum_{k=n+1}^{2n} |\sin(k\pi\alpha)|^{ka_k \ln k} \leq \sum_{k=n+1}^{2n} \left(\frac{2n}{Ck}\right)^{\varepsilon} \leq \left(\frac{2n}{C}\right)^{\varepsilon} \int_n^{2n} \frac{dx}{x^{\varepsilon}} \leq C_1 n$$

Thus

$$\sum_{n=1}^{\infty} \frac{a_n}{n(\ln n)^{\delta}} \sum_{k=n+1}^{2n} \frac{1}{|\sin(k\pi\alpha)|^{ka_k \ln k}} = \sum_{n=1}^{n'} \frac{a_n}{n(\ln n)^{\delta}} \sum_{k=n+1}^{2n} \frac{1}{|\sin(k\pi\alpha)|^{ka_k \ln k}}$$

$$+ \sum_{n=n'+1}^{\infty} \frac{a_n}{n(\ln n)^{\delta}} \sum_{k=n+1}^{2n} \frac{1}{|\sin(k\pi\alpha)|^{ka_k \ln k}}$$

Now remains to prove that the last term in the above expression converges. Indeed,

$$\sum_{n=n'+1}^{\infty} \frac{a_n}{n(\ln n)^{\delta}} \sum_{k=n+1}^{2n} \frac{1}{|\sin(k\pi\alpha)|^{ka_k \ln k}} \leq \sum_{n=n'+1}^{\infty} \frac{C_1 a_n}{(\ln n)^{\delta}} \leq C_1 \varepsilon \sum_{n=n'+1}^{\infty} \frac{C_1}{n(\ln n)^{1+\delta}}$$

which converges as can be easily proven.

20. *Proposed by Ovidiu Furdui, Cluj, Romania.* Let $p > 1/2$ be a real number. Calculate

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{\binom{n}{k}^p} \right)^{n^p}$$

Solution by the proposer. The limit equals e^2. Let $a_n = \sum\limits_{k=1}^{n} \frac{1}{\binom{n}{k}^p}$. Then, for $n \geq 3$, is

$$a_n \geq \frac{1}{\binom{n}{n}^p} + \frac{1}{\binom{n}{1}^p} + \frac{1}{\binom{n}{n-1}^p} = 1 + \frac{2}{n^p}$$

Also, for $n \geq 6$, we have

$$a_n = \frac{1}{\binom{n}{n}^p} + \left(\frac{1}{\binom{n}{1}^p} + \frac{1}{\binom{n}{n-1}^p}\right) + \left(\frac{1}{\binom{n}{2}^p} + \frac{1}{\binom{n}{n-2}^p}\right) + \left(\frac{1}{\binom{n}{3}^p} + \cdots + \frac{1}{\binom{n}{n-3}^p}\right)$$
$$\leq 1 + \frac{2}{n^p} + \frac{2^{p+1}}{n^p(n-1)^p} + \frac{n-5}{\binom{n}{3}^p},$$

since $\binom{n}{k} \geq \binom{n}{3}$ for $3 \leq k \leq n-3$. Thus,

$$1 + \frac{2}{n^p} \leq a_n \leq 1 + \frac{2}{n^p} + \frac{2^{p+1}}{n^p(n-1)^p} + \frac{6^p(n-5)}{(n(n-1)(n-2))^p}.$$

Hence,

$$\left(1 + \frac{2}{n^p}\right)^{n^p} \leq a_n^{n^p} \leq \left(1 + \frac{2}{n^p} + \frac{2^{p+1}}{n^p(n-1)^p} + \frac{6^p(n-5)}{(n(n-1)(n-2))^p}\right)^{n^p},$$

and the result follows taking limits when $n \to +\infty$ in the preceding inequality.

21. *Proposed by Mihály Bencze, Braşov, Romania.* Prove that

$$\operatorname{tg}\Big(\sum_{k=1}^{n} \operatorname{tg}^{-1} \frac{k^2+k-1}{(k^2+k+1)(k^2+k+2)}\Big) = \frac{n^2}{2n^2+5n+5}$$

Solution by Joaquín Rivero Rodríguez, I.E.S. Antonio de Nebrija, Zalamea de la Serena, Spain. We will argue by induction. For $n = 1$ the statement trivially holds. So, assume that for any arbitrary positive integer n, the identity holds. To prove it for $n+1$, we use the well-known addition tangent formulae. That is,

$$\operatorname{tg}\left(\sum_{k=1}^{n+1} \operatorname{tg}^{-1} \frac{k^2+k-1}{(k^2+k+1)(k^2+k+2)}\right)$$
$$= \operatorname{tg}\left(\sum_{k=1}^{n} \operatorname{tg}^{-1} \frac{k^2+k-1}{(k^2+k+1)(k^2+k+2)} + \operatorname{tg}^{-1} \frac{(n+1)^2+n}{((n+1)^2+n+2)((n+1)^2+n+3)}\right)$$
$$= \frac{(n+1)^2(n^4+4n^3+9n^2+10n+5)}{(2n^2+9n+12)(n^4+4n^3+9n^2+10n+5)} = \frac{(n+1)^2}{2n^2+9n+12}$$
$$= \frac{(n+1)^2}{2(n+1)^2+5(n+1)+5}$$

and by the PMI the statement is proven.

Also solved by Neculai Stanciu, Buzău, Romania; Moubinool Omarjee, Paris, France; Perfetti Paolo, Tor Vergata University, Roma, Italy; and the proposer.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

16. A number of three digits is written as xyz in base 7 and as zxy in base 9. Find the number in base 10.

17. A regular convex polygon of $L+M+N$ sides must be colored using three colors: red, yellow and blue, in such a way that L sides must be red, M yellow and N blue. Give the necessary and sufficient conditions, using inequalities, to obtain a colored polygon with no two consecutive sides of the same color.

18. Let $ABDC$ be a cyclic quadrilateral inscribed in a circle $\mathcal{C}$. Let M and N be the midpoints of the arcs AB and CD which do not contain C and A respectively. If MN meets side AB at P, then show that

$$\frac{AP}{BP}=\frac{AC+AD}{BC+BD}$$

19. Place n points on a circle and draw in all possible chord joining these points. If no three chord are concurrent, find (with proof) the number of disjoint regions created.

20. Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{\frac{1-a}{bc}}\left(\frac{1+b}{c+a}\right)^{\frac{1-b}{ca}}\left(\frac{1+c}{a+b}\right)^{\frac{1-c}{ab}}}\geq 64$$

Solutions

11. Let n be a positive integer. Compute the following sum

$$\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}$$

(Longlist VJIMC Ostrava 2011)

Solution 1 by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. We have

$$\begin{aligned}
\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k} &= \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} \left(1+\frac{1}{k+3}\right) \binom{n}{k} \\
&= \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} \binom{n}{k} + \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)(k+3)} \binom{n}{k} \\
&= \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \frac{(n+1)(n+2)}{(k+1)(k+2)} \binom{n}{k} \\
&+ \frac{1}{(n+1)(n+2)(n+3)} \sum_{k=0}^{n} \frac{(n+1)(n+2)(n+3)}{(k+1)(k+2)(k+3)} \binom{n}{k} \\
&= \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \binom{n+2}{k+2} \\
&+ \frac{1}{(n+1)(n+2)(n+3)} \sum_{k=0}^{n} \binom{n+3}{k+3}
\end{aligned}$$

Taking into account the Binomial theorem, we have

$$\sum_{k=0}^{n} \binom{n+2}{k+2} = 2^{n+2} - \binom{n+2}{0} - \binom{n+2}{1} = 2^{n+2} - n - 3$$

and

$$\sum_{k=0}^{n} \binom{n+3}{k+3} = 2^{n+3} - \binom{n+3}{0} - \binom{n+3}{1} - \binom{n+3}{2} = \frac{1}{2}(2^{n+4} - n(n+7) - 14)$$

from which follows

$$\begin{aligned}
\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k} &= \frac{2^{n+2}-n-3}{(n+1)(n+2)} + \frac{2^{n+4}-n(n+7)-14}{2(n+1)(n+2)(n+3)} \\
&= \frac{2^{n+3}(n+5) - 3n^2 - 19n - 32}{2(n+1)(n+2)(n+3)},
\end{aligned}$$

and we are done.

Solution 2 by Paolo Perfetti, Tor Vergata University, Rome, Italy. We have,

$$\frac{k+4}{(k+1)(k+2)(k+3)} = \frac{3}{2}\frac{1}{k+1} - \frac{2}{k+2} + \frac{1}{2}\frac{1}{k+3}$$

Integrating in the interval $[0,1]$ both sides of the identity

$$\sum_{k=0}^{n}\binom{n}{k}x^k=(1+x)^n,$$

we get

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}=\int_0^1\sum_{k=0}^{n}\binom{n}{k}x^k dx=\int_0^1(1+x)^n dx=\frac{2^{n+1}-1}{n+1}=I_1$$

Likewise, from $\sum_{k=0}^{n}\binom{n}{k}x^{k+1}=x(1+x)^n$, we obtain

$$\int_0^1 x(1+x)^n dx=\int_0^1\sum_{k=0}^{n}\binom{n}{k}x^{k+1}dx=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+2}$$

and

$$\int_0^1 x(1+x)^n dx=\int_0^1(1+x)^{n+1}dx-\int_0^1(1+x)^n dx=\frac{2^{n+2}-1}{n+2}-\frac{2^{n+1}-1}{n+1}=I_2$$

Finally, from $\sum_{k=0}^{n}\binom{n}{k}x^{k+2}=x^2(1+x)^n$, we get

$$\int_0^1 x^2(1+x)^n dx=\int_0^1\sum_{k=0}^{n}\binom{n}{k}x^{k+2}dx=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+3}$$

and

$$\begin{aligned}\int_0^1 x^2(1+x)^n dx &= \int_0^1(1+x)^{n+2}dx-2\int_0^1 x(1+x)^n dx-\int_0^1(1+x)^n dx\\ &= \frac{2^{n+3}-1}{n+3}-2\,\frac{2^{n+2}-1}{n+2}+2\,\frac{2^{n+1}-1}{n+1}-\frac{2^{n+1}-1}{n+1}=I_3\end{aligned}$$

Taking into account the preceding, yields

$$\sum_{k=0}^{n}\frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k}=\frac{3}{2}\,I_1-2\,I_2+\frac{1}{2}\,I_3=\frac{2^{n+3}(n+5)-3n^2-19n-32}{2(n+1)(n+2)(n+3)}$$

Also solved by Henry Ricardo, New York, USA.

12. Let $\alpha>0$ be a real number and let $f:[-\alpha,\alpha]\to\mathbb{R}$ be a continuous function two times derivable in $(-\alpha,\alpha)$ such that $f(0)=0$ and f'' is bounded in $(-\alpha,\alpha)$. Show that the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_n=\begin{cases}\displaystyle\sum_{k=1}^{n}f\left(\frac{k}{n^2}\right), & n>\frac{1}{\alpha};\\ 0, & n\leq\frac{1}{\alpha}\end{cases}$$

is convergent and determine its limit.

(Training Spanish Team for IMC 2008)

Solution by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain. First, we observe that if $n > \frac{1}{\alpha}$, then $\frac{k}{n^2} \le \frac{1}{n} < \alpha$ for $(1 \le k \le n)$, and $[0, \frac{k}{n^2}] \subset (-\alpha, \alpha)$. Applying Taylor's formula, we get

$$f\left(\frac{k}{n^2}\right) = f(0) + \frac{f'(0)}{1!}\left(\frac{k}{n^2}\right) + \frac{f''(c_k)}{2!}\left(\frac{k}{n^2}\right)^2, \qquad \left(0 < c_k < \frac{k}{n^2}\right)$$

and

$$x_n = \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = f'(0)\sum_{k=1}^{n}\frac{k}{n^2} + f''(c_k)\sum_{k=1}^{n}\frac{k^2}{2n^4}$$

From the above immediately follows

$$\left|x_n - f'(0)\sum_{k=1}^{n}\frac{k}{n^2}\right| = \left|\sum_{k=1}^{n}\frac{k^2}{2n^4}f''(c_k)\right| \le \sum_{k=1}^{n}\frac{k^2}{2n^4}\,|f''(c_k)| \le M\sum_{k=1}^{n}\frac{k^2}{2n^4},$$

where $(0 < M < +\infty)$. Taking into account the well known close form of the sums of the first and second powers of positive integers yields

$$\left|x_n - f'(0)\frac{n(n+1)}{2n^2}\right| \le \frac{Mn(n+1)(2n+1)}{12n^4}$$

When $n \to \infty$, from the preceding we obtain

$$\lim_{n\to\infty} x_n = \frac{1}{2}f'(0)$$

This completes the proof and we are done.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paolo Perfetti, Tor Vergata University, Rome, Italy; and José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.

13. *Let n be a positive integer. Compute*

$$\sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\frac{2^k}{(i_1+1)(i_2+1)\ldots(i_k+1)}$$

(Longlist Mediterranean 2008)

Solution 1 by Paolo Perfetti, Tor Vergata University, Rome, Italy. Let us denote by $a_k = \dfrac{2}{k+1}$, $A_n = \displaystyle\sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\frac{2^k}{(i_1+1)(i_2+1)\ldots(i_k+1)}$. Let

$$P_n(x) = \prod_{k=1}^{n}(x + a_k) = x^n + \sum_{k=1}^{n} b_k x^{n-k}$$

Then,

$$P_n(1) = 1 + \sum_{k=1}^{n} b_k = 1 + A_n = \prod_{k=1}^{n}\left(1 + \frac{2}{k+1}\right) = \frac{(n+2)(n+3)}{6}$$

from which follows

$$A_n = \sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\frac{2^k}{(i_1+1)(i_2+1)\ldots(i_k+1)} = \frac{n(n+5)}{6}$$

Solution 2 by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. For all function f for which $f(k) \neq 0, (1 \leq k \leq n)$ we have

$$\prod_{k=1}^{n}\left(1+\frac{1}{f(k)}\right) = 1+\sum_{k=1}^{n}\frac{1}{f(k)} + \sum_{1\leq i_1<i_2\leq n}\frac{1}{f(i_1)f(i_2)} + \cdots + \frac{1}{f(1)f(2)\ldots f(n)}$$

$$= 1+\sum_{k=1}^{n}\sum_{1\leq i_1<\ldots<i_k\leq n}\frac{1}{f(i_1)f(i_2)\ldots f(i_k)}$$

Putting $f(x) = \dfrac{x+1}{2}$ into the preceding expression, we get

$$1+\sum_{k=1}^{n}\sum_{1\leq i_1<\ldots<i_k\leq n}\frac{2^k}{(i_1+1)(i_2+1)\ldots(i_k+1)} = \prod_{k=1}^{n}\left(1+\frac{2}{k+1}\right)$$

We claim that $\displaystyle\prod_{k=1}^{n}\left(1+\frac{2}{k+1}\right) = \frac{(n+2)(n+3)}{6}$. To prove our claim, we argue by induction. The case when $n = 1$ trivially holds. Assume that the identity holds for n. We have to prove

$$\prod_{k=1}^{n+1}\left(1+\frac{2}{k+1}\right) = \frac{(n+3)(n+4)}{6}$$

Indeed,

$$\prod_{k=1}^{n+1}\left(1+\frac{2}{k+1}\right) = \prod_{k=1}^{n}\left(1+\frac{2}{k+1}\right)\left(1+\frac{2}{n+2}\right)$$

$$= \left(\frac{(n+2)(n+3)}{6}\right)\left(1+\frac{2}{n+2}\right) = \frac{(n+3)(n+4)}{6}$$

Therefore,

$$\sum_{k=1}^{n}\sum_{1\leq i_1<\ldots<i_k\leq n}\frac{2^k}{(i_1+1)(i_2+1)\ldots(i_k+1)} = \frac{n(n+5)}{6}$$

and we are done.

Also solved by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.

14. *Let F_n be the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$ and for all $n \geq 2, F_n = F_{n-1} + F_{n-2}$. Prove that*

$$\frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{T_k}{F_k}\right)^2 \geq \frac{T_{n+1}^2}{9F_nF_{n+1}},$$

where T_k is the k^{th} triangular number defined by $T_k = \binom{k+1}{2}$ for all $k \geq 1$.

(IMAC-2007)

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. From the trivial inequality $(bx - ay)^2 \geq 0$ or equivalently $\dfrac{bx^2}{a} + \dfrac{ay^2}{b} \geq 2xy$

is easy to get $\left(\frac{a+b}{a}\right)x^2+\left(\frac{a+b}{b}\right)y^2 \geq (x+y)^2$ from which immediately follows

$$\frac{x^2}{a}+\frac{y^2}{b} \geq \frac{(x+y)^2}{a+b}$$

Applying the above recursively to the positive numbers $a_1, a_2, \cdots, a_n$ and the reals $x_1, x_2, \cdots, x_n$, we have

$$\begin{aligned}\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\ldots+\frac{x_n^2}{a_n} &\geq \frac{(x_1+x_2)^2}{a_1+a_2}+\frac{x_3^2}{a_3}+\ldots+\frac{x_n^2}{a_n}\\ &\geq \frac{(x_1+x_2+x_3)^2}{a_1+a_2+a_3}+\frac{x_4^2}{a_4}+\ldots+\frac{x_n^2}{a_n}\\ &\geq \frac{(x_1+x_2+\cdots+x_n)^2}{a_1+a_2+\cdots+a_n}\end{aligned}$$

Setting $x_k = T_k$ and $a_k = F_k^2$ $(1 \leq k \leq n)$ into the preceding inequality yields,

$$\sum_{k=1}^{n}\left(\frac{T_k}{F_k}\right)^2 \geq \frac{(T_1+T_2+\ldots+T_k)^2}{F_1^2+F_2^2+\ldots+F_n^2}$$

Since $T_1+T_2+\ldots+T_k = \dfrac{n(n+1)(n+2)}{6}$ and $F_1^2+F_2^2+\ldots+F_n^2 = F_nF_{n+1}$, as can be easily proved by induction, then the preceding inequality becomes

$$\sum_{k=1}^{n}\left(\frac{T_k}{F_k}\right)^2 \geq \frac{n^2(n+1)^2(n+2)^2}{36F_nF_{n+1}}$$

from which the statement immediately follows. Notice that equality holds when $n=1$ and we are done.

Also solved by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.

15. *Prove that*

$$\frac{1}{2}+\int_0^1 \sqrt[3]{x+\ln(1+x)}\,dx \int_0^1 \sqrt[3]{(x+\ln(1+x))^2}\,dx < 2\ln 2$$

(József Wildt Competition 2006)

Solution 1 by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. We begin with a lemma.

Lema 1. *Let* $f:[0,1]\to\mathbb{R}^+$ *be a continuous function. Then, the following inequality*

$$\int_0^1 f(x)\,dx \int_0^1 f^2(x)\,dx \leq \int_0^1 f^3(x)\,dx$$

holds.

Proof. First, we observe that $f^2(x)$ and $f^3(x)$ are also continuous (integrable). Now, we set $a_k = f(k/n)$ and $b_k = f^2(k/n)$, $(1 \le k \le n)$, into Chebyshev's inequality, namely,

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_k - \frac{1}{n}\sum_{k=1}^{n} a_k \ \frac{1}{n}\sum_{k=1}^{n} b_k \ge 0$$

and we get

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\ \frac{1}{n}\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right) \le \frac{1}{n}\sum_{k=1}^{n} f^3\left(\frac{k}{n}\right)$$

and the proof follows taking limits when n goes to infinity. □

Finally, setting $f(x) = \sqrt[3]{x + \ln(1+x)}$ into the preceding lemma, we have

$$\int_0^1 \sqrt[3]{x + \ln(1+x)}\, dx \int_0^1 \sqrt[3]{(x + \ln(1+x))^2}\, dx$$

$$< \int_0^1 (x + \ln(1+x))\, dx = \left[\frac{x^2}{2} + (1+x)(\ln(1+x) - 1)\right]_0^1 = 2\ln 2 - \frac{1}{2}$$

and we are done.

Solution 2 by Paolo Perfetti, Tor Vergata University, Rome, Italy. The inequality claimed immediately follows from Chebyshev's inequality for integrals. Namely, if f and g are defined on $[a, b]$ and have the same monotonicity, then

$$\int_a^b f(x)dx \int_a^b g(x)dx \le (b-a)\int_a^b f(x)g(x)dx$$

Since $f(x) = x + \ln(1+x)$ is increasing in $[0, 1]$, then $g(x) = \sqrt[3]{x + \ln(1+x)}$ and $h(x) = \sqrt[3]{(x + \ln(1+x))^2}$ are also increasing in $[0, 1]$. So, applying Chebyshev's inequality, yields

$$\int_0^1 \sqrt[3]{x + \ln(1+x)}\, dx \int_0^1 \sqrt[3]{(x + \ln(1+x))^2}\, dx \le \int_0^1 (x + \ln(1+x))\, dx = 2\ln 2 - \frac{1}{2}$$

from which we obtain the inequality claimed.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; and José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.