Mathproblems
ISSN: 2217-446X, url: http://www.mathproblems-ks.com
Volume 1, Issue 3 (2011), Pages 17–31

PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before 2 August 2011

Problems

15. *Proposed by Valmir Bucaj, Texas Lutheran University, Seguin, TX.* Consider the set given by

$$L_{a,r} = \{ar^n | n \in \mathbb{Z}_+\}, \text{ where } r = \frac{1}{q}, q \in \mathbb{Z}_+, \text{ and } \gcd(a,q) = 1.$$

Show that

$$B_q = \{L_{a,r} | r = \frac{1}{q}, q \in \mathbb{Z}_+, q \geq 2 \text{ and } \gcd(a,q) = 1\},$$

forms a basis for a topology on $\mathbb{Q}_+$.

16. *Proposed by José Luis Díaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain.* Find all triplets (x, y, z) of real numbers which are solutions of the following system of equations

$$\left.\begin{array}{r} x^2(y+z)^2 = (3x^2 - x + 1)y^2z^2 \\ y^2(z+x)^2 = (4y^2 - y + 1)z^2x^2 \\ z^2(x+y)^2 = (5z^2 - z + 1)x^2y^2 \end{array}\right\}$$

17. *Proposed by José Luis Díaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain.* Let $A(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \neq 0)$ be a nonconstant polynomial with complex coefficients. Prove that all its zeros lie in the ring shaped region $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{4^k \binom{n}{k}}{5^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{1/k} \quad \text{and} \quad r_2 = \max_{1 \leq k \leq n} \left\{ \frac{5^n - 1}{4^k \binom{n}{k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}$$

18. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.* If, $f(x) = \frac{1-\sqrt{1-2x}}{2}$, and $f_n^{-1} = \underbrace{f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}}_{n}$, then evaluate:

$$\lim_{n \to \infty} \int_0^1 f_n^{-1}(x)dx.$$

19. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let $\{a_k\}_{k \geq 1}$ be a monotonic sequence of real positive numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Moreover, $\{a_k\}_{k \geq 1}$ fulfills the condition

$$a_k - a_{k+1} \geq 2^{-n} a_{2^{n+1}}, \quad \text{for all} \quad k \quad \text{with} \quad 2^n \leq k \leq 2^{n+1} - 1.$$

Let α be a quadratic irrational. Prove that the following series converges for any $\delta > 0$.

$$\sum_{n=1}^{\infty} \frac{a_n}{n(\ln n)^{\delta}} \sum_{k=n+1}^{2n} \frac{1}{|\sin(k\pi\alpha)|^{k a_k \ln k}}$$

20. *Proposed by Ovidiu Furdui, Cluj, Romania.* Let $p > 1/2$ be a real number. Calculate

$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{\binom{n}{k}^p} \right)^{n^p}$$

21. *Proposed by Mihály Bencze, Braşov, Romania.* Prove that

$$\tan\Big(\sum_{k=1}^{n} \arctan \frac{k^2 + k - 1}{(k^2 + k + 1)(k^2 + k + 2)} \Big) = \frac{n^2}{2n^2 + 5n + 5}$$

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

8. *Proposed by Valmir Krasniqi and Armend Sh. Shabani, Department of Mathematics, University of Prishtina, Republic of Kosova.* If f is a nonnegative function on $[0,1]$ and $f'(x) \geq 1$, then

$$\int_0^1 f^3(x)dx \geq \left(\int_0^1 f(x)dx\right)^2$$

Solution by Ovidiu Furdui, Cluj, Romania. We prove that, under the hypothesis of the problem, one has that for $x \in [0,1]$, the following stronger inequality holds

$$\int_0^x f^3(t)dt - \left(\int_0^x f(t)dt\right)^2 \geq f^2(0)\int_0^x f(t)dt$$

When $x = 1$ this implies that

$$\int_0^1 f^3(x)dx - \left(\int_0^1 f(x)dx\right)^2 \geq f^2(0)\int_0^1 f(x)dx$$

Let $F : [0,1] \to \mathbb{R}$ be the function defined by

$$F(x) = \int_0^x f^3(t)dt - \left(\int_0^x f(t)dt\right)^2$$

Then $F'(x) = f(x)\left(f^2(x) - 2\int_0^x f(t)dt\right)$. Let g be the function defined on $[0,1]$ by $g(x) = f^2(x) - 2\int_0^x f(t)dt$. A calculation shows that $g'(x) = 2f(x)(f'(x) - 1) \geq 0$. Hence g increases and it follows that $g(x) \geq g(0) = f^2(0)$. It follows that $F'(x) \geq f^2(0)f(x)$, and this implies that $(F(x) - f^2(0)\int_0^x f(t)dt)' \geq 0$. Thus, the function $x \to F(x) - f^2(0)\int_0^x f(t)dt$ increases. It follows that $F(x) - f^2(0)\int_0^x f(t)dt \geq F(0) = 0$, and the problem is solved.

Comment by Henry Ricardo, USA: This problem is a special case of results found in a paper by Mohamed Akkouchi (Some integral inequalities, *Divulgaciones Matemticas* 11 (2003), 121-125), which generalizes some results of Feng Qi (Several integral inequalities, *Journal of Inequalities in Pure and Applied Mathematics* vol. 1, issue 2, Article 19, 2000).

Also solved by Henry Ricardo, USA; Arnau Massegué Buisan, Technical University of Barcelona, Spain, Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, Valmir Bucaj, Texas Lutheran University, Seguin, TX and the proposers.

9. *Proposed by Roberto Tauraso, Department of Mathematics, Tor Vergata University, Rome, Italy.* Show that for any prime p and for any non-negative integer

$n,\ p\,|\,L_{pn} - L_n,$ where L_n is the n-th Lucas number defined by $L_0 = 2, L_1 = 1$ and for $n \geq 2, L_n = L_{n-1} + L_{n-2}$.

Solution by the proposer. If $p = 2$ then $L_2 = 3 \equiv 1 = L_1 \pmod 2$. Now we assume that p is an odd prime. Since p divides $\binom{p}{k}$ for $k = 1, \ldots, p-1$, by Fermat's Little Theorem

$$L_p = \left(\frac{1+\sqrt{5}}{2}\right)^p + \left(\frac{1-\sqrt{5}}{2}\right)^p = \frac{1}{2^{p-1}} \sum_{\substack{0 \leq k \leq p \\ k \equiv 0 \pmod 2}} \binom{p}{k} 5^{k/2}$$

$$= \frac{1}{2^{p-1}} + \frac{1}{2^{p-1}} \sum_{\substack{1 \leq k \leq p-1 \\ k \equiv 0 \pmod 2}} \binom{p}{k} 5^{k/2} \equiv 1 = L_1 \pmod p$$

Moreover, by the recurrence $L_{n+m} = L_m L_n - (-1)^m L_{n-m}$, we have that for any prime p

$$L_{(n+1)p} \equiv L_p L_{np} + L_{(n-1)p} \equiv L_{np} + L_{(n-1)p} \pmod p$$

Hence by letting $a_n = L_{np} - L_n$, it follows that

$$a_0 \equiv 0,\ a_1 \equiv 0, \text{ and } a_{n+1} = a_n + a_{n-1} \pmod p \text{ for } n \geq 1,$$

which means that $a_n \equiv 0 \pmod p$ for all $n \geq 0$.

10. *Proposed by Roberto Tauraso, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let $n = 2010^{100}$. Compute the cardinality of the set

$$S_n = \{d \ :\ d \in [1, n] \cap \mathbb{N},\ d | n^2,\ d \nmid n\}$$

Solution by Valmir Bucaj, Texas Lutheran University, Seguin, TX. We will solve the more general problem instead: Let n be an integer given in the standard prime factorization form: $n = p^{e_1} p_2^{e_2} \cdots p_v^{e_v}$. Compute the cardinality of

$$S_m = \{d : d \in [1, m] \cap \mathbb{N}, d | m^2, d \nmid m\},$$

where $m = n^k$ for some positive integer k. Let $\phi(m)$ be the number of divisors of m excluding m. Then, we have $\phi(m) = (ke_1 + 1)(ke_2 + 1) \cdots (ke_v + 1) - 1$ and $\phi(m^2) = (2ke_1 + 1)(2ke_2 + 1) \cdots (2ke_v + 1) - 1$, as it is well-known. Also, since given an integer n, half of its divisors are less than $\sqrt{n}$ and half greater than $\sqrt{n}$, in our specific case follows that half of the divisors of m^2 are less than m. Therefore,

$$|S_m| = \frac{\phi(m^2)}{2} - \phi(m)$$

For the original problem we have: $n = 2010, k = 100$, and $m = 2010^{100}$. So, since $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, we have

$$\phi(m) = 101^4 - 1 \qquad \textit{and} \qquad \phi(m^2) = 201^4 - 1.$$

Therefore,

$$|S_m| = \frac{\phi(m^2)}{2} - \phi(m) = \frac{201^4 - 1}{2} - 101^4 + 1 = 712060000$$

Also solved by the proposer.

11. *Proposed by Roberto Tauraso, Department of Mathematics, Tor Vergata University, Rome, Italy.* Find a closed formula for

$$\sum_{\substack{A\subset\{1,\ldots,n\}\\ A\neq\emptyset}}\ \sum_{\substack{B\subset\{1,\ldots,n\}\\ B\neq\emptyset}}\ \sum_{x\in A\cup B} x$$

Solution 1 by the proposer. We first note that for any $x \in \{1,\ldots,n\}$ the number of subsets of $\{1,\ldots,n\}$ which contain the element x is 2^{n-1}. Therefore

$$\begin{aligned} g(n) &= \sum_{\substack{A\subset\{1,\ldots,n\}\\ A\neq\emptyset}}\ \sum_{x\in A} x = \sum_{x\in\{1,\ldots,n\}} x \sum_{\substack{A\subset\{1,\ldots,n\}\\ A\neq\emptyset}} [x\in A] \\ &= 2^{n-1}\sum_{x\in\{1,\ldots,n\}} x = 2^{n-1}\binom{n+1}{2} \end{aligned}$$

where $[P]$ is 1 if P is true and 0 otherwise. Similarly, for any $x \in \{1,\ldots,n\}$ the number of couples of subsets of $\{1,\ldots,n\}$ whose union set contains the element x is $3\cdot 2^{n-1}\cdot 2^{n-1}$ (3 because x could belong to $A\setminus B$, $A\cap B$ or $B\setminus A$). Hence

$$\begin{aligned} f(n) &= \sum_{A\subset\{1,\ldots,n\}}\ \sum_{B\subset\{1,\ldots,n\}}\ \sum_{x\in A\cup B} x = \sum_{x\in\{1,\ldots,n\}} x \sum_{A\subset\{1,\ldots,n\}}\ \sum_{B\subset\{1,\ldots,n\}} [x\in A\cup B] \\ &= 3\cdot 4^{n-1}\sum_{x\in\{1,\ldots,n\}} x = 3\cdot 4^{n-1}\binom{n+1}{2} \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\substack{A\subset\{1,\ldots,n\}\\ A\neq\emptyset}}\ \sum_{\substack{B\subset\{1,\ldots,n\}\\ B\neq\emptyset}}\ \sum_{x\in A\cup B} x &= f(n) - \sum_{\substack{A\subset\{1,\ldots,n\}\\ A\neq\emptyset}}\ \sum_{B=\emptyset}\ \sum_{x\in A\cup B} x - \sum_{A=\emptyset}\ \sum_{\substack{B\subset\{1,\ldots,n\}\\ B\neq\emptyset}}\ \sum_{x\in A\cup B} x \\ &= f(n) - 2g(n) = \left(3\cdot 4^{n-1} - 2^n\right)\binom{n+1}{2} \end{aligned}$$

Solution 2 by the Joaquín Rivero Rodríguez, I.E.S. Antonio de Nebrija, Zalamea de la Serena, Spain. We have

$$\begin{aligned} \sum_{\substack{A\subset\{1,\ldots,n\}\\ A\neq\emptyset}}\ \sum_{\substack{B\subset\{1,\ldots,n\}\\ B\neq\emptyset}}\ \sum_{x\in A\cup B} x &= \sum_{\substack{A\subset\{1,\ldots,n\}\\ A\neq\emptyset}}\ \sum_{\substack{B\subset\{1,\ldots,n\}\\ B\neq\emptyset}} \left(\sum_{x\in A} x + \sum_{x\in B} x - \sum_{x\in A\cap B} x\right) \\ &= 2\cdot \sum_{\substack{A\subset\{1,\ldots,n\}\\ A\neq\emptyset}}\ \sum_{\substack{B\subset\{1,\ldots,n\}\\ B\neq\emptyset}}\ \sum_{x\in A} x - \sum_{\substack{A\subset\{1,\ldots,n\}\\ A\neq\emptyset}}\ \sum_{\substack{B\subset\{1,\ldots,n\}\\ B\neq\emptyset}}\ \sum_{x\in A\cap B} x \\ &= 2\cdot 2^{n-1}\cdot(2^n-1)\cdot\sum_{k=1}^{n} k - 2^{n-1}\cdot 2^{n-1}\cdot\sum_{k=1}^{n} k \\ &= \left[2^n\cdot(2^n-1) - 2^{2n-2}\right]\frac{n(n+1)}{2} \\ &= \left(2^{2n-1} - 2^{2n-3} - 2^{n-1}\right) n(n+1) \\ &= \left(3\cdot 4^{n-1} - 2^n\right)\binom{n+1}{2} \end{aligned}$$

and we are done.

12. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let a, b, c be positive numbers. Prove that

$$\sum_{\text{cyc}} \sqrt{\frac{5a^2+5c^2+8b^2}{4ac}} \geq 3 \cdot \sqrt[9]{\frac{8(a+b)^2(b+c)^2(c+a)^2}{(abc)^2}}$$

Solution by the proposer. Since $5a^2+5c^2+8b^2 = 4a^2+(a^2+4b^2)+(4b^2+c^2)+5c^2 \geq 4(a^2+ab+bc+c^2)$, as can be easily proven, then

$$\sum_{\text{cyc}} \sqrt{\frac{5a^2+5c^2+8b^2}{4ac}} \geq \sum_{\text{cyc}} \sqrt{\frac{a+b}{c}+\frac{b+c}{a}} \geq 3 \cdot \sqrt[9]{\frac{8(a+b)^2(b+c)^2(c+a)^2}{(abc)^2}}$$

Setting

$$x = \frac{b+c}{a},\ y = \frac{a+c}{b},\ z = \frac{b+a}{c} \implies a = \frac{1}{1+x},\ b = \frac{1}{1+y},\ c = \frac{1}{1+z}$$

from which follows $x+y+z+2 = xyz$. The inequality claimed becomes

$$\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x} \geq 3 \cdot \sqrt[3]{2}(xyz)^{2/9} \qquad \text{if} \qquad x+y+z+2 = xyz$$

Squaring and using $\sqrt{(a+b)(a+c)} \geq a+\sqrt{bc}$ (Cauchy–Schwarz), we obtain

$$4(x+y+z)+2\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right) \geq 9 \cdot 2^{2/3}(xyz)^{4/9}$$

By the AGM we show that

$$4(x+y+z)+6(xyz)^{1/3} \geq 9 \cdot 2^{2/3}(xyz)^{4/9}$$

Now put $r = (xyz)^{1/9}$ and use $x+y+z+2 = xyz$ to get

$$4r^9 - 9 \cdot 2^{2/3}r^4 + 6r^3 - 8$$

$$= (r-2^{\frac{1}{3}})(4r^8+4\cdot 2^{\frac{1}{3}}r^7+4\cdot 2^{\frac{2}{3}}r^6+8r^5+8\cdot 2^{\frac{1}{3}}r^4-2^{\frac{2}{3}}r^3+4r^2+4\cdot 2^{\frac{1}{3}}r+4\cdot 2^{\frac{2}{3}}) \geq 0 \quad (1)$$

The constraint $x+y+z+2 = xyz$ yields $xyz \geq 2$. Indeed

$$xyz = 2+x+y+z \geq 2+3(xyz)^{1/3}$$

thus by defining $p = (xyz)^{\frac{1}{3}} \geq 2$, we get $p^3-3p-2 = (p-2)(p+1)^2 \geq 0$ that is $p \geq 2$ implying $r \geq 2^{\frac{1}{3}}$. Since

$$8 \cdot 2^{\frac{1}{3}}r^4 - 2^{\frac{2}{3}}r^3 \geq 0$$

then (1) is proved and we are done.

Editorial comment: We claim that equality is never achieved.

13. *Proposed by Mihály Bencze, Braşov, Romania.* Let a_k, $1 \leq k \leq n$, be any positive numbers. Prove that

$$(n-1)\left(\sum_{k=1}^{n} a_k + \frac{1}{\prod_{k=1}^{n} a_k}\right) \geq \frac{\left(n-1+2\sum_{1\leq i<j\leq n} a_i a_j\right)^2}{\sum_{1\leq i<j\leq n} a_i a_j(a_i+a_j)+(n-1)\sum_{k=1}^{n} a_k^2}$$

Solution by the proposer. Using the Cauchy-Schwarz inequality we get:

$$\Big(\sum_{k=1}^{n} a_k + \frac{1}{\prod\limits_{k=1}^{n} a_k}\Big)\Big(\sum a_1 \cdot a_p^2 + \prod_{k=1}^{n} a_k\Big) \geq \Big(\sum a_1 a_p + 1\Big)^2$$

where $p \in \{2, 3, \ldots, n\}$ or

$$\sum_{k=1}^{n} a_k + \frac{1}{\prod\limits_{k=1}^{n} a_k} \geq \frac{\Big(\sum a_1 a_p + 1\Big)^2}{\sum a_1 \cdot a_p^2 + \prod\limits_{k=1}^{n} a_k}$$

therefore

$$(n-1)\Big(\sum_{k=1}^{n} a_k + \frac{1}{\prod\limits_{k=1}^{n} a_k}\Big) = \sum_{p=2}^{n}\Big(\sum_{k=1}^{n} a_k + \frac{1}{\prod\limits_{k=1}^{n} a_k}\Big) \geq$$

$$\geq \sum_{p=2}^{n} \frac{\Big(\sum a_1 a_p + 1\Big)^2}{\sum a_1 \cdot a_p^2 + \prod\limits_{k=1}^{n} a_k} \geq \frac{\Big(\sum\limits_{p=2}^{n}\Big(\sum a_1 a_p + 1\Big)\Big)^2}{\sum\limits_{p=2}^{n}\Big(\sum a_1 \cdot a_p^2 + \prod\limits_{k=1}^{n} a_k\Big)}$$

$$= \frac{\Big(n - 1 + 2\sum\limits_{1 \leq i < j \leq n} a_i a_j\Big)^2}{\sum\limits_{1 \leq i < j \leq n} a_i a_j (a_i + a_j) + (n-1)\prod\limits_{k=1}^{n} a_k}$$

14 (Correction). *Proposed by Mihály Bencze, Braşov, Romania.* Solve the equation

$$64^x - 27 = 343^{x-1} + \frac{3}{7} \cdot 28^x$$

Solution by the proposer. Setting $a = 3, b = -4^x, c = 7^{x-1}$ the given equation becomes $a^3 + b^3 + c^3 - 3abc = 0$ or equivalently,

$$(a+b+c)(\sum_{cyc} a^2 - \sum_{cyc} ab) = 0$$

Now we consider the following two possibilities:

(1) $\sum\limits_{cyc} a^2 - \sum\limits_{cyc} ab = 0 \Leftrightarrow a = b = c$ which is impossible.

(2) $a + b + c = 0 \Rightarrow 3 - 4^x + 7^{x-1} = 0$ or $7^{x-1} - 4^{x-1} = 3 \cdot (4^x - 1)$. Applying Lagrange Theorem to the function $f(t) = t^{x-1}$ we have that there exist $\alpha \in (1, 4)$ and $\beta \in (4, 7)$ such that

$$f(7) - f(4) = 3(x-1)\beta^{x-2} \quad \text{and} \quad f(4) - f(1) = 3(x-1)\alpha^{x-2}$$

From the preceding immediately follows that $3(x-1)\beta^{x-2} = 3(x-1)\alpha^{x-2} \Rightarrow x = 1$ is the unique solution.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

11. Let n be a positive integer. Compute the following sum

$$\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}$$

12. Let $\alpha > 0$ be a real number and let $f : [-\alpha, \alpha] \to \mathbb{R}$ be a continuous function two times derivable in $(-\alpha, \alpha)$ such that $f(0) = 0$ and f'' is bounded in $(-\alpha, \alpha)$. Show that the sequence $\{x_n\}_{n \geq 1}$ defined by

$$x_n = \begin{cases} \displaystyle\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right), & n > \frac{1}{\alpha}; \\ 0, & n \leq \frac{1}{\alpha} \end{cases}$$

is convergent and determine its limit.

13. Let n be a positive integer. Compute

$$\sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \frac{2^k}{(i_1+1)(i_2+1)\ldots(i_k+1)}$$

14. Let F_n be the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$ and for all $n \geq 2, F_n = F_{n-1} + F_{n-2}$. Prove that

$$\frac{1}{n^2} \sum_{k=1}^{n} \left(\frac{T_k}{F_k}\right)^2 \geq \frac{T_{n+1}^2}{9F_nF_{n+1}},$$

where T_k is the k^{th} triangular number defined by $T_k = \binom{k+1}{2}$ for all $k \geq 1$.

15. Prove that

$$\frac{1}{2} + \int_0^1 \sqrt[3]{x + \ln(1+x)}\, dx \int_0^1 \sqrt[3]{(x + \ln(1+x))^2}\, dx < 2 \ln 2$$

Solutions

6. *Let a, b, c be the lengths of the sides of a triangle ABC with circumradius r and area $\mathcal{A}$. Compute*

$$\frac{\cos A-\cos B}{\mathcal{A}-rc}+\frac{\cos B-\cos C}{\mathcal{A}-ra}+\frac{\cos C-\cos A}{\mathcal{A}-rb}$$

(Spanish First Stage 2007)

Solution by José Gibergans-Báguena and José Luis Díaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain. Since $\mathcal{A}=pr$, (p represents the semi-perimeter of $\triangle ABC$), then $\mathcal{A}-ra=rp-ra=r(p-a)$ (cyclic) and the given expression can be written as

$$\begin{aligned}&\frac{\cos A-\cos B}{r(p-c)}+\frac{\cos B-\cos C}{r(p-a)}+\frac{\cos C-\cos A}{r(p-b)}\\ &=\frac{1}{r}\left(\frac{\cos A-\cos B}{p-c}+\frac{\cos B-\cos C}{p-a}+\frac{\cos C-\cos A}{p-b}\right)\\ &=\frac{2}{r}\left(\frac{\cos A-\cos B}{a+b-c}+\frac{\cos B-\cos C}{b+c-a}+\frac{\cos C-\cos A}{a-b+c}\right)\end{aligned}$$

Applying the Law of Cosine, yields

$$\begin{aligned}\frac{\cos A-\cos B}{a+b-c} &= \frac{1}{a+b-c}\left(\frac{c^2+b^2-a^2}{2bc}-\frac{c^2+a^2-b^2}{2ac}\right)\\ &= \frac{b(c^2+b^2-a^2)-a(c^2+a^2-b^2}{2abc(a+b-c)}\\ &= \frac{(a-b)[c^2-(a+b)^2]}{2abc(a+b-c)}=\frac{2p(b-a)}{abc}\end{aligned}$$

Likewise, $\dfrac{\cos B-\cos C}{b+c-a}=\dfrac{2p(c-b)}{abc}$ and $\dfrac{\cos C-\cos A}{a-b+c}=\dfrac{2p(a-c)}{abc}$. Hence,

$$\begin{aligned}&\frac{\cos A-\cos B}{\mathcal{A}-rc}+\frac{\cos B-\cos C}{\mathcal{A}-ra}+\frac{\cos C-\cos A}{\mathcal{A}-rb}\\ &=\frac{2}{r}\left(\frac{\cos A-\cos B}{a+b-c}+\frac{\cos B-\cos C}{b+c-a}+\frac{\cos C-\cos A}{a-b+c}\right)\\ &=\frac{2}{r}\left(\frac{2p(b-a)}{abc}+\frac{2p(c-b)}{abc}+\frac{2p(a-c)}{abc}\right)=0\end{aligned}$$

and we are done.

7. *Let $\ln a, \ln b$ and $\ln c$ be the lengths of the sides of a triangle ABC. Prove that*

$$\frac{3}{5}\leq\frac{\ln a}{\ln(ab^2c^2)}+\frac{\ln b}{\ln(a^2bc^2)}+\frac{\ln c}{\ln(a^2b^2c)}<1$$

(Shortlist XIX Ibero 2004)

Solution 1 by Valmir Bucaj, Texas Lutheran University, Seguin, TX. Since the sum of any two sides of a triangle is greater than the third, then we have

$$\begin{aligned}\sum_{cyclic}\frac{\ln a}{\ln(ab^2c^2)} &= \sum_{cyclic}\frac{\ln a}{\ln a+2(\ln b+\ln c)} < \sum_{cyclic}\frac{\ln a}{\ln a+2\ln a}\\ &= \sum_{cyclic}\frac{\ln a}{3\ln a} = \sum_{cyclic}\frac{1}{3} = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1\end{aligned}$$

To prove the LHS inequality, we suppose that $\ln a \geq \ln b \geq \ln c$. Since the conditions for Chebyshev's Inequality are satisfied, then applying Chebyshev's Inequality and $AM - HM$ successively, we get

$$\begin{aligned}\sum_{cyclic}\frac{\ln a}{\ln(ab^2c^2)} &\geq \frac{1}{3}\ln(abc)\left(\sum_{cyclic}\frac{1}{\ln(ab^2c^2)}\right)\\ &\geq \frac{\ln(abc)}{3}\,9\left(\sum_{cyclic}\ln(ab^2c^2)\right)^{-1}\\ &= \frac{3\ln(abc)}{5\ln(abc)} = \frac{3}{5}.\end{aligned}$$

This completes the proof.

Solution 2 by José Luis Díaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain. First, we write the statement in the most convenient form

$$\frac{3}{5} \leq \frac{\ln a}{\ln a+2(\ln b+\ln c)} + \frac{\ln b}{\ln b+2(\ln c+\ln a)} + \frac{\ln c}{\ln c+2(\ln a+\ln b)} < 1$$

RHS inequality trivially holds from

$$\frac{\ln b+\ln c}{\ln a} > 1, \quad \frac{\ln c+\ln a}{\ln b} > 1 \text{ and } \frac{\ln a+\ln b}{\ln c} > 1$$

To prove LHS inequality, we put

$$x = \frac{\ln a}{\ln a+\ln b+\ln c}, \quad y = \frac{\ln b}{\ln a+\ln b+\ln c}, \quad z = \frac{\ln c}{\ln a+\ln b+\ln c},$$

and we have

$$\begin{aligned}&\frac{\ln a}{\ln a+2(\ln b+\ln c)} + \frac{\ln b}{\ln b+2(\ln c+\ln a)} + \frac{\ln c}{\ln c+2(\ln a+\ln b)}\\ &= \frac{x}{2-x}+\frac{y}{2-y}+\frac{z}{2-z}\end{aligned}$$

Since $\left(\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z}\right)\Big[(2-x)+(2-y)+(2-z)\Big] \geq 9$, or equivalently,

$$\left(\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z}\right)\Big[6-(x+y+z)\Big] \geq 9,$$

then

$$\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z} \geq \frac{9}{5}$$

on account of the fact that $x+y+z=1$. Therefore, from the preceding we have

$$\frac{x}{2-x}+\frac{y}{2-y}+\frac{z}{2-z}=\frac{x+2-2}{2-x}+\frac{y+2-2}{2-y}+\frac{z+2-2}{2-z}$$
$$=2\left(\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z}\right)-3\geq\frac{3}{5}$$

Notice that equality holds when $\ln a=\ln b=\ln c$. That is, when $\triangle ABC$ is equilateral.

Also solved by José Gibergans-Báguena, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain, Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.

8. *Suppose that the three roots of the equation* $t^3-at^2+t-b=0$ *are positive real numbers. Show that* $9b^2(1+6ab)\leq 1$.

(Longlist OME 2006)

Solution by José Luis Díaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain. First, we observe that the coefficients b and c are nonzero. Dividing by b^3 both sides of $9b^2(1+6ab)\leq 1$ and taking the cubic root of both sides of the resulting inequality, yields

$$9b^2(1+6ab)\leq 1\quad\Leftrightarrow\quad 3\sqrt[3]{\frac{1}{b}+6a}\leq\frac{\sqrt[3]{3}}{b}$$

Let x,y,z be the roots of $t^3-at^2+t-b=0$. On account of Cardan's formulae, we have

$$\begin{aligned}x+y+z&=a,\\ xy+yz+zx&=1,\\ xyz&=b,\end{aligned}$$

and the last inequality becomes

$$3\sqrt[3]{\frac{1}{xyz}+6(x+y+z)}\leq\frac{\sqrt[3]{3}}{xyz},$$

or equivalently,

$$\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{1}{xyz}+6(x+y+z)}\leq\frac{1}{xyz}$$

We have

$$\begin{aligned}\frac{1}{xyz}+6(x+y+z)&=\frac{1+6x^2yz+6xy^2z+6xyz^2}{xyz}\\ &=\frac{1+3xy(xz+yz)+3yz(yx+zx)+3zx(xy+yz)}{xyz}\end{aligned}$$

Taking into account that $xy+yz+zx=1$, we get

$$\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{4-3[(xy)^2+(yz)^2+(zx)^2]}{xyz}}\leq\frac{1}{xyz}$$

Now, from $xy + yz + zx = 1$ and

$$3[(xy)^2 + (yz)^2 + (zx)^2] \geq (xy + yz + zx)^2 = 1,$$

to prove the last inequality it suffices to prove

$$\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{3}{xyz}} \leq \frac{1}{xyz}$$

which follows immediately from AM-GM inequality. Indeed,

$$x^2y^2z^2 = (xy)(yz)(zx) \leq \left(\frac{xy + yz + zx}{3}\right)^3 = \frac{1}{3^3}$$

Multiplying both sides by xyz and reordering terms, yields

$$\frac{3^3}{xyz} \leq \frac{1}{(xyz)^3} \quad \Leftrightarrow \quad \frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{3}{xyz}} \leq \frac{1}{xyz}$$

Equality holds when $x = y = z = \dfrac{1}{\sqrt{3}}$. That is, when $a = \sqrt{3}$ and $b = \dfrac{\sqrt{3}}{9}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; and José Gibergans-Báguena, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain, Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.

9. *Let* $\{a_n\}_{n\geq 0}$ *be the sequence defined by* $a_0 = 1, a_1 = 2, a_2 = 1$ *and for all* $n \geq 3$, $a_n^3 = a_{n-1}a_{n-2}a_{n-3}$. *Find* $\lim\limits_{n\to\infty} a_n$.

(Longlist IMC 2006)

Solution by José Gibergans-Báguena and José Luis Díaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain. Setting $b_n = \log_2 a_n$, we get the sequence $\{b_n\}_{n\geq 0}$ defined by $b_0 = 0, b_1 = 1, b_2 = 0$ and for all $n \geq 3$, $3b_n = b_{n-1} + b_{n-2} + b_{n-3}$. The characteristic equation of b_n is $3t^3 - t^3 - t - 1 = 0$ which roots are 1, $-\frac{1}{3}(1 - i\sqrt{2})$, $-\frac{1}{3}(1 + i\sqrt{2})$. Therefore,

$$b_n = a + b\left(-\frac{1}{3}(1 - i\sqrt{2})\right)^n + c\left(-\frac{1}{3}(1 + i\sqrt{2})\right)^n$$

Taking into account of the initial conditions, we get the system of equations

$$\begin{aligned} a + b + c &= 0 \\ a + b\left(-\frac{1}{3}(1 - i\sqrt{2})\right) + c\left(-\frac{1}{3}(1 + i\sqrt{2})\right) &= 1 \\ a + b\left(-\frac{1}{3}(1 - i\sqrt{2})\right)^2 + c\left(-\frac{1}{3}(1 + i\sqrt{2})\right)^2 &= 0 \end{aligned}$$

with solutions $a = \dfrac{1}{3}, b = -\dfrac{1}{6}(1 - 5i\sqrt{2}/2)$ and $c = -\dfrac{1}{6}(1 + 5i\sqrt{2}/2)$. Thus, if $\lim_{n\to\infty} a_n = L$, then

$$\log_2 L = \lim_{n\to\infty} \log_2 a_n = \lim_{n\to\infty} b_n = \frac{1}{3} \Rightarrow L = \sqrt[3]{2}$$

and we are done.

Also solved by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.

10. *Let x, y, z be three distinct positive real numbers. Prove that*

$$\frac{1}{\max\{x,y,z\}} < \sum_{cyclic} \frac{\ln x^{2x}}{(x-y)(x-z)} < \frac{1}{\min\{x,y,z\}}$$

(Longlist IMC 2009)

Solution by José Luis Díaz-Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain. The inequality claimed is equivalent to

$$\frac{1}{2\max\{x,y,z\}} < \frac{x\ln x}{(x-y)(x-z)} + \frac{y\ln y}{(y-x)(y-z)} + \frac{z\ln z}{(z-x)(z-y)} < \frac{1}{2\min\{x,y,z\}}$$

Taking into account that

$$\frac{x}{(x-y)(x-z)} + \frac{y}{(y-x)(y-z)} + \frac{z}{(z-x)(z-y)} = 0,$$

as can be easily proven, we have that the last inequality is equivalent to

$$\frac{1}{2\max\{x,y,z\}} < \frac{x(\ln x-1)}{(x-y)(x-z)} + \frac{y(\ln y-1)}{(y-x)(y-z)} + \frac{z(\ln z-1)}{(z-x)(z-y)} < \frac{1}{2\min\{x,y,z\}}$$

Applying the well-known result [1] from the theory of divided differences

$$f[z_0, z_1, \cdots, z_n] = \sum_{j=0}^{n} f(z_j) \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{1}{z_j - z_k}$$

to the function $f(t) = t(\ln t - 1)$, we get

$$\begin{aligned} f[x,y,z] &= \frac{f(x)}{(x-y)(x-z)} + \frac{f(y)}{(y-x)(y-z)} + \frac{f(z)}{(z-x)(z-y)} \\ &= \frac{x(\ln x-1)}{(x-y)(x-z)} + \frac{y(\ln y-1)}{(y-x)(y-z)} + \frac{z(\ln z-1)}{(z-x)(z-y)} \end{aligned}$$

Now we need the following result.

Lema 1. *Let $f : [a,b] \to \mathbb{R}$ be a real valued function with second derivative $f''(x)$ continuous in $[a,b]$ and $x,y,z \in [a,b]$. Then there exists $c \in \Big[\min\{x,y,z\}, \max\{x,y,z\}\Big]$ such that $f[x,y,z] = \dfrac{f''(c)}{2}$.*

Proof. Since $f''(x)$ is continuous in $[a,b]$, then it has a maximum and a minimum in $[a,b]$. Let $m = \min_{a\le x\le b} f''(x)$ and $M = \max_{a\le x\le b} f''(x)$. Then from the integral representation of $f[x,y,z]$, we have

$$m\int_0^1 dt_1 \int_0^{t_1} dt_2 \le f[x,y,z] \le M \int_0^1 dt_1 \int_0^{t_1} dt_2$$

and

$$\frac{m}{2} \le f[x,y,z] \le \frac{M}{2} \quad \text{or} \quad m \le 2f[x,y,z] \le M$$

Since $f''(x)$ is continuous, by applying the intermediate value theorem to it, we have $2f[x,y,z] = f''(c)$ for some $c \in \Big[\min\{x,y,z\}, \max\{x,y,z\}\Big]$ and the proof is complete. $\square$

Applying Lemma 1 to the function $f(t) = t(\ln t - 1)$ to which $f''(t)$ is the decreasing function $f''(t) = 1/t$ then there exists $c \in \Big[\min\{x,y,z\}, \max\{x,y,z\}\Big]$ such that $f[x,y,z] = \dfrac{1}{2}f''(c) = \dfrac{1}{2c}$. So, $\dfrac{1}{2\max\{x,y,z\}} \le \dfrac{1}{2c} \le \dfrac{1}{2\min\{x,y,z\}}$. Since x,y,z are distinct then the statement follows and we are done.

References

[1] E. Isaacson and H. B. Keller. *Analysis of Numerical Methods.* Dover, New York, 1994.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. Both the LHS and the RHS are symmetric and homogeneous. The symmetry is evident while for the homogeneity we multiply each variable by α obtaining

$$\frac{1}{\max\{\alpha x, \alpha y, \alpha z\}} < \sum_{\text{cyclic}} \frac{2\alpha x(\ln\alpha + \ln x)}{\alpha^2(x-y)(x-z)} < \frac{1}{\min\{\alpha x, \alpha y, \alpha z\}}$$

Now we observe that

$$\sum_{\text{cyclic}} \frac{x}{(x-y)(x-z)} = 0$$

thus we remain with

$$\frac{1}{\alpha\max\{x,y,z\}} < \sum_{\text{cyclic}} \frac{2\ln x}{\alpha(x-y)(x-z)} < \frac{1}{\alpha\min\{x,y,z\}}$$

that is the homogeneity. By symmetry we can set $x \le y \le z$ and by homogeneity $x = 1$ thus we have

$$\frac{1}{z} < \frac{2y\ln y}{(y-1)(y-z)} + \frac{2z\ln z}{(z-1)(z-y)} < \frac{1}{x} = 1$$

The RHS becomes

$$\frac{2z\ln z}{z-1} - z \le \frac{2y\ln y}{y-1} - y \tag{2}$$

Let $f(z) = \dfrac{2z\ln z}{z-1} - z$, $z \ge 1$. Then,

$$f'(z) = -\frac{z^2 - 4z + 3 + 2\ln z}{(z-1)^2} \le 0 \iff z^2 - 4z + 3 + 2\ln z \ge 0$$

$$(z^2 - 4z + 3 + 2\ln z)|_{z=1} = 0, \quad (z^2 - 4z + 3 + 2\ln z)' = 2z - 4 + \frac{2}{z} > 0 \ \forall\ z \ge 1$$

It follows that $f(z)$ does not increase yielding (2) and concluding the proof of the RHS.

As for the LHS of (2) we need to prove that

$$\frac{z-y}{z} < \frac{2z\ln z}{z-1} - \frac{2y\ln y}{y-1} \tag{3}$$

Let $f(\xi) = \dfrac{2\xi \ln \xi}{\xi - 1}$. By the Lagrange theorem $f(z) - f(y) = f'(c)(z - y)$ where $y < c < z$ and therefore $f'(\xi) = 2\,\dfrac{\xi - 1 - \ln \xi}{(\xi - 1)^2}$. We conclude the computation writing a Lemma

Lema 2. *For all $\xi > 1$ holds* $2\,\dfrac{\xi - 1 - \ln \xi}{(\xi - 1)^2} > \dfrac{1}{\xi}$.

Proof. It is equivalent to show

$$F(\xi) = 2\,\frac{\xi^2 - \xi - \ln \xi}{(\xi - 1)^2} - 1 > 0,\ F(1) = \lim_{\xi \to 1} f(\xi) = 0$$

$$F'(\xi) = \frac{2(\ln(\xi)\xi + \ln(\xi) - 2\xi + 2)}{(x - 1)^3} \geq 0 \iff h(\xi) = (\ln(\xi)\xi + \ln(\xi) - 2\xi + 2) \geq 0$$

$$h(1) = 0, \quad h'(\xi) = \frac{2(-\xi + \ln(\xi)\xi + 1)}{\xi} \geq 0 \iff k(\xi) = -\xi + \ln(\xi)\xi + 1 \geq 0$$

$$k(1) = 0, \quad k'(\xi) = \ln(\xi) \geq 0, \quad \xi \geq 1$$

thus the assertion of the Lemma. q.e.d. □

Now the c in $f(z) - f(y) = f'(c)(z - y)$ satisfies $y < c < z$ and then by the Lemma $f'(c) \geq \frac{1}{c} > \frac{1}{z}$ proving (3) and concluding the proof.

Also solved by José Gibergans-Báguena, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain.