MATHPROBLEMS
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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before 08.04.2011

Problems

8. *Proposed by Valmir Krasniqi and Armend Sh. Shabani, Department of Mathematics, University of Prishtina, Republic of Kosova.* If f is a non-negative function on $[0,1]$ and $f'(x) \geq 1$. Prove that

$$\int_0^1 [f(x)]^3 \, dx \geq \left[\int_0^1 f(x) dx \right]^2$$

9. *Proposed by Roberto Tauraso, Department of Mathematics, Tor Vergata University, Rome, Italy.* Show that for any prime p and for any non-negative integer n,

$$p \,|\, L_{pn} - L_n,$$

where L_n is the n-th Lucas number defined by $L_0 = 2, L_1 = 1$ and for $n \geq 2, L_n = L_{n-1} + L_{n-2}$.

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10. *Proposed by Roberto Tauraso, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let $n = 2010^{100}$. Compute the cardinality of the set

$$S_n = \left\{ d \ : \ d \in [1,n] \cap \mathbb{N},\, d | n^2,\, d \nmid n \right\}$$

11. *Proposed by Roberto Tauraso, Department of Mathematics, Tor Vergata University, Rome, Italy.* Find a closed formula for

$$\sum_{\substack{A \subset \{1,\ldots,n\} \\ A \neq \emptyset}} \ \sum_{\substack{B \subset \{1,\ldots,n\} \\ B \neq \emptyset}} \ \sum_{x \in A \cup B} x$$

12. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let a, b, c be positive numbers. Prove that

$$\sum_{\text{cyc}} \sqrt{\frac{5a^2 + 5c^2 + 8b^2}{4ac}} \geq 3 \cdot \sqrt[9]{\frac{8(a+b)^2(b+c)^2(c+a)^2}{(abc)^2}}$$

13. *Proposed by Mihály Bencze, Braşov, Romania.* Let a_k, $1 \leq k \leq n$, be any positive numbers. Prove that

$$(n-1)\left(\sum_{k=1}^{n} a_k + \frac{1}{\prod\limits_{k=1}^{n} a_k}\right) \geq \frac{\left(n - 1 + 2 \sum\limits_{1 \leq i < j \leq n} a_i a_j\right)^2}{\sum\limits_{1 \leq i < j \leq n} a_i a_j (a_i + a_j) + (n-1) \sum\limits_{k=1}^{n} a_k^2}$$

14. *Proposed by Mihály Bencze, Braşov, Romania.* Solve the equation

$$64^x - 17 = 343^{x-1} + \frac{9}{7} \cdot 28^x$$

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

1. *Proposed by Valmir Krasniqi, Department of Mathematics, University of Prishtina, Republic of Kosova.* Let be $f:(0,\infty)\to\mathbb{R}$. Show that the function $g(x)=f\left(\frac{1}{x}\right)$ is convex in $(0,\infty)$, if and only if the function $h(x)=xf(x)$ is convex in $(0,\infty)$.

Solution by Ovidiu Furdui, Cluj, Romania. First we assume that g is convex on $(0,\infty)$ and we prove that h is convex. This implies that for all $x,y>0$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ holds

$$f\left(\frac{1}{\alpha x+\beta y}\right)\leq \alpha f\left(\frac{1}{x}\right)+\beta f\left(\frac{1}{y}\right). \tag{1}$$

We need to prove that for all $u,v>0$ and $\alpha',\beta'\geq 0$ with $\alpha'+\beta'=1$ one has that

$$(\alpha' u+\beta' v)\, f(\alpha' u+\beta' v)\leq \alpha' u f(u)+\beta' v f(v). \tag{2}$$

Setting $x=1/u$, $y=1/v$, $\alpha=\frac{\alpha' u}{\alpha' u+\beta' v}$, and $\beta=\frac{\beta' v}{\alpha' u+\beta' v}$ in (1) we get that (2) holds. To prove the other implication put $\alpha'=\frac{\alpha x}{\alpha x+\beta y}$, $\beta'=\frac{\beta y}{\alpha x+\beta y}$, $u=1/x$, and $v=1/y$ in (2) and inequality (1) follows.

Also solved by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Arnau Massegué Buisan, Spain, and the proposer.

2. *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain.*

Find all $n-$tuples $(x_1,x_2,\ldots,x_n)$ of real numbers such that

$$\left.\begin{array}{r} x_1^2+\sqrt{x_2^2+7}=\sqrt{x_2^2+160},\\ x_2^2+\sqrt{x_3^2+7}=\sqrt{x_3^2+160},\\ \cdots\cdots\\ x_{n-1}^2+\sqrt{x_n^2+7}=\sqrt{x_n^2+160},\\ x_n^2+\sqrt{x_1^2+7}=\sqrt{x_1^2+160}. \end{array}\right\}$$

Solution by Arnau Massegué Buisan, Spain. Putting $x_i^2=t_i$, $1\leq i\leq n$, we obtain

$$\left.\begin{array}{r} t_1+\sqrt{t_2+7}=\sqrt{t_2+160},\\ t_2+\sqrt{t_3+7}=\sqrt{t_3+160},\\ \cdots\cdots\\ t_{n-1}+\sqrt{t_n+7}=\sqrt{t_n+160},\\ t_n+\sqrt{t_1+7}=\sqrt{t_1+160}. \end{array}\right\}\Leftrightarrow \left.\begin{array}{r} t_1=\sqrt{t_2+160}-\sqrt{t_2+7},\\ t_2=\sqrt{t_3+160}-\sqrt{t_3+7},\\ \cdots\cdots\\ t_{n-1}=\sqrt{t_n+160}-\sqrt{t_n+7},\\ t_n=\sqrt{t_1+160}-\sqrt{t_1+7}. \end{array}\right\}$$

Now we consider the function $f : [0, +\infty) \to \mathbb{R}$ defined by

$$f(t) = \sqrt{t+160} - \sqrt{t+7} = \frac{153}{\sqrt{t+160} + \sqrt{t+7}}$$

Since for $0 \leq u < v$ is

$$f(u) = \frac{153}{\sqrt{u+160} + \sqrt{u+7}} > \frac{153}{\sqrt{v+160} + \sqrt{v+7}} = f(v),$$

then f is increasing and the same holds with $f(\cdots(f(f(t))))$, as it is well-known. On the other hand, from $f(t_2) = t_1, f(t_3) = t_2, \ldots, f(t_1) = t_n$ it follows that $f(\cdots(f(f(t_1)))) = t_1$. The preceding holds if and only if $f(t_1) = t_1$, as can be easily checked. So, we have to find the fixed points of f. That is, we have to solve the equation $\sqrt{t+160} - \sqrt{t+7} = t$ or equivalently, $153 - t^2 = 2t\sqrt{t+7}$. Since $153 - t^2 \geq 0$, then $t \in [0, 3\sqrt{17}]$. Squaring the preceding equation yields,

$$t^4 - 4t^3 - 334t^2 + 23409 = (t-9)(t^3 + 5t^2 - 289t - 2601) = 0$$

Let $g : [0, 3\sqrt{17}] \to \mathbb{R}$ be defined by $g(t) = t^3 + 5t^2 - 289t - 2601$. Using elementary calculus we have that $g(t) < 0$ for all $t \in [0, 3\sqrt{17}]$. Therefore, the only fixed point of f is $t = 9$, from which follows that $x_1^2 = x_2^2 = \ldots = x_n^2 = 9$ and the set of real $n-$tuples solution of the system is

$$\{(3,3,\cdots,3), (-3,3,\cdots,3), (3,-3,\cdots,3)\ldots(3,3,\cdots-3)\ldots(-3,-3,\cdots,-3)\}$$

Notice that it has 2^n elements, and we are done.

Also solved by Ovidiu Furdui, Cluj, Romania and the proposer.

3. *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain.*

Let $a_1, a_2, \ldots, a_n$ be n positive real numbers and let $k \leq m$ be positive integers. Prove that

$$\sum_{i=1}^{n} F_i^2 a_i^m \geq \frac{1}{F_n F_{n+1}} \left(\sum_{i=1}^{n} F_i^2 a_i^k \right) \left(\sum_{i=1}^{n} F_i^2 a_i^{m-k} \right),$$

where F_k is the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$, and for all $n \geq 2, F_n = F_{n-1} + F_{n-2}$.

Solution 1 by Arnau Massegué Buisan, Spain. Using the well-known identity $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ which can be easily proven by induction on n, we can rewrite the inequality stated as

$$\left(\sum_{i=1}^{n} F_i^2 \right) \left(\sum_{i=1}^{n} F_i^2 a_i^m \right) \geq \left(\sum_{i=1}^{n} F_i^2 a_i^k \right) \left(\sum_{i=1}^{n} F_i^2 a_i^{m-k} \right)$$

After expanding the products and canceling equal terms the inequality becomes equivalent to

$$\sum_{i<j}^{n} F_i^2 F_j^2 (a_i^m + a_j^m) \geq \sum_{i<j}^{n} F_i^2 F_j^2 (a_i^k a_j^{m-k} + a_i^{m-k} a_j^k)$$

So, since $F_i^2 F_j^2 \geq 0$ it is enough to show that $a_i^m + a_j^m \geq a_i^k a_j^{m-k} + a_i^{m-k} a_j^k$, but it is a straightforward consequence of rearrangement inequality.

Solution 2 by the proposer. To prove our claim, we need the following result

Lema 1. *Let* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *be sequences of positive numbers. Then, holds*

$$\left(\sum_{i=1}^{n} a_i^m b_i\right)\left(\sum_{i=1}^{n} b_i\right) - \left(\sum_{i=1}^{n} a_i^k b_i\right)\left(\sum_{i=1}^{n} a_i^{m-k} b_i\right) = \sum_{1\le i<j\le n} b_i b_j \left(a_i^k - a_j^k\right)\left(a_i^{m-k} - a_j^{m-k}\right) \ge 0,$$

where $k \le m$ *are positive integers.*

Proof. We have

$$\begin{aligned}\left(\sum_{i=1}^{n} a_i^m b_i\right)\left(\sum_{i=1}^{n} b_i\right) &= \left(a_1^m b_1 + a_2^m b_2 + \ldots + a_n^m b_n\right)\left(b_1 + b_2 + \ldots + b_n\right)\\ &= \left(a_1^m b_1^2 + a_1^m b_1 b_2 + \ldots + a_1^m b_1 b_n\right) + \left(a_2^m b_2 b_1 + a_2^m b_2^2 + \ldots + a_2^m b_2 b_n\right)\\ &\quad + \cdots + \left(a_n^m b_n b_1 + a_n^m b_n b_2 + \ldots + a_n^m b_n^2\right) \qquad (3)\end{aligned}$$

and

$$\begin{aligned}\left(\sum_{i=1}^{n} a_i^k b_i\right)\left(\sum_{i=1}^{n} a_i^{m-k} b_i\right) &= \left(a_1^k b_1 + a_2^k b_2 + \ldots + a_n^k b_n\right)\left(a_1^{m-k} b_1 + a_2^{m-k} b_2 + \ldots + a_n^{m-k} b_n\right)\\ &= \left(a_1^m b_1^2 + a_1^k a_2^{m-k} b_1 b_2 + \ldots + a_1^k a_n^{m-k} b_1 b_n\right) + \left(a_2^k a_1^{m-k} b_2 b_1 + a_2^m b_2^2 + \ldots + a_2^k a_n^{m-k} b_2 b_n\right)\\ &\quad + \cdots + \left(a_n^k a_1^{m-k} b_n b_1 + a_n^k a_2^{m-k} b_n b_2 + \ldots + a_n^m b_n^2\right) \qquad (4)\end{aligned}$$

Subtracting (3) from (4), we get

$$\begin{aligned}&\left(\sum_{i=1}^{n} a_i^m b_i\right)\left(\sum_{i=1}^{n} b_i\right) - \left(\sum_{i=1}^{n} a_i^k b_i\right)\left(\sum_{i=1}^{n} a_i^{m-k} b_i\right)\\ &= b_1 b_2\left(a_1^m - a_1^k a_2^{m-k} - a_2^k a_1^{m-k} + a_2^m\right) + \cdots + b_{n-1} b_n\left(a_{n-1}^m - a_{n-1}^k a_n^{m-k} - a_n^k a_{n-1}^{m-k} + a_n^m\right)\\ &= \sum_{1\le i<j\le n} b_i b_j\left(a_i^m - a_i^k a_j^{m-k} - a_j^k a_i^{m-k} + a_j^m\right)\\ &= \sum_{1\le i<j\le n} b_i b_j\left(a_i^k - a_j^k\right)\left(a_i^{m-k} - a_j^{m-k}\right) \ge 0\end{aligned}$$

and the proof is complete. □

Putting $b_i = F_i^2$, $1 \le i \le n$, in the previous lemma and taking into account that $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$ (as can be easily proven by induction), we get

$$F_n F_{n+1}\left(\sum_{i=1}^{n} F_i^2 a_i^m\right) - \left(\sum_{i=1}^{n} F_i^2 a_i^k\right)\left(\sum_{i=1}^{n} F_i^2 a_i^{m-k}\right) = \sum_{1\le i<j\le n} F_i^2 F_j^2\left(a_i^k - a_j^k\right)\left(a_i^{m-k} - a_j^{m-k}\right) \ge 0$$

Equality holds when $a_1 = a_2 = \ldots = a_n$ and this completes the proof.

4. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let x, y be positive real numbers. Prove that

$$\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}}\le\sqrt{xy}+\frac{x+y}{2}+\frac{(\frac{x+y}{2}-L(x,y))^2}{\frac{x+y}{2}}$$

where $L(x,y)=(x-y)/(\ln(x)-\ln(y))$ if $x\neq y$ and $L(x,x)=x$.

Solution by the proposer. Let $A=(x+y)/2$ and $G=\sqrt{xy}$. On account of the well-known inequality $L\le(2G+A)/3\le A$, we can insert a term and to prove

$$\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}}\le\sqrt{xy}+\frac{x+y}{2}+\frac{(\frac{x+y-2\sqrt{xy}}{3})^2}{(\frac{x+y}{2})}\le\sqrt{xy}+\frac{x+y}{2}+\frac{(\frac{x+y}{2}-L(x,y))^2}{\frac{x+y}{2}}$$

RHS inequality trivially holds. To prove LHS inequality, we observe that symmetry allows us to consider $x/y\ge 1$ and the homogeneity to write the inequality in terms of the variable $t=x/y$. That is,

$$\frac{4}{9}\frac{\left(\frac{t+1}{2}-\sqrt{t}\right)^2}{\left(\frac{t+1}{2}\right)}+\frac{1+t}{2}+\sqrt{t}\ge\frac{2t}{1+t}+\sqrt{\frac{1+t^2}{2}}$$

Clearing the denominators the preceding inequality is equivalent to

$$\frac{13}{18}(t+1)^2-\frac{10}{9}t\ge\frac{1}{2}(t+1)\sqrt{2+2t^2}-\frac{1}{9}(t+1)\sqrt{t}$$

Putting $t=z^2$ in the preceding, we get

$$\frac{13}{18}(z^2+1)^2-\frac{10}{9}z^2\ge\frac{1}{2}(z^2+1)\sqrt{2+2z^4}-\frac{1}{9}(z^2+1)z$$

That is,

$$\left(\frac{13}{18}(z^2+1)^2-\frac{10}{9}z^2+\frac{1}{9}(z^2+1)z\right)^2-\frac{1}{4}(z^2+1)^2(2+2z^4)\ge 0$$

or

$$P(z)=\frac{7}{324}+\frac{13}{81}z-\frac{41}{81}z^2+\frac{19}{81}z^3+\frac{29}{162}z^4+\frac{19}{81}z^5-\frac{41}{81}z^6+\frac{13}{81}z^7+\frac{7}{324}z^8\ge 0$$

We have $P^{(j)}(1)=0$ for any $0\le j\le 3$, where $P^{(j)}(1)$ is the j-th derivative of $P(z)$ at $z=1$. Moreover, $P^{(k)}(1)>0$ for any $4\le k\le 7$ and $P^{(8)}(t)>0$. It follows that $P(t)>0$ for any $t\neq 1$ and $P(1)=0$. More specifically, we have

$$P^{(4)}(1)=64/3,\ \ P^{(5)}(1)=640/3,\ \ P^{(6)}(1)=880,\ \ P^{(7)}(1)=1680,\ \ P^{(8)}(t)=7840/9$$

Finally, we will prove that $L\le(2G+A)/3\le A$. The inequality $(2G+A)/3\le A$ trivially holds on account of AM-GM inequality. Using the variable $t=x/y$ again, LHS inequality becomes

$$\frac{t-1}{\ln t}\le\frac{2}{3}\sqrt{t}+\frac{1+t}{6}$$

Now we consider the function f defined by

$$f(t)=\ln t-6\frac{t-1}{4\sqrt{t}+1+t}$$

Since $f(1)=0$ and $f'(t)=\frac{2(t-1)^4}{t(4t+1+t^2)^2}\ge 0$, then $f(t)\ge 0$. Equality holds when $x=y$, and the proof is complete.

5. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let $[a]$ be the integer part of a. Evaluate

$$\int_0^1 \int_0^{1-x} \frac{dxdy}{\left(\left[\frac{x}{y}\right]+1\right)^2}$$

Solution by Ovidiu Furdui, Cluj, Romania. More generally, we prove that if $k \geq 1$ is an integer, then

$$\int_0^1 \int_0^{1-x} \frac{dxdy}{\left(\left[\frac{x}{y}\right]+1\right)^k} = \frac{1}{2}\left((-1)^k - \sum_{j=2}^{k+1} (-1)^{k+j}\zeta(j)\right),$$

where ζ denotes the **Zeta** function. When $k = 2$, we have

$$\int_0^1 \int_0^{1-x} \frac{dxdy}{\left(\left[\frac{x}{y}\right]+1\right)^2} = \frac{1}{2}\left(\zeta(3) + 1 - \zeta(2)\right)$$

Using the substitution $y = xt$, we have

$$I = \int_0^1 \int_0^{1-x} \frac{dxdy}{\left(\left[\frac{x}{y}\right]+1\right)^k} = \int_0^1 x \left(\int_0^{(1-x)/x} \frac{dt}{\left(\left[\frac{1}{t}\right]+1\right)^k}\right) dx$$

Integrating by parts with

$$f(x) = \int_0^{(1-x)/x} \frac{dt}{\left(\left[\frac{1}{t}\right]+1\right)^k}, \quad f'(x) = -\frac{1}{x^2}\cdot\frac{1}{\left(\left[\frac{x}{1-x}\right]+1\right)^k}, \quad g'(x) = x, \quad g(x) = x^2/2,$$

we get

$$\begin{aligned} I &= \frac{x^2}{2}\int_0^{(1-x)/x} \frac{dt}{\left(\left[\frac{1}{t}\right]+1\right)^k}\Bigg|_{x=0}^{x=1} + \frac{1}{2}\int_0^1 \frac{dx}{\left(\left[\frac{x}{1-x}\right]+1\right)^k} \\ &= \frac{1}{2}\int_0^1 \frac{dx}{\left(\left[\frac{x}{1-x}\right]+1\right)^k} = \frac{1}{2}\int_0^1 \frac{dx}{\left(\left[\frac{1-x}{x}\right]+1\right)^k} \\ &= \frac{1}{2}\int_0^1 \frac{dx}{\left(\left[\frac{1}{x}\right]\right)^k} = \frac{1}{2}\int_1^\infty \frac{dt}{t^2[t]^k} = \frac{1}{2}\sum_{m=1}^\infty \int_m^{m+1} \frac{dt}{t^2 m^k} \\ &= \frac{1}{2}\sum_{m=1}^\infty \frac{1}{m^k}\left(\frac{1}{m} - \frac{1}{m+1}\right) = \frac{1}{2}\zeta(k+1) - \frac{1}{2}\sum_{m=1}^\infty \frac{1}{m^k(m+1)} \end{aligned}$$

Let $S_k = \sum_{m=1}^\infty \frac{1}{m^k(m+1)}$. Since $\frac{1}{m^k(m+1)} = \frac{1}{m^k} - \frac{1}{m^{k-1}(m+1)}$, then $S_k = \zeta(k) - S_{k-1}$. This implies that $(-1)^k S_k = (-1)^k\zeta(k) + (-1)^{k-1}S_{k-1}$, and by iteration, it follows that

$$S_k = (-1)^{k+1} + \sum_{j=2}^{k} (-1)^{k+j}\zeta(j).$$

Thus,

$$\begin{aligned} I &= \frac{1}{2}\left(\zeta(k+1) + (-1)^k - \sum_{j=2}^{k}(-1)^{k+j}\zeta(j)\right) \\ &= \frac{1}{2}\left((-1)^k - \sum_{j=2}^{k+1}(-1)^{k+j}\zeta(j)\right), \end{aligned}$$

and we are done.

Also solved by the proposer

6. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let $\{a_k\}_{k\geq 1}$ be a sequence of real positive numbers. Define $S_n = \sum_{k=1}^{n} a_k$. Prove that if $a_{k+1} \leq a_k e^{a_{k+1}}$ then

$$\lim_{n\to+\infty} n a_n e^{-S_n} = 0$$

Solution by the proposer. The condition $a_{k+1} \leq a_k e^{a_{k+1}}$ is equivalent to $a_{k+1}e^{-S_{k+1}} \leq a_k e^{-S_k}$ that is the monotonicity of the sequence $\{a_k e^{-S_k}\}_{k\geq 1}$. The series $\sum_{k=1}^{\infty} a_k e^{-S_k}$ is convergent. Indeed,

$$\sum_{k=1}^{\infty} a_k e^{-S_k} = \sum_{k=1}^{\infty}(S_k - S_{k-1})e^{-S_k} \leq \sum_{k=1}^{\infty}\int_{S_{k-1}}^{S_k} e^{-x}dx \leq \int_0^{\infty} e^{-x}dx < +\infty$$

Now we use the well known result according to which a convergent series $\sum_{k=1}^{\infty} a_k$ of general term not increasing and positive, implies

$$\lim_{k\to\infty} k a_k = 0$$

This result is a standard application of the Cauchy property of convergent sequences. Namely,

$$\sum_{k=1}^{\infty} b_k \text{ converges} \iff \forall\, \varepsilon\ \exists\, n_\varepsilon :\ n, m > n_\varepsilon \ \Rightarrow \left|\sum_{k=m}^{n} b_k\right| < \varepsilon$$

As a consequence, we have

$$(n-m+1)a_n < \sum_{k=m}^{n} a_k < \varepsilon$$

that is the conclusion. The monotonicity of $\{a_k e^{-S_k}\}_{k\geq 1}$ and the convergence of $\sum_{k=1}^{\infty} a_k e^{-S_k}$ completes the proof.

Also solved by Moubinool Omarjee, France

7. *Proposed by Ovdiu Furdui and Alina Sîntămărian, Cluj, Rumania.* Let $k \geq 1$ and $p \geq 2$ be positive integers and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim\limits_{n\to\infty} \frac{x_n}{\sqrt[p]{n}} = L > 0$. Find the value of,

$$\lim_{n\to\infty} \frac{x_n + x_{n+1} + \cdots + x_{kn}}{n x_n}$$

Solution by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. Since $\lim_{n\to\infty} \frac{x_n}{\sqrt[p]{n}} = L$, then

$$\forall\ \varepsilon > 0\ \exists\ n_\varepsilon :\ n > n_\varepsilon \implies (L-\varepsilon)\sqrt[p]{n} < x_n < (L+\varepsilon)\sqrt[p]{n}$$

Thus,

$$(L-\varepsilon)\sum_{j=n}^{kn}\sqrt[p]{j} < \sum_{j=n}^{kn} x_j < (L+\varepsilon)\sum_{j=n}^{kn}\sqrt[p]{j}$$

The monotonicity of $x^{1/p}$ for $x > 0$ yields

$$\int_{n-1}^{kn} x^{1/p}dx \le \sum_{j=n}^{kn}\sqrt[p]{j} \le \int_{n}^{kn+1} x^{1/p}dx,$$

and therefore

$$\frac{p}{p+1}\left((kn)^{\frac{p+1}{p}} - (n-1)^{\frac{p+1}{p}}\right) < \sum_{j=n}^{kn}\sqrt[p]{j} < \frac{p}{p+1}\left((kn+1)^{\frac{p+1}{p}} - n^{\frac{p+1}{p}}\right)$$

This implies that

$$\frac{\frac{(L-\epsilon)p}{p+1}\left((kn)^{\frac{p+1}{p}} - (n-1)^{\frac{p+1}{p}}\right)}{n\sqrt[p]{n}}\frac{\sqrt[p]{n}}{x_n} \le \frac{\sum\limits_{j=n}^{kn} x_j}{n\cdot x_n}$$

$$\le \frac{\frac{(L+\epsilon)p}{p+1}\left((kn+1)^{\frac{p+1}{p}} - n^{\frac{p+1}{p}}\right)}{n\sqrt[p]{n}}\frac{\sqrt[p]{n}}{x_n} \qquad (5)$$

Computing the limits of the first and third terms of the preceding expression, yields

$$\frac{\left((kn+1)^{\frac{p+1}{p}} - n^{\frac{p+1}{p}}\right)}{n\sqrt[p]{n}}\cdot\frac{\sqrt[p]{n}}{x_n} \to \frac{\left(k^{\frac{p+1}{p}}-1\right)}{L},$$

and

$$\frac{\left((kn)^{\frac{p+1}{p}} - (n-1)^{\frac{p+1}{p}}\right)}{n\sqrt[p]{n}}\cdot\frac{\sqrt[p]{n}}{x_n} \to \frac{\left(k^{\frac{p+1}{p}}-1\right)}{L}.$$

Letting $n \to \infty$ in (5) we get

$$\frac{(L-\epsilon)}{L}\cdot\frac{p}{p+1}\left(k^{\frac{p+1}{p}}-1\right) \le \lim_{n\to\infty}\frac{\sum\limits_{j=n}^{kn} x_j}{nx_n} \le \frac{(L+\epsilon)}{L}\cdot\frac{p}{p+1}\left(k^{\frac{p+1}{p}}-1\right),$$

and since $\epsilon > 0$ is arbitrary, then the result follows.

Also solved by Arnau Massegué Buisan, Spain; Moubinool Omarjee, France and the proposers.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

6. Let a, b, c be the lengths of the sides of a triangle ABC with circumradius r and area $\mathcal{A}$. Compute

$$\frac{\cos A-\cos B}{\mathcal{A}-rc}+\frac{\cos B-\cos C}{\mathcal{A}-ra}+\frac{\cos C-\cos A}{\mathcal{A}-rb}$$

7. Let $\ln a, \ln b$ and $\ln c$ be the lengths of the sides of a triangle ABC. Prove that

$$\frac{3}{5}\leq\frac{\ln a}{\ln(ab^2c^2)}+\frac{\ln b}{\ln(a^2bc^2)}+\frac{\ln c}{\ln(a^2b^2c)}<1$$

8. Suppose that the three roots of the equation $t^3-at^2+t-b=0$ are positive real numbers. Show that $9b^2(1+6ab)\leq 1$.

9. Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0=1, a_1=2, a_2=1$ and for all $n\geq 3$, $a_n^3=a_{n-1}a_{n-2}a_{n-3}$. Find $\lim\limits_{n\to\infty} a_n$.

10. Let x, y, z be three distinct positive real numbers. Prove that

$$\frac{1}{\max\{x,y,z\}}<\sum_{cyclic}\frac{\ln x^{2x}}{(x-y)(x-z)}<\frac{1}{\min\{x,y,z\}}$$

Solutions

1. Let n be an even positive integer. Find all triples (x, y, z) of real numbers such that

$$x^n y + y^n z + z^n x = xy^n + yz^n + zx^n$$

(BMO 2000)

Solution by Arnau Massegué Buisan, Spain. If $n = 0$ all x, y, z verifies the equality. If $n > 0$ and $x = y$, $y = z$ or $z = x$ the equality also holds. To see that these are the only solutions consider $n > 0$ and take y, z fixed, with $y \neq z$. Let $f(x) = x^n y + y^n z + z^n x - xy^n - yz^n - zx^n$. Clearly $f'(x) = nx^{n-1}(y-z) + (z^n - y^n)$ and $f''(x) = n(n-1)x^{n-2}(y-z)$. Since clearly f'' has only one zero and f'' has constant sign, then f' is monotone so it has at most one zero, which implies that f has at most two different zeroes. But $y = x$ and $y = z$ are two different zeroes of f, then f does not have any other zero. In conclusion, there does not exist any solution of the form $x \neq y$, $y \neq z$ and $z \neq x$, for $n > 0$ and n even.

Also solved by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain.

2. Let $A_1, A_2, \ldots, A_n$ be the vertices of a cyclic $n-$gon $\mathcal{P}$. Suppose that the lengths of the sides of $\mathcal{P}$ satisfy the inequalities $A_nA_1 > A_1A_2 > A_2A_3 > \ldots > A_{n-1}A_n$. Prove that $\widehat{A}_1 < \widehat{A}_2 < \widehat{A}_3 < \cdots < \widehat{A}_{n-1}$ and $\widehat{A}_{n-1} > \widehat{A}_n > \widehat{A}_1$, where $\widehat{A}_i$, $1 \leq i \leq n$, are the interior angles of $\mathcal{P}$.

(VI Spanish Math Olympiad 1968)

Solution by by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$, be the central angles corresponding to the sides $A_nA_1, A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$. We have

$$\alpha_1 + \alpha_2 > \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 > \alpha_3 + \alpha_4, \ldots, \alpha_{n-2} + \alpha_{n-1} > \alpha_{n-1} + \alpha_n$$

On the other hand,

$$\widehat{A}_1 = 180^\circ - \frac{\alpha_1 + \alpha_2}{2}, \widehat{A}_2 = 180^\circ - \frac{\alpha_2 + \alpha_3}{2}, \ldots, \widehat{A}_n = 180^\circ - \frac{\alpha_{n-1} + \alpha_n}{2}$$

from which follows

$$180^\circ - \frac{\alpha_1 + \alpha_2}{2} < 180^\circ - \frac{\alpha_2 + \alpha_3}{2} < \ldots < 180^\circ - \frac{\alpha_{n-1} + \alpha_n}{2},$$

or equivalently, $\widehat{A}_1 < \widehat{A}_2 < \widehat{A}_3 < \cdots < \widehat{A}_{n-1}$. Since $\alpha_{n-1} > \alpha_1$, then

$$A_{n-1} = 180^\circ - \frac{\alpha_{n-1} + \alpha_n}{2} > 180^\circ - \frac{\alpha_n + \alpha_1}{2} = A_n$$

Likewise, from $\alpha_n < \alpha_2$, we get

$$A_n = 180^\circ - \frac{\alpha_n + \alpha_1}{2} > 180^\circ - \frac{\alpha_1 + \alpha_2}{2} = A_1,$$

and we are done.

3. Let m_a, m_b, m_c and R be the medians and the circum-radii of a triangle ABC, respectively. Prove that

$$\frac{m_a^2 + m_b^2 + m_c^2}{R^2(\sin^2 A + \sin^2 B + \sin^2 C)}$$

is a positive integer and determine its value.

(Catalonian Math Olympiad 2008)

Solution by Ercole Suppa, Teramo, Italy

By using the Apollonius's formula

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \text{(cyclic)},$$

we have

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right) \tag{6}$$

On the other hand, on account of Sine's Law, yields $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$. Therefore,

$$R^2\left(\sin^2 A + \sin^2 B + \sin^2 C\right) = \frac{1}{4}\left(a^2 + b^2 + c^2\right) \tag{7}$$

From (6) and (7) it follows

$$\frac{m_a^2 + m_b^2 + m_c^2}{R^2\left(\sin^2 A + \sin^2 B + \sin^2 C\right)} = 3,$$

and the proof is complete.

Also solved by Arnau Massegué Buisan, Spain; Ricardo Barroso Campos, Spain, and José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain.

4. Given 5 points of a sphere of radius r, show that two of the points are a distance less than or equal to $r\sqrt{2}$ apart.

(II Barzilian Math Olympiad 1980)

Solution by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain. We argue by contradiction. So, assume that we can find 5 points with the distance between any two of them greater than $r\sqrt{2}$. Then the angle subtended by any two at the center of the sphere is greater than 90°. Take one of the points to be at the north pole. Then the other four must all be south of the equator. Two must have longitude differing by at most 90°. Now we claim that these two points subtend an angle at most 90° at the center. Indeed, we may take rectangular coordinates with origin at the center of the sphere so that both points have all their coordinates non-negative. Suppose one point is (x, y, z) and the other (u, v, w). Since both lie on the sphere, then

$$x^2 + y^2 + z^2 = u^2 + v^2 + w^2 = r^2,$$

and the square of the distance between them is

$$(x-u)^2 + (y-v)^2 + (z-w)^2 \le (x^2 + y^2 + z^2) + (u^2 + v^2 + w^2) = 2r^2,$$

so the distance between them is at most $r\sqrt{2}$, as required.

5. Let n be a positive integer. Prove that

$$F_n^4 F_{n+1}^4 \le \left(\sum_{k=1}^{n} F_k F_{2k}\right)\left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt[3]{L_k}}\right)^3$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers respectively.

(XVI József Wildt International Math Competition 2006)

Solution by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain. We will use Jensen's inequality. Namely,

$$f\left(\sum_{k=1}^{n} q_k x_k\right) \le \sum_{k=1}^{n} q_k f(x_k)$$

valid for all set of nonnegative numbers $q_1, q_2, \ldots, q_n$ of sum one and $x_1, x_2, \ldots, x_n \in I$ the domain where f is convex. (When f is concave the inequality reverses).

Setting $f(x) = \dfrac{1}{\sqrt[3]{x}}$, that is convex in $(0, +\infty)$, $q_k = \dfrac{F_k^2}{F_n F_{n+1}}$, $1 \le k \le n$, and $x_k = L_k$, $1 \le k \le n$, in Jensen's inequality, yields

$$\begin{aligned} f\left(\sum_{k=1}^{n} q_k x_k\right) = \left(\sum_{k=1}^{n} \frac{F_k^2 L_k}{F_n F_{n+1}}\right)^{-1/3} &= (F_n F_{n+1})^{1/3}\left(\sum_{k=1}^{n} F_k^2 L_k\right)^{-1/3} \\ \le \sum_{k=1}^{n} \frac{F_k^2}{F_n F_{n+1}}\left(\frac{1}{L_k}\right)^{1/3} &= \sum_{k=1}^{n} q_k f(x_k) \end{aligned}$$

From the preceding expression immediately follows

$$(F_n F_{n+1})^{1/3}\left(\sum_{k=1}^{n} F_k^2 L_k\right)^{-1/3} \le \frac{1}{F_n F_{n+1}} \sum_{k=1}^{n} \frac{F_k^2}{\sqrt[3]{L_k}}$$

Taking into account the well known identity $F_k L_k = F_{2k}$ and rearranging terms, we have

$$(F_n F_{n+1})^{4/3} \le \left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt[3]{L_k}}\right)\left(\sum_{k=1}^{n} F_k F_{2k}\right)^{1/3}$$

from which the statement immediately follows. Notice that equality holds when $n = 1$ and we are done.