LINEAR INDEPENDENCE OF THE RENORMALIZED HIGHER POWERS OF WHITE NOISE

L. ACCARDI

Centro Vito Volterra, Università di Roma Tor Vergata
via Columbia 2, 00133 Roma, Italy, E-mail: accardi@volterra.mat.uniroma2.it
http://volterra.mat.uniroma2.it

A. BOUKAS

Department of Mathematics and Natural Sciences, American College of Greece
Aghia Paraskevi, Athens 15342, Greece, E-mail: andreasboukas@acgmail.gr

The connection between the Lie algebra of the Renormalized Higher Powers of White Noise (RHPWN) and the centerless Virasoro (or Witt)-Zamolodchikov-w_∞ Lie algebras of conformal field theory, as well as the associated Fock space construction, have recently been established in Ref.[1–6] In this note we prove the linear independence of the RHPWN Lie algebra generators.

1. Introduction: Renormalized Higher Powers of White Noise

The quantum white noise functionals $a_t^\dagger$ and a_t satisfy the Boson commutation relations

$$[a_t, a_s^\dagger] = \delta(t-s), \; ; \; [a_t^\dagger, a_s^\dagger] = [a_t, a_s] = 0 \tag{1}$$

where $t, s \in \mathbb{R}$ and δ is the Dirac delta function, as well as the duality relation

$$(a_s)^* = a_s^\dagger \tag{2}$$

Here (and in what follows) $[x, y] := xy - yx$ is the usual operator commutator. For all $t, s \in \mathbb{R}$ and integers $n, k, N, K \geq 0$ we have (Ref.[6])

$$[{a_t^\dagger}^n a_t^k, {a_s^\dagger}^N a_s^K] = \tag{3}$$

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$$\epsilon_{k,0}\,\epsilon_{N,0}\sum_{L\geq 1}\binom{k}{L}N^{(L)}\,{a_t^\dagger}^n\,{a_s^\dagger}^{N-L}\,a_t^{k-L}\,a_s^K\,\delta^L(t-s)$$

$$-\epsilon_{K,0}\,\epsilon_{n,0}\sum_{L\geq 1}\binom{K}{L}n^{(L)}\,{a_s^\dagger}^N\,{a_t^\dagger}^{n-L}\,a_s^{K-L}\,a_t^k\,\delta^L(t-s)$$

where for $n,k\in\{0,1,2,...\}$ we have used the notation $\epsilon_{n,k}:=1-\delta_{n,k}$, where $\delta_{n,k}$ is Kronecker's delta and $x^{(y)}=x(x-1)\cdots(x-y+1)$ with $x^{(0)}=1$. In order to consider the smeared fields defined by the higher powers of a_t and $a_t^\dagger$, for a test function f and $n,k\in\{0,1,2,...\}$ we define the sesquilinear form

$$B_k^n(f):=\int_{\mathbb{R}}f(t)\,{a_t^\dagger}^n\,a_t^k\,dt \tag{4}$$

with involution

$$(B_k^n(f))^*=B_n^k(\bar{f}) \tag{5}$$

In[1] and[2] we introduced the convolution type renormalization of the higher powers of the Dirac delta function

$$\delta^l(t-s)=\delta(s)\,\delta(t-s)\quad;\quad l=2,3,.... \tag{6}$$

By multiplying both sides of (3) by test functions $f(t)\,g(s)$ such that $f(0)=g(0)=0$ and then formally integrating the resulting identity (i.e. taking $\int\int\ldots\,dsdt$ of both sides), using (6), we obtained the RHPWN Lie algebra commutation relations

$$[B_k^n(f),B_K^N(g)]_{RHPWN}:=(k\,N-K\,n)\;B_{k+K-1}^{n+N-1}(f\,g) \tag{7}$$

As shown in[1] and,[2] for $n,k\in\mathbb{Z}$ with $n\geq 2$, the white noise operators

$$\hat{B}_k^n(f):=\int_{\mathbb{R}}f(t)\,e^{\frac{k}{2}(a_t-a_t^\dagger)}\left(\frac{a_t+a_t^\dagger}{2}\right)^{n-1}e^{\frac{k}{2}(a_t-a_t^\dagger)}\,dt \tag{8}$$

with involution

$$\left(\hat{B}^n_k(f)\right)^* = \hat{B}^n_{-k}(\bar{f}) \tag{9}$$

satisfy the commutation relations of the second quantized Virasoro-Zamolodchikov-w_∞ Lie algebra (Ref.[7]), namely

$$[\hat{B}^n_k(f), \hat{B}^N_K(g)]_{w_\infty} = (k\,(N-1) - K\,(n-1))\;\hat{B}^{n+N-2}_{k+K}(f\,g) \tag{10}$$

In particular,

$$\hat{B}^2_k(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)} \left(\frac{a_t + a_t^\dagger}{2}\right) e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt \tag{11}$$

is the white noise form of the centerless Virasoro algebra generators.

We may analytically continue the parameter k in the definition of $\hat{B}^n_k(f)$ to an arbitrary complex number $k \in \mathbb{C}$ and to $n \geq 1$ and we can show (Ref.[3]) that the RHPWN and w_∞ Lie algebras are connected through

$$\hat{B}^n_k(f) = \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^p\, \frac{k^{p+q}}{p!\,q!}\, B^{m+p}_{n-1-m+q}(f) \tag{12}$$

and

$$B^n_k(f) = \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} \frac{(-1)^\rho}{2^{\rho+\sigma}}\, \frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0}\, \hat{B}^{k+n+1-(\rho+\sigma)}_z(f) \tag{13}$$

For $n \geq 1$ we define the n-th order RHPWN $*$–Lie algebras $\mathcal{L}_n$ as follows: (i) $\mathcal{L}_1$ is the $*$–Lie algebra generated by B^1_0 and B^0_1 i.e., $\mathcal{L}_1$ is the linear span of $\{B^1_0, B^0_1, B^0_0\}$ (ii) $\mathcal{L}_2$ is the $*$–Lie algebra generated by B^2_0 and B^0_2 i.e., $\mathcal{L}_2$ is the linear span of $\{B^2_0, B^0_2, B^1_1\}$ (iii) For $n \in \{3, 4, ...\}$, $\mathcal{L}_n$ is the $*$–Lie algebra generated by B^n_0 and B^0_n through repeated commutations and linear combinations. It consists of linear combinations of creation/annihilation operators of the form B^x_y where $x - y = k\,n$, $k \in \mathbb{Z} - \{0\}$, and of number operators B^x_x with $x \geq n-1$. Through white noise and norm compatibility considerations, the action of the RHPWN operators on Φ was defined in[4] as

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$$B^n_k(f)\,\Phi := \begin{cases} 0 & \text{if } n<k \text{ or } n\cdot k<0 \\ B^{n-k}_0(f)\,\Phi & \text{if } n>k\geq 0 \\ \frac{1}{n+1}\int_{\mathbb{R}} f(t)\,dt\,\Phi & \text{if } n=k \end{cases} \tag{14}$$

In what follows, for all integers n,k we will use the notation $B^n_k := B^n_k(\chi_I)$ where I is some fixed subset of $\mathbb{R}$ of finite measure $\mu := \mu(I) > 0$. Moreover, for all $t \in [0,+\infty)$ and for all integers n,k we will use the notation $B^n_k(t) := B^n_k(\chi_{[0,t]})$.

To avoid ghosts (i.e., vectors of negative norm) appearing in the cases $n \geq 3$ in the Fock kernels $\langle (B^n_0)^k\,\Phi, (B^n_0)^k\,\Phi\rangle$ where $k \geq 0$, in[4] we defined

$$B^{n-1}_{n-1}\,(B^n_0)^k\,\Phi := \left(\frac{\mu}{n} + k\,n\,(n-1)\right)\,(B^n_0)^k\,\Phi \tag{15}$$

and were able to show that for all $k,n \geq 1$

$$\langle (B^n_0)^k\,\Phi, (B^n_0)^m\,\Phi\rangle = \delta_{m,k}\,k!\,n^k \prod_{i=0}^{k-1}\left(\mu + \frac{n^2\,(n-1)}{2}\,i\right) \tag{16}$$

Therefore, the $\mathcal{F}_n$ inner product $\langle \psi_n(f), \psi_n(g)\rangle_n$ of the exponential vectors

$$\psi_n(\phi) := \prod_i e^{a_i\,B^n_0(\chi_{I_i})}\,\Phi \tag{17}$$

where $\phi := \sum_i a_i\,\chi_{I_i}$ is a test function, for $n=1$ is

$$\langle \psi_1(f), \psi_1(g)\rangle_1 := e^{\int_{\mathbb{R}} \bar{f}(t)\,g(t)\,dt} \tag{18}$$

while for $n \geq 2$ it is

$$\langle \psi_n(f), \psi_n(g)\rangle_n := e^{-\frac{2}{n^2\,(n-1)}\int_{\mathbb{R}} \ln\left(1-\frac{n^3\,(n-1)}{2}\,\bar{f}(t)\,g(t)\right)\,dt} \tag{19}$$

where $|f(t)| < \frac{1}{n}\sqrt{\frac{2}{n\,(n-1)}}$ and $|g(t)| < \frac{1}{n}\sqrt{\frac{2}{n\,(n-1)}}$.

The n-th order truncated RHPWN (or TRHPWN) Fock space $\mathcal{F}_n$ is the Hilbert space completion of the linear span of the exponential vectors $\psi_n(f)$

under the inner product $\langle\cdot,\cdot\rangle_n$. The full TRHPWN Fock space $\mathcal{F}$ is the direct sum of the $\mathcal{F}_n$'s.

The Fock representation of the TRHPWN generators B_0^n and B_n^0 obtained in[4] is

$$B_n^0(f)\,\psi_n(g) = n\int_{\mathbb{R}} f(t)\,g(t)\,dt\;\;\psi_n(g) + \frac{n^3\,(n-1)}{2}\,\frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi_n(g+\epsilon\,f\,g^2) \tag{20}$$

$$B_0^n(f)\,\psi_n(g) = \frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi_n(g+\epsilon\,f) \tag{21}$$

where $f := \sum_i a_i\,\chi_{I_i}$ and $g := \sum_i b_i\,\chi_{I_i}$ with $I_i \cap I_j = \oslash$ for $i \neq j$ and $f(0) = g(0) = 0$.

As shown in,[4] for all $s \in [0,\infty)$

$$\langle e^{s\,(B_0^1(t)+B_1^0(t))}\,\Phi,\Phi\rangle_1 = e^{\frac{s^2}{2}\,t} \tag{22}$$

i.e., $\{x_1(t) := B_0^1(t) + B_1^0(t)\}_{t\geq 0}$ is Brownian motion, while for $n \geq 2$

$$\langle e^{s\,(B_0^n(t)+B_n^0(t))}\,\Phi,\Phi\rangle_n = \left(\sec\left(\sqrt{\frac{n^3\,(n-1)}{2}}\,s\right)\right)^{\frac{2\,n\,t}{n^3\,(n-1)}} \tag{23}$$

i.e., for each $n \geq 2$, $\{x_n(t) := B_0^n(t) + B_n^0(t)\}_{t\geq 0}$ is a continuous binomial/Beta process.

2. Linear independence of the RHPWN generators

Lemma 2.1.

For all integers $m \geq 0$

$$\sum_{n=0}^{m} c_n\,B_0^n(f_n) = 0 \implies c_n = 0 \;\; \forall n \in \{0,1,...,m\} \tag{24}$$

where we assume that the test functions f_n are such that for all $n \in \{0,1,...,m\}$

$$\int_{\mathbb{R}} f_n(t)\,a_t^{\dagger^n}\,dt \not\equiv 0 \tag{25}$$

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Proof.

For $m = 0$,

$$c_0 \, B_0^0(f_0) = 0 \Longrightarrow c_0 \int_{\mathbb{R}} f_0(t) \, dt = 0 \Longrightarrow c_0 = 0 \tag{26}$$

and so (24) holds. Suppose that it holds for $m = M$. We will show that it is true for $m = M + 1$ also. So suppose that

$$\sum_{n=0}^{M+1} c_n \, B_0^n(f_n) = 0 \tag{27}$$

Then

$$\sum_{n=0}^{M+1} c_n \, [B_1^0(g), B_0^n(f_n)] = 0 \tag{28}$$

where g is any test function such that

$$\int_{\mathbb{R}} g(t) \, f_n(t) \, a_t^{\dagger^n} \, dt \not\equiv 0 \tag{29}$$

for all n, i.e.,

$$\sum_{n=0}^{M+1} n \, c_n \, B_0^{n-1}(g \, f_n) = 0 \tag{30}$$

which is equivalent to

$$\sum_{n=1}^{M+1} n \, c_n \, B_0^{n-1}(g \, f_n) = 0 \tag{31}$$

or, letting $N := n - 1$, to

$$\sum_{N=0}^{M} (N + 1) \, c_{N+1} \, B_0^N(g \, f_{N+1}) = 0 \tag{32}$$

which, by the induction hypothesis, implies that

$$(N + 1) \, c_{N+1} = 0 \Longrightarrow c_{N+1} = 0 \Longrightarrow c_n = 0 \tag{33}$$

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for all $n \in \{1, 2, ..., M+1\}$. But then (27) reduces to $c_0\, B_0^0(f_0) = 0$ which, as we have already seen, implies that $c_0 = 0$ as well. □

Lemma 2.2. *For all integers* $m \geq 0$

$$\sum_{k=0}^{m} c_k\, B_k^0(f_k) = 0 \implies c_k = 0 \;\; \forall k \in \{0, 1, ..., m\} \tag{34}$$

where we assume that the arbitrary test functions f_k *are such that for all* $k \in \{0, 1, ..., m\}$

$$\int_{\mathbb{R}} f_k(t)\, a_t^k\, dt \not\equiv 0 \tag{35}$$

Proof. Taking the adjoint of equation (34) we obtain

$$\sum_{k=0}^{m} \bar{c}_k\, B_0^k(\bar{f}_k) = 0 \tag{36}$$

which by Lemma 2.1 implies that $\bar{c}_k = 0$, and so $c_k = 0$, for all $k \in \{0, 1, ..., m\}$. □

Theorem 2.1. *The generators* $B_k^n(f)$ *of the RHPWN Lie algebra are linearly independent, i.e., for all integers* $m \geq 0$

$$\sum_{n=0}^{m} \sum_{k=0}^{m} c_{n,k}\, B_k^n(f_{n,k}) = 0 \implies c_{n,k} = 0 \;\; \forall n, k \in \{0, 1, ..., m\} \tag{37}$$

where we assume that the arbitrary test functions $f_{n,k}$ *are such that*

$$\int_{\mathbb{R}} f_{n,k}(t)\, a_t^{\dagger^n}\, a_t^k\, dt \not\equiv 0 \tag{38}$$

Note: By filling in with zero coefficients if necessary, every finite linear combination of the RHPWN generators can be put in the form

$$\sum_{n=0}^{m} \sum_{k=0}^{m} c_{n,k}\, B_k^n(f_{n,k}) \tag{39}$$

Proof. We will proceed by induction on m. For $m = 0$, equation (37) becomes

$$c_{0,0}\, B_0^0(f_{0,0}) = 0 \Longrightarrow c_{0,0} = 0 \tag{40}$$

which is true by (38). Suppose that equation (37) holds for $m = M$. We will show that it is true for $m = M + 1$ also. So suppose that

$$\sum_{n=0}^{M+1} \sum_{k=0}^{M+1} c_{n,k}\, B_k^n(f_{n,k}) = 0 \tag{41}$$

Taking the commutator of (41) first with $B_0^1(g)$ and then with $B_1^0(g)$, where g is any test function such that

$$\int_{\mathbb{R}} g(t)\, f_{n,k}(t)\, a_t^{\dagger\, n}\, a_t^k\, dt \not\equiv 0 \tag{42}$$

for all n, k, we obtain

$$\sum_{n=0}^{M+1} \sum_{k=0}^{M+1} k\, n\, c_{n,k}\, B_{k-1}^{n-1}(g^2\, f_{n,k}) = 0 \tag{43}$$

which is equivalent to

$$\sum_{n=1}^{M+1} \sum_{k=1}^{M+1} k\, n\, c_{n,k}\, B_{k-1}^{n-1}(g^2\, f_{n,k}) = 0 \tag{44}$$

which, letting $N := n - 1$ and $K := k - 1$, is equivalent to

$$\sum_{N=0}^{M} \sum_{K=0}^{m} (K + 1)\, (N + 1)\, c_{N+1,K+1}\, B_K^N(g^2\, f_{N+1,K+1}) = 0 \tag{45}$$

which, by the induction hypothesis, implies that

$$(K + 1)\, (N + 1)\, c_{N+1,K+1} = 0 \Longrightarrow c_{N+1,K+1} = 0 \Longrightarrow c_{n,k} = 0 \tag{46}$$

for all $n, k \in \{1, 2, ..., M + 1\}$. If $n = 0$ and/or $k = 0$ then equation (41) reduces to

$$c_{0,0}\, B_0^0(f_{0,0}) + \sum_{n=1}^{m+1} c_{n,0}\, B_0^n(f_{n,0}) + \sum_{k=1}^{m+1} c_{0,k}\, B_k^0(f_{0,k}) = 0 \tag{47}$$

Taking the commutator of (47) with $B_0^1(g)$, where g is as above, we obtain

$$\sum_{k=1}^{m+1} k\, c_{0,k}\, B_{k-1}^0(g\, f_{0,k}) = 0 \tag{48}$$

which by Lemma 2.2 implies that $k\, c_{0,k} = 0$ for all $k \in \{1, 2, ..., M+1\}$ and so $c_{0,k} = 0$ for all $k \in \{1, 2, ..., M+1\}$. Similarly, taking the commutator of (47) with $B_1^0(g)$ we obtain

$$\sum_{n=1}^{m+1} n\, c_{n,0}\, B_0^{n-1}(g\, f_{n,0}) = 0 \tag{49}$$

which by Lemma 2.1 implies that $n\, c_{n,0} = 0$ for all $n \in \{1, 2, ..., M+1\}$ and so $c_{n,0} = 0$ for all $n \in \{1, 2, ..., M+1\}$. So, (41) reduces to

$$c_{0,0}\, B_0^0(f_{0,0}) = 0 \tag{50}$$

which by (38) implies that $c_{0,0} = 0$. Therefore $c_{n,k} = 0$ for all $n, k \in \{0, 1, 2, ..., M+1\}$. □

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