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WHITE NOISE HEISENBERG EVOLUTION AND EVANS–HUDSON FLOWS

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We study white noise Heisenberg equations giving rise to flows which are $*$-automorphisms of the observable algebra, but not necessarily inner automorphisms. We prove that the causally normally ordered form of these white noise Heisenberg equations are equivalent to Evans–Hudson flows. This gives in particular, the microscopic structure of the maps defining these flows, in terms of the original white noise derivations.

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1. Introduction

1.1. *Statement of the problem*

The possibility to write stochastic differential equations as white noise Hamiltonian equations is one of the main advantages of white noise calculus with respect to stochastic calculus.

This approach evidences the role of regularization (through the choice of the constant in the causal commutation relations, cf. Sec. 2.2 below) which, in the classical case, is essentially equivalent to the choice of a notion of stochastic integral (Itô, Stratonovich, ...).

Moreover, the formal unitarity condition, for white noise equations, simply amounts to the formal self-adjointness of the white noise Hamiltonian, in agreement with the physical intuition. In the bounded case (the only one considered here), formal unitarity implies unitarity.

Usual Hamiltonian equations generate one-parameter unitary groups (unitary cocycles, in the interaction representation) and these define inner Heisenberg evolutions, i.e. inner one-parameter automorphism groups of the algebra of observables.

However, there are many important physical systems whose Heisenberg dynamics is not inner (e.g. infinite lattice systems), i.e. the generator of the dynamics is a $*$-derivation which is not expressible as the commutator with an element of the algebra of observables (an outer derivation).

Since white noise Hamiltonian equations generate inner Heisenberg evolutions, it is natural to ask oneself if one can introduce some non-inner white noise Heisenberg equations so to obtain non-inner dynamics.

In this paper we prove that the answer to the above question is affirmative and that the causally normally ordered form of a white noise Heisenberg equation (cf. Definition 2.2 below) is an Evans–Hudson flow.[16,18] As a corollary we obtain the microscopic structure of the Evans–Hudson structure maps in terms of the original white noise derivation just as Accardi, Lu and Volovich obtained the microscopic structure of the coefficients of an Hudson–Parthasarathy equation in terms of the original white noise Hamiltonian.

In both cases the coefficients of the stochastic equations are nontrivial (in particular nonlinear) functions of the coefficients of the white noise equations and, while the latter coefficients have a direct physical interpretation, the former in general have not.

The plan of this paper is the following. In Sec. 1.2 we introduce some basic notations. In Sec. 2.1 we prove a weak form of the time consecutive principle which is sufficient for the purposes of this paper. This result is used, in Sec. 2.2 to extend to maps the original result obtained by Accardi, Lu and Volovich[5] on the causal normal order of white noise equations and extended by J. Gough to more general (not necessarily causal) Stratonovich type white noise integrals.[12,17]

In Sec. 2.3, in order to clarify the difference between Schrödinger or Heisenberg evolutions and their white noise analogue, we recall some terminology concerning various types of Heisenberg evolutions: inner, outer, backward, forward, ...

In Sec. 2.4 we rewrite an inner white noise Heisenberg equation in such a way to suggest a natural outer generalization (cf. Definition 3.1). The problem here is to separate the outer part, which is entirely due to system operators, from the white noise part, which is inner.

Once this is done, using the time consecutive principle, we can put in causal normal order the white noise Heisenberg equation (cf. Sec. 3).

At this point we can use the equivalence between stochastic equations and normally ordered white noise equations (equivalence principle). However, what we get with this equivalence is not yet an Evans–Hudson flow but its backward version. In Sec. 4 we deduce the structure equations for the homomorphic backward white noise Heisenberg equations.

On the other hand, using the equivalence principle, one can immediately write the normally ordered white noise equation associated to an Evans–Hudson flow (cf. Sec. 5). We prove that the homomorphism conditions for these flows coincide with the Evans–Hudson condition and with the homomorphism conditions for the backward Heisenberg evolutions.

We conjecture that there is a full identification of the Evans–Hudson flows with the causally normally ordered form of a white noise forward Heisenberg evolution. This problem will be discussed elsewhere. The difficulty lies in the fact that, the forward equations are more delicate than the backward: their causually normally ordered form cannot be deduced from the time consecutive principle without a series expansion (which is not discussed here).

Sections 6 and 7 are included to help the reader in bridging the outer and the, now well known, inner case. We have included a proof of the unitarity condition: here the difference between forward and backward evolution is reflected by the difference of the approaches needed to prove the isometry and the co-isometry conditions.

1.2. *Notations*

This paper is the second one of a series where we continue the program of giving a rigorous mathematical basis to the white noise approach to stochastic calculus. For this reason here we will keep notations to a minimum and we refer to Accardi, Ayed and Ouerdiane,[6] for a detailed description of the quantities involved.

All operators here act on the Hilbert space $\mathcal{H} := \mathcal{H}_S \otimes \Gamma$ where Γ is the Boson Fock space over $L^2(\mathbb{R}; \mathcal{K})$ and $\mathcal{K}$, $\mathcal{H}_S$ are Hilbert spaces (always complex separable unless explicitly stated) called respectively initial (or system) space and multiplicity (or polarization) space. The initial algebra $\mathcal{B}_S$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H}_S)$ (the algebra of all bounded operators on $\mathcal{H}_S$). The observable algebra $\mathcal{B}_S$ is $\mathcal{A} := \mathcal{B}_S \otimes \mathcal{B}_R$ where $\mathcal{B}_R := \mathcal{B}(\Gamma)$ is called the noise algebra and the tensor product is closed under the natural topology on $\mathcal{H}_S \otimes \Gamma$. We enlarge the noise algebra so as to include the polynomials, in the sense of distributions, in the field operators[6] for a detailed description of this algebra and of the domain where it acts.

If $b_S \in \mathcal{B}_S$ and $b_R \in \mathcal{B}_R$, we will often write $b_S b_R$ instead of $b_S \otimes b_R$.

A bounded linear map $Z\mathcal{A} := \mathcal{B}_S \otimes \mathcal{B}_R \to \mathcal{A}$ on the observable algebra is called of system type if it commutes with the noise, i.e. if it satisfies:

$$Z(b_R x b'_R) = b_R Z(x) b'_R \,; \qquad x \in \mathcal{A} \,; \qquad b'_R, b_R \in \mathcal{B}_R \,.$$

This is equivalent to saying that Z has the form $Z = \hat{Z} \otimes \mathrm{id}_{\mathcal{B}_R}$ for some linear map $Z : \mathcal{B}_S \to \otimes\mathcal{B}_S$. Noise type linear maps are defined analogously. Notice that, if α_ε

is a system type map and X_R is a noise type map, then

$$\alpha_\varepsilon \circ X_R = X_R \alpha_\varepsilon \,.$$

2. Causal Normal Ordering

2.1. *The time consecutive principle*

Definition 2.1. Let

$$D_\varepsilon = \begin{cases} A, & \varepsilon = -1\,, \\ B, & \varepsilon = +1\,, \\ C, & \varepsilon = 0\,, \\ T, & \varepsilon = 2\,. \end{cases} \tag{2.1}$$

be

(i) either bounded linear operators on the initial space $\mathcal{H}_S$,
(ii) or bounded linear maps of system type.

A solution of the white noise equation:

$$\partial_t U_t = -i(Ab_t + Bb_t^+ + b_t^+ T b_t + C)U_t = b_t^\varepsilon D_\varepsilon U_t\,, \qquad U_0 = 1\,, \tag{2.2}$$

where

$$b_t^\varepsilon = \begin{cases} b_t\,, & \varepsilon = -1\,, \\ b_t^+\,, & \varepsilon = +1\,, \\ 1\,, & \varepsilon = 0\,, \\ b_t^+ b_t\,, & \varepsilon = 2\,, \end{cases} \tag{2.3}$$

is a white noise adapted process U_t (in the sense of Sec. 4 of Ref. 6 satisfying the identity:

$$U_t = 1 - i\int_0^t (Ab_s + Bb_s^+ + b_s^+ T b_s + C)U_s ds = 1 - i\int_0^t b_s^\varepsilon D_\varepsilon U_s ds \tag{2.4}$$

in the sense that the white noise integral on the right-hand side exists on the maximal algebraic domain $\mathcal{D}_{MA}$ and the identity holds.

Definition 2.2. A normally ordered form of Eq. (2.2) is an equation of the type

$$\partial_t U_t = -i\hat{A}U_t b_t - i\hat{B}U_t b_t^+ - i b_t^+ \hat{T} U_t b_t - i\hat{C}U_t\,, \qquad U_0 = 1 \tag{2.5}$$

which has the same solutions as (2.2). In case (i), if the symmetry conditions:

$$A^* = B\,, \qquad T = T^*\,, \qquad C = C^* \tag{2.6}$$

are satisfied, (2.2) is called a white noise Hamiltonian equation.

Remark. Notice that the symmetry conditions (2.6) are equivalent to the formal self-adjointness of the operator valued distribution:

$$H_t := Ab_t + Bb_t^+ + b_t^+ T b_t + C\,.$$

Such an operator valued distribution will be called a white noise Hamiltonian.

Remark. In Ref. 6, the authors obtained the estimates on the white noise integrals and used them to prove the existence and uniqueness theorem and *a priori* bounds on the solutions, for white noise equations with bounded coefficients in the non-normally ordered case. Once these estimates are given and keeping the boundedness assumption, the existence proof is based on routine arguments on the iterated series and can be extended without difficulties to the case (ii) and to non-normally ordered equations. In the following we will freely use such extensions without spelling out, due to space constraint, the simple modifications required in the two above-mentioned cases.

The normal order problem consists in giving prescriptions which, given an equation of the form (2.2), allow to write it in the form (2.5) and to explicitly compute the new coefficients $\hat{A}$, $\hat{B}$, $\hat{T}$, $\hat{C}$.

Definition 2.3. The pair $\{b_t^+, b_t\}$ is said to satisfy the causal commutation relations with parameter $\gamma_- \in \mathbb{C}$

$$\mathrm{Re}\gamma_- > 0 \tag{2.7}$$

if $\forall\, t$, for any pair of white noise adapted processes F_s, G_s and for any x_s, $y_s \in \{b_s, b_s^+, 1\}$:

$$\int_0^t x_s F_s [b_t, b_s^+] G_s y_s ds = \gamma_- \int_0^t x_s F_s \delta_+(t-s) G_s y_s ds = \gamma_- x_t F_t G_t y_t\,. \tag{2.8}$$

Moreover, all the remaining commutators are zero under the integral sign. In this case we write:

$$[b_s, b_t^+] = \gamma_- \delta_+(t-s)\,; \qquad \forall\, s, t \in \mathbb{R};\ s < t\,. \tag{2.9}$$

Remark. For the origins of the causal commutation relations (2.9) we refer to Accardi, Lu and Volovich.[1] The meaning of the symbol $\delta_+(t-s)$ is defined in Sec. IX of Ref. 5. In this paper we will use this symbol only in the identity (2.8) which can therefore be considered as a definition of the right-hand side of (2.9).

The Hudson–Parthasarathy calculus corresponds to the choice:

$$\gamma_- = \frac{1}{2}\,.$$

In the following, we will write Eq. (2.2) in the form

$$\partial_t U_t = b_t^\varepsilon \dot{U}_{\varepsilon,t} = b_t^\varepsilon D_\varepsilon U_t\,, \qquad U_0 = 1 \tag{2.10}$$

where summation over the repeated index $\varepsilon \in \{-1, 0, +1, 2\}$ is understood and where, by definition

$$\dot{U}_{-1,s} = -iAU_s\,; \qquad \dot{U}_{+1,s} = -iBU_s\,; \qquad \dot{U}_{0,s} = -iCU_s\,; \qquad \dot{U}_{2,s} = -iTU_s\,. \tag{2.11}$$

Theorem 2.1. (Time consecutive principle: weak form) *Let U_t be the unique solution of the white noise equation* (2.10) *with bounded coefficients D_ε. Then, for any $t \in \mathbb{R}_+$ and any ε, $\theta \in \{\pm 1, 0, 2\}$*

$$\int_0^t b_s^\varepsilon [b_t^\theta, \dot{U}_{\varepsilon,s}] ds = \int_0^t D_\varepsilon b_s^\varepsilon [b_t^\theta, U_s] ds = 0\,. \tag{2.12}$$

Proof. Clearly (2.12) will follow from

$$\int_0^t b_{t_1}^\varepsilon [b_t^\theta, U_{t_1}] dt_1 = 0\,; \qquad \forall\, \varepsilon, \theta \in \{\pm 1, 0, 2\} \tag{2.13}$$

and for U_t the unique solution of (2.10). Since $[b_t^\theta, 1] = 0$, (2.13) is equivalent to

$$\begin{aligned}
&\int_0^t b_{t_1}^\varepsilon \int_0^{t_1} [b_t^\theta, b_{t_2}^{\varepsilon_1} D_\varepsilon U_{t_2}] dt_1 dt_2 \\
&\quad = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n b_{t_1}^\varepsilon [b_t^\theta, D_{\varepsilon_2} \cdots D_{\varepsilon_n} b_{t_2}^{\varepsilon_2} \cdots b_{t_n}^{\varepsilon_n} U_{t_n}] \\
&\quad = D_{\varepsilon_2} \cdots D_{\varepsilon_n} \int_{\Delta_n^t} dt_1 \cdots dt_n b_{t_1}^\varepsilon [b_t^\theta, b_{t_2}^{\varepsilon_1} \cdots b_{t_n}^{\varepsilon_n}] U_{t_n} \\
&\qquad + D_{\varepsilon_2} \cdots D_{\varepsilon_n} \int_{\Delta_n^t} dt_1 \cdots dt_n b_{t_1}^\varepsilon b_{t_2}^{\varepsilon_2} \cdots b_{t_n}^{\varepsilon_n} [b_t^\theta, U_{t_n}] =: I_n + J_n\,.
\end{aligned}$$

The integral in I_n is a sum of terms each of which contains a

$$\delta_+(t - t_j)\,; \qquad j \geq 2\,.$$

From the theory of distributions on the standard simplex (see Ref. 5), we know that, since

$$t \geq t_1 \geq t_2 \geq \cdots \geq t_j\,,$$

then any term containing $\delta_+(t - t_j)$ vanishes. Thus

$$I_n = 0\,. \tag{2.14}$$

Now consider the norm of the matrix element of the term J_n with respect to two vectors φ, ψ of the maximal algebraic domain. This is less than or equal to a finite sum of integrals of the form

$$\int_{\Delta_n^t} dt_1 \cdots dt_n f_1(t_1) \cdots f_n(t_n) \langle \xi, [b_t^\theta, U_{t_n}] \eta \rangle\,, \tag{2.15}$$

where $\xi \in J(\varphi)$, $\eta \in J(\psi)$ (in the Accardi, Fagnola and Quaegebeur notations[9]). In particular both ξ, $\eta \in \mathcal{D}_{MA}$ and the $f_1, \ldots, f_n$ are positive functions that can be assumed to be bounded. Therefore (2.15) is less than or equal to

$$c^n \int_{(\Delta^t)_n} dt_1 \cdots dt_n \{\|\langle b_t^{\theta +}\xi, U_{t_n}\eta\rangle\| + \|\langle \xi, U_{t_n} b_t^{\theta}\eta\rangle\|\}\,.$$

Since ξ, $\eta \in \mathcal{D}_{MA}$, this term is a sum of terms of the form

$$c^n f(t) \int_{\Delta_n^t} dt_1 \cdots dt_n \|\langle \xi_1, U_{t_n}\eta\rangle\|$$

or of the form

$$c^n g(t) \int_{\Delta_n^t} dt_1 \cdots dt_n \|\langle \xi, U_{t_n}\eta_1\rangle\|\,,$$

where f, g are bounded functions, and $\xi_1 \in J(\xi)$, $\eta_1 \in J(\eta)$.

But, from the *a priori* estimates on the solutions of Eq. (2.10) given in Theorem 5.1 of Ref. 6, we know that, for ξ', $\eta' \in \mathcal{D}_{MA}$ there exists a constant $c_{\xi',\eta'}(t)$ such that:

$$\|\langle \xi', U_t \eta'\rangle\| \leq c_{\xi',\eta'}(t)\,; \qquad \forall\, s \in [0, t]\,.$$

Since, in our case, the ξ', η' can vary in a finite set, we finally obtain that (2.15) is less than or equal to

$$c_{\varphi,\psi}(t) c^n \int_{\Delta_n^t} dt_1 \cdots dt_n = c_{\varphi,\psi}(t) \frac{(ct)^n}{n!}\,.$$

Since this is true for any $n \in \mathbb{N}$, we conclude that the matrix element of the integrals

$$\int_0^t b_s^{\varepsilon}[b_t^{\theta}, U_s] ds\,; \qquad \forall\, \varepsilon, \theta \in \{\pm 1, 0, 2\}$$

are zero for any pair of vectors φ, $\psi \in \mathcal{D}_{MA}$ and this proves (2.12). □

2.2. *Causal normal order of white noise equations*

Corollary 2.1. *If* U_t *is a solution of Eq.* (2.2), *then one has*:

$$[b_t, U_t] = -i\gamma_-(BU_t + Tb_tU_t)\,, \tag{2.16}$$

$$[b_t^+, U_t^*] = -i\bar{\gamma}_-(U_t^* B^* + U_t^* b_t^+ T^*)\,, \tag{2.17}$$

$$[b_t, U_t^*] = i\gamma_- U_t^* A^* + i\gamma_- U_t^* b_t T^*\,, \tag{2.18}$$

$$[b_t^+, U_t] = i\bar{\gamma}_- A U_t + i\bar{\gamma}_- T b_t^+ U_t\,. \tag{2.19}$$

Proof.

$$\begin{aligned}
[b_t, U_t] &= \left[b_t, 1 - i\int_0^t (Ab_s + b_s^+ B + b_s^+ T b_s + C) U_s ds\right] \\
&= -iA\int_0^t [b_t, b_s U_s] ds - iB\int_0^t [b_t, b_s^+ U_s] ds - iT\int_0^t [b_t, b_s^+ b_s U_s] ds \\
&\quad - iC\int_0^t [b_t, U_s] ds\,.
\end{aligned}$$

Developing the commutators with the Leibnitz rule and using Theorem 2.1 we see that this is equal to

$$\begin{aligned}
&-iB\int_0^t [b_t, b_s^+] U_s ds - iT\int_0^t [b_t, b_s^+] b_s U_s ds \\
&\quad = -i\gamma_- B\int_0^t \delta_+(t-s) U_s ds - i\gamma_- T\int_0^t \delta_+(t-s) b_s U_s ds \\
&\quad = -i\gamma_-(BU_t + Tb_t U_t)\,,
\end{aligned}$$

$$[b^+, U] = -iA\int_0^t [b_t^+, b_s] U_s ds - iT\int_0^t b_s^+ [b_t^+, b_s] u_s ds = i\gamma_- AU_t + i\gamma_- T b_t^+ U_t\,,$$

$$b_t^+ U_t = U_t b_t^+ + i\bar{\gamma}_- AU_t + i\bar{\gamma}_- T b_t^+ U_t\,,$$

$$(1 - i\bar{\gamma}_- T) b_t^+ U_t = U_t b_t^+ + i\bar{\gamma}_- AU_t$$

and this proves (2.16). Similar arguments applied to the commutator $[b_t, U_t^*]$ lead to:

$$\begin{aligned}
[b_t, U_t^*] &= \left[b_t, 1 + i\int_0^t U_s^*(b_s^+ A^* + B^* b_s + b_s^+ T^* b_s + C^*) ds\right] \\
&= i\int_0^t [b_t, U_s^* b_s^+] A^* ds + i\int_0^t [b_t, U_s^* B^* b_s] ds \\
&\quad + i\int_0^t [b_t, U_s^* b_s^+ T^* b_s] ds + i\int_0^t [b_t, U_s^* C^*] ds \\
&= i\int_0^t U_s^* [b_t, b_s^+] A^* ds + i\int_0^t U_s^* [b_t, b_s^+] T^* b_s ds \\
&= i\gamma_-\int_0^t \delta_+(t-s) U_s^* A^* ds + i\gamma_-\int_0^t \delta_+(t-s) U_s^* T^* b_s ds \\
&= i\gamma_- U_t^* A^* + i\gamma_- U_t^* b_t T^*
\end{aligned}$$

and this proves (2.18). The remaining two identities are proved by taking the adjoint. □

Corollary 2.2. *In the above notations, if* $(1+i\gamma_- T)$ *and* $(1+i\gamma_- T^*)$ *are invertible, then denoting*:

$$K = (1 + i\gamma_- T)^{-1}\,; \qquad K' = (1 + i\gamma_- T^*) \tag{2.20}$$

one has

$$b_t U_t = -i\gamma_- KBU_t + KU_t b_t\,, \tag{2.21}$$

$$U_t b_t^+ = -i\bar{\gamma}_- AU_t + K^{'*} b_t^+ U_t\,. \tag{2.22}$$

Proof. From Corollary 2.1 one deduces that:

$$b_t U_t = U_t b_t - i\gamma_-(BU_t + Tb_t U_t)$$

or equivalently

$$(1 + i\gamma_- T)b_t U_t = -i\gamma_- BU_t + U_t b_t$$

and this proves (2.21). Similarly

$$b_t U_t^* = U_t^* b_t + i\gamma_- U_t^* A^* + i\gamma_- U_t^* b_t T^* = i\gamma_- U_t^* A^* + U_t^* b_t(1 + i\gamma_- T^*)$$

or equivalently

$$b_t U_t^* = i\gamma_- U_t^* A^* + U_t^* b_t K'$$

and this proves (2.22). □

2.3. *Forward and backward evolutions*

A forward Heisenberg evolution is a two-parameter family of automorphisms of $\mathcal{A}$

$$j_{s,t}^+ : \mathcal{A} \to \mathcal{A}\,, \qquad s \le t, s, t \in \mathbb{R}$$

satisfying

$$j_{r,s}^+ \circ j_{s,t}^+ = j_{r,t}^+\,, \qquad r \le s \le t$$

$$j_{s,s}^+ = \mathrm{id}_{\mathcal{A}}\,.$$

The inverse of a forward Heisenberg evolution

$$j_{s,t}^- := (j_{s,t}^+)^{-1}$$

is called a backward Heisenberg evolution. It satisfies

$$j_{s,t}^- \circ j_{r,s}^- = j_{r,t}^-\,, \qquad r \le s \le t$$

$$j_{s,s}^- = \mathrm{id}_{\mathcal{A}}\,.$$

A smooth forward Heisenberg evolution satisfies an equation of the form

$$\partial_t j_{s,t}^+ = i j_{s,t}^+ \circ \delta_t\,, \qquad j_{s,s}^+ = \mathrm{id}_{\mathcal{A}}\,; \tag{2.23}$$

where $i^2 = -1$ and δ_t is a $*$-derivation on $\mathcal{A}$ (or on an appropriate subspace). The associated backward evolution satisfies the equation

$$\partial_t j^-_{s,t} = -i\delta_t \circ j_{s,t}\,, \qquad j^-_{s,s} = \mathrm{id}_{\mathcal{A}}\,. \tag{2.24}$$

A forward Heisenberg evolution $j^+_{s,t}$ is called inner if each of the automorphisms $j^+_{s,t}$ is inner, i.e. if for any $s \le t$ there exists a unitary operator $U_{s,t} \in \mathcal{A}$ such that

$$j^+_{s,t}(x) = U^*_{s,t} x U_{s,t}\,, \qquad x \in \mathcal{A}\,. \tag{2.25}$$

The associated backward evolution $j^-_{s,t}$ is also inner and satisfies

$$j^-_{s,t}(x) = U_{s,t} x U^*_{s,t}\,, \qquad x \in \mathcal{A}\,. \tag{2.26}$$

One way to produce such evolutions is to start from a forward unitary operator evolution, i.e. a two-parameter family $U_{s,t}$ of unitary operators in $\mathcal{A}$ satisfying

$$U^+_{s,t} U^+_{r,s} = U^+_{r,t}\,, \qquad r \le s \le t\,,$$

and to define $j^+_{s,t}$ using (2.25).

If such an evolution is smooth, it satisfies an equation of the form

$$\partial_t U^+_{s,t} = -iH_t U^+_{s,t}\,, \qquad U^+_{s,s} = 1\,. \tag{2.27}$$

The associated backward evolution is defined by

$$U^-_{s,t} := (U^+_{s,t})^*$$

through (2.26) and, in the smooth case, it satisfies the equation

$$\partial_t U^-_{s,t} = U^-_{s,t} iH_t\,, \qquad U^-_{s,s} = 1\,. \tag{2.28}$$

For inner evolutions the Heisenberg equations (2.23), (2.24) become respectively

$$\partial_t j^+_{s,t}(x) = j^+_{s,t}(i[x, H_t])\,, \qquad j^+_{s,s}(x) = x$$
$$\partial_t j^-_{s,t}(x) = i[H_t, j^-_{s,t}(x)]\,,$$

where $x \in \mathcal{A}$.

When H_t is a self-adjoint operator, Eq. (2.27) is called a Schrödinger equation. When H_t has the form

$$H_t = Ab^+_t + Bb_t + Tb^+_t b_t + C\,;$$

where $b^\pm_t$ is a boson Fock Brownian motion and A, B, C, T are system operators satisfying

$$A^* = B\,, \qquad T^* = T\,, \qquad C^* = C\,,$$

then (2.27) is called a white noise Hamiltonian equation.

2.4. *Inner white noise Heisenberg evolutions*

Lemma 2.1. *If* $\{\cdot,\cdot\}$ *denotes the anticommutator, then the following identities hold*:

$$\{a, xy\} = \{a, x\}y + x[y, a]\,, \tag{2.29}$$

$$\{a, xy\} = x\{y, a\} + [a, x]y\,. \tag{2.30}$$

Proof. Equation (2.29) follows from the identity

$$\{a, xy\} = axy + xya = \{axy + xay\} - xay + xya\,,$$

and (2.30) follows from the identity:

$$\{a, xy\} = axy + xya = x\{ya + ay\} - xay + axy\,.$$

Recall that, in the notations (2.3), (2.1) a backward inner white noise Heisenberg equation has the form:

$$\partial_t j_t(x) = \delta_{I,t}(j_t(x)) = i[H_t, j_t(x)] = [D_\varepsilon b_t^\varepsilon, j_t(x)] = [D_\varepsilon, j_t(x)]b_t^\varepsilon + D_\varepsilon[b_t^\varepsilon, j_t(x)]\,.$$

But, since $D_\varepsilon b_t^\varepsilon = b_t^\varepsilon D_\varepsilon$, one also has

$$\partial_t j_t(x) = [b_t^\varepsilon D_\varepsilon, j_t(x)] = [b_t^\varepsilon, j_t(x)]D_\varepsilon + b_t^\varepsilon[D_\varepsilon, j_t(x)]\,.$$

Therefore, summing the two, we obtain:

$$\partial_t j_t(x) = \frac{1}{2}\{[D_\varepsilon, j_t(x)]b_t^\varepsilon + b_t^\varepsilon[D_\varepsilon, j_t(x)] + [b_t^\varepsilon, j_t(x)]D_\varepsilon + D_\varepsilon[b_t^\varepsilon, j_t(x)]\}\,.$$

Now denote

$$x_t := j_t(x)\,,$$

and notice that

$$\{D_\varepsilon, [b_t^\varepsilon, x_t]\} = [b_t^\varepsilon, \{D_\varepsilon, x_t\}]\,.$$

Then

$$\partial_t x_t = \left[\frac{1}{2}D_\varepsilon, x_t\right] b_t^\varepsilon + b_t^\varepsilon \left[\frac{1}{2}D_\varepsilon, x_t\right] + [b_t^\varepsilon, \{D_\varepsilon, x_t\}]\,.$$

Therefore, with the notation

$$\delta_\varepsilon(z) := \left[\frac{1}{2}D_\varepsilon, z\right]\,, \qquad \alpha_\varepsilon(z) = \{D_\varepsilon, z\}\,,$$

we see that the δ_ε, are derivations and the equation satisfied by x_t becomes

$$\partial_t x_t = \delta_\varepsilon(x_t)b_t^\varepsilon + b_t^\varepsilon\delta_\varepsilon(x_t) + [b_t^\varepsilon, \alpha_\varepsilon(x_t)]\,,$$

equivalently

$$\partial_t x_t = \{b_t^\varepsilon, \delta_\varepsilon(x_t)\} + [b_t^\varepsilon, \alpha_\varepsilon(x_t)] = \delta_t(x_t)\,.$$

Notice that both α_ε and δ_ε act on the system algebra. $\square$

3. Normally Ordered Backward Heisenberg Evolutions

Definition 3.1. A bounded white noise derivation is a set of bounded linear maps $\{\delta_\varepsilon, \alpha_\varepsilon\}$ ($\varepsilon \in \{\pm 1, 0, 2\}$) on $\mathcal{A}$, both of system type and such that each δ_ε is a $*$-derivation on $\mathcal{B}_S$ and:

$$\{b_t^\varepsilon, \delta_\varepsilon(\cdot)\} + [b_t^\varepsilon, \alpha_\varepsilon(\cdot)] =: i\delta_{I,t}$$

is a $*$-derivation on $\mathcal{B}_S \otimes \mathcal{B}_R$.

Lemma 3.1. *The maps* $\{\delta_\varepsilon, \alpha_\varepsilon\}$ ($\varepsilon \in \{\pm 1, 0, 2\}$) *are a white noise derivation if and only if, for any* x, $y \in \mathcal{A}$ *and any* $\varepsilon \in \{\pm 1, 0, 2\}$,

$$[b_t^\varepsilon, \alpha_\varepsilon(xy)] - [b_t^\varepsilon, \alpha_\varepsilon(x)]y - x[b_t^\varepsilon, \alpha_\varepsilon(y)] = \delta_\varepsilon(x)[b_t^\varepsilon, y] + [x, b_t^\varepsilon]\delta_\varepsilon(y)\,. \tag{3.1}$$

Proof. The derivation property of $\delta_{I,t}$ gives:

$$\delta_{I,t}(xy) = \delta_{I,t}(x)y + x\delta_{I,t}(y) \tag{3.2}$$

$$= \{b_t^\varepsilon, \delta_\varepsilon(xy)\} + [b_t^\varepsilon, \alpha_\varepsilon(xy)]\,. \tag{3.3}$$

Using (2.29) and the derivation property of δ_ε

$$\begin{aligned}&\{b_t^\varepsilon, \delta_\varepsilon(x)y\} + \{b_t^\varepsilon, x\delta_\varepsilon(y)\} + [b_t^\varepsilon, \alpha_\varepsilon(xy)]\\ &\quad = \{b_t^\varepsilon, \delta_\varepsilon(x)\}y + \delta_\varepsilon(x)[y, b_t^\varepsilon] + x\{b_t^\varepsilon, \delta_\varepsilon(y)\} + [b_t^\varepsilon, x]\delta_\varepsilon(y) + [b_t^\varepsilon, \alpha_\varepsilon(xy)]\,.\end{aligned}$$

On the other hand,

$$\delta_{I,t}(x)y + x\delta_{I,t}(y) = [\{b_t^\varepsilon, \delta_\varepsilon(x)\} + [b_t^\varepsilon, \alpha_\varepsilon(x)]]y + x[\{b_t^\varepsilon, \delta_\varepsilon(y)\} + [b_t^\varepsilon, \alpha_\varepsilon(y)]]\,.$$

This is equivalent to

$$\delta_\varepsilon(x)[y, b_t^\varepsilon] + [b_t^\varepsilon, x]\delta_\varepsilon(y) + [b_t^\varepsilon, \alpha_\varepsilon(xy)] = [b_t^\varepsilon, \alpha_\varepsilon(x)]y + x[b_t^\varepsilon, \alpha_\varepsilon(y)]\,,$$

and this is equivalent to (3.1). □

Theorem 3.1. *In the notations of Definition* 3.1, *assume that the map* $1-\gamma_-(\delta_2 + \alpha_2)$ *is invertible and define*

$$\tau_+ = (1 - \gamma_-(\delta_2 + \alpha_2))^{-1}\,, \tag{3.4}$$

$$\tau_- = 1 + \gamma_-(\delta_2 - \alpha_2)\,. \tag{3.5}$$

Then the normally ordered form of the white noise backward Heisenberg equation:

$$\partial_t j_t(x) = \delta_{I,t}(j_t(x)) = \{b_t^\varepsilon, \delta_\varepsilon(j_t(x))\} + [b_t^\varepsilon, \alpha_\varepsilon(j_t(x))] \tag{3.6}$$

is:

$$\begin{aligned}&\partial_t j_t(x)\\ &\quad = b_t^+[\delta_1\{(1 + \tau_+^*\tau_-^*)j_t(x)\} + \alpha_1\{(1 - \tau_+^*\tau_-^*)j_t(x)\} + 2\gamma_-(\delta_2 + \alpha_2)(\tau_+\delta_1(j_t(x)))]\end{aligned}$$

$$+ [\delta_{-1}\{(\tau_+\tau_- + 1)j_t(x)\} + \alpha_{-1}\{(\tau_+\tau_-1)j_t(x)\} + 2\bar{\gamma}_-(\delta_2 - \alpha_2)(\tau_+^*\delta_1^*(j_t(x)))]b_t$$
$$+ b_t^+[(\delta_2 + \alpha_2)(\tau_+\tau_- j_t(x)) + (\delta_2 - \alpha_2)(\tau_+^*\tau_-^* j_t(x))]b_t$$
$$+ 2\gamma_-(\delta_{-1} + \alpha_{-1})(\tau_+\delta_1(j_t(x))) + 2\bar{\gamma}_-(\delta_1 - \alpha_1)(\tau_+^*\delta_1^*(j_t(x))) + 2\delta_0(j_t(x)) . \tag{3.7}$$

Proof. Equation (3.6) is equivalent to

$$\begin{aligned} j_t(x) &= x + \int_0^t ds\delta_{I,s}(j_s(x)) \\ &= x + \int_0^t ds\{b_s^\varepsilon, \delta_\varepsilon(j_s(x))\} + [b_s^\varepsilon, \alpha_\varepsilon(j_s(x))] \\ &= x + \int_0^t ds(\{b_s^+, \delta_1(j_s(x))\} + [b_s^+, \alpha_1(j_s(x))] + \{b_s, \delta_{-1}(j_s(x))\} \\ &\quad + [b_s, \alpha_{-1}(j_s(x))] + \{b_s^+ b_s, \delta_2(j_s(x))\} + [b_s^+ b_s, \alpha_2(j_s(x))] \\ &\quad + \{1, \delta_0(j_s(x))\} + [1, \alpha_0(j_s(x))]) . \end{aligned}$$

To put the last equation in normally ordered form, we use the time consecutive principle and the relation $[b_t, \delta_\varepsilon(j_s(x))] = \delta_\varepsilon([b_t, j_s(x)])$, then by applying same technic as Corollary 2.2, we get:

$$(1 - \gamma_-(\delta_2 + \alpha_2))b_t j_t(x) = 2\gamma_-\delta_1(j_t(x)) + (1 + \gamma_-(\delta_2 - \alpha_2))j_t(x)b_t . \tag{3.8}$$

From the definition of (3.4), (3.5) we obtain

$$b_t j_t(x) = 2\gamma_-\tau_+\delta_1(j_t(x)) + \tau_+\tau_- j_t(x)b_t . \tag{3.9}$$

Then

$$j_t(x)b_t^+ = 2\bar{\gamma}_-\tau_+^*\delta_1^*(j_t(x)) + \tau_+^*\tau_-^* b_t^+ j_t(x) . \tag{3.10}$$

Now, let us use (3.9) and (3.10) to put (3.6) in normally ordered form:

$$\begin{aligned} \partial_t j_t(x) = {} & \delta_1(b_t^+ j_t(x)) + \delta_1(j_t(x)b_t^+) + \alpha_1(b_t^+ j_t(x)) - \alpha_1(j_t(x)b_t^+) \\ & + \delta_{-1}(b_t j_t(x)) + \delta_{-1}(j_t(x)b_t) + \alpha_{-1}(b_t j_t(x)) - \alpha_{-1}(j_t(x)b_t) \\ & + \delta_2(b_t^+ b_t j_t(x)) + \delta_2(j_t(x)b_t^+ b_t) + \alpha_2(b_t^+ b_t j_t(x)) \\ & - \alpha_2(j_t(x)b_t^+ b_t) + 2\delta_0(j_t(x)) , \end{aligned}$$

$$\begin{aligned} \partial_t j_t(x) = {} & \delta_1(b_t^+ j_t(x)) + \delta_1(2\bar{\gamma}_-\tau_+^*\delta_1^*(j_t(x)) + \tau_+^*\tau_-^* b_t^+ j_t(x)) \\ & + \alpha_1(b_t^+ j_t(x)) - \alpha_1(2\bar{\gamma}_-\tau_+^*\delta_1^*(j_t(x)) + \tau_+^*\tau_-^* b_t^+ j_t(x)) \\ & + \delta_{-1}(2\gamma_-\tau_+\delta_1(j_t(x)) + \tau_+\tau_- j_t(x)b_t) + \delta_{-1}(j_t(x)b_t) \\ & + \alpha_{-1}(2\gamma_-\tau_+\delta_1(j_t(x)) + \tau_+\tau_- j_t(x)b_t) - \alpha_{-1}(j_t(x)b_t) \end{aligned}$$

$$
\begin{aligned}
&+\delta_2(b_t^+[2\gamma_-\tau_+\delta_1(j_t(x))+\tau_+\tau_-j_t(x)b_t])\\
&+\delta_2([2\bar{\gamma}_-\tau_+^*\delta_1^*(j_t(x))+\tau_+^*\tau_-^*b_t^+j_t(x)]b_t)\\
&+\alpha_2(b_t^+[2\gamma_-\tau_+\delta_1(j_t(x))+\tau_+\tau_-j_t(x)b_t])\\
&-\alpha_2([2\bar{\gamma}_-\tau_+^*\delta_1^*(j_t(x))+\tau_+^*\tau_-^*b_t^+j_t(x)]b_t)+2\delta_0(j_t(x))\,,
\end{aligned}
$$

this leads to

$$
\begin{aligned}
\partial_t j_t(x) = {} & b_t^+\delta_1(j_t(x)) + 2\bar{\gamma}_-\delta_1(\tau_+^*\delta_1^*(j_t(x))) + b_t^+\delta_1(\tau_+^*\tau_-^*j_t(x))\\
&+b_t^+\alpha_1(j_t(x)) - 2\bar{\gamma}_-\alpha_1(\tau_+^*\delta_1^*(j_t(x))) + b_t^+\alpha_1(\tau_+^*\tau_-^*j_t(x))\\
&+2\gamma_-\delta_{-1}(\tau_+\delta_1(j_t(x))) + \delta_{-1}(\tau_+\tau_-j_t(x))b_t + \delta_{-1}(j_t(x))b_t\\
&+2\gamma_-\alpha_{-1}(\tau_+\delta_1(j_t(x))) + \alpha_{-1}(\tau_+\tau_-j_t(x))b_t - \alpha_{-1}(j_t(x))b_t\\
&+2\gamma_-b_t^+\delta_2(\tau_+\delta_1(j_t(x))) + b_t^+\delta_2(\tau_+\tau_-j_t(x))b_t + 2\bar{\gamma}_-\delta_2(\tau_+^*\delta_1^*(j_t(x)))b_t\\
&+b_t^+\delta_2(\tau_+^*\tau_-^*j_t(x))b_t + 2\gamma_-b_t^+\alpha_2(\tau_+\delta_1(j_t(x))) + b_t^+\alpha_2(\tau_+\tau_-j_t(x))b_t\\
&-2\bar{\gamma}_-\alpha_2(\tau_+^*\delta_1^*(j_t(x)))b_t - b_t^+\alpha_2(\tau_+^*\tau_-^*j_t(x))b_t + 2\delta_0(j_t(x))\,.
\end{aligned}
$$

Then, the normally ordered form of (3.6) is equivalent to Eq. (3.7). □

4. Homomorphic White Noise Backward Heisenberg Evolutions

Theorem 4.1. *Let δ_2, δ_1, δ_{-1} and δ_0 be bounded linear maps acting on the bounded operators on the initial space and let us denote with the same symbols the linear extensions of these operators to $\mathcal{B}(\mathcal{H}_S\otimes\Gamma)\equiv\mathcal{B}(\mathcal{H}_S)\otimes\mathcal{B}(\Gamma)$ characterized by:*

$$
b_S\otimes b_\Gamma\in\mathcal{B}(\mathcal{H}_S)\otimes\mathcal{B}(\Gamma)\mapsto\delta_\varepsilon(b_S)\otimes b_\Gamma\in\mathcal{B}(\mathcal{H}_S)\otimes\mathcal{B}(\Gamma)\,;\qquad \varepsilon\in\{\pm1,0,2\}\,.
$$

The unique solution of the backward flow equation

$$
\partial_t j_t(x) = b_t^+\delta_2(j_t(x))b_t + b_t^+\delta_1(j_t(x)) + \delta_{-1}(j_t(x))b_t + \delta_0(j_t(x))\,; \tag{4.1}
$$

$$
j_0(x)=x\qquad\forall\, x\in B(\mathcal{H}_S)\,,
$$

is an identity preserving $*$-homomorphism if and only if the maps δ_ε ($\varepsilon\in\{\pm1,0,2\}$) satisfy the following conditions:

$$
\delta_2(1)=\delta_1(1)=\delta_0(1)=\delta_{-1}(1)=0\,, \tag{4.2}
$$

$$
\delta_{-1}(x) = \delta_1(x^*)^*\,, \tag{4.3}
$$

$$
\delta_1(x) = \delta_{-1}(x^*)^*\,, \tag{4.4}
$$

$$
\delta_0(x) = \delta_0(x^*)^*\,, \tag{4.5}
$$

$$
\delta_2(x) = \delta_2(x^*)^*\,, \tag{4.6}
$$

$$\delta_2(xy) = \delta_2(x)y + x\delta_2(y) + 2\,\mathrm{Re}\,\gamma_-\delta_2(x)\delta_2(y)\,, \tag{4.7}$$

$$\delta_1(xy) = \delta_1(x)y + x\delta_1(y) + 2\,\mathrm{Re}\,\gamma_-\delta_2(x)\delta_1(y)\,, \tag{4.8}$$

$$\delta_{-1}(xy) = \delta_{-1}(x)y + x\delta_{-1}(y) + 2\,\mathrm{Re}\,\gamma_-\delta_{-1}(x)\delta_2(y)\,, \tag{4.9}$$

$$\delta_0(xy) = \delta_0(x)y + x\delta_0(y) + 2\,\mathrm{Re}\,\gamma_-\delta_{-1}(x)\delta_1(y)\,. \tag{4.10}$$

Proof. The condition $j_t(1) = 1$, is equivalent to $\partial_t j_t(1) = 0$ and leads to:

$$b_t^+\delta_2(j_t(1))b_t + b_t^+\delta_1(j_t(1)) + \delta_{-1}(j_t(1))b_t + \delta_0(j_t(1)) = 0\,.$$

The independence of the basic noises then implies that

$$\delta_2(1) = \delta_1(1) = \delta_0(1) = \delta_{-1}(1) = 0$$

and, evaluating this at $t = 0$, we find (4.2). Replacing x by x^* (5.1) becomes:

$$\partial_t j_t(x^*) = b_t^+\delta_2(j_t(x^*))b_t + b_t^+\delta_1(j_t(x^*)) + \delta_{-1}(j_t(x^*))b_t + \delta_0(j_t(x^*))\,. \tag{4.11}$$

On the other hand, by the $*$-homomorphism condition, this must be equal to:

$$\partial_t j_t(x)^* = b_t^+\delta_2(j_t(x))^*b_t + \delta_1(j_t(x))^*b_t + b_t^+\delta_{-1}(j_t(x))^* + \delta_0(j_t(x))^*\,. \tag{4.12}$$

By the same argument as above we obtain the conditions (4.3)–(4.6).

To exploit the homomorphism condition we calculate the causal commutators $[b_t, j_t(x)]$ and $[b_t, j_t(x^*)]$. The former gives:

$$[b_t, j_t(x)] = \gamma_-\delta_2(j_t(x))b_t + \gamma_-\delta_1(j_t(x))\,,$$

or equivalently:

$$b_t j_t(x) = (1 + \gamma_-\delta_2)(j_t(x))b_t + \gamma_-\delta_1(j_t(x)) = \rho(j_t(x))b_t + \gamma_-\delta_1(j_t(x))\,;$$

where we denote

$$\rho := (1 + \gamma_-\delta_2)\,.$$

Similarly, one has:

$$[b_t, j_t(x)^*] = \gamma_-\delta_2(j_t(x))^*b_t + \gamma_-\delta_{-1}(j_t(x))^*\,,$$

and, taking adjoint:

$$j_t(x)b_t^+ = b_t^+(1 + \bar\gamma_-\delta_2)(j_t(x)) + \bar\gamma_-\delta_{-1}(j_t(x)) = b_t^+\rho'(j_t(x)) + \bar\gamma_-\delta_{-1}(j_t(x))\,;$$

where we denote

$$\rho' = (1 + \bar\gamma_-\delta_2)\,.$$

Equation (5.1), applied to xy gives:

$$\partial_t j_t(xy) = b_t^+\delta_2(j_t(xy))b_t + b_t^+\delta_1(j_t(xy)) + \delta_{-1}(j_t(xy))b_t + \delta_0(j_t(xy))\,,$$

but, from $j_t(xy) = j_t(x)j_t(y)$ and the Leibnitz rule for white noise derivatives, we deduce also:

$$\begin{aligned}
\partial_t j_t(xy) &= \partial_t j_t(x) j_t(y) + j_t(x)\partial_t j_t(y) \\
&= \{b_t^+ \delta_2(j_t(x))b_t + b_t^+\delta_1(j_t(x)) + \delta_{-1}(j_t(x))b_t + \delta_0(j_t(x))\}j_t(y) \\
&\quad + j_t(x)\{b_t^+\delta_2(j_t(y))b_t + b_t^+\delta_1(j_t(y)) + \delta_{-1}(j_t(y))b_t + \delta_0(j_t(y))\} \\
&= b_t^+\delta_2(j_t(x))\{\rho(j_t(y))b_t + \gamma_-\delta_1(j_t(y))\} + b_t^+\delta_1(j_t(x))j_t(y) \\
&\quad + \delta_{-1}(j_t(x))\{\rho(j_t(y))b_t + \gamma_-\delta_1(j_t(y))\} + \delta_0(j_t(x))j_t(y) \\
&\quad + \{b_t^+\rho'(j_t(x)) + \bar\gamma_-\delta_{-1}(j_t(x))\}\delta_2(j_t(y))b_t \\
&\quad + \{b_t^+\rho'(j_t(x)) + \bar\gamma_-\delta_{-1}(j_t(x))\}\delta_1(j_t(y)) \\
&\quad + j_t(x)\delta_{-1}(j_t(y))b_t + j_t(x)\delta_0(j_t(y)) \\
&= b_t^+\{\delta_2(j_t(x))\rho(j_t(y)) + \rho'(j_t(x))\delta_2(j_t(y))\}b_t \\
&\quad + b_t^+\{\gamma_-\delta_2(j_t(x))\delta_1(j_t(y)) + \delta_1(j_t(x))j_t(y) + \rho'(j_t(x))\delta_1(j_t(y))\} \\
&\quad + \{\delta_{-1}(j_t(x))\rho(j_t(y)) + \bar\gamma_-\delta_{-1}(j_t(x))\delta_2(j_t(y)) + j_t(x)\delta_{-1}(j_t(y))\}b_t \\
&\quad + \{\gamma_-\delta_{-1}(j_t(x))\delta_1(j_t(y)) + \delta_0(j_t(x))j_t(y) \\
&\quad + \bar\gamma_-\delta_{-1}(j_t(x))\delta_1(j_t(y)) + j_t(x)\delta_0(j_t(y))\}\,.
\end{aligned}$$

Using the independence of the basic noises as before, one then gets:

$$\begin{aligned}
\delta_2(xy) &= \delta_2(x)\rho(y) + \rho'(x)\delta_2(y) \\
\delta_1(xy) &= \gamma_-\delta_2(x)\delta_1(y) + \delta_1(x)y + \rho'(x)\delta_1(y) \\
\delta_{-1}(xy) &= \delta_{-1}(x)\rho(y) + \bar\gamma_-\delta_{-1}(x)\delta_2(y) + x\delta_{-1}(y) \\
\delta_0(xy) &= \gamma_-\delta_{-1}(x)\delta_1(y) + \delta_0(x)y + \bar\gamma_-\delta_{-1}(x)\delta_1(y) + x\delta_0(y)\,.
\end{aligned}$$

Eventually, replacing ρ and ρ' by their expressions,we obtain (4.7)–(4.10). $\square$

5. White Noise Evans–Hudson Flows

The following theorem is a white noise extension of the the Evans–Hudson structure equations which are recovered when $\mathrm{Re}\,\gamma_- = 1/2$.

Theorem 5.1. *Let δ_2, δ_1, δ_{-1} and δ_0 be norm bounded linear maps acting on the algebra of all bounded operators on the initial space. Then the unique solution of the normally ordered forward flow equation*:

$$\partial_t j_t(x) = b_t^+ j_t(\delta_2(x))b_t + b_t^+ j_t(\delta_1(x)) + j_t(\delta_{-1}(x))b_t + j_t(\delta_0(x))\,, \tag{5.1}$$

and

$$j_0(x) = x \qquad \forall\, x \in B(\mathcal{H}_S)\,,$$

is an identity preserving $*$-homomorphism of $\mathcal{B}(\mathcal{H}_S)$ if and only if the maps δ_ε ($\varepsilon \in \{\pm 1, 0, 2\}$) satisfy the same conditions as Theorem 4.1.

Proof. The steps of the proof are similar to those of the proof of Theorem 4.1. □

6. Inner White Noise Heisenberg Equations

Corollary 6.1. *The causally normally ordered form of the equation*

$$\partial_t U_t = -i(Ab_t + Bb_t^+ + b_t^+ T b_t + C)U_t\,; \qquad U_0 = 1\,, \tag{6.1}$$

is, in the notation (2.20)

$$\partial_t U_t = -ib_t^+ KBU_t - iAKU_t b_t - iTKb_t^+ U_t b_t + (-\gamma_- AKB - iC)U_t\,. \tag{6.2}$$

Proof. Equation (2.2) can be written:

$$\begin{aligned}\partial_t U_t &= -iAb_t U_t - iBb_t^+ U_t - ib_t^+ T b_t U_t - iCU_t \\ &= -iBb_t^+ U_t - iA[b_t, U_t] - iAU_t b_t - ib_t^+ T[b_t, U_t] - ib_t^+ TU_t b_t - iCU_t\,.\end{aligned}$$

Therefore, from Corollary 2.2, we obtain:

$$\partial_t U_t = -iBb_t^+ U_t - iA(-i\gamma_- KBU_t + KU_t b_t) - ib_t^+ T(-i\gamma_- KBU_t + KU_t b_t) - iCU_t\,,$$

equivalently,

$$\partial_t U_t = b_t^+(-iB - \gamma_- TKB)U_T - iAKU_t b_t - iTKb_t^+ U_t b_t + (-\gamma_- AKB - iC)U_t\,. \tag{6.3}$$

Therefore (6.3) becomes (6.2). □

6.1. *The forward inner Langevin equation*

Proposition 6.1. *Let U_t be the unique solution of Eq.* (2.2). *Define, for any bounded operator x on the initial space $\mathcal{H}_S$ and for any $t \in \mathbb{R}_+$:*

$$j_t(x) := U_t^* x U_t\,. \tag{6.4}$$

The equation satisfied by $j_t(x)$ is called the forward inner Langevin equation. Its causally normally ordered form is:

$$\begin{aligned}\partial_t j_t(x) = b_t^+ j_t(&iK^*A^*x + \gamma_- K^*T^*xKB - iK^*xB - \gamma_- K^*xTKB) \\ &+ j_t(iB^*xK - \bar\gamma_- B^*K^*T^*xK - ixAK + \bar\gamma_- B^*K^*xTK)b_t \\ &+ b_t^+ j_t(iK^*T^*xK - iK^*xTK)b_t + j_t(\gamma_- B^*xKB - \bar\gamma_- B^*K^*A^*x \\ &+ i|\gamma_-|^2 B^*K^*T^*xKB + iC^*x + \bar\gamma_- B^*K^*xB - \gamma_- xAKB \\ &- i|\gamma_-|^2 B^*K^*xTKB - ixC)\,;\end{aligned} \tag{6.5}$$

where K is given by (2.20).

Proof. The white noise equation satisfied by $j_t(x)$ is:

$$\begin{aligned}\partial_t j_t(x) &= \partial_t(U_t^* x U_t)\\ &= U_t^*(iB^*b_t + ib_t^+A^* + ib_t^+T^*b_t + iC^*)xU_t\\ &\quad + U_t^*x(-iBb_t^+ - iAb_t - ib_t^+Tb_t - iC)U_t\,.\end{aligned}$$

From Corollary 2.2 one deduces that:

$$\begin{aligned}\partial_t j_t(x) &= U_t^*(iB^*x)(-i\gamma_-KBU_t + KU_tb_t) + (i\bar\gamma_-U_t^*B^*K^* + b_t^+U_t^*K^*)(iA^*x)U_t\\ &\quad + (b_t^+U_t^*K^* + i\bar\gamma_-U_t^*B^*K^*)(iT^*x)(-i\gamma_-KBU_t + KU_tb_t) + U_t^*(iC^*x)U_t\\ &\quad + (i\bar\gamma_-U_t^*B^*K^* + b_t^+U_t^*K^*)(-ixB)U_t + U_t^*(-iAx)(-i\gamma_-KBU_t + KU_tb_t)\\ &\quad + (i\bar\gamma_-U_t^*B^*K^* + b_t^+U_t^*K^*)(-iTx)(-i\gamma_-KBU_t + KU_tb_t) + U_t^*(-iCx)U_t\\ &= j_t(\gamma_-B^*xKB) + j_t(iB^*xK)b_t + b_t^+j_t(iK^*A^*x) + j_t(-\bar\gamma_-B^*K^*A^*x)\\ &\quad + b_t^+j_t(\gamma_-K^*T^*xKB) + b_t^+j_t(iK^*T^*xK)b_t + j_t(i|\gamma_-|^2B^*K^*T^*xKB)\\ &\quad + j_t(-\bar\gamma_-B^*K^*T^*xK)b_t + j_t(iC^*x) + b_t^+j_t(-iK^*xB) + j_t(\bar\gamma_-B^*K^*xB)\\ &\quad + j_t(-\gamma_-xAKB) + j_t(-ixAK)b_t + b_t^+j_t(-\gamma_-K^*xTKB)\\ &\quad + b_t^+j_t(-iK^*xTK)b_t + j_t(-i|\gamma_-|^2B^*K^*xTKB)\\ &\quad + j_t(\bar\gamma_-B^*K^*xTK)b_t + j_t(-ixC)\,,\end{aligned}$$

and this is equivalent to (6.5). □

6.2. *White noise backward inner Heisenberg evolutions*

In the previous section we have discussed the normally ordered form of the equation satisfied by the forward flow $(U_t^*xU_t)$, associated to the white noise Hamiltonian equation (6.1). In this section we solve the same problem for the backward flow associated to the same equation.

Theorem 6.1. *Consider the white noise Hamiltonian $H_I(t)$ given by:*

$$H_I(t) = Db_t^+ + b_tD^+ + Tb_t^+b_t + C\,; \tag{6.6}$$

where D, T and C are elements of $B(\mathcal{H}_S)$ such that T and C are self-adjoint. Then the causally normally ordered form of the white noise Heisenberg equation:

$$\partial_t j_t(x) = -i[H_I(t), j_t(x)]\,, \tag{6.7}$$

is:

$$\begin{aligned}\partial_t j_t(x) &= b_t^+(-iKDj_t(x) + iK^{*-1}j_t(x)K^*D + \gamma_-TKj_t(x)D)\\ &\quad + (ij_t(x)D^+K^* - iD^+Kj_t(x)K^{-1} + \bar\gamma_-D^+j_t(x)K^*T)b_t\\ &\quad + b_t^+(-iTKj_t(x)K^{-1} + iK^{*-1}j_t(x)K^*T)b_t\\ &\quad + (-\bar\gamma_-[j_t(x), D^+]K^*D - \gamma_-D^+K[D, j_t(x)] - iCj_t(x) + ij_t(x)C)\,.\end{aligned} \tag{6.8}$$

Proof. Consider the following white noise Hamiltonian equation:

$$\partial_t j_t(x) = -i[H_I(t), j_t(x)]\,. \tag{6.9}$$

We want to put in causal normal order the equation:

$$\begin{aligned}\partial_t j_t(x) &= -i(Db_t^+ j_t(x) - j_t(x) Db_t^+ + b_t D^+ j_t(x)\\ &\quad - j_t(x) D^+ b_t + Tb_t^+ b_t j_t(x) - j_t(x) b_t^+ b_t T + Cj_t(x) - j_t(x)C)\,.\end{aligned}$$

To this goal we calculate the commutators $[b_t, j_t(x)]$ and $[b_t, j_t(x^*)]$ using the time consecutive principle. This gives:

$$[b_t, j_t(x)] = -i\gamma_-[D, j_t(x)] - i\gamma_-(Tb_t j_t(x) - j_t(x) b_t T)\,.$$

In the notation (2.20) (i.e. $K = (1 + i\gamma_- T)^{-1}$) this is equivalent to

$$b_t j_t(x) = -i\gamma_- K[D, j_t(x)] + Kj_t(x) b_t K^{-1}\,.$$

From the equation for $j_t(x^*)$ we deduce:

$$b_t j_t(x^*) = i\gamma_- K[j_t(x^*), D] + Kj_t(x^*) b_t K^{-1}\,,$$

it follows that:

$$j_t(x) b_t^+ = +i\bar{\gamma}_-[j_t(x), D^+]K^* + K^{*-1} b_t^+ j_t(x) K^*\,.$$

Then

$$\begin{aligned}\partial_t j_t(x) &= -i(b_t^+ D j_t(x) - \{i\bar{\gamma}_-[j_t(x), D^+]K^* + K^{*-1} b_t^+ j_t(x) K^*\}D\\ &\qquad + D^+\{-i\gamma_- K[D, j_t(x)] + Kj_t(x) b_t K^{-1}\} - j_t(x) D^+ b_t\\ &\qquad + Tb_t^+\{-i\gamma_- K[D, j_t(x)] + Kj_t(x) b_t K^{-1}\}\\ &\qquad - \{i\bar{\gamma}_-[j_t(x), D^+]K^* + K^{*-1} b_t^+ j_t(x) K^*\} b_t T - iCj_t(x) + ij_t(x)C)\\ &= b_t^+(-iDj_t(x) + iK^{*-1} j_t(x) K^* D - \gamma_- TK[D, j_t(x)])\\ &\qquad + (ij_t(x) D^+ - iD^+ K j_t(x) K^{-1} - \bar{\gamma}_-[j_t(x), D^+]K^* T) b_t\\ &\qquad + b_t^+(-iTKj_t(x) K^{-1} + iK^{*-1} j_t(x) K^* T) b_t\\ &\qquad (-\bar{\gamma}_-[j_t(x), D^+]K * D - \gamma_- D^+ K[D, j_t(x)] + Cj_t(x) - j_t(x)C)\end{aligned}$$

and this is equivalent to (6.8). □

The following is the analogue of Theorem 8.1 for backward flows.

6.3. *The structure maps in terms of the Hamiltonian coefficients*

Combining the results of Theorems 4.1 and 6.1, we obtain the expression of the structure maps as functions of the original Hamiltonian coefficients.

Corollary 6.2. *In the notations of Theorem* 6.1, *the structure maps are given by*:

$$\delta_2(x) = -iTKxK^{-1} + iK^{*-1}xK^*T\,,$$

$$\delta_1(x) = -iKDx + iK^{*-1}xK^*D + \gamma_- TKxD\,,$$

$$\delta_{-1}(x) = ixD^+K^* - iD^+KxK^{-1} + \bar{\gamma}_- D^+xK^*T\,,$$

$$\delta_0(x) = -\bar{\gamma}_-[x, D^+]K^*D - \gamma_- D^+K[D,x] + Cx - xC\,.$$

6.4. *The backward inner Langevin equation*

Proposition 6.2. *The backward flow is defined, for any bounded operator x on the initial space $\mathcal{H}_S$ and for any $t \in \mathbb{R}$ by*:

$$j_t(x) = U_t x U_t^*\,. \tag{6.10}$$

The causally normally ordered form of the backward inner Langevin equation satisfied by $j_t(x)$ is:

$$\begin{aligned}\partial_t j_t(x) = {} & b_t^+(-iB - \gamma_- TKB)j_t(x) + \gamma_- b_t^+ TKj_t(x)A^* + iK'^* b_t^+ j_t(x)K^*A^* \\ & - iAKj_t(x)b_tK' + j_t(x)(iB^* - \bar{\gamma}_- B^*K^*T^*)b_t + \bar{\gamma}_- Aj_t(x)K^*T^*b_t \\ & - ib_t^+TKj_t(x)b_tK' + iK'^*b_t^+j_t(x)K^*T^*b_t \\ & - iA(-i\gamma_- KBj_t(x) + i\gamma_- Kj_t(x)A^*) \\ & - iCj_t(x) - \bar{\gamma}_- j_t(x)B^*K^*A^* + \bar{\gamma}_- Aj_t(x)K^*A^* + ij_t(x)C^*\,.\end{aligned} \tag{6.11}$$

Proof. Differentiating by the Leibnitz rule the product $U_txU_t^*$ one finds:

$$\begin{aligned}\partial_t j_t(x) &= \partial_t U_t x U_t^* = (\partial_t U_t)xU_t^* + U_tx(\partial_t U_t^*) \\ &= (-iBb_t^+ - iAb_t - ib_t^+Tb_t - iC)U_txU_t^* \\ &\quad + U_txU_t^*(iB^*b_t + ib_t^+A^* + ib_t^+T^*b_t + iC^*)\,.\end{aligned}$$

From Corollary 2.2, and in the same notations, we obtain:

$$\begin{aligned}b_tj_t(x) &= b_tU_txU_t^* = (-i\gamma_- KBU_t + KU_tb_t)xU_t^* \\ &= -i\gamma_- KBj_t(x) + KU_tx(i\gamma_- U_t^*A^* + U_t^*b_tK') \\ &= -i\gamma_- KBj_t(x) + i\gamma_- Kj_t(x)A^* + Kj_t(x)b_tK'\,.\end{aligned}$$

Therefore:

$$\begin{aligned}\partial_t j_t(x) &= -iBb_t^+j_t(x) - iA(-i\gamma_- KBj_t(x) + i\gamma_- Kj_t(x)A^* + Kj_t(x)b_tK') \\ &= -ib_t^+T(-i\gamma_- KBj_t(x) + i\gamma_- Kj_t(x)A^* + Kj_t(x)b_tK') \\ &= -iCj_t(x) + ij_t(x)b_tB^* + i(i\bar{\gamma}_- j_t(x)B^*K^* - i\bar{\gamma}_- Aj_t(x)K^*\end{aligned}$$

$$+ k'^* b_t^+ j_t(x) K^*) A^*$$
$$= +i(i\bar{\gamma}_- j_t(x) B^* K^* - i\bar{\gamma}_- A j_t(x) K^* + k'^* b_t^+ j_t(x) K^*) T^* b_t + i j_t(x) C^* .$$

By grouping together the homogeneous terms we find (6.11). □

7. The Unitarity Conditions

7.1. *The isometricity condition*

The following theorem shows that the isometry condition on an adapted solution of the equation

$$\partial_t U_t = -i(Ab_t + Bb_t^+ + b_t^+ T b_t + C) U_t , \qquad U_0 = 1 , \tag{7.1}$$

is equivalent to the formal self-adjointness of the operator valued distribution:

$$Ab_t + Bb_t^+ + b_t^+ T b_t + C .$$

Theorem 7.1. *Let A, B, C and T be bounded operators on the initial space $\mathcal{H}_S$ then, for the unique solution U_t of Eq.* (7.1), *the following statements are equivalent*:

(i) *U_t is isometric, i.e.*

$$U_t^* U_t = 1$$

and both $(1 + i\gamma_- T)$ and $(1 + i\gamma_- T^*)$ are invertible.

(ii) *The coefficients A, B, T and C satisfy the relations*:

$$T = T^* , \tag{7.2}$$

$$B^* = A , \tag{7.3}$$

$$C = C^* \tag{7.4}$$

so that Eq. (7.1) *takes the form*

$$\partial_t U_t = -i(Ab_t + A^* b_t^+ + b_t^+ T b_t + C) U_t , \qquad U_0 = 1 . \tag{7.5}$$

Proof. The isometry condition is equivalent to

$$j_t(1) = 1 \qquad \forall\, t ; \tag{7.6}$$

where j_t is given by (6.4). Thus condition (7.6) is equivalent to the fact that the right-hand side of (6.5) vanishes for all t. The independence of the basic noises implies then that the coefficients of the various terms on the right-hand side of (6.5) vanish separately. The vanishing of the $b_t^+ b_t$-term gives:

$$K^* T^* K = K^* T K \Leftrightarrow K^*(T^* - T) K = 0 ,$$

with K given by (2.20). Since K is invertible by assumption this is equivalent to (7.2).

The vanishing of the b_t-term gives:

$$0 = iB^* K - \bar{\gamma}_- B^* K^* T^* K - iAK + \bar{\gamma}_- B^* K^* T K .$$

Using again the invertibility of K and its explicit form (2.20), this is equivalent to (7.3).

The vanishing of the constant term gives:

$$0 = \gamma_- B^* K B - \bar{\gamma}_- B^* K^* A^* + i|\gamma_-|^2 B^* K^* T^* K B \\ + iC^* + \bar{\gamma}_- B^* K^* B - \gamma_- A K B - i|\gamma_-|^2 B^* K^* T K B - iC\,,$$

and, since the conditions (7.2) and (7.3) are satisfied, it is equivalent to (7.4).

Conversely, suppose that conditions (7.2)–(7.4) are satisfied. Then, since $\mathrm{Re}\,\gamma_- \neq 0$, it follows that $1 + i\gamma_- T$ is always invertible. In fact, assuming the contrary, the identity

$$1 + i\gamma_- T = i\gamma_- \left(\frac{1}{i\gamma_-} + T\right)$$

and the fact that $\gamma_- \neq 0$ would imply that also $1/(i\gamma_-) + T$ is not invertible, i.e. $1/(i\gamma_-)$ should be an element of the spectrum of T. But this is a contradiction because T is self-adjoint. Therefore the same arguments as in the first part of the proof show that the right-hand side of (6.5) is identically zero and, since $j_0(1) = 1$ by assumption, U_t is an isometry for every t. This proves the statement. □

7.2. *The co-isometricity condition*

There are two ways to handle the co-isometries condition for U: one is through the multiplicativity of the forward flow (6.4); another is through the conservativity of the backward flow (6.10) we begin to discuss the latter.

Theorem 7.2. *Let A, B, C and T be bounded operators on the initial space $\mathcal{H}_S$ then, for the unique solution U_t of Eq.* (7.1), *the following statements are equivalent*:

(i) *U_t is co-isometric, i.e.*

$$U_t U_t^* = 1$$

and both $(1 + i\gamma_- T)$ and $(1 + i\gamma_- T^*)$ are invertible.

(ii) *The coefficients A, B, T and C satisfy the relations* (7.2)–(7.4) *so that Eq.* (7.1) *takes the form* (7.5).

Proof. The co-isometricity condition is equivalent to:

$$j_t(1) = 1 \qquad \forall\, t\,; \tag{7.7}$$

where j_t is given by (6.10). Thus condition (7.7) is equivalent to the fact that the right-hand side of (6.11) vanishes for all t. The independence of the basic noises implies then that the coefficients of the various terms on the right-hand side of (6.11) vanish separately. The vanishing of the $b_t^+ b_t$-term gives:

$$TKK' = K'^* K^* T^* \tag{7.8}$$

with K and K' defined by (2.20). From (51) and (73), under our assumptions this is equivalent to:

$$TKK' = T(1+i\gamma_-T)^{-1}(1+i\gamma_-T^*) = (1+i\gamma_-T)^{-1}T(1+i\gamma_-T^*)$$
$$= K'^*K^*T^* = (1-i\bar{\gamma}_-T)(1-i\bar{\gamma}_-T^*)^{-1}T^* = (1-i\bar{\gamma}_-T)T^*(1-i\bar{\gamma}_-T^*)^{-1}\,,$$

which is equivalent to:

$$(1+i\gamma_-T)(1-i\bar{\gamma}_-T)T^* = T(1+i\gamma_-T^*)(1-i\bar{\gamma}_-T^*)$$
$$\Leftrightarrow (1-|\gamma_-|^2TT^*)T^* = T(1-|\gamma_-|^2TT^*)\,.$$

Therefore for any polynomial P

$$P(1-|\gamma_-|^2TT^*)T^* = TP(1-|\gamma_-|^2TT^*)\,.$$

This implies, by approximation, that for any measurable function f

$$f(1-|\gamma_-|^2TT^*)T^* = Tf(1-|\gamma_-|^2TT^*)\,.$$

Since f is arbitrary, one also has

$$f(|T^*|)T^* = Tf(|T^*|)\,.$$

Now, let

$$T^* = |T^*|V$$

be the polar decomposition of T^*, so that

$$T = V^*|T^*|\,.$$

Then

$$f(|T^*|)|T^*|V = V^*|T^*|f(|T^*|) = V^*f(|T^*|)|T^*|\,.$$

Since f is arbitrary this implies that, for any measurable g one has

$$g(|T^*|)V = V^*g(|T^*|)\,.$$

In particular the choice $g(x) = x$ gives

$$T^* = |T^*|V = V^*|T^*| = T\,.$$

The b_t^+-term gives, again using $T = T^*$:

$$-iB - \gamma_-TKB + \gamma_-TKA^* + iK'^*K^*A^* = 0\,. \tag{7.9}$$

Since

$$-i - \gamma_-TK = -i(1-i\gamma_-TK) = -i\left(\frac{1-i\gamma_-T}{1+i\gamma_-T}\right)$$
$$= -i\left(\frac{1+i\gamma_-T-i\gamma_-T}{1+i\gamma_-T}\right) = -i\left(\frac{1}{1+i\gamma_-T}\right) = -iK\,,$$

it follows that:

$$-iB - \gamma_-TKB = -iKB\,.$$

Combining this with $KK' = 1$ and with (7.9), we find

$$-iKB + \gamma_- TKA^* + iA^* = 0\,,$$

which implies

$$-iKB+(\gamma_- TK+i)A^* = -iKB+(\gamma_- T(1+i\gamma_- T)^{-1}+i)A^* = -iKB+iKA^* = 0\,,$$

and this too is identically satisfied because K is invertible and $B = A^*$ holds by assumption (7.3).

Using the above results, the vanishing of the drift term becomes equivalent to $C = C^*$ which is also true by assumption. This completes the proof. □

Remark. Notice the asymmetry in the proof of the isometry and of the co-isometry conditions even in the bounded case.

8. Expression of the Hudson–Parthasarathy Coefficients in Terms of the White Noise Hamiltonian

Corollary 8.1. *Let* A, C *and* T *be bounded operators on the initial space* $\mathcal{H}_S$ *satisfying the unitarity conditions* (7.2), (7.4). *denoting*:

$$S := \frac{1 - i\bar{\gamma}_- T}{1 + i\gamma_- T}\,, \tag{8.1}$$

$$D^+ := iA\frac{1}{1 + i\gamma_- T}\,, \tag{8.2}$$

the causally normally ordered form of the white noise Hamiltonian equation

$$\partial_t U_t = -i(Ab_t + A^* b_t^+ + b_t^+ T b_t + C)U_t\,, \qquad U_0 = 1\,, \tag{8.3}$$

is

$$\begin{aligned}\partial_t U_t = {} & SDb_t^+ U_t - D^+ U_t b_t + \frac{1}{2Re(\gamma_-)}(S-1)b_t^+ U_t b_t \\ & + (-\gamma_- D^+ D + i|\gamma_-|^2 D^+ TD - iC)U_t\,,\end{aligned} \tag{8.4}$$

which is equivalent to the stochastic differential equation

$$\begin{aligned}dU_t = \Bigg(& SDdB_t^+ - D^* dB_t + \frac{1}{2\mathrm{Re}(\gamma_-)}(S-1)dN_t \\ & + (-\gamma_- D^+ D + i|\gamma_-|^2 D^+ TD - iC)dt\Bigg)U_t\,.\end{aligned} \tag{8.5}$$

Remark. It is known that Eq. (8.5) is the most general unitary stochastic differential equation in the sense of Hudson–Parthasarathy.

Proof. We have seen that the causally normally ordered form of Eq. (6.1) is, in the notation (2.20):

$$\partial_t U_t = -ib_t^+ KA^* U_t - iAKU_t b_t - iTKb_t^+ U_t b_t + (-\gamma_- AKA^* - iC)U_t \,. \tag{8.6}$$

Now notice that

$$\begin{aligned}
-iA^* - \gamma_- TKA^* &= -i(1 - i\gamma_- TK)A^* = -i\left(1 - \frac{i\gamma_- T}{1 + i\gamma_- T}\right)A^* \\
&= -i\left(\frac{1 + i\gamma_- T - i\gamma_- T}{1 + i\gamma_- T}\right)A^* = -i\left(\frac{1}{1 + i\gamma_- T}\right)A^* = -iKA^* \,.
\end{aligned}$$

Let S be defined by (8.1) so that $S := KK^{*-1}$, then

$$\begin{aligned}
S - 1 &= \frac{1 - i\bar{\gamma}_- T}{1 + i\gamma_- T} - 1 = \frac{1 - i\bar{\gamma}_- T - 1 - i\gamma_- T}{1 + i\gamma_- T} \\
&= \frac{-i(\bar{\gamma}_- + \gamma_-)T}{1 + i\gamma_- T} = \frac{-2i\mathrm{Re}\,(\gamma_-)T}{1 + i\gamma_- T} = -2i\mathrm{Re}(\gamma_-)TK \,.
\end{aligned} \tag{8.7}$$

Therefore, using this, (8.6) becomes

$$\partial_t U_t = -ib_t^+ KA^* U_t - iAKU_t b_t + \frac{1}{2\mathrm{Re}(\gamma_-)}(S - 1)b_t^+ U_t b_t + (-\gamma_- AKA^* - iC)U_t \,. \tag{8.8}$$

In the notations (8.2) one has:

$$D = -i\frac{1}{1 - i\bar{\gamma}_- T}A^* \,,$$

this leads to:

$$-iKA^* = -i\frac{1}{1 + i\gamma_- T}A^* = -i\left(\frac{1 - i\bar{\gamma}_- T}{1 + i\gamma_- T}\right)\frac{1}{1 - i\bar{\gamma}_- T}A^* = SD \,.$$

For the drift term in (8.8), one has:

$$\begin{aligned}
-\gamma_- AKA^* &= -\gamma_- A\frac{1}{1 + i\gamma_- T}A^* = -\gamma_- A\frac{1 - i\bar{\gamma}_- T}{|1 + i\gamma_- T|^2}A^* \\
&= \gamma_- A\frac{1}{|1 + i\gamma_- T|^2}A^* + i|\gamma_-|^2 A\frac{T}{|1 + i\gamma_- T|^2}A^* \\
&= -\gamma_- D^+ D + i|\gamma_-|^2 A\frac{T}{|1 + i\gamma_- T|^2}A^* \,.
\end{aligned}$$

Then (8.8) is a rewriting of (8.4). Since (8.4) is a normally ordered white noise equation, it is equivalent to the stochastic differential equation (8.5). □

Theorem 8.1. *Let $j_t(x) := U_t^* x U_t$ be the forward inner flow associated to the white noise Hamiltonian equation* (6.1) *and define*:

$$\sigma(x) := S^* x S \,, \tag{8.9}$$

$$\delta^+(x) := \sigma(x)D - Dx\,, \tag{8.10}$$

$$\delta^-(x) := D^+\sigma(x) - xD^+\,, \tag{8.11}$$

$$\delta_T := i[T, x] = iTx - ixT\,, \tag{8.12}$$

$$\delta_H(x) := i[H, x] = iHx - ixH\,, \tag{8.13}$$

$$H := i\,\mathrm{Im}(\gamma_-)D^+D - |\gamma_-|^2 D^+TD + C\,, \tag{8.14}$$

$$L(x) := 2\,\mathrm{Re}(\gamma_-)\left(D^+\sigma(x)D - \frac{1}{2}\{D^+D, x\}\right) + \delta_H(x)\,. \tag{8.15}$$

Then, with K *given by* (2.20), $j_t(x)$ *satisfies the following causally normally ordered equation*:

$$\partial_t j_t(x) = b_t^+ j_t(\delta_T(K^*xK))b_t + j_t(\delta^-(x))b_t + b_t^+ j_t(\delta^+(x)) + j_t(L(x))\,, \tag{8.16}$$

which is equivalent to the stochastic differential equation:

$$dj_t(x) = j_t(\delta_T(K^*xK))dN_t + j_t(\delta^-(x))dB_t + j_t(\delta^+(x))dB_t^+ + j_t(L(x))dt\,. \tag{8.17}$$

Proof. Using Eq. (8.4), one has:

$$\begin{aligned}
\partial_t j_t(x) &= \partial_t(U_t^* x U_t) = \partial_t U_t^* x U_t + U_t^* x \partial_t U_t \\
&= \Bigg[U_t^* b_t D^+ S^* - b_t^+ U_t^* D + \frac{1}{2\mathrm{Re}(\gamma_-)} b_t^+ U_t^* b_t (S^* - 1) \\
&\quad + U_t^*(-\bar{\gamma}_- D^+ D - i|\gamma_-|^2 D^+TD + iC)\Bigg] x U_t \\
&\quad + U_t^* x \Bigg[SDb_t^+ U_t - D^+ U_t b_t + \frac{1}{2\mathrm{Re}(\gamma_-)}(S-1) b_t^+ U_t b_t \\
&\quad + (-\gamma_- D^+ D + i|\gamma_-|^2 D^+TD - iC)U_t\Bigg]\,,
\end{aligned}$$

using the identities:

$$b_t U_T = \gamma_- SDU_t + KU_t b_t\,,$$

$$U_t^* b_t^+ = \bar{\gamma}_- U_t^* D^+ S^* + b_t^+ U_t^* K^*\,,$$

one finds

$$\begin{aligned}
\partial_t j_t(x) &= U_t^* D^+ S^* x(\gamma_- SDU_t + KU_t b_t) - b_t^+ U_t^* D x U_t \\
&\quad + \frac{1}{2\mathrm{Re}(\gamma_-)} b_t^+ U_t^*(S^* - 1)x(\gamma_- SDU_t + KU_t b_t)
\end{aligned}$$

$$
\begin{aligned}
&+ (\bar{\gamma}_- U_t^* D^+ S^* + b_t^+ U_t^* K^*) x S D U_t \\
&- U_t^* x D^+ U_t b_t + \frac{1}{2\mathrm{Re}(\gamma_-)} (\bar{\gamma}_- U_t^* D^+ S^* + b_t^+ U_t^* K^*) x (S-1) U_t b_t \\
&+ U_t^* (-\bar{\gamma}_- D^+ D - i|\gamma_-|^2 D^+ T D + iC) x U_t \\
&+ U_t^* x (-\gamma_- D^+ D + i|\gamma_-|^2 D^+ T D - iC) U_t \\
= {}& b_t^+ j_t \left(\frac{1}{2\mathrm{Re}(\gamma_-)} (S^* - 1) x K + \frac{1}{2\mathrm{Re}(\gamma_-)} K^* x (S-1) \right) b_t \\
&+ j_t (D^+ S^* x K - x D^+ + \frac{\bar{\gamma}_-}{2\mathrm{Re}(\gamma_-)} D^+ S^* x (S-1)) b_t \\
&+ b_t^+ j_t \left(-Dx + K^* x S D + \frac{\gamma_-}{2\mathrm{Re}(\gamma_-)} (S^* - 1) x S D \right) \\
&+ j_t (\gamma_- D^+ S^* x S D) + j_t (\bar{\gamma}_- D^+ S^* x S D) \\
&+ j_t (-\bar{\gamma}_- D^+ D x - i|\gamma_-|^2 D^+ T D x + iCx) \\
&+ j_t (-\gamma_- x D^+ D + i|\gamma_-|^2 x D^+ T D - ixC) \,,
\end{aligned}
$$

therefore using (8.7), (8.1), one has:

$$
\begin{aligned}
\partial_t j_t(x) = {}& b_t^+ j_t (iK^* T x K - iK^* x T K) b_t + j_t (D^+ S^* x K - x D^+ - i\bar{\gamma}_- D^+ S^* x T K) b_t \\
&+ b_t^+ j_t (-Dx + K^* x S D + i(\gamma_-) K^* T x S D) + j_t (2\mathrm{Re}(\gamma_-) D^+ S^* x S D) \\
&+ j_t (-\mathrm{Re}(\gamma_-) \{D^+ D, x\} + i\,\mathrm{Im}(\gamma_-) [D^+ D, x]) + j_t (-i[|\gamma_-|^2 D^+ T D - C, x]) \\
= {}& b_t^+ j_t (i[T, K^* x K]) b_t + j_t (D^+ S^* x (1 - i\bar{\gamma}_- T) K - x D^+) b_t \\
&+ b_t^+ j_t (-Dx + K^* (1 + i(\gamma_-) T) x S D) \\
&+ j_t (2\mathrm{Re}(\gamma_-) \left(-\frac{1}{2} \{D^+ D, x\} + D^+ S^* x S D \right) \\
&+ i\,\mathrm{Im}(\gamma_-) [D^+ D, x]) + j_t (-i[|\gamma_-|^2 D^+ T D - C, x]) \\
= {}& b_t^+ j_t (i[T, K^* x K]) b_t + j_t ([D^+ S^*, x] S) b_t + b_t^+ j_t (S^{-1} [x, S D]) \\
&+ j_t (2\mathrm{Re}(\gamma_-) \left(-\frac{1}{2} \{D^+ D, x\} + D^+ S^* x S D \right) + i\,\mathrm{Im}(\gamma_-) [D^+ D, x]) \\
&+ j_t (-i[|\gamma_-|^2 D^+ T D - C, x]) \,.
\end{aligned}
$$

In the notations (8.9) and (8.10) we see that:

$$
\begin{aligned}
\delta^+(xy) &= \sigma(xy) D - Dxy = \sigma(x)\sigma(y) D - Dxy \\
&= \sigma(x)[\sigma(y) D - Dy] + \sigma(x) D y - Dxy \\
&= \sigma(x)[\sigma(y) D - Dy] + [\sigma(x) D - Dx] y = \delta^+(x) y + \sigma(x) \delta^+(y) \,.
\end{aligned}
$$

Thus δ^+ is a right-σ-derivation. Similarly δ^-, defined by (8.11) is a left-σ-derivation. In fact

$$\begin{aligned}\delta^-(xy) &= D^+\sigma(xy) - xyD^+ = D^+\sigma(x)\sigma(y) - xyD^+ \\ &= [D^+\sigma(y) - yD^+]\sigma(y) + xD^+\sigma(y) - xyD^+ = \delta^-(x)\sigma(y) + x\delta^-(y)\,.\end{aligned}$$

It follows that:

$$\begin{aligned}\partial_t j_t(x) &= b_t^+ j_t(i[T, K^*xK])b_t + j_t(D^+\sigma(x) - xD^+)b_t + b_t^+ j_t(\sigma(x)D - Dx) \\ &\quad + j_t(2\mathrm{Re}(\gamma_-)\left(-\frac{1}{2}\{D^+D, x\} + D^+\sigma(x)D\right) + i\,\mathrm{Im}(\gamma_-)[D^+D, x]) \\ &\quad + j_t(-i[|\gamma_-|^2 D^+TD - C, x])\,, \end{aligned} \tag{8.18}$$

which, in the notations introduced above, coincides with (8.16). □

Remark 8.1. Combining the results of Theorems 8.1 and 5.1 we obtain the expression of the structure maps as functions of the original Hamiltonian coefficients. In fact, in the notations of Theorem 8.1, these are given by:

$$\begin{aligned}\delta_2(x) &:= \delta_T(K^*xK)\,, \\ \delta_1(x) &:= \delta^+(x)\,, \\ \delta_{-1}(x) &:= \delta^-(x)\,, \\ \delta_0(y) &:= 2\,\mathrm{Re}(\gamma_-)\left(-\frac{1}{2}\{D^+D, x\} + D^+\sigma(x)D\right) + \delta_H(x)\,.\end{aligned}$$

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