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ON THE FOCK REPRESENTATION OF THE CENTRAL EXTENSIONS OF THE HEISENBERG ALGEBRA

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ABSTRACT. We examine the possibility of a direct Fock representation of the recently obtained non-trivial central extensions $CEHeis$ of the Heisenberg algebra, generated by elements $a, a^\dagger, h$ and E satisfying the commutation relations $[a, a^\dagger]_{CEHeis} = h$, $[h, a^\dagger]_{CEHeis} = z\,E$ and $[a, h]_{CEHeis} = \bar{z}\,E$, where a and $a^\dagger$ are dual, h is self-adjoint, E is the non-zero self-adjoint central element and $z \in \mathbb{C} \setminus \{0\}$. We define the exponential vectors associated with the $CEHeis$ Fock space, we compute their Leibniz function (inner product), we describe the action of a, $a^\dagger$ and h on the exponential vectors and we compute the moment generating and characteristic functions of the classical random variable corresponding to the self-adjoint operator $X = a + a^\dagger + h$.

Key words and phrases: Heisenberg algebra, Central extension, Fock representation.

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1. THE CENTRALLY EXTENDED HEISENBERG $*$–LIE ALGEBRA.

The generators a, $a^\dagger$ and h of the Heisenberg algebra $Heis$ satisfy the Lie algebra commutation relations

$$[a,a^\dagger]_{Heis}=h \quad ; \quad [a,h]_{Heis}=[h,a^\dagger]_{Heis}=0 \tag{1.1}$$

and the duality relations (throughout this paper we use x^* to denote the dual of x)

$$(a)^*=a^\dagger \; ; \; h^*=h \tag{1.2}$$

As shown in [1], the Heisenberg algebra can be centrally extended to the $*$–Lie algebra $CEHeis$ generated by $\{a,a^\dagger,h,E\}$ with (non-zero) commutation relations among generators

$$[a,a^\dagger]_{CEHeis}=h+\lambda\,E \; ; \; [h,a^\dagger]_{CEHeis}=z\,E \; ; \; [a,h]_{CEHeis}=\bar z\,E \tag{1.3}$$

where $\lambda\in\mathbb{R}$, $z=\Re z+i\,\Im z\in\mathbb{C}$, and $E\not\equiv 0$ is the self-adjoint central element. The central extension $CEHeis$ of $Heis$ is trivial if and only if $z=0$. Duality relations (1.2) still hold. $CEHeis$ is a nilpotent and thus solvable $*$-Lie algebra.

Renaming $h+\lambda\,E$ to just h in (1.3) we obtain the equivalent commutation relations

$$[a,a^\dagger]_{CEHeis}=h \quad ; \quad [h,a^\dagger]_{CEHeis}=z\,E \quad ; \quad [a,h]_{CEHeis}=\bar z\,E \tag{1.4}$$

From now on we will use (1.4) and (1.2) as the defining commutation relations of $CEHeis$.

2. REPRESENTATIONS OF $CEHeis$

As shown in [2], the generators a, $a^\dagger, h$ and E of $CEHeis$ can be expressed in terms of the generators of the Schrödinger $*$–Lie algebra generated by b, $b^\dagger$, b^2, ${b^\dagger}^2$, $b^\dagger\,b$ and 1 where $b^\dagger$, b and 1 are the generators of a Boson Heisenberg algebra with

$$[b,b^\dagger]=1 \quad ; \quad (b^\dagger)^*=b \tag{2.1}$$

and $CEHeis$ can therefore be represented (as a proper sub–algebra of the Schrödinger algebra) on the usual Heisenberg Fock space defined as the Hilbert space completion of the linear span of the exponential vectors $\{y(\lambda)=e^{\lambda\,b^\dagger}\,\Phi\,;\,\lambda\in\mathbb{C}\}$ (where Φ is the vacuum vector such that $b\,\Phi=0$ and $||\Phi||=1$) with respect to the inner product

$$\langle y(\lambda),y(\mu)\rangle=e^{\bar\lambda\,\mu} \tag{2.2}$$

by using the well-known representation for non-negative integers n and k

$$ {b^\dagger}^n\,b^k\,y(\lambda)=\lambda^k\,\frac{\partial^n}{\partial\,\epsilon^n}|_{\epsilon=0}\,y(\lambda+\epsilon) \tag{2.3}$$

In this section we examine the possibility of constructing a direct Fock representation of $CEHeis$ in a manner similar to that used for the (non-extended) Heisenberg algebra.

Definition 2.1. A $*$–representation of $CEHeis$ as linear, densely defined operators on a Hilbert space $\mathcal{H}$ with a cyclic unit vector Φ satisfying

$$a\,\Phi=0 \tag{2.4}$$

and such that Φ is in the domain of all the operators of the form (2.12) below, where the exponentials are meant in the sense of series expansion, is called a Fock representation.

In what follows we replace the central element E by the multiplication identity "1" and we simply write $[\cdot,\cdot]$ instead of $[\cdot,\cdot]_{CEHeis}$. Notice that if the central extension of the Heisenberg algebra is not trivial, i.e. if $z \neq 0$, then Φ cannot be an eigenvector of h with eigenvalue $\lambda_h \in \mathbb{R}$ since then, denoting by $\langle\cdot,\cdot\rangle$ the (linear in the first, conjugate linear in the second argument) Fock space inner product normalized to $\langle\Phi,\Phi\rangle = 1$, we have

$$0 \neq \bar{z} = \langle\Phi, [a,h]\,\Phi\rangle = \langle\Phi, a\,h\,\Phi\rangle = \langle\Phi, a\,\lambda_h\,\Phi\rangle = \lambda_h\,\langle\Phi, a\,\Phi\rangle = 0 \tag{2.5}$$

Therefore we cannot set $h\,\Phi = \lambda_h\,\Phi$ where $\lambda_h \in \mathbb{R}$. That, in particular, excludes the option of setting $h\,\Phi = 0$.

As shown in [2], for all $\lambda, \mu \in \mathbb{C}$ we have that

$$e^{\lambda\,a}\,e^{\mu\,a^\dagger} = e^{\mu\,a^\dagger}\,e^{\lambda\,a}\,e^{\lambda\,\mu\,h}\,e^{\frac{\lambda\,\mu}{2}\,(\mu\,z-\lambda\,\bar{z})} \tag{2.6}$$

$$a\,e^{\mu\,a^\dagger} = e^{\mu\,a^\dagger}\,(a+\mu\,h+\frac{\mu^2\,z}{2}) \tag{2.7}$$

$$e^{\lambda\,a}\,e^{\mu\,h} = e^{\mu\,h}\,e^{\lambda\,a}\,e^{\lambda\,\mu\,\bar{z}} \tag{2.8}$$

$$e^{\mu\,h}\,e^{\lambda\,a^\dagger} = e^{\lambda\,a^\dagger}\,e^{\mu\,h}\,e^{\lambda\,\mu\,z} \tag{2.9}$$

$$a\,e^{\mu\,h} = e^{\mu\,h}\,(a+\mu\,\bar{z}) \tag{2.10}$$

and

$$h\,e^{\lambda\,a^\dagger} = e^{\lambda\,a^\dagger}\,(h+\lambda\,z) \tag{2.11}$$

In general, for $u,v,w,y \in \mathbb{C}$ the centrally extended Heisenberg group elements

$$g(u,v,w,y) := e^{u\,a^\dagger}\,e^{v\,h}\,e^{w\,a}e^{yE} \tag{2.12}$$

obey (see [2] for a proof) the nonlinear group law

$$g(\alpha,\beta,\gamma,\delta)\,g(A,B,C,D) = \tag{2.13}$$

$$= g(\alpha+A, \beta+B+\gamma\,A, \gamma+C, \left(\frac{\gamma\,A^2}{2}+\beta\,A\right)\,z+\left(\frac{\gamma^2\,A}{2}+\gamma\,B\right)\,\bar{z}+\delta+D)$$

Definition 2.2. For $\alpha,\beta \in \mathbb{C}$ we define the exponential vector $\psi(\alpha,\beta)$ by

$$\psi(\alpha,\beta) = e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi \tag{2.14}$$

In the following proposition we compute the sesquilinear form ("Fock space inner product") associated with two such exponential vectors. In analogy with [5] we refer to that as the "Leibniz function".

Proposition 2.1. *(Leibniz function) For* $w \in \mathbb{C}$ *let* $f_h(w) = \langle \Phi, e^{w\,h}\,\Phi\rangle$*. Then, for all* $\alpha, \beta, A, B \in \mathbb{C}$

$$\begin{aligned}\langle \psi(\alpha,\beta), \psi(A,B)\rangle &= e^{\left((\frac{\bar{\alpha}^2 A}{2} + \bar{\alpha}\,B)\,\bar{z} + (\frac{\bar{\alpha}\,A^2}{2} + A\,\bar{\beta})\,z\right)} \langle \Phi, e^{(\bar{\alpha}\,A + B + \bar{\beta})\,h}\,\Phi\rangle \\ &= e^{\left((\frac{\bar{\alpha}^2 A}{2} + \bar{\alpha}\,B)\,\bar{z} + (\frac{\bar{\alpha}\,A^2}{2} + A\,\bar{\beta})\,z\right)} f_h(\bar{\alpha}\,A + B + \bar{\beta})\end{aligned}$$

and

$$||\psi(\alpha,\beta)||^2 = \langle \psi(\alpha,\beta), \psi(\alpha,\beta)\rangle = e^{\Re\left((\bar{\alpha}\,\alpha^2 + 2\,\alpha\,\bar{\beta})\,z\right)} f_h(|\alpha|^2 + 2\,\Re\,\beta) \tag{2.15}$$

Proof. Using (2.13) and the fact that $e^{\bar{\alpha}\,a}\,\Phi = \Phi$ we have

$$\begin{aligned}\langle \psi(\alpha,\beta), \psi(A,B)\rangle &= \langle e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi, e^{A\,a^\dagger}\,e^{B\,h}\,\Phi\rangle \\ &= \langle e^{\beta\,h}\,\Phi, e^{\bar{\alpha}\,a}\,e^{A\,a^\dagger}\,e^{B\,h}\,\Phi\rangle \\ &= \langle e^{\beta\,h}\,\Phi, e^{A\,a^\dagger}\,e^{\bar{\alpha}\,a}\,e^{\bar{\alpha}\,A\,h}\,e^{\frac{\bar{\alpha}\,A}{2}\,(A\,z - \bar{\alpha}\,\bar{z})}\,e^{B\,h}\,\Phi\rangle \\ &= e^{\frac{\bar{\alpha}\,A}{2}\,(A\,z - \bar{\alpha}\,\bar{z})} \langle e^{\beta\,h}\,\Phi, e^{A\,a^\dagger}\,e^{\bar{\alpha}\,a}\,e^{(\bar{\alpha}\,A + B)\,h}\,\Phi\rangle \\ &= e^{\frac{\bar{\alpha}\,A}{2}\,(A\,z - \bar{\alpha}\,\bar{z})} \langle e^{\beta\,h}\,\Phi, e^{A\,a^\dagger}\,e^{(\bar{\alpha}\,A + B)\,h}\,e^{\bar{\alpha}\,a}\,e^{\bar{\alpha}\,(\bar{\alpha}\,A + B)\,\bar{z}}\,\Phi\rangle \\ &= e^{\left(\frac{\bar{\alpha}^2 A}{2} + \bar{\alpha}\,B\right)\bar{z} + \frac{\bar{\alpha}\,A^2}{2}\,z} \langle e^{\beta\,h}\,\Phi, e^{A\,a^\dagger}\,e^{(\bar{\alpha}\,A + B)\,h}\,\Phi\rangle \\ &= e^{\left(\frac{\bar{\alpha}^2 A}{2} + \bar{\alpha}\,B\right)\bar{z} + \frac{\bar{\alpha}\,A^2}{2}\,z} \langle e^{\bar{A}\,a}\,e^{\beta\,h}\,\Phi, e^{(\bar{\alpha}\,A + B)\,h}\,\Phi\rangle \\ &= e^{\left(\frac{\bar{\alpha}^2 A}{2} + \bar{\alpha}\,B\right)\bar{z} + \frac{\bar{\alpha}\,A^2}{2}\,z} \langle e^{\beta\,h}\,e^{\bar{A}\,a}\,e^{\bar{A}\,\beta\,\bar{z}}\,\Phi, e^{(\bar{\alpha}\,A + B)\,h}\,\Phi\rangle \\ &= e^{\left((\frac{\bar{\alpha}^2 A}{2} + \bar{\alpha}\,B)\,\bar{z} + (\frac{\bar{\alpha}\,A^2}{2} + A\,\bar{\beta})\,z\right)} \langle e^{\beta\,h}\,\Phi, e^{(\bar{\alpha}\,A + B)\,h}\,\Phi\rangle \\ &= e^{\left((\frac{\bar{\alpha}^2 A}{2} + \bar{\alpha}\,B)\,\bar{z} + (\frac{\bar{\alpha}\,A^2}{2} + A\,\bar{\beta})\,z\right)} \langle \Phi, e^{(\bar{\alpha}\,A + B + \bar{\beta})\,h}\,\Phi\rangle \\ &= e^{\left((\frac{\bar{\alpha}^2 A}{2} + \bar{\alpha}\,B)\,\bar{z} + (\frac{\bar{\alpha}\,A^2}{2} + A\,\bar{\beta})\,z\right)} f_h(\bar{\alpha}\,A + B + \bar{\beta})\end{aligned}$$

which for $A = \alpha$ and $B = \beta$ yields

$$||\psi(\alpha,\beta)||^2 = \langle \psi(\alpha,\beta), \psi(\alpha,\beta)\rangle = e^{\Re\left((\bar{\alpha}\,\alpha^2 + 2\,\alpha\,\bar{\beta})\,z\right)} f_h(|\alpha|^2 + 2\,\Re\,\beta)$$

∎

Minimal requirements on $f_h : \mathbb{C} \to \mathbb{C}$ so that the Leibniz function $\langle \psi(\alpha,\beta), \psi(A,B)\rangle$ of Proposition 2.1 is positive semi-definite is that f_h is an analytic function such that:

(i) $f_h(0) = 1$

(ii) $f_h(w) > 0$ for all $w \in \mathbb{R}$

(iii) $\overline{f_h(w)} = f_h(\bar{w})$ for all $w \in \mathbb{C}$ (so that the Leibniz function is Hermitian)

(iv) $f_1(w) = e^w$ for all $w \in \mathbb{C}$ (so that we recover the Heisenberg algebra Fock space)

(v) $\frac{\partial}{\partial\,w^k}\,|_{w=0}\,f(w) = \langle \Phi, h^k\,\Phi\rangle \geq 0$ for all $k \geq 0$ (so that $||{a^\dagger}^n\,h^m\,\Phi|| \geq 0$ for all $n, m \geq 0$, see Corollary 2.2 below)

Corollary 2.2. *For all* $n, k \geq 0$

$$||a^{\dagger^n}\, h^k\, \Phi||^2 = \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \sum_{\theta=0}^{k\wedge\sigma} \delta_{\rho+\sigma, 2\,\theta} \binom{k}{\rho} \binom{n}{\sigma} \binom{k}{\theta} \frac{|z|^{2\,\theta}}{2^{\sigma-\theta}}\, \sigma^{(\theta)}\, n!\, \langle \Phi, h^{n+2\,k-3\,\theta}\, \Phi\rangle \tag{2.16}$$

where $x^{(y)} = x(x-1)\cdots(x-y+1)$ *with* $x^{(0)} = 1$. *By condition (v) on* f_h*,*

$$||a^{\dagger^n}\, h^k\, \Phi|| > 0 \tag{2.17}$$

Proof. By Proposition 2.1, for $\alpha, \beta, A, B \in \mathbb{R}$

$$\begin{aligned} ||a^{\dagger^n}\, h^k\, \Phi||^2 &= \frac{\partial^{n+k}}{\partial\alpha^n\, \partial\beta^k}|_{\alpha=\beta=0}\, \frac{\partial^{n+k}}{\partial A^n\, \partial B^k}|_{A=B=0}\, \langle \psi(\alpha,\beta), \psi(A,B)\rangle \\ &= \frac{\partial^{n+k}}{\partial\alpha^n\, \partial\beta^k}|_{\alpha=\beta=0}\, \frac{\partial^{n+k}}{\partial A^n\, \partial B^k}|_{A=B=0}\, e^{\left((\frac{\alpha^2\,A}{2}+\alpha\,B)\,\bar{z}+(\frac{\alpha\,A^2}{2}+A\,\beta)\,z\right)}\, \langle \Phi, e^{(\alpha\,A+B+\beta)\,h}\, \Phi\rangle \end{aligned}$$

from which the result follows with the use of the Leibniz rule for derivatives.

∎

Unlike the non-extended Heisenberg case, vectors of the form $a^{\dagger^k}\,\Phi$ are not orthogonal. For example, $\langle a^{\dagger^2}\,\Phi, a^{\dagger}\,\Phi\rangle = z \not\equiv 0$. Of course, in the Heisenberg algebra case $z = 0$. In general:

Proposition 2.3. *For all* $n \geq k \geq 0$

$$\langle a^{\dagger^n}\,\Phi, a^{\dagger^k}\,\Phi\rangle = \sum_{\rho=0}^{k} \sum_{\sigma=0}^{k-\rho} \delta_{n,\,2\,k-\rho-\frac{3\,\sigma}{2}} \binom{k}{\rho} \binom{k-\rho}{\sigma} \frac{\alpha(\sigma)\,n!}{2^{k-\rho-\sigma}}\, z^{\frac{\sigma}{2}}\, \bar{z}^{\,k-\rho-\sigma}\, \langle \Phi, h^{\rho}\,\Phi\rangle \tag{2.18}$$

where

$$\alpha(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is odd} \\ 1 & \text{if } \sigma = 0 \\ 3\cdot 5\cdot 7\cdot \ldots \cdot(2\,\sigma-1) & \text{if } \sigma \text{ is even} \end{cases} \tag{2.19}$$

Proof. By Proposition 2.1, for $\lambda, \mu \in \mathbb{R}$

$$\begin{aligned} \langle a^{\dagger^n}\,\Phi, a^{\dagger^k}\,\Phi\rangle &= \frac{\partial^{n+k}}{\partial\lambda^n\,\partial\mu^k}|_{\lambda=\mu=0}\, \langle e^{\lambda\,a^{\dagger}}\,\Phi, e^{\mu\,a^{\dagger}}\,\Phi\rangle \\ &= \frac{\partial^{n+k}}{\partial\lambda^n\,\partial\mu^k}|_{\lambda=\mu=0}\, \langle \psi(\lambda,0)\,\Phi, \psi(\mu,0)\,\Phi\rangle \\ &= \frac{\partial^{n+k}}{\partial\lambda^n\,\partial\mu^k}|_{\lambda=\mu=0}\, \langle \Phi, e^{\lambda\,\mu\,h}\,\Phi\rangle\, e^{\frac{\lambda\,\mu^2}{2}\,z+\frac{\lambda^2\,\mu}{2}\,\bar{z}} \end{aligned}$$

and the result follows by making repeated use of the Leibniz rule for derivatives and the fact that

$$\begin{aligned}
\frac{\partial^\rho}{\partial \mu^\rho}|_{\mu=0}\,\langle \Phi, e^{\lambda\,\mu\,h}\,\Phi\rangle &= \lambda^\rho\,\langle \Phi, h^\rho\,\Phi\rangle \\
\frac{\partial^\sigma}{\partial \mu^\sigma}|_{\mu=0}\,e^{\frac{\lambda\,\mu^2}{2}\,z} &= \alpha(\sigma)\,z^{\frac{\sigma}{2}}\,\lambda^{\frac{\sigma}{2}} \\
\frac{\partial^{k-\rho-\sigma}}{\partial \mu^{k-\rho-\sigma}}|_{\mu=0}\,e^{\frac{\lambda^2\mu}{2}\,\bar{z}} &= \frac{\lambda^{2\,(k-\rho-\sigma)}\,\bar{z}^{k-\rho-\sigma}}{2^{k-\rho-\sigma}} \quad \text{(with } 0^0 := 1)
\end{aligned}$$

∎

The Leibniz function of Proposition 2.1 does not define an inner product for arbitrary z and h. If it did, then we could apply the Cauchy-Schwartz inequality to $\psi(\alpha,\beta) = e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi$ and $\psi(0,0) = \Phi$, and we would have that

$$|\langle \psi(\alpha,\beta), \Phi\rangle| \leq \|\langle \psi(\alpha,\beta)\|\,\|\Phi\| \tag{2.20}$$

which, by Proposition 2.1 and the fact that $\|\Phi\| = 1$, becomes

$$|f_h(\bar{\beta})| \leq e^{\Re\left((|\alpha|^2+2\,\bar{\beta})\,\alpha\,z\right)}\,f_h(|\alpha|^2 + 2\,\Re\beta) \tag{2.21}$$

and so, by condition (ii) on f_h,

$$e^{\Re\left((|\alpha|^2+2\,\bar{\beta})\,\alpha\,z\right)} \geq \frac{|f_h(\bar{\beta})|}{f_h(|\alpha|^2 + 2\,\Re\beta)} \tag{2.22}$$

which, for $\beta = 0$ and $\alpha = 1$, implies that

$$e^{\Re z} \geq \frac{1}{\langle \Phi, e^h\,\Phi\rangle} \tag{2.23}$$

while, for $\beta = 0$ and $\alpha = i$, it implies that

$$e^{\Im z} \leq \langle \Phi, e^h\,\Phi\rangle \tag{2.24}$$

Therefore, (2.23) and (2.24) are necessary conditions for the Leibniz function of Proposition 2.1 to define an inner product.

The problem of finding examples of f_h for which the Leibniz function of Proposition 2.1 defines an inner product is open. The natural choices $f_h(w) = \cosh(w)$ and $f_h(w) = e^{c\,w}$, where $c > 0$, do not work since in both cases we can find $c_1, c_2, \alpha_1, \beta_1, \alpha_2, \beta_2$ for which $\|c_1\,\psi(\alpha_1,\beta_1) + c_2\,\psi(\alpha_1,\beta_1)\|^2$ is either negative or has non-zero imaginary part. For example, for $f_h(w) = e^w$ and $z = 1$ we find that $\|-\psi(-2,-1) + 2\,\psi(1,-2)\|^2 < 0$. Similarly, for $f_h(w) = \cosh(w)$ and $z = 1$ we find that $\|\psi(-1,1) - \psi(1,-1) - \psi(-1,-1)\|^2 < 0$.

The action of a, $a^\dagger$ and h on the exponential vectors $\psi(\alpha,\beta)$ is described in the following:

Proposition 2.4. *(The action of a, $a^\dagger$ and h on the exponential vectors) For all $\alpha,\beta \in \mathbb{C}$*

$$\begin{aligned}
a^\dagger\,\psi(\alpha,\beta) &= \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,\psi(\alpha+\epsilon,\beta)\\
a\,\psi(\alpha,\beta) &= \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,\psi(\alpha,\epsilon\,\alpha+\beta)+\left(\frac{\alpha^2\,z}{2}+\beta\,\bar z\right)\,\psi(\alpha,\beta)\\
h\,\psi(\alpha,\beta) &= \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,\psi(\alpha,\beta+\epsilon)+\alpha\,z\,\psi(\alpha,\beta)
\end{aligned}$$

In the Heisenberg case, corresponding to $h=1$, $\beta=0$ and $z=0$, letting $y(\alpha)=\psi(\alpha,0)$ we are reduced to the well known representation

$$\begin{aligned}
a^\dagger\,y(\alpha) &= \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,y(\alpha+\epsilon)\\
a\,y(\alpha) &= \alpha\,y(\alpha)\\
1\,y(\alpha) &= y(\alpha)
\end{aligned}$$

Proof. We have that

$$a^\dagger\,\psi(\alpha,\beta)=a^\dagger\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi=\frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,e^{(\alpha+\epsilon)\,a^\dagger}\,e^{\beta\,h}\,\Phi=\frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,\psi(\alpha+\epsilon,\beta)$$

Similarly, by (2.13) and the fact that $a\,\Phi=0$,

$$\begin{aligned}
a\,\psi(\alpha,\beta) &= a\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi=e^{\alpha\,a^\dagger}\,(a+\alpha\,h+\frac{\alpha^2\,z}{2})\,e^{\beta\,h}\,\Phi\\
&= e^{\alpha\,a^\dagger}\,a\,e^{\beta\,h}\,\Phi+\alpha\,e^{\alpha\,a^\dagger}\,h\,e^{\beta\,h}\,\Phi+\frac{\alpha^2\,z}{2}\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi\\
&= e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,(a+\beta\,\bar z)\,\Phi+\alpha\,e^{\alpha\,a^\dagger}\,h\,e^{\beta\,h}\,\Phi+\frac{\alpha^2\,z}{2}\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi\\
&= e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\beta\,\bar z\,\Phi+\alpha\,e^{\alpha\,a^\dagger}\,h\,e^{\beta\,h}\,\Phi+\frac{\alpha^2\,z}{2}\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi\\
&= \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,e^{\alpha\,a^\dagger}\,e^{(\epsilon\,\alpha+\beta)\,h}\,\Phi+\left(\frac{\alpha^2\,z}{2}+\beta\bar z\right)\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi\\
&= \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,\psi(\alpha,\epsilon\,\alpha+\beta)+\left(\frac{\alpha^2\,z}{2}+\beta\bar z\right)\,\psi(\alpha,\beta)
\end{aligned}$$

and also, again by (2.13),

$$\begin{aligned}
h\,\psi(\alpha,\beta) &= h\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi=e^{\alpha\,a^\dagger}\,(h+\alpha\,z)\,e^{\beta\,h}\,\Phi\\
&= e^{\alpha\,a^\dagger}\,h\,e^{\beta\,h}\,\Phi+e^{\alpha\,a^\dagger}\,\alpha\,z\,e^{\beta\,h}\,\Phi\\
&= \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,e^{\alpha\,a^\dagger}\,e^{(\epsilon+\beta)\,h}\,\Phi+\alpha\,z\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi\\
&= \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,\psi(\alpha,\beta+\epsilon)+\alpha\,z\,\psi(\alpha,\beta)
\end{aligned}$$

∎

Proposition 2.5. *On the linear span of the exponential vectors of Definition 2.2, the operators $a, a^\dagger$ and h defined in Proposition 2.4 satisfy $[a,a^\dagger]=h$, $[h,a^\dagger]=z$, $[a,h]=\bar z$, $(a^\dagger)^*=a$ and $h^*=h$.*

Proof. We have

$$\begin{aligned}
\langle \psi(\alpha,\beta), a^\dagger \psi(A,B)\rangle &= \frac{\partial}{\partial \epsilon}\,|_{\epsilon=0}\,\langle \psi(\alpha,\beta), \psi(A+\epsilon,B)\rangle \\
&= \frac{\partial}{\partial \epsilon}\,|_{\epsilon=0}\, e^{\left((\frac{\bar\alpha^2\,(A+\epsilon)}{2}+\bar\alpha\,B)\,\bar z + (\frac{\bar\alpha\,(A+\epsilon)^2}{2}+(A+\epsilon)\,\bar\beta)\,z\right)}\,\langle \Phi, e^{(\bar\alpha\,(A+\epsilon)+B+\bar\beta)\,h}\,\Phi\rangle \\
&= \frac{\partial}{\partial \epsilon}\,|_{\epsilon=0}\, e^{\left((\frac{\bar\alpha^2\,A}{2}+\bar\alpha\,B)\,\bar z + (\frac{\bar\alpha\,A^2}{2}+A\,(\epsilon\bar\alpha+\bar\beta))\,z\right)}\,\langle \Phi, e^{(\bar\alpha\,A+B+\epsilon\bar\alpha+\bar\beta)\,h}\,\Phi\rangle \\
&\quad + \left(\frac{\bar\alpha^2\,\bar z}{2}+\bar\beta z\right)\, e^{\left((\frac{\bar\alpha^2\,A}{2}+\bar\alpha\,B)\,\bar z + (\frac{\bar\alpha\,A^2}{2}+A\,\bar\beta)\,z\right)}\,\langle \Phi, e^{(\bar\alpha\,A+B+\bar\beta)\,h}\,\Phi\rangle \\
&= \frac{\partial}{\partial \epsilon}\,|_{\epsilon=0}\,\langle \psi(\alpha,\epsilon\,\alpha+\beta), \psi(A,B)\rangle + \left(\frac{\bar\alpha^2\,\bar z}{2}+\bar\beta\,z\right)\,\langle \psi(\alpha,\beta), \psi(A,B)\rangle \\
&= \langle a\,\psi(\alpha,\beta), \psi(A,B)\rangle
\end{aligned}$$

Similarly $\langle \psi(\alpha,\beta), h\,\psi(A,B)\rangle = \langle h\,\psi(\alpha,\beta), \psi(A,B)\rangle$. Thus $(a^\dagger)^* = a$ and $h^* = h$. To prove that the extended Heisenberg commutation relations (1.4) are satisfied on the exponential domain, we notice that using (2.13) to put expressions that involve $a^\dagger$, h and a in "normal order" i.e. $a^\dagger$ is on the left, h is in the middle and a is on the right, we find that

$$[a,a^\dagger]\,\psi(\alpha,\beta) = (a\,a^\dagger - a^\dagger\,a)\,\psi(\alpha,\beta) = (a\,a^\dagger - a^\dagger\,a)\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi = e^{\alpha\,a^\dagger}\,h\,e^{\beta\,h}\,\Phi + \alpha\,z\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi$$

and also

$$h\,\psi(\alpha,\beta) = h\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi = e^{\alpha\,a^\dagger}\,h\,e^{\beta\,h}\,\Phi + \alpha\,z\,e^{\alpha\,a^\dagger}\,e^{\beta\,h}\,\Phi$$

Therefore $[a,a^\dagger]\,\psi(\alpha,\beta) = h\,\psi(\alpha,\beta)$. Similarly, $[h,a^\dagger]\,\psi(\alpha,\beta) = z\,\psi(\alpha,\beta)$ and $[a,h]\,\psi(\alpha,\beta) = \bar z\,\psi(\alpha,\beta)$.
∎

3. RANDOM VARIABLES

If $s \in \mathbb{R}$, Φ is the Fock vacuum vector and X is a self-adjoint operator on a Fock space then $\langle \Phi, e^{s\,X}\,\Phi\rangle$ and $\langle \Phi, e^{i\,s\,X}\,\Phi\rangle$ can be viewed, respectively, as the moment generating and characteristic functions of a classical random variable. In this section we compute the moment generating and characteristic functions of the self-adjoint operator $X = a + a^\dagger + h$ with respect to the sesquilinear form of Proposition 2.1.

Lemma 3.1. *For all* $X, Y \in span\{a, a^\dagger, h, E\}$

$$e^{X+Y} = e^X\,e^Y\,e^{-\frac{1}{2}\,[X,Y]}\,e^{\frac{1}{6}\,(2\,[Y,[X,Y]]+[X,[X,Y]])}$$

Proof. This is a special case of the general Zassenhaus formula (converse of the BCH formula). See [2] for a proof. ∎

Lemma 3.2. *For all* $s \in \mathbb{R}$

$$e^{s\,(a+a^\dagger+h)} = e^{s\,a^\dagger}\,e^{s\,a}\,e^{\left(\frac{s^2}{2}+s\right)\,h}\,e^{\frac{s^3}{6}\,(z-2\,\bar z)+\frac{s^2}{2}\,(z-\bar z)} \tag{3.1}$$

Proof. By Lemma 3.1, with $X = s\,(a+a^\dagger)$ and $Y = s\,h$, we have

$$\begin{aligned} e^{s\,(a+a^\dagger+h)} &= e^{s\,(a+a^\dagger)}\, e^{s\,h}\, e^{-\frac{1}{2}\,[s\,(a+a^\dagger), s\,h]}\, e^{\frac{1}{6}\,(2\,[s\,h,[s\,(a+a^\dagger),s\,h]]+[s\,(a+a^\dagger),[s\,(a+a^\dagger),s\,h]])} \\ &= e^{s\,(a+a^\dagger)}\, e^{s\,h}\, e^{-\frac{s^2}{2}\,[a+a^\dagger,h]}\, e^{\frac{s^3}{6}\,(2\,[h,[a+a^\dagger,h]]+[a+a^\dagger,[a+a^\dagger,h]])} \\ &= e^{s\,(a+a^\dagger)}\, e^{s\,h}\, e^{-\frac{s^2}{2}\,(\bar z - z)} \end{aligned}$$

Similarly,

$$e^{s\,(a+a^\dagger)} = e^{s\,(a^\dagger+a)} = e^{s\,a^\dagger}\, e^{s\,a}\, e^{\frac{s^2}{2}\,h}\, e^{\frac{1}{6}\,(-2\,s^3\,\bar z + s^3\,z)}$$

Therefore

$$\begin{aligned} e^{s\,(a+a^\dagger+h)} &= e^{s\,a^\dagger}\, e^{s\,a}\, e^{\frac{s^2}{2}\,h}\, e^{\frac{1}{6}\,(-2\,s^3\,\bar z+s^3\,z)}\, e^{s\,h}\, e^{-\frac{s^2}{2}\,(\bar z - z)} \\ &= e^{s\,a^\dagger}\, e^{s\,a}\, e^{\left(\frac{s^2}{2}+s\right)\,h}\, e^{\frac{s^3}{6}\,(z-2\,\bar z)+\frac{s^2}{2}\,(z-\bar z)} \end{aligned}$$

∎

Proposition 3.3. *(Moment generating and Characteristic functions) (i) For all* $s \in \mathbb{R}$

$$\langle \Phi, e^{s\,(a+a^\dagger+h)}\,\Phi\rangle = e^{\left(\frac{s^3}{3}+s^2\right)\,\Re z}\, f_h\left(\frac{s^2}{2}+s\right) \tag{3.2}$$

(ii) For all $s \in \mathbb{R}$

$$\langle \Phi, e^{i\,s\,(a+a^\dagger+h)}\,\Phi\rangle = e^{-\left(i\,\frac{s^3}{3}+s^2\right)\,\Re z}\, f_h\left(-\frac{s^2}{2}-i\,s\right) \tag{3.3}$$

where f_h *is as in Proposition 2.1.*

Proof. By Lemma 3.2 and the fact that $e^{s\,a}\,\Phi = \Phi$ we have

$$\begin{aligned} \langle \Phi, e^{s\,(a+a^\dagger+h)}\,\Phi\rangle &= \langle \Phi, e^{s\,a^\dagger}\, e^{s\,a}\, e^{\left(\frac{s^2}{2}+s\right)\,h}\, e^{\frac{s^3}{6}\,(z-2\,\bar z)+\frac{s^2}{2}\,(z-\bar z)}\,\Phi\rangle \\ &= e^{\frac{s^3}{6}\,(z-2\,\bar z)+\frac{s^2}{2}\,(z-\bar z)}\,\langle \Phi, e^{s\,a}\, e^{\left(\frac{s^2}{2}+s\right)\,h}\,\Phi\rangle \\ &= e^{\frac{s^3}{6}\,(z-2\,\bar z)+\frac{s^2}{2}\,(z-\bar z)}\,\langle \Phi, e^{\left(\frac{s^2}{2}+s\right)\,h}\, e^{s\,a}\, e^{\left(\frac{s^2}{2}+s\right)\,s\,\bar z}\,\Phi\rangle \\ &= e^{\left(\frac{s^3}{3}+s^2\right)\,\Re z}\,\langle \Phi, e^{\left(\frac{s^2}{2}+s\right)\,h}\,\Phi\rangle \\ &= e^{\left(\frac{s^3}{3}+s^2\right)\,\Re z}\, f_h\left(\frac{s^2}{2}+s\right) \end{aligned}$$

The proof of (ii) is similar. It can also be directly obtained from (i) by replacing s by $i\,s$.

∎

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