Junior problems

J55. Let $a_0 = 1$ and $a_{n+1} = a_0 \cdot ... \cdot a_n + 4, n \geq 0$. Prove that $a_n - \sqrt{a_{n+1}} = 2$ for all $n \geq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Andrea Munaro, Italy

Observe that $a_i > 0, \forall i \geq 0$. We know that $a_0 \cdot ... \cdot a_{n-1} + 4 = a_n$, after multiplying both sides by $a_0 \cdot ... \cdot a_{n-1}$ and adding 4 to both sides we obtain

$$(a_0 \cdots a_{n-1})^2 + 4(a_0 \cdots a_{n-1}) + 4 = a_0 \cdots a_n + 4.$$

This simplifies to $(a_0 \cdot ... \cdot a_{n-1} + 2)^2 = a_{n+1}$, which in turn is equivalent to $(a_n - 4 + 2)^2 = a_{n+1}$ or $a_n - 2 = \sqrt{a_{n+1}}$, and we are done.

Second solution by Jose Hernandez Santiago, UTM Oaxaca, Mexico

By the principle of mathematical induction we show that the equality

$$a_{n+1} = (a_n - 2)^2 \tag{1}$$

holds for every $n \in \mathbb{N}$. This in turn will settle the original question. The relation in (1) clearly holds if $n = 1$. Suppose the statement is true for n, then

$$\begin{aligned}
a_{n+2} &= a_0 \cdot \ldots \cdot a_n \cdot a_{n+1} + 4 \\
&= (a_0 \cdot \ldots \cdot a_n)(a_{n+1}) + 4 \\
&= (a_{n+1} - 4)(a_{n+1}) + 4 \\
&= a_{n+1}^2 - 4a_{n+1} + 4 \\
&= (a_{n+1} - 2)^2,
\end{aligned}$$

it follows that (1) remains true for $n + 1$. Therefore, $a_{n+1} = (a_n - 2)^2$ for every $n \in \mathbb{N}$ and we are done.

Third solution by Vishal Lama, Southern Utah University, USA

We first note that (a_n) is an increasing sequence of positive integers with $a_i > 2$ for all $i \in N$. Now for all $n \geq 0$, we have

$$\begin{aligned} a_{n+1} &= a_0 \cdot \ldots \cdot a_n + 4 \\ &\Rightarrow a_{n+1}^2 - 4a_{n+1} = a_0 \cdot \ldots \cdot a_n \cdot a_{n+1} \\ &\Rightarrow a_{n+1}^2 - 4a_{n+1} + 4 = a_0 \cdot \ldots \cdot a_n \cdot a_{n+1} + 4 \\ &\Rightarrow (a_{n+1} - 2)^2 = a_{n+2} \\ &\Rightarrow a_{n+1} - \sqrt{a_{n+2}} = 2 \end{aligned}$$

The above statement is equivalent to $a_n - \sqrt{a_{n+1}} = 2$ for all $n \geq 1$, and we are done. In fact, the above relation is true for any $a_0 = k$, where $k \in N$.

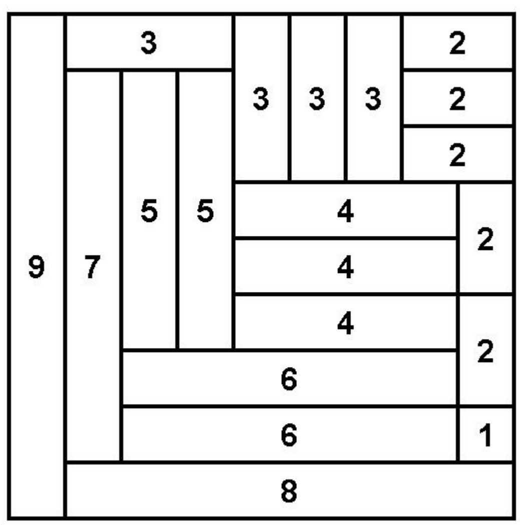
Also solved by Daniel Campos Salas, Costa Rica; Dzianis Pirshtuk, School No.41, Belarus; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Son Hong Ta, High School at Ha Noi University of Education, Vietnam

J56. Two players, A and B, play the following game: player A divides a 9×9 square into strips of unit width and various lengths. After that player B picks an integer k, $1 \leq k \leq 9$, and removes all strips of length k. Find the largest area K that B can remove, regardless the way A divides the square into strips.

Proposed by Iurie Boreico, Harvard University, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let us assume that A may force B into obtaining less than 12 area units. Then, B cannot obtain more than 11 units of area, hence there may be at most 1 strip of lengths 9, 8, 7 and 6, and at most 2 strips of lengths 5 and 4, at most 3 strips of length 3, at most 5 strips of length 2, and at most 11 strips of length 1, for a total of $9 + 8 + 7 + 6 + 10 + 8 + 9 + 10 + 11 = 78 < 9^2$ area units. Therefore, B will always obtain no less than 12 area units. But A may force B into obtaining no more than 12 area units by dividing the 9×9 square three strips of lengths 9, 8 and 7, two strips of lengths 6 and 5, three strips of length 4, four strips of length 3, five strips of length 2 and one strip of length 1, for a total of $9 + 8 + 7 + 12 + 10 + 12 + 12 + 10 + 1 = 81 = 9^2$ are units, as shown in the figure:



J57. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq 16.$$

Proposed by Mircea Becheanu, Bucharest, Romania

First solution by Ashwath Rabindranath, Vidya Mandir, India

$$\begin{aligned}
&\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\\
&\quad= a^2+b^2+c^2+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\\
&\quad= \sum_{\text{cyclic}}\left(a^2+\frac{1}{a^2}\right)+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\\
&\quad= \sum_{\text{cyclic}}\left(a^2+\frac{ab+bc+ca}{a^2}\right)+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\\
&\quad= \sum_{\text{cyclic}}\left(a^2+\frac{b}{a}+\frac{bc}{a^2}+\frac{c}{a}\right)+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\\
&\quad= \sum_{\text{cyclic}}\left(a^2+\frac{bc}{a^2}+3\frac{c}{a}+\frac{b}{a}\right)\\
&\quad\geq 1+3+9+3 \text{ (By the AM-GM inequality)}\\
&\quad= 16
\end{aligned}$$

Second solution by Daniel Campos Salas, Costa Rica

Let $E(a, b, c)$ be the left-hand side of the inequality. Note that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$, thus

$$E(a,b,c) \geq 6+(a^2+b^2+c^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

From the AM-GM inequality we obtain $a^2+\frac{1}{9a^2} \geq \frac{2}{3}$, it follows that

$$E(a,b,c) \geq 8+\frac{8}{9}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

In order to prove that $E(a,b,c) \geq 16$, it is enough to prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq 9$. From the AM-GM inequality and Cauchy-Schwarz inequality we have

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} \geq \frac{9}{ab+bc+ca} = 9,$$

and we are done.

Third solution by Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy

Taking the square root and employing the concavity of the function $\sqrt{x}$, we are left to prove that

$$\left(a+\frac{1}{b}\right)+\left(b+\frac{1}{c}\right)+\left(c+\frac{1}{a}\right) \geq 4\sqrt{3} \qquad (1)$$

Let $f(a,b,c)$ be the LHS of (1) and consider this function on the set $D \doteq \{a,b,c\colon\ ab+ac+bc=1, \quad \alpha \leq a \leq 1/\alpha,\ \alpha \leq b \leq 1/\alpha,\ \alpha \leq c \leq 1/\alpha,\ \alpha < 1/4\sqrt{3}\}$. The Weierstrass theorem on the continuous functions on compact sets guarantees the existence of both maximum and minimum of $f(a,b,c)$ on D. Moreover $f(a,b,c) > 4\sqrt{3}$ on the boundary of D. The function $F(a,b,c,\lambda) = f(a,b,c) - \lambda(ab+bc+ac-1)$ has the only critical point $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\sqrt{3})$ which corresponds to a constrained critical point of $f(a,b,c)$. Being $f(a,b,c) > 4\sqrt{3}$ on the boundary of D, the point $(a,b,c) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a minimum. Alternatively one can study the quadratic form determined by the hessian of F respect to the variables (a,b,c) and restricted to vectors tangent to the constraint $ab+bc+ca=1$. The proof is completed.

Also solved by Dzianis Pirshtuk, School No.41, Belarus; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Son Hong Ta, High School at Ha Noi University of Education, Vietnam; Vishal Lama, Southern Utah University, USA

J58. Let ABC be a triangle and let AA_1, BB_1, CC_1 be the cevians that pass through point P. Denote by X, Y, Z the midpoints of B_1C_1, A_1C_1, A_1B_1, respectively. Prove that AX, BY, CZ are concurrent.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Son Hong Ta, High School at Ha Noi University of Education, Vietnam

We will prove the generalization of this problem.

> *Let ABC be a triangle and let AA_1, BB_1, CC_1 be the cevians that pass through point P. Let P_1 the point interior triangle $A_1B_1C_1$ and denote by A_2, B_2, C_2 the intersections of A_1P_1, B_1P_1, C_1P_1 with B_1C_1, C_1A_1 and A_1B_1, respectively. Then we have AA_2, BB_2, CC_2 are concurrent.*

Proof. Denote $A_3 = AA_2 \cap BC$, $B_3 = BB_2 \cap CA$ and $C_3 = CC_2 \cap AB$. And denote $[XYZ]$ be the area of triangle XYZ.
We have

$$\frac{A_3B}{A_3C} = \frac{[AA_2B]}{[AA_2C]} = \frac{\frac{C_1A_2}{C_1B_1} \cdot [AB_1B]}{\frac{B_1A_2}{B_1C_1} \cdot [AC_1C]} = \frac{C_1A_2}{B_1A_2} \cdot \frac{\frac{AB_1}{AC} \cdot [ABC]}{\frac{AC_1}{AB} \cdot [ABC]} = \frac{C_1A_2}{B_1A_2} \cdot \frac{AB_1}{AC_1} \cdot \frac{AB}{AC}$$

Similarly, we also have two relations

$$\frac{B_3C}{B_3A} = \frac{A_1B_2}{C_1B_2} \cdot \frac{BC_1}{BA_1} \cdot \frac{BC}{BA}$$

$$\frac{C_3A}{C_3B} = \frac{B_1C_2}{A_1C_2} \cdot \frac{CA_1}{CB_1} \cdot \frac{CA}{CB}$$

Multiply these equalities and notice that AA_1, BB_1, CC_1 are concurrent and A_1A_2, B_1B_2, C_1C_2 are concurrent, thus, we get $\frac{A_3B}{A_3C} \cdot \frac{B_3C}{B_3A} \cdot \frac{C_3A}{C_3B} = 1$, i.e. AA_2, BB_2 and CC_2 are concurrent. The proof is complete.

Second solution by Francisco Javier Garcia Capitan, Spain

We use barycentric coordinates. The midpoint $B_1 = (u : 0 : w)$ with sum $u + w$ and $C_1 = (u : v : 0)$ with sum $u + v$ is $X = (u + v)B_1 + (u + w)C_1 = (u + v)(u :$

$0 : w) + (u + w)(u : v : 0) = (u(2u + u + v) : v(u + w) : w(u + v)$. Then the line AX is $(u + v)wy - v(u + w)z = 0$. We make the same calculations for the lines BY and CZ. Observe that the determinant

$$\begin{vmatrix} 0 & (u+v)w & -v(u+w) \\ -\ (u+v)w & 0 & u(v+w) \\ (u+w)v & -u(v+w) & 0 \end{vmatrix}$$

is zero, and we are done. In addition, if we calculate the intersection points of AX, BY and CZ, we get the point $Q = (u(v + w) : v(w + u) : w(u + v))$. We find a special case when $uv + vw + wu = 0$ (the sum of coordinates of Q). In this case, P is on the Steiner circumellipse of ABC and the lines AX, BY and CZ are parallel and Q is an infinity point.

Third solution by Mihai Miculita, Romania

In a barycentric system of coordinates (x, y, z), having the base triangle ABC we have: $A(1, 0, 0), B(0, 1, 0), C(0, 0, 1)$. Thus if $P(l, m, n)$, then:

$$A_1\left(0, \frac{m}{m+n}, \frac{n}{m+n}\right), B_1\left(\frac{l}{l+n}, 0, \frac{n}{l+n}\right), C_1\left(\frac{l}{l+m}, \frac{m}{m+l}, 0\right) \Rightarrow$$

the points X, Y, and Z will have the coordinates:

$$X\left(\frac{l}{2}\cdot\left(\frac{l}{l+n}+\frac{l}{l+m}\right), \frac{m}{2\cdot(l+m)}, \frac{n}{2\cdot(l+n)}\right);$$
$$Y\left(\frac{l}{2\cdot(l+m)}, \frac{m}{2}\cdot\left(\frac{l}{m+n}+\frac{l}{l+m}\right), \frac{n}{2\cdot(m+n)}\right);$$
$$Z\left(\frac{l}{2\cdot(l+n)}, \frac{m}{2\cdot(m+n)}, \frac{n}{2}\cdot\left(\frac{l}{l+n}+\frac{l}{n+m}\right)\right);$$

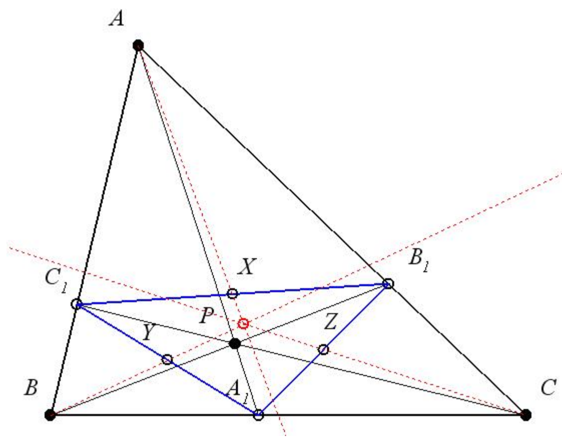
and thus the lines AX, BY, CZ will have the equations:

$$AX : \frac{(l+m)y}{m} = \frac{(l+n)z}{n}; BY : \frac{(l+m)x}{l} = \frac{(m+n)z}{n}; CZ : \frac{(l+n)x}{l} = \frac{(m+n)u}{m};$$
$$\Rightarrow \frac{(l+m)(l+n)x}{l} = \frac{(l+m)(m+n)y}{m} = \frac{(n+m)(l+n)z}{n}.$$

We can conclude that these lines are concurrent at the point:

$$P_0\left(\frac{l}{(l+m)(l+n)}, \frac{m}{(l+m)(m+n)}, \frac{n}{(n+m)(l+n)}\right),$$

and we are done.



Also solved by Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain

J59. Consider an $n \times n$ square grid. We color some of the squares in black. Prove that we can find a connected black figure consisting of three squares if

(a) $\frac{n^2}{2} + 1$ squares are colored for even n,

(b) $\frac{n(n+1)}{2}$ squares are colored for odd n.

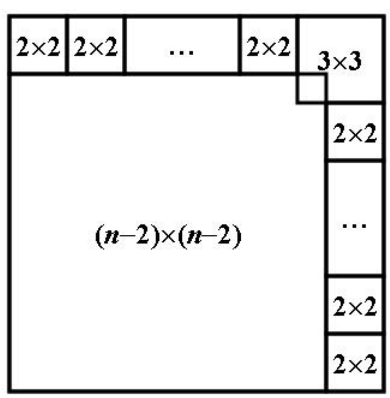
Proposed by Ivan Borsenco, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

(a) Let us assume n is even. Divide the $n \times n$ square grid into $\frac{n^2}{4}$ 2×2 squares and we have $\frac{n^2}{2} + 1$ unit squares are colored black. By the Pigeonhole Principle there will be a 2×2 square with three black squares. These squares form a connected black figure, and we are done in this case.

(b) Assume that we may color 6 squares in a 3×3 square, so that there are no three connected black squares. Since no row or column with three colored squares (they would be connected), each row and each column has exactly two black squares. Consider the top 2×3 rectangle, which contains exactly four black squares. Then, at least one of the 2×1 columns is fully colored, and the columns to its left and right cannot contain any colored square. Thus the only possiblity is that the left and right columns in the top 2×3 rectangle are fully colored. But then we may color only the central square in the bottom row, a contradiction. So as soon as $6 = \frac{3 \cdot 4}{2}$ squares are colored in a 3×3 square, a connected black figure with 3 squares appeares.

We will complete the proof for any odd n by induction. Assume that the result is true for $n-2$ and for 3. Then, consider the decomposition of the $n \times n$ square as shown in the picture:



Clearly, there is a total of $(n-3)$ 2×2 squares, one 3×3 square, and one $(n-2) \times (n-2)$ square (which has one square in common with the 3×3 square). If the total number of colored squares is

$$\frac{n^2+n}{2} = 2(n-3) + 5 + \frac{(n-2)(n-1)}{2},$$

then unless there are at least $\frac{(n-2)(n-1)}{2}$ squares are colored in the $(n-2) \times (n-2)$ square (and hence by hypothesis of induction three connected black squares would appear in this square), or at least 6 colored squares in the 3×3 square (producing the same result), or there would be more than $2(n-3)$ colored squares in the $n-3$ 2×2 squares. Therefore from the Pigeonhole principle, at least one of them would contain a connected black figure with three squares, and the conclusion follows.

Also solved by Andrea Munaro, Italy

J60. Let a, b, c be positive real numbers. Prove that

$$\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

First solution by Magkos Athanasios, Kozani, Greece

We have

$$\frac{bc}{a^2+bc}=1-\frac{a^2}{a^2+bc}$$

and analogous expressions for the other fractions. Therefore the given inequality is equivalent to

$$\sum\frac{a^2}{a^2+bc}+\sum\frac{a}{b+c}\ge 3\Leftrightarrow\sum\frac{a^2}{a^2+bc}+\sum\frac{a^2}{ab+ca}\ge 3$$

where the sums are cyclic over a, b, c. At this point we employ the well known inequality

$$\sum\frac{x_i^2}{y_i}\ge\frac{(\sum x_i)^2}{\sum y_i},$$

where i runs from 1 to $n\in N$ and $y_i>0$. Hence, the left side of the inequality is greater than or equal to

$$\frac{4(a+b+c)^2}{a^2+b^2+c^2+3(ab+bc+ca)}$$

and it suffices to prove that this is at least 3, which is true since

$$\frac{4(a+b+c)^2}{a^2+b^2+c^2+3(ab+bc+ca)}\ge 3\Leftrightarrow a^2+b^2+c^2\ge ab+bc+ca.$$

Second solution by Son Hong Ta, High School at Ha Noi University of Education, Vietnam

Write the inequality as follows

$$\sum\left(\frac{a^2}{a^2+bc}+\frac{bc}{a^2+bc}\right)\le\sum\left(\frac{a^2}{a^2+bc}+\frac{a}{b+c}\right)$$

$$\Longleftrightarrow\ 3\le\sum a^2\cdot\left(\frac{1}{a^2+bc}+\frac{1}{ab+ac}\right)$$

Using the inequality $\frac{1}{x}+\frac{1}{y}\geq\frac{4}{x+y}$, it suffices to show that

$$\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)}\geq\frac{3}{4}$$

$$\iff a^2(b+c)+b^2(c+a)+c^2(a+b)\geq\frac{3}{4}\cdot(a+b)(b+c)(c+a)$$

$$\iff \left(a^2b+bc^2\right)+\left(b^2c+ca^2\right)+\left(c^2a+ab^2\right)\geq 6\cdot abc$$

$$\iff a\cdot(b-c)^2+b\cdot(c-a)^2+c\cdot(a-b)^2\geq 0$$

Hence the problem is solved and equality occurs if and only if $a=b=c$

Third solution by Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy

We employ the result of the problem J38, piblished in this journal

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{a^2+bc}{(a+b)(a+c)}+\frac{b^2+ac}{(b+a)(b+c)}+\frac{c^2+ab}{(c+a)(c+b)}$$

and prove that

$$\frac{bc}{c^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}\leq\frac{a^2+bc}{(a+b)(a+c)}+\frac{b^2+ac}{(b+a)(b+c)}+\frac{c^2+ab}{(c+a)(c+b)}$$

Clearing the denominators we obtain

$$\sum_{\text{sym}}a^5b^4+\sum_{\text{sym}}a^5b^3c^2+2\sum_{\text{sym}}a^4b^3c^2\geq\sum_{\text{sym}}a^3b^3c^3+\sum_{\text{sym}}a^4b^4c+2\sum_{\text{sym}}a^5b^2c^2.$$

Now

$$\sum_{\text{sym}}a^5b^3c^2+\sum_{\text{sym}}a^4b^3c^2\geq 2\sum_{\text{sym}}a^5b^2c^2$$

is a consequence of the Schur's inequality:

$$\sum_{\text{sym}}x^3+\sum_{\text{sym}}xyz\geq 2\sum_{\text{sym}}x^2y$$

The inequality

$$\sum_{\text{sym}}a^5b^4+\sum_{\text{sym}}a^4b^3c^2\geq\sum_{\text{sym}}a^3b^3c^3+\sum_{\text{sym}}a^4b^4c$$

is what remains to prove, and it follows by Muirhead's theorem using $[5,4,0]\succ[4,4,1]$ and $[4,3,2]\succ[3,3,3]$. The proof is completed.

Also solved by Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Dzianis Pirshtuk, School No.41, Belarus

Senior problems

S55. Let $X = \{x_1, x_2, ..., x_n\}$ be a set of positive real numbers. Prove that there exist no more than $\frac{2^n}{\sqrt{n}}$ subsets of X, whose sum of elements is equal to 1.

Proposed by Iurie Boreico, Harvard University, USA

Solution by Jose H. Nieto S., Universidad del Zulia, Venezuela

We'll prove a stronger result, namely that the number of subsets of X, whose sum of elements is equal to 1, is no more than $\sqrt{\frac{2}{\pi}}\frac{2^n}{\sqrt{n}}$. First note that different sum 1 subsets are non-comparable with respect to the inclusion order, hence by a well known result (Sperner's lemma) its number is at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Now we recall from the proof of Wallis' formula that

$$\frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2} = \int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx < \int_0^{\frac{\pi}{2}} \sin^{2k-1} x\,dx = \frac{(2k-2)!!}{(2k-1)!!},$$

hence

$$\frac{(2k-1)!!}{(2k)!!} < \frac{1}{\sqrt{k\pi}}.$$

Now, if $n = 2k$ we have

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{2k}{k} = \frac{(2k)!^2}{(k)!^2} = \frac{2^{2k}(2k-1)!!}{(2k)!!} < \frac{2^{2k}}{\sqrt{k\pi}} = \sqrt{\frac{2}{\pi}}\frac{2^n}{\sqrt{n}},$$

while if $n = 2k+1$

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{2k+1}{k} = \frac{2k+1}{k+1}\binom{2k}{k} < \frac{2^{2k+1}}{\sqrt{k\pi}} = \sqrt{\frac{2}{\pi}}\frac{2^{2k+1}}{\sqrt{2k}} < \sqrt{\frac{2}{\pi}}\frac{2^n}{\sqrt{n}}.$$

S56. Let G be the centroid of triangle ABC. Prove that

$$\sin\angle GBC + \sin\angle GCA + \sin\angle GAB \le \frac{3}{2}.$$

Proposed by Tran Quang Hung, Ha Noi National University, Vietnam

Solution by Daniel Campos Salas, Costa Rica

Let S be the area of triangle ABC, and let m_a, m_b, m_c and h_a, h_b, h_c be the medians and altitudes from A, B, C, respectively. We wil prove the following lemma first.

Lemma. In a triangle ABC the following inequality holds:

$$(h_a^2 + h_b^2 + h_c^2) \cdot \left(\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2}\right) \le 9.$$

Proof. Let $(x, y, z) = (a^2, b^2, c^2)$. Note that

$$\begin{aligned} h_a^2 + h_b^2 + h_c^2 &= 4S^2 \cdot \frac{xy + yz + zx}{xyz} \\ &= \frac{(2(xy + yz + zx) - x^2 - y^2 - z^2)(xy + yz + zx)}{4xyz}, \end{aligned}$$

and that

$$\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2} = \frac{4(xy + yz + zx)}{(2x + 2y - z)(2y + 2z - x)(2z + 2x - y)}.$$

So we have to prove that

$$\begin{gathered} 2(xy + yz + zx)^3 \le \\ 9xyz(2x + 2y - z)(2y + 2z - x)(2z + 2x - y) + (x^2 + y^2 + z^2)(xy + yz + zx)^2. \end{gathered}$$

After expanding this inequality it follows that it is equivalent to
$(x - y)^2(y - z)^2(z - x)^2 \ge 0$, which is clearly true. $\square$

From the lemma and Cauchy-Schwarz it follows that

$$\frac{h_a}{m_b} + \frac{h_b}{m_c} + \frac{h_c}{m_a} \le 3. \tag{2}$$

Let D be the foot of the perpendicular from G to BC. Since $[AGB] = [BGA] = [CGA]$, it follows $GD = \frac{h_a}{3}$, and since $BG = \frac{2m_b}{3}$, we get $\sin\angle GBC = \frac{h_a}{2m_b}$. Analogously we write for other terms and summing up the terms we obtain the desired inequality.

S57. Suppose we have a graph with six vertices. The edges of a graph are colored in two colors. Prove that one can always find three different monochromatic cycles in it.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

It will be shown that there are actually three monochromatic cycles of the same color. A graph with 6 vertices has 15 edges. By the Pigeonhole Principle there exist 8 edges of the same color. We prove that this graph has at least three cycles of that color. Let A, B, C, D, E, F be the vertices of our graph and without loss of generality

$$\deg(A) \geq \deg(B) \geq \deg(C) \geq \deg(D) \geq \deg(E) \geq \deg(F).$$

If $\deg(F) \geq 3$, then $\frac{3 \cdot 6}{2} > 8$, thus $\deg(F) \leq 2$. Consider two cases:

1) $\deg(F) = 0, 1$. Elimination of F, would produce a graph with 5 vertices and at least 7 edges. Observe that $\deg(E) \leq 2$, otherwise $\frac{3 \cdot 5}{2} > 7$. Elimination of E, would produce a graph with 4 vertices and at least 5 edges. Thus we have $K_4 \backslash \{e\}$ that clearly contains three monochromatic cycles.

2) $\deg(F) = 2$. Without loss of generality assume that F is joined to A and B.

2a) If AB is not in the graph, we may eliminate vertex F and its corresponding edges, adding edge AB. We get a graph with 5 vertices and 7 edges, and we are sure to find at least three cycles (see case 1). In these cycles A and B will be connected through AF and BF.

2b) If AB is in the graph, we find cycle $ABFA$. Eliminating F and its corresponding edges, we need to find at least 2 cycles in a graph with 5 vertices and 6 edges. If there is one vertex with a degree less than or equal to 1, then its elimination would produce a graph with 4 vertices and 5 edges, which we know contains at least three cycles (see case 1). Because $\frac{5 \cdot 3}{2} > 6$, there exist a vertex with degree 2. Call this vertex T and assume T is connected to vertices P and S, but not to vertices Q and R. If the edge PS is in the graph, we have found a second cycle $PSTP$, and eliminating T and its corresponding edges leaves a graph with 4 vertices and 4 edges, where we are sure to find a third cycle. Finally, if PS is not in the graph, we eliminate T and edges PT and ST, but we add edge PT. This produces a graph with 4 vertices and 5 edges, where we are sure to find three cycles (see case 1), and this completes the proof.

S58. Let M, N be the midpoints of AB and CD of a cyclic quadrilateral $ABCD$. The circumscircles of triangles BAN and CMD intersect CD and AB at points P and Q, respectively. Prove that PQ passes through the intersection of the diagonals AC and BD.

Proposed by Ciupan Andrei, Bucharest, Romania

First solution by Son Hong Ta, High School at Ha Noi University of Education, Vietnam

Denote the points $I = AC \cap BD$, $E = AB \cap CD$, $F = AD \cap BC$ and $P' = FI \cap CD$.

Observe that $(EDP'C)$ form a harmonic division. Thus $ED \cdot EC = EP' \cdot EN (1)$. Furthermore, the quadrilateral $ABCD$ is cyclic, so $ED \cdot EC = EA \cdot EB \quad (2)$.

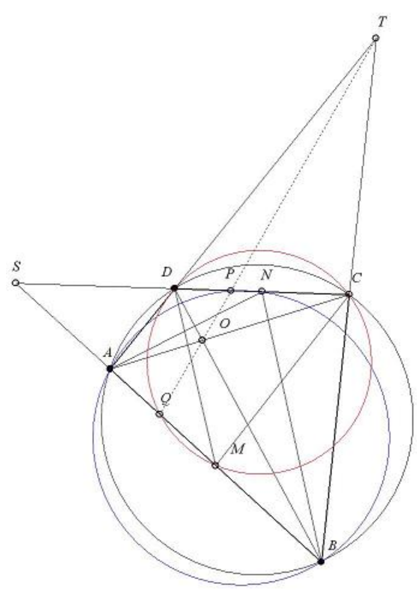
From (1) and (2) we get $EP' \cdot EN = EA \cdot EB$, it follows that P' lies on the circumcircle of triangle ABN, which implies that $P' \equiv P$, i.e. $P \in IF \quad (*)$. Similarly, we also have Q lies on the line $IF \quad (**)$

From $(*)$ and $(**)$ we have P, Q, and I are collinear, and we are done.

Second solution by Mihai Miculita, Romania

We know that $ABCD, ABNP, ABGM$ are inscribed quadrilaterals, thus

$$\overrightarrow{SA} \cdot \overrightarrow{SB} = \overrightarrow{SC} \cdot \overrightarrow{SD}$$
$$\overrightarrow{SA} \cdot \overrightarrow{SB} = \overrightarrow{SN} \cdot \overrightarrow{SP}$$
$$\overrightarrow{SC} \cdot \overrightarrow{SD} = \overrightarrow{SM} \cdot \overrightarrow{SQ},$$



which impies that $\overrightarrow{SA}\cdot\overrightarrow{SB}=\overrightarrow{SC}\cdot\overrightarrow{SD}=\overrightarrow{SM}\cdot\overrightarrow{SQ}=\overrightarrow{SN}\cdot\overrightarrow{SP}$, which means that the line PQ is the polar of the point S with respect to the circle circumscribed to $ABCD$. This in turn means that PQ passes through O, the intersection of the diagonals of $ABCD$, and through $T=AD\cap BC$, and we are done.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The result is clearly true when AB and CD are parallel, then $ABCD$ is an isosceles trapezoid. Line MN is its axis of symmetry, and the circumcircles of BAN and CMD are respectively tangent to CD and AB. Therefore $MN=PQ$ and diagonals AC and BD meet on this symmetry axis PQ, qed.

If AB and CD are not parallel, let F, E be the points where AB and CD, and AC and BD, respectively, meet, and let us assume wlog that F is closer to A and D than to B and C, respectively.

The powers of F with respect to the circumcircles of $ABCD$, $ABNP$ and $CDQM$ may be written, respectively, as $FA{\cdot}FB=FC{\cdot}FD$, $FA{\cdot}FB=FN{\cdot}FP$ and $FC\cdot FD=FQ\cdot FM$, or

$$\frac{FP}{QF}=\frac{FM}{FN}.$$

Trivially, ΔBFP and ΔNFA are similar because $BAPN$ is cyclic. Then,

$$DP=FP-FD=\frac{FA\cdot FB}{FN}-\frac{FA\cdot FB}{FC}=\frac{FA\cdot FB\,(FC-FN)}{FC\cdot FN}$$
$$=\frac{FA\cdot FB\cdot CD}{2FC\cdot FN},$$

since N is the midpoint of CD. Since ΔCFQ and ΔMFD are also similar,

$$BQ=FB-FQ=\frac{FC\cdot FD}{FA}-\frac{FC\cdot FD}{FM}=\frac{FC\cdot FD\cdot AB}{2FA\cdot FM},$$

or

$$\frac{BQ}{PD}=\frac{AB\cdot FC\cdot FN}{CD\cdot FA\cdot FM}=\frac{AB\cdot BC\cdot FN}{CD\cdot AD\cdot FM},$$

where we have used that ΔAFD and ΔCFB are similar.

Finally, using the theorem of the sine and the fact that $ABCD$ is cyclic, we find

$$\frac{DC\sin\angle ACD}{ED}=\sin\angle DEC=\sin\angle AEB=\frac{AB\sin\angle BAC}{EB},$$
$$\frac{DE}{EB}=\frac{CD\sin\angle ACD}{AB\sin\angle BAC}=\frac{AD\cdot CD}{AB\cdot CB}.$$

Thus

$$\frac{FP}{PD}\cdot\frac{DE}{EB}\cdot\frac{BQ}{QF}=1,$$

and due to the reciprocal of Menelaus' theorem, we get that P, E, Q are collinear.

S59. Consider the family of those subsets of $\{1, 2, ..., 3n\}$ whose sum of the elements is a multiple of 3. For each subset of this family compute the square of the sum of its elements. Find the sum of the numbers obtained in this way.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Fo each k in $\{0, 1, 2\}$, let us call

$$S_k(n) = \sum \binom{n}{3i+k},$$

where the sum is taken from $i = 0$ to the largest value of i such that $3i + k \leq n$. It is clear that $S_k(n)$ is the number of subsets from the set $\{1, 4, 7, ..., 3n-2\}$ whose sum is congruent k modulus 3. Also $S_k(n)$ is equal to the number of subsets out of $\{2, 5, 8, ..., 3n-1\}$ whose sum is congruent $2k$ modulus 3. Now, since any subset from the set $\{3, 6, 9, ..., 3n\}$ has sum of elements multiple of 3, then a subset of $\{1, 2, ..., 3n\}$ has sum of elements multiple of 3, iff it is the union, for some k, of a set from $S_k(n)$ subsets from $\{1, 4, 7, ..., 3n-2\}$ whose sum is congruent k modulus 3, and a set from $S_k(n)$ subsets of $\{2, 5, 8, ..., 3n-1\}$ whose sum is congruent $2k$ modulus 3, and one set from 2^n subsets of $\{3, 6, 9, ..., 3n\}$. The total number of subsets of $\{1, 2, ..., 3n\}$ whose sum of the elements is a multiple of 3 is then given by

$$2^n \left(S_0^2(n) + S_1^2(n) + S_2^2(n)\right).$$

Now,

$$S_k(n) = \binom{n}{k} + \binom{n}{3+k} + \binom{n}{6+k} + ...$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k} + \binom{n-1}{3+k-1} + \binom{n-1}{3+k} + \binom{n-1}{6+k-1} + \binom{n-1}{6+k} + ...$$

$$= 2^{n-1} - S_{k+1}(n),$$

or

$$\sum_{k=0}^{2} (S_k(n))^2 = 3 \cdot 2^{2n-2} - 2^n \sum_{k=0}^{2} S_k(n-1) + \sum_{k=0}^{2} (S_k(n-1))^2$$

$$= 2^{2n-2} + \sum_{k=0}^{2} (S_k(n-1))^2.$$

It is not difficult to prove by induction that, since $S_0(2) = \binom{2}{0} = 1$, $S_1(2) = \binom{2}{1} = 2$, $S_2(2) = \binom{2}{2} = 1$, then

$$\sum_{k=0}^{2} (S_k(2))^2 = 6 = \frac{2^4 + 2}{3},$$

$$\sum_{k=0}^{2} (S_k(n))^2 = \frac{2^{2n} + 2}{3} = 2^{2n-2} + \frac{2^{2n-2} + 2}{3},$$

and the total number of subsets is then $\frac{2^{3n}+2^{n+1}}{3}$.

Since the sum of elements of $\{1, 2, ..., 3n\}$ is $\frac{3n(3n+1)}{2}$ which is a multiple of 3, a subset has sum of elements multiple of 3 iff its complementary also has sum of elements multiple of 3. The sum of the squares of the elements of any set and the elements of its complementary is the sum of the squares of all the elements of $\{1, 2, ..., 3n\}$, i.e.,

$$\sum_{i=1}^{3n} i^2 = \frac{3n(3n+1)(6n+1)}{6} = \frac{n(3n+1)(6n+1)}{2}.$$

Since this way we count each subset twice (when we count it and we we count its complementary), the result we are looking for is half the product of the number of subsets times the sum of squares of the elements of $\{1, 2, ..., 3n\}$, i.e., the sum that we are looking for is

$$\frac{n(3n+1)(6n+1)\left(2^{3n-2} + 2^{n-1}\right)}{3}.$$

S60. Consider triangle ABC and let $\alpha(I_a), \beta(I_b), \gamma(I_c)$ be the excircles corresponding to the vertices A, B, C, respectively. Let P a point in the interior of the triangle ABC and consider its cevian AA_1, BB_1, CC_1. Denote by X, Y, Z the tangents from A', B', C' to the excircles $\alpha(I_a), \beta(I_b), \gamma(I_c)$, respectively, such that ($X \notin BC$, $Y \notin CA$, $Z \notin AB$). Prove that the lines AX, BY, CZ are concurrent.

Proposed by Cosmin Pohoata, Bucharest, Romania

Solution by Francisco Javier Garcia Capitan, Spain

We use barycentric coordinates. Let $P = (u : v : w)$ an arbitrary point (not neccesarily an interior point) of the plane of triangle ABC and $A_1B_1C_1$, with $A_1 = (0 : v : w)$, $B_1 = (u : 0 : w)$, $C_1 = (u : v : 0)$, its cevian triangle with respect to ABC. We know that the excenters are $I_a = (-a : b : c)$, $I_b = (a : -b : c)$ and $I_c = (a : b : -c)$. The excircles $\alpha(I_a)$, $\beta(I_b)$, $\gamma(I_c)$ touch the sides BC, CA, AB at $D = (0 : s-b : s-c)$, $E = (s-a : 0 : s-c)$ and $F = (s-a : s-b : 0)$, respectively, where s stands, as usual, for the semiperimeter of ABC. We can get the point X as the reflection of D with respect to the line I_aA_1, so we can calculate its coordinates, giving

$$X = \left(-((s-c)v-(s-b)w)^2, s(s-c)v^2, s(s-b)w^2\right),$$

and in a similar way for Y and Z. So, what we that the determinant

$$\begin{vmatrix} 0 & -(s-b)w^2 & (s-c)v^2 \\ (s-a)w^2 & 0 & -(s-c)u^2 \\ -(s-a)v^2 & (s-b)u^2 & 0 \end{vmatrix}$$

vanishes, and this is clearly true. In addition, we can calculate the coordinates of the intersection point Q of AX, BY, CZ, being $Q = \left(\frac{u^2}{s-a} : \frac{v^2}{s-b} : \frac{w^2}{s-c}\right)$, and we have a particular case when $P = Q = (s-a : s-b : s-c)$, the Nagel point of triangle ABC.

Undergraduate problems

U55. Let $f : [0,1] \to \mathbb{R}$ be a bijective differentiable function. Prove that there exist $c \in (0,1)$ such that

$$\int_{f(0)}^{f(1)} f^{-1}(x)dx = \frac{1}{2} \cdot f'(c).$$

Proposed by Cezar Lupu, University of Bucharest, Romania

Solution by G.R.A.20 Problem Solving Group, Roma, Italy.

First, let us use the following change of variables: $x = f(t)$. Then by the mean value theorem for integration applied with respect to the positive measure $\mu(A) = \int_A tdt$

$$\begin{aligned}\int_{f(0)}^{f(1)} f^{-1}(x)dx &= \int_0^1 td(f(t)) = \int_0^1 f'(t)d\left(\frac{t^2}{2}\right) \\ &= \int_0^1 f'(t)d\mu = f'(c) \cdot \mu([0,1]) = \frac{1}{2} \cdot f'(c)\end{aligned}$$

for some $c \in (0,1)$.

U56. Let x, y, z be positive real numbers. Prove that

$$\frac{3\sqrt{3}}{2} \leq \sqrt{x+y+z}\left(\frac{\sqrt{x}}{y+z}+\frac{\sqrt{y}}{x+z}+\frac{\sqrt{z}}{x+y}\right).$$

Proposed by Byron Schmuland, University of Alberta, Canada

First solution by Pham Huu Duc, Ballajura, Australia

Applying the Holder inequality, we get

$$(x+y+z)\left(\sum_{\text{cyc}}\frac{\sqrt{x}}{y+z}\right)^2 \geq \left[\sum_{\text{cyc}}\sqrt[3]{\left(\frac{x}{y+z}\right)^2}\right]^3.$$

Further, by the AM-GM inequality

$$2(x+y+z) = 2x+(y+z)+(y+z) \geq 3\sqrt[3]{2x(y+z)^2},$$

thus

$$\sum_{\text{cyc}}\sqrt[3]{\left(\frac{x}{y+z}\right)^2} \geq \sum_{(cyc)}\frac{3x}{\sqrt[3]{2^2}(x+y+z)} = \frac{3}{\sqrt[3]{2^2}}.$$

Hence

$$(x+y+z)\left(\sum_{\text{cyc}}\frac{\sqrt{x}}{y+z}\right)^2 \geq \left(\frac{3}{\sqrt[3]{2^2}}\right)^3 = \frac{3^3}{2^2}.$$

and by taking the square-root, the desired result follows.

Second solution by Dzianis Pirshtuk, School No.41, Belarus

Denote $a=\sqrt{x}$, $b=\sqrt{y}$, $c=\sqrt{z}$. We have to prove

$$f(a,b,c) = \sqrt{a^2+b^2+c^2}\left(\frac{a}{b^2+c^2}+\frac{b}{a^2+c^2}+\frac{c}{a^2+b^2}\right) \geq \frac{3\sqrt{3}}{2}.$$

Without loss of generality assume that $a^2+b^2+c^2=3$, because $f(a,b,c) = f(\lambda a, \lambda b, \lambda c)$ for any positive real λ. Our inequality is equivalent to

$$\frac{a}{3-a^2}+\frac{b}{3-b^2}+\frac{c}{3-c^2} \geq \frac{3}{2}.$$

From $a^2+b^2+c^2=3$ we also have $a, b, c < \sqrt{3}$.

Observe that $\frac{x}{3-x^2} \geq \frac{x^2}{2}$ for any $x \in \left(0, \sqrt{3}\right)$, as it is equivalent to $\frac{(x-1)^2(x+2)}{3-x^2} \geq 0$. Thus we get

$$\frac{a}{3-a^2} + \frac{b}{3-b^2} + \frac{c}{3-c^2} \geq \frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} = \frac{a^2+b^2+c^2}{2} = \frac{3}{2},$$

and we are done.

Third solution by Magkos Athanasios, Kozani, Greece

Since the inequality is homogeneous, without loss of generality we can assume that $x + y + z = 3$. We have to prove

$$\sum \frac{\sqrt{x}}{3-x} \geq \frac{3}{2}.$$

The arithmetic-geometric mean inequality gives

$$(\sqrt{x})^3 + 2 = (\sqrt{x})^3 + 1 + 1 \geq 3\sqrt{x}$$

and this is equivalent to $\frac{\sqrt{x}}{3-x} \geq \frac{1}{2}x$. Hence

$$\sum \frac{\sqrt{x}}{3-x} \geq \frac{1}{2}(x+y+z) = \frac{3}{2}.$$

Fourth solution by Vardan Verdiyan, Yerevan, Armenia

Since our inequality is homogeneous, without loss of generality we can assume that $x + y + z = 1$. Thus it is equivalent to

$$\frac{3\sqrt{3}}{2} \leq \sum \frac{\sqrt{x}}{1-x}.$$

Let us consider the function $f(a) = \dfrac{\sqrt{a}}{1-a} = \dfrac{1}{a^{-\frac{1}{2}} - a^{\frac{1}{2}}}$, where a is a positive real number $\Rightarrow f''(a) = \left(\dfrac{1}{a^{-1} - 2 + a^1}\right)^2 \geq 0$. By Jensen's inequality we have

$$f(x) + f(y) + f(z) \geq 3f\left(\frac{x+y+z}{3}\right) = 3f\left(\frac{1}{3}\right) \Rightarrow$$

$$\sum \frac{\sqrt{x}}{1-x} \geq 3\frac{\sqrt{\frac{1}{3}}}{1-\frac{1}{3}} = \frac{3\sqrt{3}}{2},$$

which completes our proof.

Also solved by Kee-Wai Lau, Hong Kong, China; Daniel Campos Salas, Costa Rica; Dzianis Pirshtuk; Daniel Lasaosa, Universidad Publica de Navarra, Spain

U57. Solve in positive integers the following equation: $x^3 - y^2 = 2$.

Proposed by Juan Ignacio Restrepo, Columbia

Solution by Ivan Borsenco, University of Texas at Dallas, USA

Let us write this equations as $x^3 = (y + i\sqrt{2})(y - i\sqrt{2})$. We claim that $\gcd(y + i\sqrt{2}, y - i\sqrt{2}) = 1$ in $\mathbb{Z}[i\sqrt{2}]$. If $m = \gcd(y + i\sqrt{2}, y - i\sqrt{2})$, then $m \mid 2i\sqrt{2}$. Thus, if $m = a + b\sqrt{2}$, then the norm of m divides the norm of $2i\sqrt{2}$. Hence $a^2 + 2b^2 \mid 8$.

1^{st} case: $a = 2, b = 0$, $m = 2$. We have $m^2 \mid y^2 + 2$, we get a contradiction modulo 4.

2^{nd} case: $a = 0, b = 2$, $m = 2i\sqrt{2}$. We have $m^2 \mid y^2 + 2$, again a contradiction.

3^{rd} case: $a = 0, b = 1$, $m = i\sqrt{2}$. We get $y = 0$, but $y > 0$.

Therefore we can say $y + i\sqrt{2} = (a + bi\sqrt{2})^3$ and $y - i\sqrt{2} = (a - bi\sqrt{2})^3$. Hence $1 = 3a^2b - 2b^3 = b(3a^2 - b)$, and thus $b = 1, a = 1$. The desired solution is $y = 5, x = 3$.

U58. Let $n \in \mathbb{N}$ and denote by $u(n)$ the number of ones in the binary representation of n. Example: $u(10) = u(1010_2) = 2$. Let $k, m, n \geq 0$ be integers such that $k \geq mn$.

Express $\displaystyle\sum_{i=0}^{2^k-1} (-1)^{u(i)} \binom{\binom{i}{m}}{n}$ in closed form.

Proposed by Josh Nichols-Barrer, Massachusetts Institute of Technology, USA

First solution by Josh Nichols-Barrer, MIT, USA

The given expression times $(-1)^k$ counts the number of sets S of subsets of $\{1, \ldots, 2^k - 1\}$ with the properties that:

- $|S| = n$,
- Each element of S has size m, and
- For each j with $0 \leq j < k$, there is at least one element N of a set in S for which $2^j \leq N < 2^{j+1}$.

We see this by Inclusion-Exclusion: For any set T,

$$\binom{\binom{|T|}{m}}{n}$$

is the number of sets of n m-element subsets of T, and so we just use I-E on the k intervals $[2^j, 2^{j+1})$, $j = 0, \ldots, k-1$. In the sum, the index i corresponds to a subset of these intervals (uniquely given by the binary representation of i), and the sign $(-1)^{k-u(i)}$ is just -1 to the power of the number of missing intervals.

But then if $k > mn$, the sum is just 0, and if $k = mn$, then the sum is $(-1)^{mn}$ times the number of ways to choose one integer from each of the k intervals above (which can be done in $2^0 \cdot 2^1 \cdots 2^{mn-1} = 2^{\binom{mn}{2}}$ ways), and then partition the resulting mn integers into n m-element subsets. The sum is therefore equal to

$$(-1)^{mn} 2^{\binom{mn}{2}} \frac{(mn)!}{n!(m!)^n}.$$

Second solution by Yufei Zhao, Massachusetts Institute of Technology, USA

This is an adapted finite difference method. Suppose that $P(x)$ is a polynomial of degree d, with leading coefficient a. Then $P(x) - P(x+r)$ is a polynomial of degree $d-1$ with leading coefficient $-rda$. Let $P_0(x)$ be the degree-mn polynomial

$$\binom{\binom{x}{m}}{n},$$

and define

$$P_{i+1}(x) = P_i(x) - P(x + 2^i).$$

It is straightforward to verify that the desired sum is simply $P_k(0)$. But if $k > mn$, $P_k(x) = 0$ for degree reasons. Moreover, the leading coefficient of $P_0(x)$ is $\frac{1}{n!(m!)^n}$. Therefore, $P_{mn}(x)$, which is a constant polynomial, will be equal to its leading coefficient, and be

$$\frac{1}{n!(m!)^n}(-1)^{mn}(mn) \cdot 2^0 \cdot (mn-1) \cdot 2^1 \cdots 1 \cdot 2^{mn-1} = (-1)^{mn} 2^{\binom{mn}{2}} \frac{(mn)!}{n!(m!)^n}.$$

U59. Let ϕ be Euler's totient function, where $\phi(1) = 1$. Prove that for all positive integers n we have

$$1 > \sum_{k=1}^{n} \frac{\phi(k)}{k} \ln \left(\frac{2^k}{2^k - 1} \right) > 1 - \frac{1}{2^n}.$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let $F(z)$ be the generating function associated to some combinatorial objects. Then the power seris

$$G(z) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \left(\frac{1}{1 - F(z^k)} \right)$$

is the generating function of the number of sequences of these objects which are not equivalent with respect to the circular shift. Take $F(z) = z$, that is the generating function of one object of size one, then the number of cycle sequences is equal to 1 for all positive sizes:

$$\sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \left(\frac{1}{1 - z^k} \right) = \sum_{j=1}^{\infty} z^j = \frac{z}{1 - z}.$$

Moreover, since

$$\log \left(\frac{1}{1 - z^k} \right) = \sum_{j=1}^{\infty} \frac{z^{jk}}{j} = z^k + o(z^k)$$

then

$$\sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \left(\frac{1}{1 - z^k} \right) = \sum_{k=1}^{n} \frac{\phi(k)}{k} \log \left(\frac{1}{1 - z^k} \right) + z^{n+1} h(z)$$

where $h(z)$ is a power series with positive coefficients, and

$$\sum_{k=1}^{n} \frac{\phi(k)}{k} \log \left(\frac{1}{1 - z^k} \right) = \sum_{j=1}^{n} z^j + z^{n+1} g(z) = \frac{z(1 - z^n)}{1 - z} + z^{n+1} g(z)$$

where $g(z)$ is a power series with positive coefficients. Hence, if $0 < z < 1$ then

$$\frac{z(1 - z^n)}{1 - z} < \sum_{k=1}^{n} \frac{\phi(k)}{k} \log \left(\frac{1}{1 - z^k} \right) < \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \left(\frac{1}{1 - z^k} \right) = \frac{z}{1 - z}.$$

Letting $z = \frac{1}{2}$ we find the desired inequalities.

U60. Let P_n and Q_n be the number of connected and disconnected graphs with n vertices, respectively.

a) Prove that $\lim\limits_{n\to\infty} \dfrac{P_n}{2^{\frac{n(n-1)}{2}}} = 1$, $\lim\limits_{n\to\infty} \dfrac{Q_n}{2n \cdot 2^{\frac{n(n-3)}{2}}} = 1$ for labeled graph.

b) Prove that $\lim\limits_{n\to\infty} \dfrac{n!P_n}{2^{\frac{n(n-1)}{2}}} = 1$, $\lim\limits_{n\to\infty} \dfrac{n!Q_n}{2n \cdot 2^{\frac{n(n-3)}{2}}} = 1$ for unlabeled graph.

Proposed by Iurie Boreico, Harvard University, USA

Solution by Iurie Boreico, Harvard University, USA

Let us first compute the asymptotic value of the number of unlabeled graphs with n vertices. We have $2^{\frac{n(n-1)}{2}}$ labeled graphs. Each accounts for the same unlabeled graph with other $n!-1$ graphs obtained by permuting its vertices, except those with the property that if we permute their vertices, we obtain the same graph. Let us estimate the number G_n of such graphs compared to the total number of graphs. Choose π to be a permutation and assume the graph G is preserved by π, i.e. $m(\pi(i),\pi(j)) = m(i,j)$ where $m(i,j) = 1$ if and only if i is connected with j by an edge. If π fixes a point, we can remove it and obtain a graph with $n-1$ vertices. Since we can remove a point in n ways, we get at most nG_{n-1} such graphs. We are left to look at the case where π fixes no point. Let us compute the number of graphs preserved by π. We are better off computing their ratio to the total possible number of labeled graphs. Let us decompose a permutation into cycles. Look at a cycle $i_1 \dots i_k$. We must have $m(i_r,i_s) = m(i_{r+1},i_{s+1})$ (the subscripts are modulo k) and from here we deduce that $m(i,i+j)$ depends only on j. As we can choose $j \le \frac{k}{2}$, we have at most $2^{[\frac{j}{2}]}$ possibilities for the edges within the cycle, whereas for an unlabeled graph we have $2^{\frac{k(k-1)}{2}}$. We get a ratio of $\frac{1}{2^{\frac{k^2-k-2[\frac{k}{2}]}{2}}}$. Let us look at the edges that connect two cycles $(i_1,i_2,\dots,i_k)$ and $(j_1,j_2,\dots,j_l)$. We get $m(i_r,j_s) = m(i_{r+1},j_{s+1})$ and from here it is not hard to deduce that $m(i_r,j_s)$ depends only on the residue of $r-s$ modulo $\gcd(k,l)$. So, analogously we get a ratio of at most $\frac{1}{2^{kl-\gcd(k,l)}}$. Now if $d_1,d_2,\dots,d_k$ are the cycles, we get the total ratio of at most $\frac{1}{AB}$ where $A = \prod 2^{\frac{d_i^2-d_i-2[\frac{d_i}{2}]}{2}}$, $B = \prod_{i\neq j} 2^{d_id_j-\gcd(d_i,d_j)}$. Now if we multiply A,B we get that the total ratio is at most

$$\frac{1}{2^{\frac{\sum_i(d_i^2-2d_i)+\sum_{i\neq j}2d_id_j-2\gcd(d_i,d_j)}{2}}}.$$

As $\sum d_i = n$, this rewrites as $\frac{2^{n+\sum_{i\neq j}\gcd(d_i,d_j)}}{2^{\frac{n(n-1)}{2}}}$. Now $\gcd(d_i,d_j) \le \frac{d_i+d_j}{2}$ and to conclude the ratio is at most $2^{n+\frac{k(k-1)}{2}n}/2^{\frac{n(n-1)}{2}}$. However $k \le \frac{n}{2}$ as $d_i \ge 2$ and we finally conclude the ratio is at most $2^{\frac{n(n-4)}{4}}$. Summing this for all permutations and adding the nG_{n-1} deduced above we get $\frac{G_n}{2^{\frac{n(n-1)}{2}}} \le \frac{n}{2^n}\frac{G_{n-1}}{2^{\frac{(n-1)(n-2)}{2}}} + \frac{n!}{2^{\frac{n(n-4)}{4}}}$ so $x_n = \frac{G_n}{2^{\frac{n(n-1)}{2}}}$ satisfies $x_n \le \frac{n}{2^n}x_{n-1} + \frac{n!}{2^{\frac{n(n-4)}{4}}}$. We can check that $x_n \to 0$, and even $2^{kn}n!x_n \to 0$ for any k, so the total number of graphs to "throw out" when we pass from labeled to unlabeled graphs is negligible, hence the total number of unlabeled graphs asymptotically tends to $\frac{2^{\frac{n(n-1)}{2}}}{n!}$.

Now let us estimate P_n, Q_n. For labeled graphs, we have
$Q_n = \sum_{i=1}^{n-1}\binom{n-1}{i-1}P_i 2^{\frac{(n-i)(n-i-1)}{2}}$. This is obtained if we look at the connected component containing 1 and having i vertices. We find

$$Q_n = 2^{\frac{(n-1)(n-2)}{2}} + (n-1)P_2\frac{(n-2)(n-3)}{2} + \dots.$$

Now we have $P_i \le 2^{\frac{i(i-1)}{2}}$ and dividing by $2^{n(n-3)}$, we get

$$\frac{Q_n}{2^{\frac{n(n-3)}{2}}} = 2 + c + (n-1)\frac{P_{n-1}}{2^{\frac{n(n-3)}{2}}},\ \text{where } 0 < c < \sum_{i=2}^{n-2}\frac{\binom{n-1}{i-1}}{2^{i(n-i)-n+1}}.$$

Clearly c tends to 0, so we have $\lim_{n\to\infty}\frac{Q_n}{2^{\frac{n(n-3)}{2}}} = 2 + (n-1)\lim_{n\to\infty}\frac{P_{n-1}}{2^{\frac{n(n-3)}{2}}}$.
Particularly, as $P_{n-1} \le 2^{\frac{(n-1)(n-2)}{2}}$, we conclude $\lim_{n\to\infty}\frac{Q_n}{2^{\frac{n(n-3)}{2}}} \le 2n$.
Hence $\lim_{n\to\infty}\frac{Q_n}{2^{\frac{n(n-1)}{2}}} = 0$ and so $\lim_{n\to\infty}\frac{P_n}{2^{\frac{n(n-1)}{2}}} = 1$ so $\lim_{n\to\infty}\frac{P_{n-1}}{2^{\frac{n(n-3)}{2}}} = 2$.
Thus we conclude that $\lim_{n\to\infty}\frac{Q_n}{2^{\frac{n(n-3)}{2}}} = 2n$.

To compute the values for unlabeled graph, we just need to divide by $n!$ as the amount of unlabeled graphs that correspond to less than $n!$ other graphs was proved to be negligible to the values in context. Hence for this case,

$$\lim_{n\to\infty}\frac{n!P_n}{2^{\frac{n(n-1)}{2}}} = 1, \lim_{n\to\infty}\frac{n!Q_n}{2n2^{\frac{n(n-3)}{2}}} = 1.$$

Olympiad problems

O55. For each positive integer k, let $f(k) = 4^k + 6^k + 9^k$. Prove that for all nonnegative integers m and n, $f(2^m)$ divides $f(2^n)$ whenever m is less than or equal to n.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Jose Hernandez Santiago, UTM Oaxaca, Mexico

We proceed by induction on n. If $n = 0$ or $n = 1$ the result clearly holds. Let us suppose that the claim $f(2^i) \mid f(2^n)$ for all $i \in \{0, 1, \ldots, n\}$. Since

$$\begin{aligned} f(2^{n+1}) &= 4^{2^{n+1}} + 6^{2^{n+1}} + 9^{2^{n+1}} \\ &= (4^{2^n} + 9^{2^n} + 6^{2^n})(4^{2^n} + 9^{2^n} - 6^{2^n}) \\ &= f(2^n)(4^{2^n} + 9^{2^n} - 6^{2^n}), \end{aligned}$$

it follows from the inductive hypothesis that $f(2^{n+1})$ is divisible by $f(2^i)$, for all $i \in \{0, 1, \ldots, n+1\}$. The conclusion follows.

Second solution by Vardan Verdiyan, Yerevan, Armenia

Let k be a positive integer. From the condition of the problem we have $f(k) = 4^k + 6^k + 9^k = 2^{2k} + 2^k 3^k + 3^{2k}$. Similarly $f(2k) = 2^{4k} + 2^{2k} 3^{2k} + 3^{4k}$. Thus

$$f(2k) = (2^{2k} + 3^{2k})^2 - 2^{2k} 3^{2k} = (2^{2k} + 3^{2k} + 2^k 3^k)(2^{2k} + 3^{2k} - 2^k 3^k) =$$

$$f(k)(2^{2k} + 3^{2k} - 2^k 3^k), \quad \Leftrightarrow f(k) \mid f(2k), \text{ for every } k \geq 0.$$

Hence for every nonnegative integer we have

$$f(2^{n-1}) \mid f(2^n), f(2^{n-2}) \mid f(2^{n-1}), \ldots f(1) \mid f(2).$$

It follows that $f(2^m)$ divides $f(2^n)$ whenever m is less than or equal to n.

Also solved by Vardan Verdiyan, Yerevan, Armenia; Daniel Campos Salas, Costa Rica; Dzianis Pirshtuk; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Andrea Munaro, Italy; G.R.A.20 Problem Solving Group, Roma, Italy

O56. We have k hedgehogs in the upper-left unit square of a $m \times n$ grid. Each of them moves toward the lower-right unit square of the grid, by moving each minute either one unit to the right or one unit down. What is the least possible number of grid squares that are not visited by any of the hedgehogs?

Proposed by Iurie Boreico, Harvard University, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Consider the following labelling of the grid: let (i, j) be the square on the i-th row and j-th column, where $1 \le i \le m$, $1 \le j \le n$, where the 1^{st} row is the highest and the m^{th} row the lowest, the 1^{st} column is the leftmost and the n^{th} column the rightmost.

It is clear that all squares may be visited by a hedgehog if either $m \le k$ or $n \le k$. In the first case, number m of the hedgehogs from 1 to m, and have hedgehog i move from $(1, 1)$ to $(i, 1)$, then to (i, n), and finally to (m, n). Hedgehog i has thus visited all squares in row i, and hence all squares are visited by the m hedgehogs. If $n \le k$, we proceed similarly exchanging rows and columns. Henceforth, we shall assume that $m, n > k$.

Since all the moves are either down or to the right, each hedgehog needs to make $m-1$ moves down and $n-1$ moves to the right, for a total of $m+n-2$ moves, or $m+n-1$ squares visited. If each square were visited by only one hedgehog, the total number of covered squares would be equal to $k(m+n-1)$. However, this is not so, because we know all hedgehogs visit both the upper-left square and the lower-right square. Let us consider the diagonal such that $i+j = r+1$ for $1 \le r \le k-1$. Clearly, there are r squares inside the grid along this diagonal (squares $(s, r+1-s)$ for $1 \le s \le r+1$). Each hedgehog needs to pass through one and only one of the squares of this diagonal, or, though there are k "hedgehog visits" to squares in this diagonal, only r squares are visited, leaving $k-r$ "unproductive" or "redundant" visits. The situation is symmetric around the lower-right corner. So, the total number of grid squares visited is actually not higher than the total number of "hedgehog visits" $k(m+n-1)$ minus the total number of "unproductive" visits:

$$k(m+n-1) - 2\sum_{r=1}^{k-1}(k-r) = k(m+n-1) - 2k(k-1) + 2\frac{k(k-1)}{2}$$

$$= k(m+n-k).$$

The number of squares left unvisited is then at least

$$m \cdot n - k(m+n-k) = (m-k)(n-k).$$

This number is equivalent to having the hedgehogs visit every square in all but $m-k$ rows and every square in all but $n-k$ columns. But this is perfectly possible, since we may have hedgehog i travel from square $(1,1)$ to square $(i,1)$, then to square $(i,n+1-i)$, then to square $(m,n+1-i)$, then to square (m,n). This way, all squares (i,j) in the first k rows are visited by at least one hedgehog (hedgehog i if $j \le n+1-i$, hedgehog $n+1-j$ otherwise, where obviously $i \le k$, and in the second case, $n+1-j < i$). Similarly, all squares (i,j) in the last k columns but not on the first k rows are visited by hedgehog $n+1-j$, completing the visits to all squares in the last k columns.

So the maximum number of squares not visited is $(m-k)(n-k)$ if $m,n > k$, and 0 otherwise.

O57. Consider a triangle ABC with the orthocenter H, incenter I, and circumcenter O. Denote by D the point of tangency of its circle with the side BC. Suppose that H_a is the midpoint of AH and M is the midpoint of BC. If $I \in H_aM$, prove that $AO \parallel HD$.

Proposed by Liubomir Chiriac, Princeton University, USA

First solution by Andrea Munaro, Italy

Consider the nine-point circle of triangle ABC and the midpoint M of BC. Then the circle passes through both H_a and M and H_aM is a diameter. Let N be its center. Then N is the midpoint of both H_aM and HO and so H_aOMH is a parallelogram and $HH_a = H_aA = OM$. On the other hand $AH_a \parallel OM$. Then $AOMH_a$ is a parallelogram and so $AO \parallel H_aM$. Since $I \in H_aM$, by the Feuerbach's Theorem, the point of tangency between the incircle and the nine-point circle can be H_a or M. Consider the first case. Since O and H are isogonal conjugates and AI is the angle bisector we have $\angle H_aIA = \angle IAO = \angle IAH_a$. Then $AH_a = H_aH = H_aI = ID$, and since $AH \parallel ID$, H_aIDH is a parallelogram. Finally $HD \parallel H_aM \parallel AO$. In the second case the point of tangency between the incircle of ABC and BC must be M and so $AB = AC$. Hence A, O, H, D are collinear.

Second solution by Daniel Campos Salas, Costa Rica

The case when $\angle B = \angle C$ is clear. Since the problem is symmetric with respect B and C we can assume without loss of generality that $\angle B > \angle C$. Note that I is in the interior of $\triangle AMC$, so, for H_a, I, M to be collinear $\angle A$ cannot be obtuse.

Let D' be the reflection of D on I, and let the A-excircle intersect BC at L. Note that there is a homothety with center A that maps the incircle to the A-excircle, then it follows that A, D', L are collinear. Since $BD = CL$ we get M is also the midpoint of DL, this implies that MI is parallel to $D'L$, or equivalently AL. Let the perpendicular to BC through M intersect AL at O'. Note that $O' \in \overline{AL}$, $AO' \parallel H_aM$ and $AH_a \parallel O'M$, then, $AO'MH_a$ is a parallelogram. This implies that $O'M = AH_a = OM$, and since A, O, O' are on the same side of BC it follows that $O' = O$. Then, $AO \parallel H_aM$, and given that H_a and I are the midpoints of AH and $D'D$, it follows that $H_aM \parallel HD$. From this we conclude that $AO \parallel HD$, as desired.

Also solved by Son Hong Ta, High School at Ha Noi University of Education, Vietnam

O58. Let a, b be positive integers such that $\gcd(a, b) = 1$. Find all pairs (m, n) of positive integers such that $a^m + b^m$ divides $a^n + b^n$.

Proposed by Dorin Andrica and Dorian Popa, Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The solution is any pair of the form $(m, (2k+1)\,m)$, where k is any non-negative integer, i.e., n must be an odd multiple of m.

Call $x_i = (-1)^i\, a^{(2k-i)m} b^{im} = a^{2km} r^i$, where k is any non-negative integer, and $r = -\left(\frac{b}{a}\right)^m$. Clearly, the sum of all x_i for $i = 0, 1, 2, ..., 2k$ is an integer, and

$$\sum_{i=0}^{2k} x_i = a^{2km} \sum_{i=0}^{2k} r^i = a^{2km} \frac{1 - r^{2k+1}}{1 - r} = \frac{a^{(2k+1)m} + b^{(2k+1)m}}{a^m + b^m}.$$

Thus $a^m + b^m$ divides $a^n + b^n$ for all $n = (2k+1)\,m$, where k is any non-negative integer. We prove that these are the only possible values of n.

Note that if a and b are relatively prime, then so are $a^m + b^m$ and ab. Let us assume that, for some integer n such that $m < n \le 2m$, $a^m + b^m$ divides $a^n + b^n$. Now,

$$(a^m + b^m)\left(a^{n-m} + b^{n-m}\right) - (a^n + b^n) = (ab)^{n-m}\left(a^{2m-n} + b^{2m-n}\right),$$

so $a^m + b^m$ must divide $a^{2m-n} + b^{2m-n}$, since it is prime with $(ab)^{n-m}$. But this is absurd, since $2m - n < m$. So the only n such that $0 \le n \le 2m - 1$ and $a^m + b^m$ divides $a^n + b^n$ is $n = m$. Let us complete our proof by showing by induction that, for all non-negative integer k, if $n = 2mk + d$ is such that $0 \le d \le 2m - 1$ and $a^m + b^m$ divides $a^n + b^n$, then $d = m$. The result is already proved for $k = 0$. Let us assume it true for some $k - 1$. Then,

$$(a^m + b^m)\left(a^{(2k-1)m+d} + b^{(2k-1)m+d}\right) - (ab)^m \cdot \left(a^{2(k-1)m+d} + b^{2(k-1)m+d}\right)$$
$$= a^{2km+d} + b^{2km+d} = a^n + b^n.$$

If $a^m + b^m$ divides $a^n + b^n$, and since $a^m + b^m$ is prime with $(ab)^m$, then $a^m + b^m$ must also divide $a^{2(k-1)m+d} + b^{2(k-1)m+d}$. But by hypothesis of induction, $d = m$, qed.

O59. Let P_n and Q_n be the number of connected and disconnected unlabeled graphs in the graph with n vertices. Prove that

$$P_n - Q_n \geq 2(P_{n-1} - Q_{n-1}).$$

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

Solution by Ivan Borsenco, University of Texas at Dallas, USA

Let G be a an unlabeled graph with n vertices. Define $f(G) = K_n \backslash G$ to be the complement graph of G with respect to the n-clique.

Observe the following fact if $G \in Q_n$, then $f(G) \in P_n$. Indeed, let $U, V \in G$ be two vertices such that there is no path that joins them. Clearly the edge $UV \in f(G)$. Consider $A_1, A_2 \in f(G)$. Note that either A_1U or A_1V belongs to $f(G)$. Similarly for A_2U and A_2V. Thus we can connect A_1 and A_2 through points U and V.

It is not difficult to see that $f : Q_n \to P_n$ is bijective. Now as $P_n \geq Q_n$ we have that $P_n - Q_n$ is exactly the set of connecting graphs which under f maps into itself. Let us prove by induction that the cardinality of the set satisfying this property is increasing at least twice from $n-1$ to n. We check the base case: for $n = 2$ we have $P_2 = 1, Q_2 = 1$; for $n = 3$, $P_3 = 2, Q_3 = 2$, and the base case is true. Assume that $n \geq 4$, to show the induction step we use the following construction. Observe that if $G \in P_{n-1} - Q_{n-1}$, then there is no vertex in G that is connected to all other vertices. Otherwise $f(G)$ will contain a vertex that is disconnected from all others. Now let $U_{\max}$ be the vertex with the greatest degree from $G \in P_{n-1} - Q_{n-1}$ (if there are more than one, pick an arbitrary one). We add the n^{th} vertex U_n to G in two ways to obtain two different graphs G_1, G_2:

1) G_1 - connect U_n to $U_{\max}$

2) G_2 - connect U_n to all vertices in G except $U_{\max}$.

Clearly G_1, G_2 and also $f(G_1), f(G_2)$ are connected, because U_n is connected to at least one vertex from $f(G)$. It is also clear that if $G, G' \in P_{n-1} - Q_{n-1}$, then $G_1 \not\equiv G_1'$ and $G_2 \not\equiv G_2'$. Therefore if we prove $G_1 \not\equiv G_2'$ we are done, because f determines uniquely the complement of the graph.

First of all from the construction, $U_{\max}$ is the vertex in G_1 that has the greatest degree and it is unique, because it has one connection with U_n in plus. Also we know $U_n' \in G_2'$ has $n-2$ connections, and this is the greatest number of connections possible. Thus if $G_1 \equiv G_2'$, then $U_{\max} \equiv U_n'$, and U_n' is the unique vertex with the greatest degree in G_2'. Take any vertex that is connected to U_n' it has degree at least two, because it is connected to U_n' and one of the vertices from G, as G is connected. But if consider the vertices that are connected to $U_{\max}$, there is a vertex, namely U_n, that has degree one, a contradiction. Thus $P_n - Q_n \geq 2(P_{n-1} - Q_{n-1})$, and we are done.

O60. Let $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$ be the subsets of the set $\{1, 2, ..., n\}$, such that for all i and j, A_i and B_j have exactly one common element and for all nonempty subsets T of $\{1, 2, ..., n\}$, there exists i such that the intersection of A_i and T has an odd number of elements. Prove or disprove that $B_1 = B_2 = ... = B_n$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by Gabriel Dospinescu, Ecole Normale Superieure, France

Consider the matrices $a_{ij} = 1_{j \in A_i}$ and $b_{ij} = 1_{i \in B_j}$ and observe that the hypothesis says precisely that $AB = R$, where R is the matrix having everywhere 1. On the other hand, the hypothesis says that the system $\sum_j a_{ij} x_j = 0$ has no nontrivial solution on the field F_2, thus A is invertible over F_2. But then $B = A^{-1}R$ and so all elements in any line of B are equal. But this means precisely that $B_1 = B_2 = ... = B_n$.