Junior problems

J49. Find the least k such that any k-element subset of $\{1, 2, \ldots, 10\}$ contains numbers whose sum is divisible by 11.

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Notice that for $k = 4$ the elements of the set {1,2,3,4} add up to 10. Hence, the sum of the elements of any subset will be a number between 1 and 10, thus not divisible by 11. We will prove that $k = 5$ is the least positive integer satisfying the condition.

For the sake of contradiction, assume that there is a 5-subset A such that the sum of any of its elements is not a multiple of 11. Taking everything modulo 11, the problem is equivalent to considering the set {-5,-4,-3,-2,-1,1,2,3,4,5}. Then, since the elements of any pair of the form $\{-n, n\}$ add up to a multiple of 11, where $n = 1, 2, 3, 4, 5$, then $-n$ is in A if and only if n is not. Without loss of generality, assume that 5 is in A, since A is a 5-subset and we can always multiply the original set by -1, leaving the conditions unchanged.

Suppose 2 is in A. Since $5 + 4 + 2 = 11$, 4 is not in A, so -4 is in A. But since $5 - 4 - 1 = 0$, -1 is not in A, so 1 is in A. Finally, since $5 + 3 + 2 + 1 = 11$, 3 cannot be in A, so -3 is in A. But $5 - 4 - 3 + 2 = 0$, so 2 cannot be in A.

Hence, 5 and -2 are in A. Since $5 - 3 - 2 = 0$, -3 is not in A, so 3 is in A. But $3 - 2 - 1 = 0$, so -1 is not in A, so 1 is in A. Finally, $5 - 4 - 2 + 1 = 0$, so -4 is not in A, so 4 is in A. But $5+4+3-2+1 = 11$, leading to a contradiction. This concludes that k is the least positive integer satisfying the required conditions.

Also solved by Daniel Campos Salas, Costa Rica; Vicente Vicario Garca, Huelva, Spain

J50. Let $\overline{abc}$ be a prime. Prove that $b^2 - 4ac$ cannot be a perfect square.

Proposed by Ivan Borsenco, University of Texas at Dallas

First solution by Son Hong Ta, High School for Gifted Students, Vietnam

Assume that $b^2 - 4ac$ is a perfect square and then let $b^2 - 4ac = k^2,\ k \in \mathbb{N}$
We have

$$\begin{aligned} 4a \cdot \overline{abc} &= 4a \cdot (100a + 10b + c) = 400a^2 + 40ab + 4ac \\ &= (20a + b)^2 - \left(b^2 - 4ac\right) = (20a + b + k)(20a + b - k) \quad (*) \end{aligned}$$

Since, a, b, $k \in \mathbb{N}$, then $(20a + b + k) \in \mathbb{Z}$ and $(20a + b - k) \in \mathbb{Z}$. Since, $\overline{abc}$ is a prime then according to $(*)$

$$\overline{abc} | 20a + b + k \quad \text{or} \quad \overline{abc} | 20a + b - k$$

It follows that $\overline{abc} \leq 20a+b+k$ or $\overline{abc} \leq 20a+b-k$. This leads to a contradiction since $20a + b + k < \overline{abc}$ and $20a + b - k < \overline{abc}$. Hence, $\overline{abc}$ cannot be a perfect square. This completes our proof.

Second solution by Vicente Vicario Garca, Huelva, Spain

It is clear that $a, b, c \in \mathbb{N}$, $a \neq 0$ and $\gcd(a, b, c) = 1$. If $x_1 = u$ and $x_2 = v$ are the solutions of the equation $ax^2 + bx + c = 0$, then we obtain the factorization $ax^2 + bx + c = a(x - u)(x - v)$. On the other hand, if the discriminant $D = b^2 - 4ac = h^2$, $h \in \mathbb{N}$ is a perfect square, the solutions of the equation $ax^2 + bx + c$ are rational. The factorization is such that

$$a\left(x - \frac{-b + h}{2a}\right)\left(x - \frac{-b - h}{2a}\right) = p,$$

where p si prime. We have $x = 10$ and $\overline{abc} = a \cdot 10^2 + b \cdot 10 + c = p$, thus

$$(2ax + b - h)(2ax + b + h) = 4ap.$$

As b and h have the same parity we get

$$\left(ax + \frac{b - h}{2}\right)\left(ax + \frac{b + h}{2}\right) = ap.$$

One of the factors on the left hand side should be divisible by p, but clearly $\left(ax + \frac{b-h}{2}\right), \left(ax + \frac{b+h}{2}\right) \leq 100$, a contradiction. Thus $b^2 - 4ac$ cannot be a perfect square.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain

J51. Let a, b, c the sides of a triangle. Prove that

$$(a+b)(b+c)(c+a)+(-a+b+c)(a-b+c)(a+b-c) \geq 9abc.$$

Proposed by Virgil Nicula and Cosmin Pohoata, Romania

Solution by Daniel Campos Salas, Costa Rica

Rewrite the inequality as

$$(a+b)(b+c)(c+a)-8abc \geq abc-(-a+b+c)(a-b+c)(a+b-c).$$

The left-hand side equals

$$\sum_{cyc} a(b-c)^2,$$

and the right hand side equals

$$\frac{1}{2}\sum_{cyc}(-a+b+c)(b-c)^2.$$

Perform the substitution $a = y+z, b = x+z, c = x+y$. The inequality is equivalent to

$$\sum_{cyc}(x+y)(x-y)^2 \geq \sum_{cyc} z(x-y)^2.$$

After expanding, it can be verified it is equivalent to Schur's inequality, and we're done.

Also solved by Arkady Alt, California, USA; Cristian Baba, University of Bucharest, Romania; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Son Hong Ta, High School for Gifted Students, Hanoi, Vietnam; Vicente Vicario Garca, Huelva, Spain

J52. In the Cartesian plane, mark the point with coordinates (x, y) if $x, y > 0$ and $x^2 + y^2$ is a prime number. Let l_n be the lines given by $x + y = n$. Find all positive integers n such that line l_n is fully marked in the first quadrant.

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

It is clear that n must be prime. Otherwise, let p be a prime divisor of n. Notice that the point $(p, n-p)$ is such that $x^2+y^2 = n^2-2pn+2p^2$, which is obviously divisible by p and different from p, thus it is not prime.

Let us assume that n is a prime number greater than 5. Then, it can be written as $2m + 1$, where $m \geq 3$, and hence $m - 2 > 0$, and $m + 3 < 2m + 1 = n$. If $m \equiv 1 \pmod 5$ or $m \equiv 3 \pmod 5$, then point $(m, m + 1)$ on line l_{2m+1} is such that $x^2 + y^2 = 2m^2 + 2m + 1 \equiv 0 \pmod 5$. If $m \equiv 0 \pmod 5$ or $m \equiv 4 \pmod 5$, then point $(m - 1, m + 2)$ on line l_{2m+1} is such that $x^2 + y^2 = 2m^2 + 2m + 5 \equiv 0 \pmod 5$. Finally, if $m \equiv 2 \pmod 5$, then the point $(m - 2, m + 3)$ on the line l_{2m+1} is such that $x^2 + y^2 = 2m^2 + 2m + 13 \equiv 0 \pmod 5$. Therefore, n must be a prime less than or equal to 5.

For $n = 5$, $1^2 + 4^2 = 17$ and $2^2 + 3^2 = 13$ are prime.

For $n = 3$, $1^2 + 2^2 = 5$ is prime.

For $n = 2$, $1^2 + 1^2 = 2$ is prime.

Hence, the line l_n is fully marked in the first quadrant for just $n = 2, 3, 5$.

J53. Consider a triangle ABC. Let I be its incenter and let M, N, P be the midpoints of triangle's sides. Prove that

$$IM^2 + IN^2 + IP^2 \geq r(R + r),$$

where R and r are the circumradius and the inradius, respectively.

Proposed by Cosmin Pohoata, Bucharest, Romania

First solution by Magkos Athanasios, Kozani, Greece

First we compute the expression $IM^2 + IN^2 + IP^2$. It is a known fact that triangle ABC and its medial triangle have the same centroid G. Leibniz's theorem implies that

$$IM^2 + IN^2 + IP^2 = 3IG^2 + \frac{MN^2 + NP^2 + PM^2}{3}.$$

We have $9IG^2 = s^2 + 5r^2 - 16Rr$ and $MN = \frac{b}{2}, NP = \frac{a}{2}, PM = \frac{c}{2}$, hence

$$IM^2 + IN^2 + IP^2 = \frac{s^2 + 3r^2 - 12Rr}{2}.$$

Now, what we have to prove is $\frac{s^2+3r^2-12Rr}{2} \geq r(R + r)$, or equivalently $s^2 \geq 14Rr - r^2$. But this is clear, recalling well known inequalities:

$$4R^2 + 4Rr + 3r^2 \geq s^2 \geq 16Rr - 5r^2.$$

Second solution by Daniel Campos Salas, Costa Rica

The distance between the tangency points of the incircle with the side and its midpoint is half the difference of the other two sides. Therefore the inequality is equivalent to

$$\frac{1}{4}((b - c)^2 + (c - a)^2 + (a - b)^2) + 3r^2 \geq r(R + r),$$

or

$$\frac{s}{4}((b - c)^2 + (c - a)^2 + (a - b)^2) \geq sr(R - 2r),$$

where a, b, c are the sidelenghts and s the semiperimeter.

We have that

$$\frac{s}{4}((b - c)^2 + (c - a)^2 + (a - b)^2) = \frac{a^3 + b^3 + c^3 - 3abc}{4},$$

and that

$$sr(R-2r) = \frac{abc}{4R}(R-2r) = \frac{abc - 8(s-a)(s-b)(s-c)}{4}.$$

The conclusion follows using Schur's inequality rewritten as

$$a^3 + b^3 + c^3 - 3abc \geq (a+b)(b+c)(c+a) - 8abc,$$

and the inequality of problem J51,

$$\begin{aligned}(a+b)(b+c)(c+a) - 8abc &\geq abc - (-a+b+c)(a-b+c)(a+b-c) \\ &= abc - 8(s-a)(s-b)(s-c),\end{aligned}$$

and we are done.

Third solution by Cristian Baba, University of Bucharest, Romania

Let $BC = a$, $AB = c$, $AC = b$, hence $AN = NB = \frac{c}{2}$, $AP = PC = \frac{b}{2}$, $BM = MC = \frac{a}{2}$. Also let m_a, m_b and m_c be the medians of the triangle and $2s = a + b + c$.

We will prove two useful lemmas:

Lemma 1. In a triangle ABC, where I is the incenter and M is a point in it's plane then:

$$\overrightarrow{MI} = \frac{a \cdot \overrightarrow{MA} + b \cdot \overrightarrow{MB} + c \cdot \overrightarrow{MC}}{a+b+c}.$$

Proof. Let $AD \cap BE = \{I\}$, where AD and BE are bisectors. Hence,

$$\frac{BD}{BC} \cdot \frac{CE}{EA} \cdot \frac{AI}{ID} = 1. \tag{1}$$

Because

$$\frac{BD}{DC} = \frac{c}{b},$$

we have

$$\frac{BD}{a} = \frac{c}{b+c},$$

hence, from (1)

$$\frac{AI}{ID} = \frac{b+c}{a}.$$

Furthermore,

$$\overrightarrow{MI} = \frac{1}{\frac{b+c}{a}+1} \cdot \overrightarrow{MA} + \frac{\frac{b+c}{a}}{\frac{b+c}{a}+1} \cdot \overrightarrow{MD} = \frac{a}{a+b+c} \cdot \overrightarrow{MA} + \frac{b+c}{a+b+c} \cdot \overrightarrow{MD}. \tag{2}$$

But,

$$\overrightarrow{MD} = \frac{b}{b+c} \cdot \overrightarrow{MB} + \frac{c}{b+c} \cdot \overrightarrow{MC}. \tag{3}$$

From (2) and (3) it follows that:

$$\overrightarrow{MI} = \frac{a \cdot \overrightarrow{MA} + b \cdot \overrightarrow{MB} + c \cdot \overrightarrow{MC}}{a+b+c}.$$

Lemma 2. In a triangle ABC, where I is the incenter and M is a point in it's plane then:

$$MI^2 = \frac{a \cdot MA^2 + b \cdot MB^2 + c \cdot MC^2 - abc}{a+b+c}.$$

Proof. It is a consequence of lemma 1. From lemma 1, it follows that:

$$MI^2 = \frac{\sum a \cdot MA^2 + 2\sum ab \cdot \overrightarrow{MA} \cdot \overrightarrow{MB}}{(a+b+c)^2}.$$

But

$$\begin{aligned} 2\overrightarrow{MA} \cdot \overrightarrow{MB} &= MA^2 + MB^2 - AB^2, \\ 2\overrightarrow{MA} \cdot \overrightarrow{MC} &= MA^2 + MC^2 - AC^2, \\ 2\overrightarrow{MB} \cdot \overrightarrow{MC} &= MB^2 + MC^2 - BC^2. \end{aligned}$$

Hence, by simple computation:

$$MI^2 = \frac{a \cdot MA^2 + b \cdot MB^2 + c \cdot MC^2 - abc}{a+b+c}.$$

We will use the following results, which are well-known:

1) $a^3 + b^3 + c^3 = 2s(s^2 - 3r^2 - 6Rr)$.
2) $GI^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr)$.

Now in lemma 2, we replace M with M, N and P respectively. We sum all three relations and get:

$$\sum MI^2 = \frac{a \cdot m_a^2 + b \cdot m_b^2 + c \cdot m_c^2 + \sum a\left(\frac{b^2}{4} + \frac{c^2}{4}\right) - 3abc}{a+b+c}.$$

Using the fact that $4m_a^2 = 2(b^2 + c^2) - a^2$ (and the respective symmetric relations), hence the above relation is equivalent to

$$\sum MI^2 = \frac{3a(b^2+c^2)+3b(c^2+a^2)+3c(a^2+b^2)-a^3-b^3-c^3-12abc}{4(a+b+c)} =$$

$$= \frac{(a+b+c)^3-2(a^3+b^3+c^3)-18abc}{4(a+b+c)}.$$

Thus we deduce that:

$$\sum IM^2 = \frac{s^2+3r^2-12Rr}{2}.$$

Thus, we need to prove that

$$s^2 \geq 14Rr - r^2.$$

From 2) we obtain that

$$s^2 \geq 16Rr - 5r^2.$$

But,

$$s^2 \geq 16Rr - 5r^2 \geq 14Rr - r^2,$$

where the last inequality is equivalent to $R \geq 2r$, which is Euler's result.

Remark. A stronger inequality can be proved:

$$IM^2 + IN^2 + IP^2 \geq r(2R - r).$$

Also solved by Arkady Alt, San Jose, USA; Son Hong Ta, High School for Gifted Students, Hanoi, Vietnam; Vicente Vicario Garca, Huelva, Spain

J54. For each positive integer k, find the exponent of 2 in the prime factorization of the numerator of

$$1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2k-1}.$$

Proposed by John Selfridge, USA

Solution by D. L. Silverman, USA

We have

$$\sum_{i=1}^{k}\frac{1}{2i-1}=\sum_{i=0}^{M-1}\left(\frac{1}{i\cdot 2^{r+1}+1}+\frac{1}{i\cdot 2^{r+1}+3}+\cdots+\frac{1}{(i+1)\cdot 2^{r+1}-1}\right)$$

where $k=2^rM$, M odd. If in each of the M bracketed terms, the 2^r constituent terms are collected using as common denominator P_i, the product of the denominators within the ith term, then the resulting numerators will each consist of 2^r terms, each of the form P_i over a distinct odd residue of 2^{r+1}. These ratios must themselves be the odd residues of 2^{r+1} in some order. Since the sum of the odd residues of 2^N is 2^{2N-2}, it follows that each of the M numerators is of the form $2^{2r}M_i$, M_i odd. The numerator of $\sum_{i=1}^{k}\frac{1}{2i-1}$ is thus of the form $2^{2r}Q$ where Q is the sum of an odd number of odd terms, hence odd. Hence if 2^r is the highest power of 2 dividing k, 2^{2r} is the highest power of 2 dividing the numerator of $\sum_{i=1}^{k}\frac{1}{2i-1}$.

Senior problems

S49. Find all pairs (x, y) of integers such that

$$xy + \frac{x^3 + y^3}{3} = 2007.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Anupa Murali, Derryfield School, USA

Manipulating the given equation we obtain

$$\begin{aligned} xy + \frac{x^3 + y^3}{3} &= 2007 \\ x^3 + y^3 + 3xy &= 6021 \\ (x+y)^3 - 3x^2y - 3xy^2 + 3xy &= 6021 \\ (x+y)^3 - 3xy(x+y-1) &= 6021 \end{aligned}$$

Subtracting 1 from both sides and factoring out $(x + y - 1)$, we obtain,

$$\begin{aligned} (x+y)^3 - 1 - 3xy(x+y-1) &= 6020 \\ (x+y-1)\left((x+y)(x+y+1) + 1 - 3xy\right) &= 2^2 \cdot 5 \cdot 7 \cdot 43 \end{aligned}$$

Since $((x + y)(x + y + 1) + 1 - 3xy)$ is always greater than $(x + y - 1)$, out of the 24 factors of 6020, only the first 12 factors from 1 to 70 can be potential candidates for $(x + y - 1)$. Also since $6020 \equiv 2 \pmod 3$, we can easily observe that only when $x + y - 1 \equiv 2 \pmod 3$, we will have integer solutions for xy. This again reduces the number of possible candidates for $x + y - 1$ to only four, namely 2, 5, 14, 20 and 35. Examining each of them, we find that only $x + y - 1 = 20$ gives integer solutions for (x, y). Hence using $x + y - 1 = 20$, we find the solutions for (x, y) as $(3, 18)$ and $(18, 3)$.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain

S50. Let $p \geq 5$ be a prime and let $q_1^{\beta_1} q_2^{\beta_2} \cdot \ldots \cdot q_n^{\beta_n}$ be the prime factorization of $(p-1)^p + 1$.

Prove that $\displaystyle\sum_{i=1}^{n} q_i \beta_i > p^2$.

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by Ivan Borsenco, University of Texas at Dallas

First of all we need the following lemma

Lemma. If $q_1^{\beta_1} q_2^{\beta_2} \cdot \ldots \cdot q_n^{\beta_n} = c$ and $\min(q_i) \geq 3$ then

$$\sum_{i=1}^{n} q_i \beta_i \geq \frac{\min(q_i)}{\ln(\min(q_i))} \cdot \ln c.$$

Proof. Consider a strictly increasing function $f : [3, +\infty) \to \mathbb{R}$, $f(x) = \frac{x}{\ln x}$, we have

$$\sum_{i=1}^{n} q_i \beta_i = \sum_{i=1}^{n} \frac{q_i}{\ln q_i} \cdot \beta_i \ln q_i \geq \frac{\min(q_i)}{\ln(\min(q_i))} \cdot \ln(q_1^{\beta_1} q_2^{\beta_2} \ldots q_n^{\beta_n}) = \frac{\min(q_i)}{\ln(\min(q_i))} \cdot \ln c.$$

Observe that $q_1 = p$ and $\beta_1 = 2$. Writing the binomial expansion of $(p-1)^p + 1$ we get

$$p^p - \binom{p}{1} p^{p-1} + \ldots - \binom{p}{p-2} p^2 + \binom{p}{p-1} \cdot p - 1 + 1 \equiv p^2 (\text{mod } p^3).$$

Now consider the primes $q_2, \ldots, q_n$. Notice that $n \geq 2$, as $(p-1)^p + 1 > p^2$, for $p \geq 5$ and clearly $\gcd(p-1, q_i) = 1$. Observe that

$$(p-1)^p = -1 (\text{mod } q_i),\ (p-1)^{2p} = 1 (\text{mod} q_i).$$

By Fermat's Little Theorem we have $(p-1)^{q_i - 1} = 1 (\text{mod } q_i)$.

If $(q_i - 1, 2p) = 2$, then $q_i | (p-1)^2 - 1$, or $q_i | p(p-2)$. As $q_i \neq p$, we have $q_i | (p-2)$. We obtain $(p-1)^p = ((p-2)+1)^p \equiv 1 (\text{mod } q_i)$, also we know $(p-1)^p = -1 (\text{mod } q_i)$, $\Rightarrow q_i = 2$, contradiction.

Thus $(q_i - 1, 2p) = 2p$ and $q_i = 2pt_i + 1$ for $i = 2, \ldots, n$. Therefore

$$(p-1)^p + 1 = p^2 \cdot q_2^{b_2} \cdot \ldots \cdot q_n^{b_n} = p^2 \cdot \prod_{i=2}^{n} (2pt_i + 1)^{\beta_i}.$$

Applying our lemma for $q_2^{\beta_2} \cdot q_3^{\beta_3} \cdot \ldots \cdot q_n^{\beta_n} = \frac{(p-1)^p+1}{p^2}$ we get

$$\sum_{i=2}^{n} q_i \beta_i \geq \frac{\min(q_i)}{\ln(\min(q_i))} \cdot \ln \frac{(p-1)^p+1}{p^2} > \frac{2p+1}{\ln(2p+1)} \cdot \ln \frac{(p-1)^p}{p^2}.$$

It is not difficult to see that $(p-1)^p \geq p^2 \cdot p^{p-3}$, because

$$p-1 > e > \left(1+\frac{1}{p-1}\right)^{p-1}.$$

The last step is to observe that $\frac{\ln p}{\ln(2p+1)} \geq \frac{2}{3}$, because $p^3 \geq (2p+1)^2$, for $p \geq 5$. Thus we get

$$\frac{2p+1}{\ln(2p+1)} \cdot \ln \frac{(p-1)^p}{p^2} \geq (2p+1)(p-3)\frac{\ln p}{\ln(2p+1)} \geq \frac{2}{3}(2p+1)(p-3).$$

Therefore,

$$\sum_{i=2}^{n} q_i \beta_i \geq \frac{2}{3}(2p+1)(p-3)$$

and

$$\sum_{i=1}^{n} q_i \beta_i \geq \frac{2}{3}(2p+1)(p-3) + 2p = \frac{4p^2-4p-6}{3} > p^2,$$

for $p \geq 7$. For $p = 5$, we can check that $4^5 + 1 = 5^2 \cdot 41$ and it satisfies our condition.

S51. Consider a quadrilateral $ABCD$ with no two sides parallel. Let O be the intersection of its diagonals and let $E \in AB \cap CD$ and $F \in AD \cap BC$. Parallels through O to the sides CD, DA, AB, BC intersect lines AB, BC, CD, DA at M, N, P, Q, respectively. Prove that M, N, P, Q are collinear and that the line that contain them is parallel to EF.

Proposed by Mihai Miculita, Oradea, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let us consider the points M, N, Q, on the sides AB, BF, FA of ABF. Since ON is parallel to DF,

$$\frac{BN}{NF} = \frac{OB}{OD}.$$

Since, OQ is parallel to CF,

$$\frac{AQ}{QF} = \frac{OA}{OC}.$$

Since OM is parallel to CE and DE,

$$\frac{AM}{MB} = \frac{AM}{ME} \cdot \frac{ME}{MB} = \frac{OA}{OC} \cdot \frac{OD}{OB}.$$

Finally, we find

$$\frac{BN}{NF} \cdot \frac{FQ}{QA} \cdot \frac{AM}{MB} = 1.$$

Hence, by the reciprocal of the Menelaus theorem, M, N, Q are collinear. In an analogous way we prove that M, N, P are collinear. Hence, M, N, P, Q are collinear. Now, since, OP is parallel to BE,

$$\frac{DP}{PE} = \frac{OD}{OB}.$$

Since, OQ is parallel to BF,

$$\frac{DQ}{QF} = \frac{OD}{OB} = \frac{DP}{PE}.$$

Hence, PQ is parallel to EF and we are done.

Also solved by Son Hong Ta, High School for Gifted Students, Hanoi, Vietnam

S52. Let a, b, c, d be prime numbers such that $a \neq b$ and $1 < a \leq c$. Suppose that for all sufficiently large n the numbers $an + b$ and $cn + d$ have the same sum of digits in all bases $2, 3, ..., a-1$. Prove that $a = c$ and $b = d$.

Proposed by Gabriel Dospinescu, "Louis le Grand" College, Paris

Solution by Gabriel Dospinescu, "Louis le Grand" College, Paris

Let x be a primitive root modulo a, chosen such that $2 \leq x \leq a-1$ (we will identify integers and their classes modulo a), write $b = x^u \pmod{a}$ for some nonnegative u. Actually, if $u = 0$ we may change it into $p-1$, so we may always assume $u > 0$. Now, define the sequence $n_k = \frac{x^{u(k\varphi(a)+1)}-b}{a}$, which is a positive integer for large k. Because $an_k + b$ has sum of digits 1 when written in base x, so does $cn_k + d$, thus $cn_k + d$ is a power of x, say x^{m_k}. Because n_k tends to ∞, so does m_k. On the other hand, $cx^{u(k\varphi(a)+1} + ad - bc = x^{m_k}$ and thus $ad - bc = 0$ because it is divisible by larger and larger powers of x. Thus $ad = bc$ and from the assumptions on a, b, c, d the conclusion follows.

S53. Let ABC be a triangle and let E, F be the feet of the angle bisectors of B and C, respectively. Denote by O the circumcenter of triangle ABC and by I_a the center of the excircle corresponding to vertex A. Prove that $OI_a \perp EF$.

Proposed by Cosmin Pohoata, Bucharest, Romania

Solution by Son Hong Ta, High School for Gifted Students, Hanoi, Vietnam

Let I be the incenter of $\triangle ABC$ and G, H be the second intersections of BI, CI with (O), respectively. Define: P, Q be the projections of O on I_aB, I_aC, respectively.

We have

$$OP \perp I_aB,\ GB \perp I_aB \Rightarrow OP \parallel BG \Rightarrow OP = \frac{1}{2}BG$$

Similarly,

$$OQ \perp I_aC,\ HC \perp I_aC \Rightarrow OQ \parallel CH \Rightarrow OQ = \frac{1}{2}CH$$

Hence,

$$\frac{OP}{OQ} = \frac{BG}{CH} \tag{1}$$

On the otherhand,

$$\begin{cases} \triangle ICE \sim \triangle BGH \\ \triangle IBF \sim \triangle CHG \end{cases} \Longrightarrow \begin{cases} IE \cdot BG = IC \cdot HB \\ IF \cdot CH = IB \cdot GC \end{cases}$$

And since $\triangle IHB \sim \triangle IGC$, we get

$$\frac{BG}{CH} = \frac{IF}{IE} \quad (2)$$

According to (1), (2) we get

$$\frac{OP}{OQ} = \frac{BH}{CH} = \frac{IF}{IE}$$

But since $OP \parallel IE$, $OQ \parallel IF$, we have that $\triangle OPQ \sim \triangle IFE \Longrightarrow \widehat{OPQ} = \widehat{IFE} \Longrightarrow \widehat{OI_aQ} = \widehat{IFE} \Longrightarrow OI_a \perp EF$, and we are done.

Second solution by Daniel Campos Salas, Costa Rica

We will use the usual triangle notations. ABC: sidelenghts a, b, c, semiperimeter s, circumradius R, and exradius r_a. It is well-known that for four distinct points W, X, Y, Z, WX is perpendicular to YZ if and only if $WY^2 + XZ^2 = WZ^2 + XY^2$, or equivalently

$$WY^2 - WZ^2 = XY^2 - XZ^2.$$

Hence, it's enough to show that the expression $OF^2 - FI_a^2$ is symmetric with respect to B and C. From the Cosine Law in triangle AOF we have that

$$\begin{aligned} OF^2 &= AO^2 + AF^2 - 2AO \cdot AF \cos\left(\left|\frac{\pi}{2} - C\right|\right) \\ &= R^2 + AF^2 - 2R \cdot AF \sin C \\ &= R^2 + AF^2 - AF \cdot c. \end{aligned}$$

In addition,

$$FI_a^2 = r_a^2 + (s - AF)^2 = r_a^2 + s^2 - (a + b + c)AF + AF^2.$$

This implies

$$\begin{aligned} OF^2 - FI_a^2 &= R^2 - AF \cdot c - r_a^2 - s^2 + (a + b + c)AF \\ &= R^2 - r_a^2 - s^2 + AF(a + b) \\ &= R^2 - r_a^2 - s^2 + bc, \end{aligned}$$

and we're done.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain

S54. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2-bc}{4a^2+4b^2+c^2}+\frac{b^2-ca}{4b^2+4c^2+a^2}+\frac{c^2-ab}{4c^2+4a^2+b^2}\geq 0$$

and find all equality cases.

Proposed by Vasile Cartoaje, University of Ploiesti,Romania

Solution by Cristian Baba, University of Bucharest, Romania

Lemma. If $A=\dfrac{a^2-bc}{4a^2+4b^2+c^2}+\dfrac{b^2-ca}{4b^2+4c^2+a^2}+\dfrac{c^2-ab}{4c^2+4a^2+b^2}$ and

$$B=\frac{1}{2}\left(\frac{(a^2-bc)^2(2b-c)^2}{(a^2+2b^2)(c^2+2a^2)(4a^2+4b^2+c^2)}+\frac{(b^2-ca)^2(2c-a)^2}{(b^2+2c^2)(a^2+2b^2)(4b^2+4c^2+a^2)}+\frac{(c^2-ab)^2(2a-b)^2}{(c^2+2a^2)(b^2+2c^2)(4c^2+4a^2+b^2)}\right)$$

then $A=B$, for all $a, b, c\in\mathbb{R}$, with $a^2+b^2\neq 0$ and $a^2+c^2\neq 0$ and $b^2+c^2\neq 0$.

Proof. By direct computation A and B are both equal to:

$$A=B=\frac{C}{D},$$

where

$$D=(a^2+4b^2+4c^2)(4a^2+b^2+4c^2)(4a^2+4b^2+c^2)$$

and

$$C=4a^6-4a^5(b+4c)+a^4(33b^2-4bc+24c^2)-a^3(20b^3+20b^2c+17bc^2+20c^3)+$$
$$+a^2(24b^4-17b^3c+60b^2c^2-20bc^3+33c^4)-a(16b^5+4b^4c+20b^3c^2+17b^2c^3+4bc^4+4c^5)+$$
$$+4b^6-4b^5c+33b^4c^2-20b^3c^3+24b^2c^4-16bc^5+4c^6.$$

By the lemma the inequality becomes trivial. And it is easy to see (from the lemma) that the equality holds if $a=1, b=1, c=1$ or $a=1, b=2, c=4$ or $a=1, b=4, c=2$ or $a=2, b=1, c=4$ or $a=2, b=4, c=1$ or $a=4, b=1, c=2$ or $a=4, b=2, c=1$.

Remark: From the lemma we can see that the condition "a, b, c positive real numbers" is extra, we can say that the inequality holds for $a, b, c\in\mathbb{R}$, with $a^2+b^2\neq 0$ and $a^2+c^2\neq 0$ and $b^2+c^2\neq 0$.

Second solution by Paolo Perfetti, Roma,Italy

$$\frac{a^2-bc}{4a^2+4b^2+c^2}+\frac{b^2-ac}{4b^2+4c^2+a^2}+\frac{c^2-ab}{4c^2+4a^2+b^2}\geq 0$$

Proof Let $4a^2+4b^2+c^2=D_1$, $4b^2+4c^2+a^2=D_2$, $4c^2+4a^2+b^2=D_3$. By the Cauchy–Schwarz inequality

$$\Big(\frac{a^2}{D_1}+\frac{b^2}{D_2}+\frac{c^2}{D_3}\Big)(a^2+b^2+c^2)\geq\Big(\frac{a^2}{\sqrt{D_1}}+\frac{b^2}{\sqrt{D_2}}+\frac{c^2}{\sqrt{D_3}}\Big)^2$$

furthermore,

$$\Big(\frac{a^2}{\sqrt{D_1}}+\frac{b^2}{\sqrt{D_2}}+\frac{c^2}{\sqrt{D_3}}\Big)^2\geq(a^2+b^2+c^2)\Big(\frac{bc}{D_1}+\frac{ac}{D_2}+\frac{ab}{D_3}\Big)$$

Now defining $a^2=x$, $b^2=y$, $c^2=z$ and taking $x+y+z=1$ by homogeneity, we obtain

$$\frac{x}{\sqrt{4-3z}}+\frac{y}{\sqrt{4-3x}}+\frac{z}{\sqrt{4-3y}}\geq\sqrt{\frac{\sqrt{yz}}{4-3z}+\frac{\sqrt{xz}}{4-3x}+\frac{\sqrt{xy}}{4-3y}}$$

The convexity of the function $1/\sqrt{x}$ and $2ab\leq a^2+b^2$ allows us to write

$$\sqrt{6}\sqrt{\frac{x^2}{4-3z}+\frac{y^2}{4-3x}+\frac{z^2}{4-3y}}\geq\sqrt{\frac{y+z}{4-3z}+\frac{x+z}{4-3x}+\frac{x+y}{4-3y}}$$

Squaring and clearing the denominators we get (use repeatedly $x+y+z=1$)

$$24(x^2+y^2+z^2)+72(x^2z+y^2x+z^2y)+54(x^3y+y^3z+z^3x)\geq$$

$$32-12(x^2+y^2+z^2)-36(xy+xz+yx)+9(xy^2+yz^2+zx^2)+27abc$$

or (use $x+y+z=1$ again and $xyz\leq\frac{x^2z+y^2x+z^2y}{3}$)

$$2+27(x^2z+y^2x+z^2y)+27(x^3y+y^3z+z^3x)-18(xy+xz+yz)\geq 0 \quad (1)$$

We study (1) by means of the Lagrange multipliers. Let $f(x,y,z)$ be the l.h.s. of (1). The only critical point of the function $F(x,y,z,\lambda)=f(x,y,z)-\lambda(x+y+z-1)$ is $(\frac{1}{3},\frac{1}{3},\frac{1}{3},1)$. On the boundary of the set $x+y+z=1$ the function $f(x,y,z)$ is strictly positive hence the critical point must be a minimum which exists by the compactness of the set $x+y+z=1$ and the Weierstrass theorem on the continuous functions on the compact sets. Alternatively one can study the quadratic form determined by the hessian of F respect to the variables (x,y,z) and restricted to vectors tangent to the constraint $x+y+z=1$. $f(\frac{1}{3},\frac{1}{3},\frac{1}{3})=0$ completes the proof.

Undergraduate problems

U49. Let $f : [0,1] \to [0,\infty)$ be an integrable function. Prove that

$$\int_0^1 f(x)dx \cdot \int_0^1 x^3 f(x)dx \geq \int_0^1 xf(x)dx \cdot \int_0^1 x^2 f(x)dx.$$

Proposed by Cezar Lupu, Bucharest and Mihai Piticari, Campulung, Romania

First solution by Arkady Alt, California, USA.

Let

$$R = \int_0^1 f(x)dx \cdot \int_0^1 x^3 f(x)dx = \int_0^1 \int_0^1 f(x)f(y)x^3 dxdy = \int_0^1 \int_0^1 f(x)f(y)y^3 dxdy$$

and

$$L = \int_0^1 xf(x)dx \cdot \int_0^1 x^2 f(x)dx = \int_0^1 \int_0^1 f(x)f(y)x^2 ydxdy = \int_0^1 \int_0^1 f(x)f(y)y^2 xdxdy.$$

Since $x^3 + y^3 \geq x^2 y + y^2 x$ then

$$\begin{aligned} 2R = & \int_0^1 \int_0^1 f(x)f(y)x^3 dxdy + \int_0^1 \int_0^1 f(x)f(y)y^3 dxdy = \\ & \int_0^1 \int_0^1 f(x)f(y)(x^3+y^3)dxdy \geq \int_0^1 \int_0^1 f(x)f(y)(x^2y+y^2x)dxdy = \\ & \int_0^1 \int_0^1 f(x)f(y)x^2ydxdy + \int_0^1 \int_0^1 f(x)f(y)y^2xdxdy = 2L \Leftrightarrow R \geq L. \end{aligned}$$

Second solution by Li Zhou, Polk Community College, Winter Haven

Let

$$D = \int_0^1 f(x)dx \cdot \int_0^1 x^3 f(x)dx - \int_0^1 xf(x)dx \cdot \int_0^1 x^2 f(x)dx.$$

Then

$$\begin{aligned} D &= \int_0^1 \int_0^1 x^3 f(x)f(y)dxdy - \int_0^1 \int_0^1 x^2 y f(x)f(y)dxdy \\ &= \int_0^1 \int_0^1 (x-y)x^2 f(x)f(y)dxdy. \end{aligned}$$

By symmetry,

$$D = \int_0^1 \int_0^1 (y-x)y^2 f(x)f(y)dxdy.$$

Hence,

$$\begin{aligned} D &= \frac{1}{2}\int_0^1 \int_0^1 [(x-y)x^2 + (y-x)y^2] f(x)f(y)dxdy \\ &= \frac{1}{2}\int_0^1 \int_0^1 (x-y)^2(x+y)f(x)f(y)dxdy \geq 0, \end{aligned}$$

completing the proof.

U50. Let A, B, C be $n \times n$ matrices such that

$$A^2 = B^2 = (AB)^2, \ A^2C = C^2A,$$

and A is invertible. Prove that $A^4 = B^4 = I_n$ and $AC = CA$.

Proposed by Magkos Athanasios, Kozani, Greece

Solution by Cristian Baba, University of Bucharest, Romania

We rewrite $A^2 = (AB)^2$ as $A^2 = ABAB$. Because A is invertible

$$A = BAB. \tag{1}$$

We multiply (1) on the left with B, thus

$$BA = B^2AB. \tag{2}$$

Since $B^2 = A^2$, (2) becomes

$$BA = A^3B. \tag{3}$$

We multiply (2) on the left with A, hence

$$ABA = A^4B. \tag{4}$$

Now we multiply (4) on the right with B and because $B^2 = A^2$ we have

$$(AB)^2 = A^6. \tag{5}$$

But, $A^2 = (AB)^2$ and (5) becomes

$$A^2 = A^6 \Leftrightarrow A^4 = I_n. \tag{6}$$

And because $A^2 = B^2$, it follows from (6) that

$$A^4 = B^4 = I_n.$$

For the other equality, we start from the fact that $A^2C = C^2A$ and multiply it on the left with A^2:

$$A^4C = A^2C^2A \Leftrightarrow C = A^2C^2A. \tag{7}$$

Now we multiply (7) on the right with A and on the left with A^2, hence

$$A^2CA = A^4C^2A^2. \tag{8}$$

But because $A^4 = I_n$, (8) is equivalent to

$$A^2C = C^2A.$$

U51. Let $P(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \in \mathbb{R}[X]$. Suppose $P(x)$ has only real zeros. Prove that $Q(X) = \frac{a_n}{n!} X^n + \frac{a_{n-1}}{(n-1)!} X^{n-1} + \cdots + a_0$ has only real zeros.

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

Solution by Gabriel Dospinescu, Ecole Normale Superieure, Paris

No solution proposed.

U52. Let m be a positive integer. Prove that

$$\sum_{k=0}^{\infty}(-1)^k\binom{2m-2k}{m-k}\binom{m-k}{k}=2^m.$$

Proposed by Gabriel Alexander Reyes, San Salvador, El Salvador

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

First note that

$$\binom{2m-2k}{m-k}\binom{m-k}{k}=\binom{m}{k}\binom{2m-2k}{m}$$

and

$$(1-x^2)^m=\sum_{k=0}^{\infty}(-1)^k\binom{m}{k}x^{2k}$$

and

$$\frac{x^m}{(1-x)^{m+1}}=\sum_{k=0}^{\infty}\binom{m}{k}x^k.$$

Therefore

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k\binom{2m-2k}{m-k}\binom{m-k}{k}&=\sum_{k=0}^{\infty}(-1)^k\binom{m}{k}\binom{2m-2k}{m}\\&=[x^{2m}](1-x^2)^m\cdot\frac{x^m}{(1-x)^{m+1}}\\&=[x^m](1+x)^m\cdot\frac{1}{1-x}\\&=\sum_{k=0}^{\infty}\binom{m}{k}=2^m\end{aligned}$$

U53. Let $f, g \in C[X]$ be two nonconstant polynomials and suppose for each $z \in C$, $f(z)$ is a root of a unity and $g(z)$ is also root of a unity, but not necessarily of the same order. What can we say about f and g?

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution by Iurie Boreico, Harvard University

By continuity, it follows that $|g| = 1$ whenever $|f| = 1$. We shall solve this general problem.

Lemma 1: if a polynomial f satisfies $|f(x)| = 1$ whenever $|x| = 1$ then $f(x) = cx^n$ where $|c| = 1$.

Proof: Let $f(x) = x^k g(x)$ where $g(0) \neq 0$. Then $|g(x)| = 1$ whenever $|x| = 1$. Let $g(x) = c_m x^m + c_{m-1}x^{m-1} + \ldots + c_0$, where $c_0 \neq 0$. As $|g(x)| = 1$, for $|x| = 1$, $g(x)\overline{g(x)} = 1$, for $|x| = 1$, therefore

$$(c_m x^m + c_{m-1}x^{m-1} + \ldots + c_0)(\overline{c_m}\frac{1}{x^m} + \overline{c_{m-1}}\frac{1}{x^{m-1}} + \ldots + \overline{c_0}) = 1,$$

for $|x| = 1$. As there are infinitely many such x, it must be identically 1. However it is not identically 1 for $m > 0$, as the coefficient of $\frac{1}{x^m}$ is $c_0\overline{c_m} \neq 0$. Thus $m = 0$, so $g(x)$ is constant. Clearly $g(x) = c$ with $|c| = 1$, hence $f(x) = cx^k$.

Lemma 2: Let $t_1, t_2, \ldots, t_n$ be the roots of the equation $f(x) = t$. Then $g(t_1)g(t_2)\ldots g(t_n) = ct^n$ for $|c| = 1$.

Proof: The expression $g(t_1)g(t_2)\ldots g(t_n)$ is symmetric thus can be written as a polynomial in the coefficients of $f(x) - t = 0$, thus as a polynomial in t. It suffices to apply lemma 1 to finish the claim. The degree of the polynomial is clearly n, as seen by the asymptotic of the expression.

Next let the dominant coefficient of f be a and of g be b. The term t^n in the polynomial $g(t_1)g(t_2)\ldots g(t_n)$ is $b^n t_1^n t_2^n \ldots t_n^n$ (just write $g(t_k) = bt_k^n + \ldots$ and open the brackets), so the leading coefficient equals $\pm\frac{b^n}{a^n}$. Thus $|b| = |a|$.

Finally let's look at the polynomial $g(t_1) + g(t_2) + \ldots + g(t_n)$. It is again a polynomial in t, being the sum of symmetric terms of form $(t_1^k + t_2^k + \ldots + t_n^k)$. Now, by Newton's formula, we know that $t_1 t_2 \ldots t_n$ does not appear in the representation of $t_1^k + \ldots + t_n^k$ for $k < n$, and appears with factor n in $t_1^n + \ldots + t_n^n$. This $g(t_1) + g(t_2) + \ldots + g(t_n) = n\frac{b}{a}t + p$.

Consider now $|t| = 1$. Then $|g(t_k)| = 1$ hence $|g(t_1) + g(t_2) + \ldots + g(t_n)| \leq n$. So $|n\frac{b}{a}t + p| \leq n$ whenever $|t| = 1$. Then $|(n\frac{b}{a}t + p) - (n\frac{b}{a}(-t) + p)| \leq n + n = 2n$. As $|\frac{b}{a}|$, this is actually an equality. Hence equalities hold everywhere, which implies $g(t_1) = g(t_2) = \ldots = g(t_n)$. Also $|t| = 1$ implies $|n\frac{b}{a}t + p| = n$, so $p = 0$, and then $g(t_k) = \frac{b}{a}t$, so $g = \frac{b}{a}f$.

As we assumed that f and g have the same degree, in the general case it is clear that we have $f^m = cg^n$ for some c on the unit circle and $m, n \in N$.

It is clear that all polynomials satisfying this relation satisfy the conditions, provided that c is also a root of unity.

U54. Find the best constant c such that for all n, if $f(x) \in \mathbb{R}[X]$ of degree n satisfies

$$\int_0^1 \int_0^1 (f(x) - f(y))^2 dx dy = 1,$$

then the function $g : [0, 1] \to \mathbb{R}$, $g(x) = x(1 - x)f'(x)$ has a Lipschitz constant at most cn^3.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

No solution proposed.

Olympiad problems

O49. Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC. Lines AA_1, BB_1, CC_1 intersect again the circumcircle of triangle ABC at A_2, B_2, C_2, respectively. Prove that

$$\frac{AA_1}{A_1A_2}+\frac{BB_1}{B_1B_2}+\frac{CC_1}{C_1C_2}\geq\frac{3s^2}{r\left(4R+r\right)},$$

where s, r, R are the semiperimeter, inradius, and circumradius of triangle ABC, respectively.

Proposed by Cezar Lupu, Romania and Darij Grinberg, Germany

First solution by Vicente Vicario Garca, Huelva, Spain

Let $\frac{BA_1}{A_1C}=\alpha$, $\frac{CB_1}{B_1A}=\beta$, $\frac{AC_1}{C_1B}=\gamma$. By Ceva's theorem the cevians are concurrent if and only if $\alpha\beta\gamma=1$. By Stewart's theorem we have:

$$\begin{aligned}
AA_1^2&=\frac{1}{1+\alpha}b^2+\frac{1}{1+\alpha}c^2-\frac{1}{1+2\alpha+\alpha^2}a^2,\\
BB_1^2&=\frac{1}{1+\beta}a^2+\frac{1}{1+\beta}c^2-\frac{1}{1+2\beta+\beta^2}b^2,\\
CC_1^2&=\frac{1}{1+\gamma}a^2+\frac{1}{1+\gamma}b^2-\frac{1}{1+2\gamma+\gamma^2}c^2.
\end{aligned}$$

On the other hand we have $A_1B\cdot A_1C=AA_1\cdot A_1A_2$ and $A_1B+A_1C=a$. Furthermore,

$$\begin{aligned}
k_1&=A_1B\cdot A_1C=\frac{1}{1+2\alpha+\alpha^2}a^2,\\
k_2&=B_1A\cdot B_1C=\frac{1}{1+2\beta+\beta^2}b^2,\\
k_3&=C_1A\cdot C_1B=\frac{1}{1+2\gamma+\gamma^2}c^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{AA_1}{A_1A_2}+\frac{BB_1}{B_1B_2}+\frac{CC_1}{C_1C_2}=\frac{AA_1^2}{k_1}+\frac{BB_1^2}{k_2}+\frac{CC_1^2}{k_3}=\\
&\frac{1+\alpha}{\alpha}\cdot\frac{b^2}{a^2}+(1+\alpha)\cdot\frac{c^2}{a^2}-1+\frac{1+\beta}{\beta}\cdot\frac{a^2}{b^2}+(1+\beta)\cdot\frac{a^2}{c^2}-1+\\
&\qquad\frac{1+\gamma}{\gamma}\cdot\frac{a^2}{c^2}+(1+\gamma)\cdot\frac{b^2}{c^2}-1=\\
&\frac{1}{\alpha}\cdot\frac{b^2}{a^2}+\alpha\cdot\frac{c^2}{a^2}+\frac{1}{\beta}\cdot\frac{a^2}{b^2}+\beta\cdot\frac{a^2}{c^2}+\frac{1}{\gamma}\cdot\frac{a^2}{c^2}+\gamma\cdot\frac{b^2}{c^2}-3.
\end{aligned}$$

By the arithmetic-geometric mean inequality and since $t+\frac{1}{t}\geq 2, t>0$:

$$\frac{AA_1}{A_1A_2}+\frac{BB_1}{B_1B_2}+\frac{CC_1}{C_1C_2}\geq 2\left[\frac{1+\alpha}{\sqrt{\alpha}}\cdot\frac{bc}{a^2}+\frac{1+\beta}{\sqrt{\beta}}\cdot\frac{ac}{b^2}+\frac{1+\gamma}{\sqrt{\gamma}}\cdot\frac{ab}{c^2}\right]-3\geq$$
$$4\left[\frac{bc}{a^2}+\frac{ac}{b^2}+\frac{ab}{c^2}-3\right].$$

On the other hand, $\frac{bc}{a^2}+\frac{ac}{b^2}+\frac{ab}{c^2}=\frac{b^3c^3+a^3b^3+a^3c^3}{a^2b^2c^2}$. In the identity $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$ substitute $x=ab, y=ac, z=bc$; the result is

$$b^3c^3+a^3c^3+a^3b^3=3(4Rrs)^2+(s^2+r^2+4Rr)[(s^2+r^2+4Rr)^2-12Rrs\cdot 2s]$$

since,$ab+bc+ca=\frac{(s-a)(s-b)(s-c)+abc+s^3}{s}$.
Hence,

$$\frac{bc}{a^2}+\frac{ac}{b^2}+\frac{ab}{c^2}=\frac{b^3c^3+a^3b^3+a^3c^3}{a^2b^2c^2}\geq 4\left[3+\frac{S^4}{16R^2r^2}-3\right]\geq\frac{3s^2}{r(4R+r)}$$

because

$$9+\frac{S^4}{16R^2r^2}\geq\frac{3s^2}{r(4R+r)}\Leftrightarrow(4R+r)s^4-12R62rs^2+36R^2r^2(4R+r)\geq 0\Leftrightarrow$$
$$(12R^2r)^2-4(4R+r)36R^2r^2(4R+r)\leq 0\Leftrightarrow R\leq 4R+r,$$

and thus we are done.

Second solution by Daniel Campos Salas, Costa Rica

Let D and E be the feet of the perpendiculars from A and A_2 to BC. Then, $\frac{AA_1}{A_1A_2}=\frac{AD}{A_2E}$. Since AD is constant, the expression $\frac{AA_1}{A_1A_2}$ is minimized when A_2E is maximized, this is, when A_2 is the midpoint of the arc BC not containing A, or AA_1 is the angle bisector. Analogously for the other terms, so it is enough to consider the case when AA_1, BB_1, CC_1 are angle bisectors.

The power of A_1 with respect to the circumcircle of ABC equals

$$AA_1\cdot A_1A_2=BA_1\cdot CA_1=\frac{a^2bc}{(b+c)^2}.$$

Then,

$$\frac{AA_1}{A_1A_2}=\frac{AA_1^2}{AA_1\cdot A_1A_2}=\frac{bc\left(1-\frac{a^2}{(b+c)^2}\right)}{\frac{a^2bc}{(b+c)^2}}=\frac{4s(s-a)}{a^2},$$

and analogously for the other terms. Let $(s-a, s-b, s-c) = (x, y, z)$, with $x, y, z > 0$. Then,

$$\frac{AA_1}{A_1A_2} = \frac{4x(x+y+z)}{(y+z)^2},$$

and,

$$r(4R+r) = \frac{abc}{s} + \frac{(s-a)(s-b)(s-c)}{s} = \frac{(x+y)(x+z)(y+z)+xyz}{x+y+z} = xy+xz+yz.$$

From these relations we obtain that the inequality reduces to prove that for arbitrary positive real numbers x, y, z the following inequality holds,

$$\sum_{cyc} \frac{4(x+y+z)x}{(y+z)^2} \geq \frac{3(x+y+z)^2}{xy+xz+yz}.$$

or equivalently,

$$4\sum_{cyc} \frac{x}{(y+z)^2} \geq \frac{3(x+y+z)}{xy+xz+yz}.$$

From Hölder's inequality and the inequality $(x+y+z)^2 \geq 3(xy+xz+yz)$, we have that

$$4\sum_{cyc} \frac{x}{(y+z)^2} \geq \frac{4(x+y+z)^3}{\left(\sum_{cyc} x(y+z)\right)^2} \geq \frac{3(x+y+z)}{xy+xz+yz},$$

that is what we wanted to prove.

Also solved by Li Zhou, Polk Community College, Winter Haven

O50. Find the least k for which there exist integers $a_1, a_2, ..., a_k$, different from -1, such that numbers $x^2 + a_i y^2, x, y \in \mathbb{Z},\ i = 1, 2, ..., k,$ cover the set of prime numbers.

Proposed by Iurie Boreico, Harvard University and Ivan Borsenco, University of Texas at Dallas

Solution by Li Zhou, Polk Community College, Winter Haven

We show that the least k is 3, with the minimal set $\{a_1, a_2, a_3\} = \{-4, \pm 2\}$, $\{1, \pm 2\}$, or $\{4, \pm 2\}$. Indeed, it is well known

1) Every prime $p \equiv 1 \pmod 4$ can be expressed in the form $x^2 + (2y)^2$, $x, y \in \mathbb{Z}$;

2) Every prime $p \equiv 3 \pmod 8$ can be expressed in the form $x^2 + 2y^2$, $x, y \in \mathbb{Z}$;

3) Every prime $p \equiv 7 \pmod 8$ can be expressed in the form $x^2 - 2y^2$, $x, y \in \mathbb{Z}$.

Also, $2 = 1^2 + 1^2 = 2^2 - 2(1)^2$ and $\left(\frac{p+1}{2}\right)^2 - 4\left(\frac{p-1}{4}\right)^2 = p$ if $p \equiv 1 \pmod 4$.

Next, suppose that $a_1, a_2, \ldots, a_k \in \mathbb{Z} \setminus \{-1, -2\}$. We claim that there is a prime $p \equiv 7 \pmod 8$ not covered by

$$C = \{x^2 + a_i y^2 : x, y \in \mathbb{Z}, i = 1, 2, \ldots, k\}.$$

Considering congruence modulo 8, we see that any prime $p \equiv 7 \pmod 8$ is not in $\{x^2 + y^2, x^2 + 2y^2, x^2 \pm 2^n y^2 : n \geq 2\}$. Hence we may assume that each a_i has an odd prime factor p_i (not necessarily distinct). For each i, let r_i be a quadratic nonresidue modulo p_i ($r_i = r_j$ if $p_i = p_j$), and let $m = \operatorname{lcm}(8, p_1, p_2, \ldots, p_k)$. By the Chinese remainder theorem, there is a set $S = \{mt + b : t \in \mathbb{Z}\}$ such that $\forall x \in S$, $x \equiv 7 \pmod 8$ and $x \equiv r_i \pmod{p_i}$ for each i. By Dirichlet's theorem, there is a prime (infinitely many in fact) $p \in S$. Clearly, this p is not covered by C.

Similarly, if $a_1, a_2, \ldots, a_k \in \mathbb{Z} \setminus \{-1, 2\}$, then there is a prime $p \equiv 3 \pmod 8$ not covered by C; and if $a_1, a_2, \ldots, a_k \in \mathbb{Z} \setminus \{\pm 4, \pm 1\}$, then there is a prime $p \equiv 5 \pmod 8$ not covered by C.

O51. Find a closed form for $p(x) = \prod_{a=1}^{M}\prod_{b=1}^{N}(x - e^{\frac{2\pi i\ a}{M}} \cdot e^{\frac{2\pi i\ b}{N}})$, where M and N are positive integers.

Proposed by Alex Anderson, New Trier High School,Winnetka, IL

Solution by Alex Anderson, New Trier High School,Winnetka, IL

I claim that:

$$p(x) = \left(x^{\mathrm{lcm}(M,N)} - 1\right)^{\gcd(M,N)}$$

The expression is a polynomial in x. Consider each root, we have that root $= f(t) = \epsilon^{2\pi i t}$ where t is in the form $\frac{a}{M} + \frac{b}{M}$ where $a \in 1, 2, \cdots, M$ and $b \in 1, 2, \cdots, N$.

Let $d = \gcd(M, N)$ and $M = dM'$, and $N = dN'$. We have that

$$t = \frac{aN' + bM'}{dM'N'}.$$

Evidently, we must consider the numerator mod $dM'N'$ since $f(t + 1) = f(t)$. We want to find which $dM'N'$th roots of unity are roots of the polynomial.

Equivalently, we want to fund which values of p there are integer solutions (x, y) for $xN' + yM' \equiv p \mod dM'N'$.

Lemma1:

Suppose N' and M' are relatively prime positive integers, d is a positive integer, $N = dN', M = dM', x \in 1, 2, \cdots, N$, and $x \in 1, 2, \cdots, N$. Then there are solutions (x, y) for each integer $1 \le p \le dM'N'$ for $xN'+yM' \equiv p \mod dM'N'$.

Proof:

In other words, given a fixed p, there exists (x, y, q) such that: $xN' + yM' = p + qdM'N'$. This can be rearranged to be: $[x]N' + [y - qdN'] = p$.

Consider the rems in the brackets. They are the only numbers not fixed. Be Bezout's lemma, there are solutions $(a, b) = (x, y - qdN')$ for any integer p because N' and M' are relatively primes and we are done.

Furthermore, suppose that (x_0, y_0) is a solutions, we also know that all solutions are in the form $(x_0 + M't, y_0 - N't)$ where t is an integer.

Lemma 2:

For the problem described in lemma 1, there are precisely d pairs of integers (x, y) for each value of p.

Solution:

We must consider the bounds on the values of $x = x_0 + M't$ and $y - qdN' = y_0 - N't$. We know that $0 \leq x_0 + M't \leq M \Rightarrow \frac{-x_0}{M'} \leq t \leq \frac{M-x_0}{d}$. It follows that there are precisely d solutions for x since there are d values of t that satisfy that inequality. We have d solutions $(a,b) = (x,y)$ for each values of p; we just need to assure that for (x,y,q), there is a solution q.

Now for each (x,y), we just need to have that $1 \leq y - qdN' \leq N$. But $N'd = N$, so there is only one q.

By lemma 1, we have p distinct remainders mod $dM'N'$, all if which exist. Equivalently, we have all of the $dM'N'$th roots of unity. By lemma 2, each of these roots occurs d times. This gives us a total of $(d)(dM'N' = MN)$ roots, which is the degree of our polynomial. Hence, we have accounted for all roots. It follows that our polynomial is:

$$p(x) = \left(x^{dM'N'} - 1\right)^d = \left(x^{\mathrm{lcm}(M,N)} - 1\right)^{\gcd(M,N)}$$

as desired.

O52. Suppose n is not a multiple of 3. Find all integer solutions of

$$(a^2-bc)^n+(b^2-ca)^n+(c^2-ab)^n=1.$$

Proposed by H. van der Berg

Solution by Gabriel Dospinescu, "Louis le Grand" College, Paris

We claim that the polynomial $f(a,b,c)=(a^2-bc)^n+(b^2-ac)^n+(c^2-ab)^n$ is divisible by $a^2+b^2+c^2-ab-bc-ca$. It will follow that for any solution the last quantity equals 1 and thus $(a-b)^2+(b-c)^2+(c-a)^2=2$, from where we can easily find all solutions. COnsider A a circulant matrix of permutation and define $B=xI+yA+zA^{-1}$. We can easily verify that we can write $\det(B)=x^n+y^n+z^n+x_1x^{n-2}yz+x_2x^{n-4}y^2z^2+...$ with integers x_i and evaluating this at $a=b=c=1$ (and taking into accound that n is not a multiple of 3, thus $\det(I+A+A^{-1})=3$) we deduce that $x_1+x_2+...=0$, thus $\det(B)=x^n+y^n+z^n+k(x^2-yz)$ for an integer k. Also, $x+y+z$ is clearly an eigenvalue of B. Now, specialize $x=a^2-bc, y=b^2-ca, z=c^2-ab$ and observe that with this choice x^2-yz is divisible by $a^2+b^2+c^2-ab-bc-ca$. This proves the claim and ends the solution.

O53. Let ABC be a triangle and let w be its incircle. Denote by D, E, F the intersections of w with BC, CA, AB, respectively. Let $T \in AD \cap w$, $M \in BT \cap w$, $N \in CT \cap w$. Let p_1 be a circle tangent to w at T, and p_2 a circle tangent to w at D, so that p_1 and p_2 intersect on chord (XY). Prove that X, Y, M, N lie on the same circle.

Proposed by Cosmin Pohoata, Bucharest, Romania

Solution by Saturnino Campo Ruiz, Instituto de Enseanza Secundaria, Spain

Denote by O the intersection of BC with the tangent line to w at T. This point is the pole of the line AD with respect to w. Consequently the cross ratio $(ODCB) = -1$. For the cevians of T, if $O' = M'N' \cap BC$, the Menelaus' theorem states $(O'CB)(N'BA)(M'AC)$=1, and the Ceva's theorem $(DCB)(N'BA)(M'AC) = -1$, (with $(QRS) = QR/QS$). Combining these two, we get $(O'CB) : (DCB) = -1$ and that $(BCDO')$ is harmonic, then $O' = O$ and the points O, M', N' are collinear. For the Brianchon's theorem the cevians AD, BE, CF are also concurrent (the Gergonne's point of the triangle) and also E, F, O are concurrent.

Lemma. MN and BC intersect at O.

Proof. The set of all lines through a given point is called a *pencil* of lines. The point O define the pencil (of lines) O^*. If $o \in O^*$ and o meets the lines TB, TC, TD in M, N^*, P^*, then $(OP^*MN^*) = -1$.

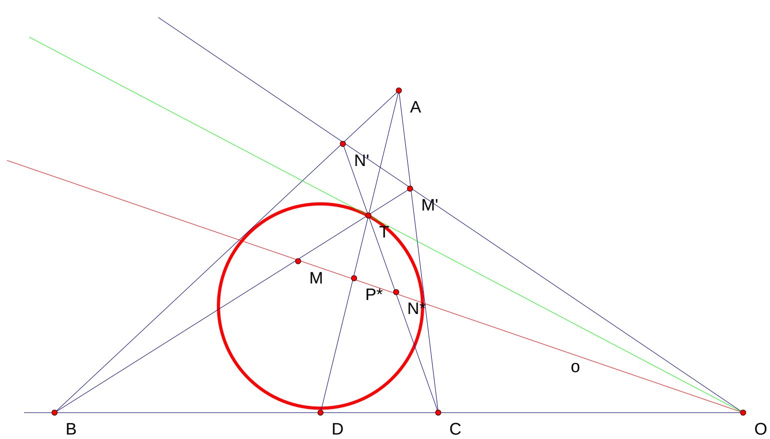

Figure 1

Also we have AD is the polar line of O respect to w, for $o \in O^*$ this line meet the conic w (a circle) in M, N" and the polar in P^*, hence $(OP^*MN") = -1$

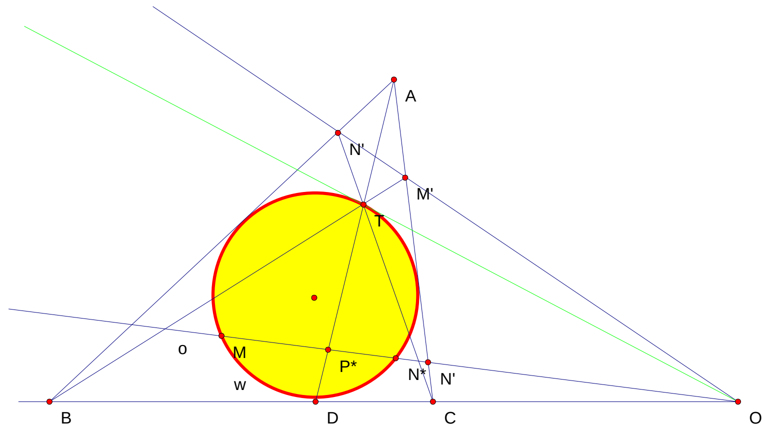

Figure 2

If $M \in BT \cap w$, the line $OM \in O^*$ yield two cross ratios: $(OP^*MN^*) = (OP^*MN") = -1$ with $N^* = OM \cap CT$ and $N" = OM \cap w$. Thus $N^* = N"$ and $N \in CT \cap w$ implies $N = N^* = N"$.

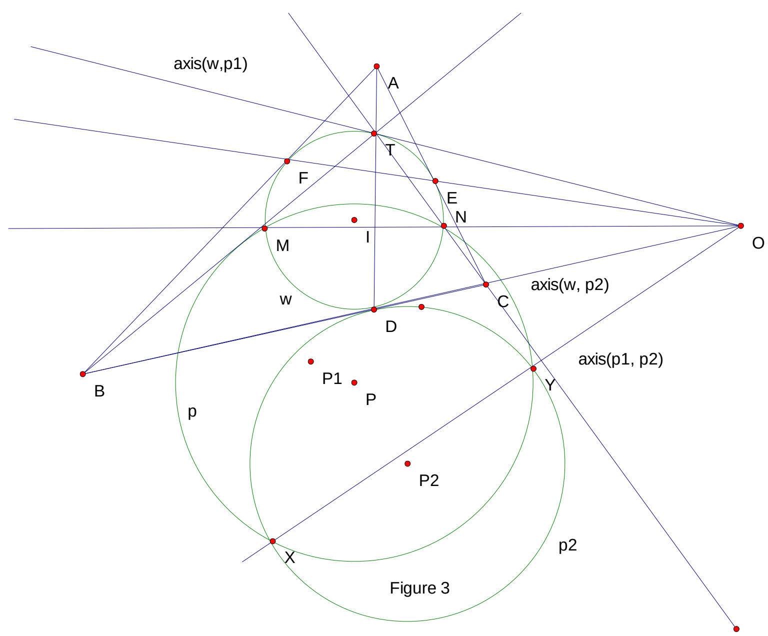

Figure 3

Now let p be the circle through X, Y, M. OT is the radical axis of w and p_1, BC is the radical axis of w and p_2. Consequently O is the radical center of these circles. XY is the radical axis of p_1 and p_2 (and p) hence X, Y, O are collinear. The power of the point O with respect to a circle p is $OX \cdot OY$. We have $O = MN \cap BC$ and is a radical Center of (w, p_1, p_2), thus $Pow(O, w) = OM = OX = Pow(O, p)$, hence $n \in p$.

O54. Let $p = 2q+1$ be a prime number greater than 3. Prove that p divides the numerator of

$$\sum_{1\le i,j\le q,\ i+j>q} \frac{1}{ij},$$

where the sum is taken over all ordered pairs (i,j).

Proposed by Iurie Boreico, Harvard University

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Since $q \ge 2$ then

$$\begin{aligned}
\sum_{1\le i,j\le q, i+j>q} \frac{1}{ij} &= H_q^2 - \sum_{k=2}^{q}\sum_{i=1}^{k-i} \frac{1}{i(k-i)} \\
&= H_q^2 - \sum_{k=2}^{q} \frac{1}{k} \sum_{i=1}^{k-i} \left(\frac{1}{i} + \frac{1}{k-i}\right) \\
&= H_q^2 - 2\sum_{k=2}^{q} \frac{H_{k-1}}{k} \\
&= H_q^2 - 2\sum_{k=2}^{q} \frac{H_k}{k} + 2\sum_{k=2}^{q} \frac{1}{k^2} \\
&= H_q^2 - 2\sum_{k=2}^{q} \frac{H_k}{k} + 2H_q^2 \\
&= H_q^2 - \left(H_q^2 + H_q^2\right) + 2H_q^2 \\
&= H_q^2.
\end{aligned}$$

The numbers $1^2, 2^2, \cdots, q^2$ are all invertible squares modulo p and since the inverse is unique $1^{-2}, 2^{-2}, \cdots, q^{-2}$ are the same numbers in a different order. Hence (note that $p > 3$)

$$H_q^2 = \sum_{k=2}^{q} k^{-2} \equiv \sum_{k=2}^{q} k^2 \equiv 6^{-1} q(q+1)(2q+1) \equiv 0 (\text{mod } p).$$

The denominator of the fraction H_q^2 is invertible modulo p, therefore p divides the numerator.