Junior problems

J43. In triangle ABC the median AM intersects the internal bisector BN at P. Denote by Q the point of intersection of lines CP and AB. Prove that triangle BNQ is isosceles.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Magkos Athanasios, Kozani, Greece

Applying Ceva Theorem for the cevians AM, BN, CQ we get

$$\frac{QA}{QB} \cdot \frac{MB}{MC} \cdot \frac{NC}{NA} = 1.$$

Because $MB = MC$ we have $\frac{QA}{QB} = \frac{NA}{NC}$, that means QN is parallel to BC. Thus $\angle QNB = \angle NBC = \angle NBQ$, where the last equation holds because BN is the angle bisector of $\angle ABC$. Therefore triangle BNQ is isosceles.

Also solved by Andrea Munaro, Italy; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Courtis G. Chryssostomos, Larissa, Greece; Vicente Vicario Garcia, Huelva, Spain; Shukurjon Shokirov, Physical-Mathematical Lyceum N1, Samarkand, Uzbekistan

J44. Consider a triangle ABC and let $g_a,\ g_b,\ g_c$ and $n_a\ n_b,\ n_c$ be the Gergonne cevians and the Nagel cevians, respectively. Prove that

$$g_a + g_b + g_c + 2\max(a, b, c) \geq n_a + n_b + n_c + 2\min(a, b, c).$$

Proposed by Mircea Lascu, Zalau, Romania

Solution by Andrea Munaro, Italy

Let G_a, G_b, G_c and N_a, N_b, N_c be the points of intersection of the Gergonne and Nagel cevians with the triangle's sides. Suppose without loss of generality that $a \geq b \geq c$. Because the Nagel point is an isotomic conjugate of the Gergonne point we have

$$G_aN_a = CG_a - BG_a, G_bN_b = CG_b - AG_b, G_cN_c = BG_c - AG_c.$$

Summing these we get $G_aN_a + G_bN_b + G_cN_c = CG_a + CG_b + BG_c - (BG_a + AG_b + AG_c) = a + CG_b - c - AG_c = a - c + (a - BG_c) - AG_c = 2(a - c)$.

By the Triangle Inequality $n_a \leq g_a + G_aN_a$, $n_b \leq g_b + G_bN_b$, $n_c \leq g_c + G_cN_c$. Finally we get $n_a + n_b + n_c \leq g_a + g_b + g_c + 2(a - c)$.

Also solved by Vicente Vicario Garcia, Huelva, Spain; Daniel Lasaosa, Universidad Publica de Navarra, Spain

J45. Let a and b be real numbers. Find all pairs (x, y) of real numbers solutions to the system

$$\begin{cases} x+y &= \sqrt[3]{a+b} \\ x^4-y^4 &= ax-by \end{cases}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas.

First solution by Arkady Alt, San Jose, California, USA

First, we will solve the original system with respect to a and b.

$$\begin{cases} a+b=(x+y)^3 \\ ax-by=x^4-y^4 \end{cases} \iff \begin{cases} a\,(x+y)=y\,(x+y)^3+x^4-y^4 \\ b\,(x+y)=x\,(x+y)^3-x^4+y^4 \end{cases} \iff$$

$$\begin{cases} a\,(x+y)=x\,(x+y)\left(x^2+3y^2\right) \\ b\,(x+y)=y\,(x+y)\left(y^2+3x^2\right) \end{cases}$$

$x+y$ can be equal to zero only if $a=-b$ and in this case the original system has infinitely many solutions $(t,-t)\,, t\in\mathbb{R}$. Supposing that $a+b\neq 0$, we have $x+y\neq 0$. Thus, we obtain

$$\begin{cases} a=x\left(x^2+3y^2\right) \\ b=y\left(y^2+3x^2\right) \end{cases} \iff \begin{cases} a+b=(x+y)^3 \\ a-b=(x-y)^3 \end{cases} \iff \begin{cases} x+y=\sqrt[3]{a+b} \\ x-y=\sqrt[3]{a-b} \end{cases}$$

$$\iff \begin{cases} x=\frac{\sqrt[3]{a+b}+\sqrt[3]{a-b}}{2} \\ y=\frac{\sqrt[3]{a+b}-\sqrt[3]{a-b}}{2} \end{cases}.$$

Second solution by Daniel Campos Salas, Costa Rica

Using that $x^4-y^4=(x-y)((x+y)^3-2xy(x+y))$, we have

$$(x-y)(a+b-2xy(x+y))=ax-by.$$

This implies that

$$bx-ay=2xy(x+y)(x-y)=2xy(x^2-y^2).$$

Then,

$$\begin{aligned} (a-b)(x+y) &= (ax-by)-(bx-ay)=(x^4-y^4)-2xy(x^2-y^2) \\ &= (x^2-y^2)(x-y)^2=(x+y)(x-y)^3. \end{aligned}$$

If $a+b\neq 0$ then $x+y\neq 0$, so

$$x - y = \sqrt[3]{a-b},$$

from where we conclude that

$$(x,y) = \left(\frac{\sqrt[3]{a+b}+\sqrt[3]{a-b}}{2}, \frac{\sqrt[3]{a+b}-\sqrt[3]{a-b}}{2}\right).$$

If $a+b=0$, then $x+y=0$, so $x^4-y^4=0=a(x+y)=ax-by$. Then, $(x,y)=(k,-k)$ is a solution, but this is a particular case of the pair mentioned above. Thus this case doesn't add new solutions. It's easy to verify that

$$(x,y) = \left(\frac{\sqrt[3]{a+b}+\sqrt[3]{a-b}}{2}, \frac{\sqrt[3]{a+b}-\sqrt[3]{a-b}}{2}\right)$$

satisfies the system.

Also solved by Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain; Courtis G. Chryssostomos, Larissa, Greece

J46. A quadrilateral is called bicentric if it can be both inscribed in a circle and circumscribed to a circle. Construct with a ruler and compass a bicentric quadrilateral with all of its sidelengths distinct.

Proposed by Ivan Borsenco, University of Texas at Dallas

First solution by Vicente Vicario Garcia, Huelva, Spain

There is a classic result that establishes that if we can construct segments with lengths a and b, then $a+b, a-b, ab, \frac{a}{b}, \sqrt{a}$ are also constructible. According to the Fuss theorem/Durrande theorem, if a quadrilateral is bicentric then

$$d = \sqrt{R^2 + r^2 - r\sqrt{4R^2 + r^2}},$$

where d is the distance between the incenter and the circumcenter of quadrilateral $ABCD$. Construct two circles with $d > 0$ that satisfy this relation. Using the famous theorem of Poncelet that says that we can start the construction choosing whatever point P on the circle of radius R and drawing tangents to the circle of radius r and will obtain a bicentric quadrilateral. Clearly if $d > 0$ there are a lot of quadrilaterals with all of their sidelengths distinct.

Second solution by Ivan Borsenco, University of Texas at Dallas

Construct a scalene triangle ABC. Let us construct the angle bisectors and find the incenter I of the triangle ABC. The projections from I onto triangle's sides intersect AB, AC at D and E, respectively. Observe that D and E are points of tangency of the incircle and AB and AC. Draw a circle with center I and radius $r = ID = IE$.

Our desire is to use the triangle's incircle as the incircle of the future bicentric quadrilateral. The idea is to construct $U \in AB$ and $V \in AC$ such that UV is antiparallel to BC and tangent to the incircle from the other side. In other words $\angle AUV = \angle ACB = \gamma$ and $\angle AVU = \angle ABC = \beta$. Thus we will obtain bicentric quadrilateral $BUVC$.

Let us analyze quadrilateral $BUVC$. We have that I is the incircle with radius r. The quadrilateral's sides are equal to

$$BC = r\cot\beta + r\cot\gamma, BU = r\cot\beta + r\tan\gamma,$$

$$CV = r\cot\gamma + r\tan\beta, UV = r\tan\gamma + r\tan\beta$$

Clearly if $\beta, \gamma \neq 45^\circ$ then all sides are distinct. To construct U and V, construct segments DU and EV equal to $r\tan\gamma$ and $r\tan\beta$.

J47. In triangle ABC let m_a and l_a be the median and the angle bisector from the vertex A, respectively. Prove that

$$0 \le m_a^2 - l_a^2 \le \frac{(b-c)^2}{2}.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Courtis G. Chryssostomos, Larissa, Greece

We know that $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$ and $l_a^2 = \frac{bc((b+c)^2-a^2)}{(b+c)^2}$. Thus

$$m_a^2 - l_a^2 = \frac{2b^2+2c^2-a^2}{4} - bc\left(1 - \frac{a^2}{(b+c)^2}\right) = \frac{2b^2+2c^2-a^2-4bc}{4} + \frac{a^2}{(b+c)^2} =$$

$$= \frac{2(b-c)^2-a^2}{4} + \frac{a^2bc}{(b+c)^2} = \frac{2(b^2-c^2)^2 - a^2(b+c)^2 + 4a^2bc}{4(b+c)^2} =$$

$$\frac{2(b-c)^2(b+c)^2 - a^2(b-c)^2}{4(b+c)^2} = \frac{(b-c)^2(2(b+c)^2-a^2)}{4(b+c)^2} \ge 0, \text{ as } b+c > a.$$

To prove the right hand side of the inequality it suffices to prove

$$\frac{(b-c)^2(2(b+c)^2-a^2)}{4(b+c)^2} \le \frac{(b-c)^2}{2},$$

or

$$2(b+c)^2 - a^2 \le 2(b+c)^2,$$

that is clearly true and we are done.

Also solved by Son Ta Hong, Ha Noi University of Education, Vietnam; Magkos Athanasios, Kozani, Greece; Arkady Alt, San Jose, California, USA; Vicente Vicario Garcia, Huelva, Spain; Daniel Lasaosa, Universidad Publica de Navarra, Spain

J48. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b(b+c)^2}+\frac{b}{c(c+a)^2}+\frac{c}{a(a+b)^2} \geq \frac{9}{4(ab+bc+ca)}.$$

Proposed by Ho Phu Thai, Da Nang, Vietnam

First solution by Jingjun Han, High School Affiliated to Fudan University, China

Using the Cauchy-Schwartz inequality we have

$$(ab+bc+ca)\left(\frac{a}{b(b+c)^2}+\frac{b}{c(c+a)^2}+\frac{c}{a(a+b)^2}\right) \geq \left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2.$$

Thus it is enough to prove that

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2 \geq \frac{9}{4} \iff$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2},$$

which is true from Nesbitt's inequality.

Second solution by Vardan Verdiyan, Yerevan, Armenia

Our inequality is equivalent to

$$\sum c \cdot a^2(a+b)^2(a+c)^2 \geq \frac{9}{4} \cdot \frac{abc}{ab+bc+ca} \cdot (a+b)^2(b+c)^2(c+a)^2.$$

Using the Cauchy-Schwartz inequality we get

$$\left(\sum c \cdot a^2(a+b)^2(a+c)^2\right) \cdot (\sum \frac{1}{c}) \geq (\sum a(a+b)(a+c))^2.$$

Thus it is enough to prove

$$\left(\sum a(a+b)(a+c)\right)^2 \geq \frac{9}{4}(a+b)^2(b+c)^2(c+a)^2$$

or

$$2(a(a+b)(a+c)+b(b+c)(b+a)+c(c+a)(c+b)) \geq 3(a+b)(b+c)(c+a).$$

The last inequality is equivalent to $\sum(a+b)(a-b)^2 \geq 0$ and we are done.

Also solved by Josep Marc Mingot; Hoang Duc Hung, Hanoi, Vietnam; Paolo Perfetti, Mathematical Department of "Tor Vergata" University, Roma, Italy; Daniel Campos Salas, Costa Rica; Courtis G. Chryssostomos, Larissa, Greece; Magkos Athanasios, Kozani, Greece; Shukurjon Shokirov, Physical-Mathematical Lyceum N1, Samarkand, Uzbekistan; Orif Ibrogimov, SamSU, Samarkand, Uzbekistan

Senior problems

S43. Consider an acute triangle ABC and let Γ be its circumcircle, centered at O. Denote by D, E, and F the midpoints of the minor arcs BC, CA, and AB, respectively. Let Γ_A be the circle through O which is tangent to Γ at D. Define analogously Γ_B and Γ_C. Let O_A be the intersection of Γ_B and Γ_C, different from O. Define analogously O_B and O_C. Prove that triangles ABC and $O_AO_BO_C$ are similar if and only if ABC is equilateral.

Proposed by Daniel Campos Salas, Costa Rica

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Since Γ_A is tangent to Γ at D, then D, O and the center of Γ_A are collinear. It follows that the center of $\Gamma_A \in OD$, therefore OD is a diameter of Γ_A. Analogously OE, OF are diameters for Γ_B, Γ_C, respectively. We have $\angle OO_AE = \angle OO_AF = \frac{\pi}{2}$, and O_A is on EF. Furthermore, Γ is the circumcircle of ΔDEF, and O_A is the foot of the perpendicular from O to EF. Thus O_A is the midpoint of EF, and similarly O_B is the midpoint of FD and O_C is the midpoint of DE. Hence $\Delta O_AO_BO_C$ is similar to ΔDEF. The problem then reduces to proving that ΔDEF and ΔABC are similar if and only if ΔABC is equilateral.

Observe that

$$\angle EOD = \angle EOC + \angle COD = \frac{\angle AOC}{2} + \frac{\angle COB}{2} = \angle B + \angle A,$$

$$\angle F = \frac{\angle EOD}{2} = \frac{\angle A + \angle B}{2},$$

and similarly $\angle E = \frac{\angle C + \angle A}{2}$ and $\angle D = \frac{\angle B + \angle C}{2}$. Assume without loss of generality that $\angle A \geq \angle B \geq \angle C$. Then $\angle F \geq \angle E \geq \angle D$, and ΔDEF and ΔABC are similar if and only if $\angle A = \angle B$ (for $\angle A$ to be equal to $\angle F$) and $\angle C = \angle B$ (for $\angle C$ to be equal to $\angle D$), or if and only if ΔABC is equilateral.

Also solved by David E. Narvaez, Universidad Tecnologica de Panama

S44. Let $C(O)$ be a circle and let P be a point outside of C. Tangents from P intersect the circle at A and B. Let M be the midpoint of AP and let $N = BM \cap C(O)$. Prove that $PN = 2MN$.

Proposed by Pohoata Cosmin, Bucharest, Romania

First solution by Francisco Javier Garcia Capitan, Cordoba, Spain

Let us calculate the power of point M with respect to the circle $C(O)$, we have $MN \cdot MB = MA^2$. Let N' be the reflection of N with respect to M. Because

$$N'M \cdot MB = MN \cdot \frac{MA^2}{MN} = MA^2 = MP \cdot MA,$$

we get that $N'ABP$ is a cyclic quadrilateral.

Now, since $PM = MA$ and $NM = MN'$, the quadrilateral $N'ANP$ is a parallelogram.

Denote by $\alpha = \angle NAP$ and $\beta = \angle NAB$. Observe that $\angle NAP = \angle N'PA = \angle N'BA = \alpha$ and $\angle APN = \angle N'AP = \angle N'BP = \angle NAB = \beta$.

Thus in the triangle $N'AB$ we have $\angle AN'B = \pi - 2\alpha - 2\beta$, and in the triangle $PN'N$ we have $\angle N'NP = \pi - 2\alpha - 2\beta$ and $\angle PN'N = \alpha + \beta$. It follows that triangle $NN'P$ is isosceles with $NP = NN'$. Therefore $NP = NN' = 2MN$ and we are done.

Second solution by Daniel Campos Salas, Costa Rica

Let $\alpha = \angle BAN = \angle PBN$ and $\beta = \angle ABN = \angle PAN$. Then $\angle ABP = \angle BAP = \alpha + \beta, \angle APB = 180 - 2(\alpha + \beta)$, and $\angle AMB = 180 - (\alpha + 2\beta)$.

Since $Area_{ABM} = Area_{PBM}$ we have that $BP \sin\alpha = AB \sin\beta$, which implies that

$$\begin{aligned} \sin\alpha &= \frac{AB}{BP} \cdot \sin\beta = \frac{\sin 2(\alpha+\beta)}{\sin(\alpha+\beta)} \cdot \sin\beta \\ &= 2\sin\beta\cos(\alpha+\beta) = \sin(\alpha+2\beta) - \sin\alpha. \end{aligned}$$

It follows that $2\sin\alpha = \sin(\alpha + 2\beta)$, or

$$\frac{1}{2} = \frac{\sin\alpha}{\sin(\alpha+2\beta)} = \frac{\sin\alpha}{\sin\beta} \cdot \frac{\sin\beta}{\sin(\alpha+2\beta)} = \frac{BN}{AN} \cdot \frac{MN}{AN}.$$

Then, $AN^2 = 2BN \cdot MN$. The power of M with respect to $C(O)$ equals

$$\frac{1}{4}PA^2 = MA^2 = MN \cdot MB = MN^2 + MN \cdot BN = MN^2 + \frac{1}{2}AN^2. \quad (1)$$

Because MN is a median, it follows that

$$MN^2 = \frac{1}{2}AN^2 + \frac{1}{2}PN^2 - \frac{1}{4}PA^2. \quad (2)$$

From (1) and (2) it follows that $2MN^2 = \frac{1}{2}PN^2$, so $PN = 2MN$.

Third solution by David E. Narvaez, Universidad Tecnologica de Panama

Let Q and R be the points of intersection of PN with AB and $C(O)$ ($R \neq N$). Let S be the midpoint of PN and let T be the point of intersection of AN with PB. We claim that BR is parallel to PA. Then

$$\angle APR = \angle PRB = \angle NAB$$

and, since $\angle NBA = \angle NAP$ (because PA is tangent to $C(O)$)

$$\angle TNB = \angle NAB + \angle NBA = \angle NPA + \angle PAN = \angle PNT$$

but $\angle TNB = \angle MNA$ and MS is parallel to AN, so

$$\angle SMN = \angle MNA = \angle PNT = \angle MSN$$

which proves that $MN = SN$.

To prove our claim, let X be the point of intersection of BR and PN. Since AB is the polar line of P with respect to $C(O)$, points P, N, Q and R are harmonic conjugates. Then P, M, A and X are harmonic conjugates too, so

$$\frac{PM}{MA} = \frac{PX}{XA} = 1$$

whice implies that X is a point at the infinity, as we wanted to prove.

Also solved by Son Ta Hong, Ha Noi University of Education, Vietnam; Courtis G. Chryssostomos, Larissa, Greece; Vicente Vicario Garcia, Huelva, Spain; Vardan Verdiyan, Yerevan, Armenia

S45. Consider two sequences of integers, (a_n) and (b_n) such that $|a_{n+2} - a_n| \leq 2$ for all n in $\mathbb{Z}$ and $a_m + a_n = b_{m^2+n^2}$, for all m, n in $\mathbb{Z}$. Prove that there exist at most 6 distinct numbers in the sequence a_n.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution by Iurie Boreico, Moldova

We start with an observation that every number divisible by 4 can be written as $(n+2)^2 - n^2$. Indeed, $4k = (k+1)^2 - (k-1)^2$.

Now let m, n have the same parity. Then $m^2 - n^2$ is divisible by 4, thus can be written as $(k+2)^2 - k^2$. Therefore $n^2 + (k+2)^2 = m^2 + k^2$ and $a_n + a_{k+2} = a_m + a_k$ or $a_n - a_m = a_{k+2} - a_k$. We deduce that $|a_m - a_n| \leq 2$. So any two numbers from the sequence $(a_{2k})_{k \in \mathbb{Z}}$ differ by at most 2. It is not difficult to conclude that they can take at most three different values (just take a the smallest value so the possible values can be only $(a, a+1, a+2)$. The same holds for the sequence $(a_{2k+1})_{k \in \mathbb{Z}}$ so there can be also at most three possible values for the members of odd indices, summing for a total of at most six values.

S46. Let ABC be a triangle and let D, E, F be the points of tangency of the incircle with the sides of the triangle. Prove that the centroid of triangle DEF and the centroid of triangle ABC are isogonal if and only if triangle ABC is equilateral.

Proposed by Pohoata Cosmin, Bucharest, Romania

Solution by Daniel Campos Salas, Costa Rica

It is not difficult to conclude that if triangle ABC is equilateral then the condition holds. Suppose that the condition holds, we will prove that the triangle is equilateral. Let a, b, c be the lengths of sides BC, CA, AB, and assume without loss of generality that A is the origin. We have that

$$\frac{BD}{CD}=\frac{s-b}{s-c}, \frac{AE}{AB}=\frac{s-a}{c}, \frac{AF}{AC}=\frac{s-a}{b},$$

which implies that

$$\overrightarrow{D}=\frac{(s-c)\overrightarrow{B}+(s-b)\overrightarrow{C}}{a}, \overrightarrow{E}=\frac{(s-a)\overrightarrow{B}}{c}, \overrightarrow{F}=\frac{(s-a)\overrightarrow{C}}{b}.$$

Let G' be the centroid of triangle DEF. Then,

$$\overrightarrow{G'}=\frac{\left(\frac{s-a}{c}+\frac{s-c}{a}\right)\overrightarrow{B}+\left(\frac{s-a}{b}+\frac{s-b}{a}\right)\overrightarrow{C}}{3}. \tag{1}$$

On the other hand, let K be the Lemoine point, which is isogonal to the centroid, and let L, M, N be the intersections of AK, BK, CK with BC, CA, AB, respectively. It is well-known that

$$\frac{AN}{BN}=\frac{b^2}{a^2}, \frac{AM}{CM}=\frac{c^2}{a^2}, \frac{BL}{CL}=\frac{c^2}{b^2}.$$

From van Aubel's theorem applied to K, we deduce that

$$\frac{AK}{KL}=\frac{AN}{BN}+\frac{AM}{CM}=\frac{b^2+c^2}{a^2}.$$

Then,

$$\overrightarrow{L}=\frac{b^2\overrightarrow{B}+c^2\overrightarrow{C}}{b^2+c^2}, \text{ and } \overrightarrow{K}=\frac{b^2\overrightarrow{B}+c^2\overrightarrow{C}}{a^2+b^2+c^2}. \tag{2}$$

We have that $G'=K$. From (1) and (2) it follows that

$$\frac{\frac{s-a}{c}+\frac{s-c}{a}}{3}=\frac{b^2}{a^2+b^2+c^2} \quad \text{and} \quad \frac{\frac{s-a}{b}+\frac{s-b}{a}}{3}=\frac{c^2}{a^2+b^2+c^2}.$$

After some manipulations it follows that these equations are equivalent to

$$6ab^2c = (a^2+b^2+c^2)(ab+2ac+bc-a^2-c^2) \quad \text{and}$$
$$6abc^2 = (a^2+b^2+c^2)(2ab+ac+bc-a^2-b^2).$$

Thus

$$\frac{b}{c} = \frac{ab+2ac+bc-a^2-c^2}{2ab+ac+bc-a^2-b^2},$$

or

$$(b-c)(a^2+b^2+c^2-2a(b+c)) = 0.$$

Suppose that $b \neq c$. Then,

$$a^2+b^2+c^2 = 2a(b+c).$$

We have that

$$\frac{\frac{s-a}{c}+\frac{s-c}{a}}{3} = \frac{b^2}{a^2+b^2+c^2}$$

is equivalent to

$$6ab^2c = (a^2+b^2+c^2)(ab+2ac+bc-a^2-c^2).$$

Then,

$$\begin{aligned} ab+2ac+bc-a^2-c^2 &= (ab+bc)+(2ac-a^2-c^2) \\ &= b(a+c)+(b^2-2ab) \\ &= b(b+c-a). \end{aligned}$$

It follows, that

$$6ab^2c = (a^2+b^2+c^2)(ab+2ac+bc-a^2-c^2)$$
$$\Leftrightarrow 6ab^2c = 2a(b+c)\cdot b(b+c-a) \Leftrightarrow 3bc = (b+c)(b+c-a)$$
$$\Leftrightarrow 2a(b+c) = 2(b^2-bc+c^2) \Leftrightarrow a^2+b^2+c^2 = 2b^2-2bc+2c^2$$
$$\Leftrightarrow a^2 = (b-c)^2,$$

which contradicts the triangle inequality. We conclude that $b = c$. Then,

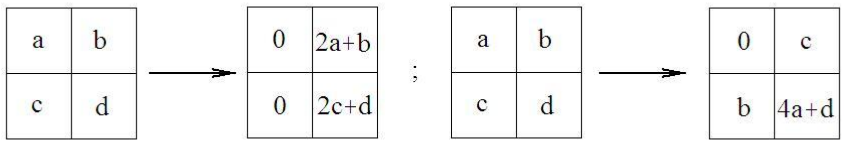
$$\frac{\frac{2b-a}{2b}+\frac{a}{2a}}{3} = \frac{b^2}{a^2+2b^2},$$

or

$$a(a-b)(a-2b) = 0.$$

If $a = 2b$, then $a = b+c$, which contradicts the triangle inequality. Then, $a = b$ from where we conclude that the triangle ABC is equilateral, and this completes the proof.

S47. Consider an $n \times n$ grid filled with ones. A move consists of taking a square with numbers (a, b, c, d) and rewriting the entries in one of the two following ways:



Prove that no matter how one makes moves, at one point there is only one nonzero entry on the table. Also prove that the value of this entry is unique.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution by Iurie Boreico, Moldova

We correct the problem: we will have a single non-zero entry unless one makes the same moves for an infinite number of times, thus sticking up the configuration.

Label the rows and columns from the lower-right corner and assign to the table the quantity $\sum_{0 \le i,j \le n} 2^{i+j-2} a_{i,j}$, where $a_{i,j}$ is the element in the square labeled (i, j). It's straightforward to check that the allowed operations preserve this quantity, so it will always be the same. In the beginning it was

$$A = 1 + 2 \cdot 2 + 4 \cdot 3 + \ldots + 2^{n-1} \cdot n + 2^n \cdot (n-1) + \ldots + 2^{2n-2}.$$

This value can actually be computed, but we only need that it is odd.

Next, note that any "normal" operation increases the sum of squares of entries of the table (a "normal" operations is one which actually produces a change in the table). From the other this sum cannot increase indefinitely because each entry in the table is at most A. Thus there is a moment when we cannot perform a "normal" operation anymore, or we have reached a situation where there is one non-zero number. Still, we note that if in the 2×2 table considered the "a" entry is non-zero then the operation is "normal". Thus all entries except the ones on the final row and column are non-zero. It is now easy to see how we can increase the sum of squares by operations if an entry except the lower-right one is non-zero.

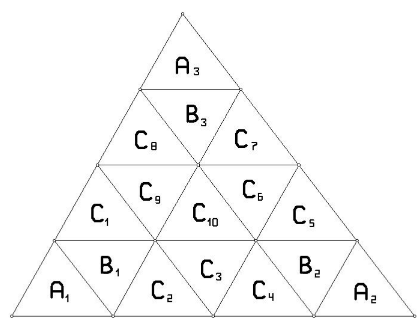
So we have proven that we have one non-zero entry at a moment. It remains to prove it is unique. Indeed, it must be placed in the lower-right square otherwise A would be negative which is wrong. And in this case it must equal to the fixed value A.

S48. Consider an equilateral triangle divided into 16 congruent equilateral triangles. Prove that no matter how we label these triangles with the numbers 1 through 16, there will be two adjacent triangles whose labels differ by at least 4.

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by Ivan Borsenco, University of Texas at Dallas

Divide the region in three regions $A = \{A_1, A_2, A_3\}$, $B = \{B_1, B_2, B_3\}$, and $C = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_9, C_{10}$. Let us define the length of path between two triangles to be the number of "steps" it would take to travel between them (a step being from one triangle to another with whom it shares a side). For example, the length of path between A_1 and A_2 is 6. Observe that the length of path between any triangles in C is 4.



Assume that there exists a configuration such that the difference between the labels of some two adjacent triangles is less or equal to 3.

The first step is to prove that in such a configuration, the labels *1* and *16* should lie in A. Suppose *1* lies in C. Observe that there exist two A_i, A_j, say A_1, A_2, with the distance no more than 4 from *1*. It follows that only triangles A_3, B_3 are at least 5 apart from *1*. By the Pigeonhole Principle one among the numbers *14, 15, 16* will be at most 4 steps from *1*. The difference between numbers is at least 13, hence there will be two adjacent triangles with difference at least 4. If *1* lies in B_i then observe that we can interchange it with the number in A_i and the big triangle still satisfies the condition.

Suppose $A_1 = 1$ and $A_2 = 16$. The second step is to prove that A_3 contains either *2* or *15*. Clearly the numbers *2, 15* cannot lie both in C, because the length of path between them is at most 4. Suppose *2* lies outside C.

1^{st} case: *2* lies in B_2, which is clearly impossible.

2^{nd} case: *2* lies in B_3, then we interchange it with the number in A_3 and we are done.

3^{rd} case: *2* lies in B_1. Then *15* should lie in C_5 or C_7. If *15* lies in C_5, C_7, otherwise *15* lies in A_3 and we are done. From here we deduce that *3* lies in C_1 or C_2. Observe that *3* cannot lie in C_2 because *16* lies in A_2 the length is 4 and the difference between them is 13. Therefore *3* lies in C_1. The length of path between C_1 and C_5 or C_7 is 4 and the difference between them is 12. It follows on this path the difference between every adjacent triangle is 3. Thus $C_9 = 6, C_{10} = 9, C_6 = 12$.

The next observation is that triangles C_2, C_3, C_4 do not contain numbers *13,14,15*, because $B_1 = 2$. They lie in B_2, C_5 and $\{A_3, B_3, C_7\}$. Therefore one of them lies in one of $\{A_3, B_3, C_7\}$. Thus *4,5* cannot lie in C_8, A_3, B_3, C_7. Also *4,5* cannot lie in C_3, C_4, B_2, C_5. It follows that we have only just one place C_1 for *4,5*, contradiction. Finally, we proved that either *15* or *2* lie in A_3.

Without loss of generality let *2* lie in A_3. Numbers in $C_1, B_1, C_2, C_7, B_3, C_2$ are not greater than 8. As *1,2* are already in places, it follows that *3,4,5,6,7,8* lie in $C_1, B_1, C_2, C_7, B_3, C_2$. Numbers in C_3, C_6, C_9 are greater than C_2, C_7, C_8, C_1 by at most 3. Impossible, because $C_3, C_6, C_9 = \{9,10,11\}$ and there will exist a pair with the difference greater than 3.

The problem is solved.

Undergraduate problems

U43. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that for any positive integer n there exists $c \in [0, 1]$ such that

$$f(c) = f(c + \frac{1}{n}).$$

Proposed by Jose Luis Diaz-Barrero, Barcelona, Spain

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The case $n = 1$ clearly is true taking $c = 0$.

Let us define, for $n \geq 2$, on the interval $\left[0, \frac{n-1}{n}\right]$, the real-valued function

$$g_n(x) = f\left(x + \frac{1}{n}\right) - f(x).$$

Since f is continuous, so is g_n. Note also that whenever $g_n(c) = 0$, $f\left(c + \frac{1}{n}\right) = f(c)$. Thus we need to prove that there exists at least one c in $\left[0, \frac{n-1}{n}\right]$ such that $g_n(c) = 0$.

Now,

$$\sum_{i=1}^{n-1} g_n\left(\frac{i}{n}\right) = f(1) - f(0) = 0.$$

This is possible if either all the elements in the sum are 0, or at least one is positive and at least one is negative. In the first case, $g_n(x) = 0$ for $x = \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}$. In the second case, there are at least two values, a and b, in $\left\{\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}\right\}$, and thus in $\left[0, \frac{n-1}{n}\right]$, such that $g_n(a)$ and $g_n(b)$ have opposite signs. By the intermediate value theorem, there is at least one value c in (a, b) such that $g_n(c) = 0$.

Also solved by Paolo Perfetti, Mathematical Department of "Tor Vergata" University, Roma, Italy, Courtis G. Chryssostomos, Larissa, Greece

U44. Let x, y be positive real numbers such that $x^y + y = y^x + x$. Prove that $x + y \leq 1 + xy$.

Proposed by Cezar Lupu, University of Bucharest, Romania

First solution by Vardan Verdiyan, Yerevan, Armenia

Without loss of generality assume $x \geq y$ and let us assume the contrary $x + y > 1 + xy$. Then $(1-x)(1-y) < 0, \Rightarrow x > 1 > y > 0$. By the Bernoulli's inequality, for every $a, b \in \mathbb{R}$ we have

(1) If $a > 1$ and $b > -1$ then $(1+b)^a > 1 + ab, \Rightarrow$

$$(1 + (y-1))^x > 1 + (y-1)x.$$

(2) If $0 < a < 1$ and $b > 0$ then $(1+b)^a < 1 + ab, \Rightarrow$

$$(1 + (x-1))^y < 1 + (x-1)y.$$

Summing up these two inequalities we get

$$(1 + (y-1))^x + 1 + (x-1)y > (1 + (x-1))^y + 1 + (y-1)x,$$

or

$$y^x + x > x^y + y,$$

contradiction and the conclusion follows.

Second solution by Paolo Perfetti, Mathematical Department of "Tor Vergata" University, Roma, Italy

Assume the contrary $x + y > 1 + xy$. We prove that if

$$x > 0,\ y > 0,\ x + y > 1 + xy, \text{ then } x^y + y - y^x - x > 0\,.$$

Observe that $x + y > 1 + xy$ is equivalent to $(1-x)(1-y) < 0$, namely to one of the two conditions

$$1)\ 0 < y < 1,\ x > 1 \qquad\qquad 2)\ y > 1,\ 0 < x < 1$$

It's enough to consider case 1), because $x^y + y - y^x - x$ is symmetric with respect to the bisector of the first and the third quadrant.

Let $0 < a < 1$ and $f(y) \doteq a^y - y^a + y - a$, where $y > 1$. Note that $\lim_{y \to 1^+} f(y) = 0$ and $f'(y) = a^y \ln a - a y^{a-1} + 1 > a \ln a - a + 1$, because $\ln a < 0$, $a^y < a$, and $y^{a-1} < 1$. Consider $g(x) = x \ln x - x + 1$, $x \in [0,1]$. We have $g'(x) = \ln x + 1 - 1 = \ln x < 0$, $g'(x)$ is decreasing on $[0,1]$, it follows that $g(x) \geq g(1) = 0$. Thus $f'(y) > a \ln a - a + 1 > 0$. It follows that $f(y)$ is increasing on $[1, +\infty)$ and $f(y) \geq f(1) = 0$. Hence we get $x^y + y - y^x - x > 0$, contradiction. It follows that $x + y \leq 1 + xy$.

Also solved by Arkady Alt, San Jose, California, USA

U45. Let $A \in M_n(R)$ be a matrix that has zeros on the main diagonal and all other entries are from the set $\{-1, 1\}$. Is it possible that $\det A = 0$ for $n = 2007$? What about for $n = 2008$?

Proposed by Aleksandar Ilic, Serbia

Solution by Vicente Vicario Garcia, Huelva, Spain

a) n is odd.

If A is antisymmetric matrix of odd order, $A = -A^t$, then $\det(A) = 0$. According to the Cauchy-Binet Theorem $\det(A \cdot B) = \det(A) \cdot \det(B)$, and $\det A = \det(A^t)$. We have

$$A = -A^t, \Rightarrow \ \det(-A) = \det(A^t), \Rightarrow \ -\det(A) = \det(A), \Rightarrow \ \det(A) = 0.$$

Now, if A is an antisymmetric matrix of odd order, that has zeros on the main diagonal and all other entries are from the set $\{-1, 1\}$, $\det(A) = 0$, and the answer is true.

b) n is even

We prove that it is not possible that a matrix A of even order that has zeros on the main diagonal and all other entries are from the set $\{-1, 1\}$ is such that $\det(A) = 0$. We really prove more. We prove that it is not possible that a matrix A of even order with entries $\{-1, 1\}$ except only one zero for every different column 2^{nd},..., n^{th} with only one zero for every row, is such $\det(A) = 0$. This determinant is an odd number. We use mathematical induction. For $n = 2$, if $\epsilon = \pm 1$, then

$$\begin{vmatrix} \epsilon & \epsilon \\ \epsilon & 0 \end{vmatrix} = \pm 1, \quad \begin{vmatrix} \epsilon & 0 \\ \epsilon & \epsilon \end{vmatrix} = \pm 1.$$

We assume that the proposition is true until $n = 2k$. Then for a matrix $n = 2k + 2$ developing the original determinant we get that it is equal to the sum of $2k + 1$ odd non-zero determinants of the anterior form. Developing its determinants again we obtain the sum of odd numbers, different from zero and we are done.

U46. Let k be a positive integer and let

$$a_n = \left\lfloor \left(k+\sqrt{k^2+1}\right)^n + \left(\frac{1}{2}\right)^n \right\rfloor, \ n \geq 0.$$

Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = \frac{1}{8k^2}.$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by by Jose Hernandez Santiago, UTM, Oaxaca, Mexico

Let us consider the sequence b_n, defined for $n \geq 0$ as follows

$$\begin{cases} b_0 &= 2 \\ b_1 &= 2k \\ b_n &= 2k(b_{n-1}) + b_{n-2}. \end{cases}$$

Through generating functions we learn that for every $n \in \mathbb{Z}^+$ the corresponding element of b_n satisfies the following explicit formula:

$$b_n = (k+\sqrt{k^2+1})^n + (k-\sqrt{k^2+1})^n.$$

Now, since the inequalities

$$b_n \leq (k+\sqrt{k^2+1})^n + \left(\frac{1}{2}\right)^n < b_n + 1$$

hold for every $n \in \mathbb{Z}^+$, we conclude that

$$a_n = \left\lfloor (k+\sqrt{k^2+1})^n + \left(\frac{1}{2}\right)^n \right\rfloor = b_n,$$

for every nonnegative whole number n. Then,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} &= \sum_{n=1}^{\infty} \frac{1}{b_{n-1}b_{n+1}} & (3) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{2k}\right)\left(\frac{1}{b_{n-1}b_n} - \frac{1}{b_nb_{n+1}}\right) & (4) \\ &= \frac{1}{2k}\sum_{n=1}^{\infty}\left(\frac{1}{b_{n-1}b_n} - \frac{1}{b_nb_{n+1}}\right) = \frac{1}{8k^2}. & (5) \end{aligned}$$

The last identity follows by telescoping our sum and using the fact that $\lim_{n\to\infty} \frac{1}{b_n} = 0$.

U47. Let P be an arbitrary point inside equilateral triangle ABC. Find the minimum value of
$$\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC}.$$

Proposed by Hung Quang Tran, Ha Noi National University, Vietnam

Solution by Hung Quang Tran, Ha Noi National University, Vietnam

We can assume that triangle ABC is an equilateral with side 1. Suppose M has barycentric coordinates (x, y, z), $x+y+z=1$ and $x, y, z \geq 0$, because M is inside the triangle ABC. Using distance formula we have

$$MA^2 = \frac{y+z}{x+y+z}a^2 - \frac{xy+yz+zx}{(x+y+z)^2}a^2.$$

Therefore, with $a=1$ and $x+y+z=1$ we get

$$MA=\sqrt{y^2+yz+z^2},\ MB=\sqrt{x^2+xz+z^2},\ MC=\sqrt{x^2+xy+y^2}.$$

Thus we need to find the minimum of

$$\frac{1}{\sqrt{y^2+yz+z^2}}+\frac{1}{\sqrt{x^2+xz+z^2}}+\frac{1}{\sqrt{x^2+xy+y^2}},$$

when $x+y+z=1$. We use Lagrange multipliers to find this minimum.

Assume without loss of generality $0 \leq x \leq y \leq z < 1$

1^{st} case: $x=0$, we want to prove

$$f(y,z)=\frac{1}{\sqrt{y^2+yz+z^2}}+\frac{1}{y}+\frac{1}{z} \geq 4+\frac{2\sqrt{3}}{3}.$$

Indeed
$$f(y,z) \geq f(\frac{y+z}{2},\frac{y+z}{2}) = 4+\frac{2\sqrt{3}}{3}.$$

2^{nd} case: $0 < x \leq y \leq z < 1$. Let

$$F(x,y,z,\lambda)=\sum \frac{1}{\sqrt{x^2+xy+y^2}}+\lambda(x+y+z-1)$$

$\frac{dF}{dx}=0$, $\frac{dF}{dy}=0$, $\frac{dF}{dz}=0$, then

$$-\frac{1}{2}\left[\frac{2x+y}{\sqrt{(x^2+xy+y^2)^3}}+\frac{2x+z}{\sqrt{(z^2+zx+x^2)^3}}\right]+\lambda=0,$$

$$-\frac{1}{2}\left[\frac{2y+x}{\sqrt{(x^2+xy+y^2)^3}}+\frac{2y+z}{\sqrt{(y^2+yz+z^2)^3}}\right]+\lambda=0,$$

$$-\frac{1}{2}\left[\frac{2z+x}{\sqrt{(z^2+zx+x^2)^3}}+\frac{2z+y}{\sqrt{(y^2+yz+z^2)^3}}\right]+\lambda=0,$$

Adding them we get

$$\lambda\frac{1}{2}\left[\frac{x+y}{\sqrt{(x^2+xy+y^2)^3}}+\frac{y+z}{\sqrt{(y^2+yz+z^2)^3}}+\frac{z+x}{\sqrt{(z^2+zx+x^2)^3}}\right].$$

Inserting λ in the first equation we obtain

$$x\sum\frac{1}{\sqrt{(x^2+xy+y^2)^3}}=\frac{1}{\sqrt{(y^2+yz+z^2)^3}}.$$

Analogously,

$$y\sum\frac{1}{\sqrt{(x^2+xy+y^2)^3}}=\frac{1}{\sqrt{(z^2+zx+x^2)^3}},$$

$$z\sum\frac{1}{\sqrt{(x^2+xy+y^2)^3}}=\frac{1}{\sqrt{(x^2+xy+y^2)^3}}.$$

Hence

$$y^2(z^2+zx+x^2)^3=z^2(x^2+xy+y^2)^3.$$

Let $y=ax$, $z=bx$, where $1\le a\le b$, we have

$$a^2(b^2+b+1)^3=b^2(a^2+a+1)^3.$$

Clearly we get $a=b$. It follows $y=z$ is a necessary condition for critical points in the interior of the region $0<x,y,z<1$. We have to prove

$$\frac{2}{\sqrt{x^2+xy+y^2}}+\frac{1}{y\sqrt{3}}\ge 4+\frac{2\sqrt{3}}{3},$$

where $x+2y=1$. Consider

$$g(y)=\frac{2}{3y^2-3y+1}+\frac{1}{y\sqrt{3}}\ge 4+\frac{2\sqrt{3}}{3},$$

where $\frac{1}{3}\le y\le\frac{1}{2}$, because $x\le y\le z$. Using differentiation it is not difficult to check that the absolute minimum of $g(y)$ on $\left[\frac{1}{3},\frac{1}{2}\right]$ is $g(\frac{1}{2})=4+\frac{2\sqrt{3}}{3}$.

Thus $\frac{1}{MA}+\frac{1}{MB}+\frac{1}{MC}$ attains its minimum value $4+\frac{2\sqrt{3}}{3}$ at $M(\frac{1}{2},\frac{1}{2},0)$ and other permutations. These points are the midpoints of the triangle's sides.

U48. Let n be an integer greater than 1 and let $k \geq 1$ be a real number. For an n dimensional simplex $X_1X_2\dots X_{n+1}$ define its k-perimeter by $\sum_{1\leq i<j\leq n+1}|X_iX_j|^k$. Take now a regular simplex $A_1A_2\dots A_{n+1}$ and consider all simplexes $B_1B_2\dots B_{n+1}$ where B_i lies on the face $A_1\dots A_{i-1}A_{i+1}\dots A_n$. Find, in terms of the k-perimeter of $A_1A_2\dots A_{n+1}$, the minimal possible k-perimeter of $B_1B_2\dots B_{n+1}$.

Proposed by Iurie Boreico, Moldova

Solution by Iurie Boreico, Moldova

Let O be the center of our simplex $A_1A_2\dots A_n$. We may assume $OA_i = 1$.

We will prove the intuitive assertion that the minimal value is obtained when B_i is the center of the corresponding face $A_1\dots A_{i-1}A_{i+1}\dots A_{n+1}$, i.e. equals $\frac{1}{n^k}$ of the k-perimeter of $A_1A_2\dots A_{n+1}$ (because in this case $B_1\dots B_{n+1}$ is obtained from $A_1A_2\dots A_{n+1}$ via a homothety through O of magnitude $\frac{1}{n}$).

Let us prove that the minimal value actually occurs. Pick up a random simplex $B_1\dots B_n$ and let S be its k-perimeter. $\overline{OB_i}$ can be written as $\sum_{j\neq i}c_j\overline{O}B_j$ where $\sum_{j\neq i}c_j = 1$. Now its projection on the face $A_1\dots A_{j-1}A_{j+1}\dots A_{n+1}$ will have length $|c_j|$ thus $B_iB_j \geq |c_j|$ so its k-perimeter will have length at least $|c_j|^k$. So if we want to have its perimeter not greater than S we infer that $|c_j| \leq \sqrt[k]{S}$. Therefore c_j is bounded and hence so are all of the B_i. Now Weierstrass Theorem implies that there is a minimal value for the k-perimeter and it is actually attained.

To proceed further, we define a well-known group of transformations on A_i. For $n = 2$ these are the rotations with respect to O by 120 degrees and the symmetries with respect to the lines OA_i. For higher spaces we define them using linear algebra.

Consider O as the origin of the coordinate system and let $v_i = \overline{OA_i}$. We know that $v_1+v_2+\ldots+v_{n+1} = 0$ and any n of these vectors span R^n. For any permutation $\pi \in S_{n+1}$ consider the linear transform T_π that take v_i to $v_{\pi(i)}$ for $i = 1,\dots,n$. As $v_1+v_2+\ldots+v_{n+1} = 0$ it also takes v_{n+1} to $v_{\pi(n+1)}$. As A_iA_j does not depend on i,j for $i \neq j$ we conclude that $v_iv_j = \frac{v_i^2+v_j^2-(v_i-v_j)^2}{2} = \frac{2-A_i^2A_j^2}{2}$ does not depend on $i \neq j$ either. It is now also pretty clear to verify directly that if $v = \sum c_iv_i, v' = \sum c_i'v_i'$ then $|v-v'|^2 = |T_\pi(v) - T_\pi(v')|^2$. So T_π preserves the distance between two points.

To proceed, we need a simple lemma.

If $u, v \in R^n$ then $\frac{(|u|^k+|v|^k)}{2} \geq |\frac{u+v}{2}|^k$. The equality holds if and only if $u = v$ for $k > 1$ and if and only if $u||v$ for $k = 1$.

Pick now a set $B_1, \ldots, B_{n+1}$ that realize the minimum k-perimeter asked. If $k=1$ among all these sets we can select the one which gives the minimum possible 2-perimeter.

Pick up a random permutation $\sigma \in S_{n+1}$ and let $C_i = T_\sigma(B_i)$. Clearly the simplex $C_1 C_2 \ldots C_{n+1}$ also satisfies our conditions and has the minimal possible k-perimeter (actually the two simplices are congruent). Then set $B'_{\sigma(i)}$ be the midpoint of the segment determined by $B_{\sigma(i)}$ and C_i. It is clear that $|B'_{\sigma(i)}B'_{\sigma(j)}|^k \leq \frac{1}{2}(|B_{\sigma(i)}B_{\sigma(j)}|^k + |C_iC_j|^k)$. We sum all these relations to conclude that the k-perimeter of $B'_1B'_2 \ldots B'_{n+1}$ is not more than the possible minimum. This implies that we have equalities everywhere. For $k>1$ we conclude that $\overline{B_{\sigma(i)}B_{\sigma(j)}} = \overline{C_iC_j}$. We make the same conclusion for $k=1$ because otherwise by the same lemma the 2-perimeter of the simplex would decrease, contradicting the fact that it is also minimal.

We have deduced that $\overline{B_{\sigma(i)}B_{\sigma(j)}} = \overline{T_\sigma(B_i)T_\sigma(B_j)}$. Thus these four points form a parallelogram so we deduce that $\overline{B_{\sigma(i)}T_\sigma(B_i)} = \overline{B_{\sigma(j)}T_\sigma(B_j)}$ for any σ. So all the vectors $\overline{B_\sigma(i)T_\sigma(B_i)}$ equal the same vector v_σ. However as $B_{\sigma(i)}$ and $T_{\sigma(i)}$ all belong to the sides $A_1A_2 \ldots A_{\sigma(i)-1}A_{\sigma(i)+1} \ldots A_{n+1}$, v_σ is parallel to it i.e. perpendicular to $v_{\sigma(i)}$. Letting i run over $1, 2, \ldots, n+1$ as $v_1, v_2, \ldots, v_{n+1}$ span R^n, we conclude $v_\sigma = 0$ so $B_{\sigma(i)} = T_\sigma(B_i)$

Choose now σ be a transposition, for example transposing n and $n+1$. Introduce the unique matrix $A \in M_{n+1}(R) = (a_{ij})$ with zeroes on the diagonal such that $\overline{OB_i} = \sum_{j=1}^{n+1} a_{ij}v_j$. We see $B_{\sigma(1)} = B_1$ so $\overline{OB_{\sigma(1)}} = \sum_{i=2}^{n+1} a_{1,i}v_i$. At the same time $T_\sigma(B_1) = \sum_{i=2}^{n-1} a_{1,i}v_i + a_{1,n+1}v_n + a_{1,n}v_{n+1}$. It implies $a_{1,n+1} = a_{1,n}$. Since $n, n+1$ were taken arbitrarily, we conclude $a_{1,2} = a_{1,3} = \ldots = a_{1,n+1}$. As there sum must be 1 for B_1 to lie on $(A_2A_3 \ldots A_{n+1})$ we deduce $a_{1,2} = a_{1,3} = \ldots = a_{1,n+1} = \frac{1}{n}$ so B_1 is the center of the face $A_2A_3 \ldots A_{n+1}$. We reason similarly for $B_2, B_3, \ldots, B_{n+1}$ to finish the problem.

Remarks:

1. The above reasoning shows that for $k>1$ the only case for equality occurs when B_i are the corresponding centers of the faces. The reasoning fails for $k=1$, however a more careful analysis of the matrix A can lead us to the same conclusion. The reader is encouraged to do this himself, if any interested reader exists.

2. The main idea of this problem is to consider the linear transformations of the plane that preserve the distance and permute the vertices of the simplex $A_1A_2 \ldots A_{n+1}$. The same idea could be used to other problems that arise when defining a k-perimeter in other way. Particularly we could define a (symmetric but non-transitive) relation $\sim$ on $1, 2, \ldots, n+1$ and define the k-perimeter as $\sum_{i \sim j} A_iA_j^k$. For such generalizations the difficulties rise considerably because we cannot take any permutation σ anymore, but only the ones that maintain

the relation $\sim$ unchanged. For example at the Mathematical Competition for Former Soviet-Union Countries in 2005 the k-perimeter of a tetrahedron $A_1A_2A_3A_4$ was defined as $A_1A_2 + A_2A_3 + A_3A_4 + A_4A_1$. The reader is encouraged to tackle this problem using the same method (but this time only cyclic permutations will work). Note that this problem was the inspiration for the mine.

Olympiad problems

O43. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}} \geq \sqrt{\frac{16(a+b+c)^3}{3(a+b)(b+c)(c+a)}}.$$

Proposed by Vo Quoc Ba Can, Can Tho University, Vietnam

First solution by Arkady Alt, San Jose, California, USA

Let us solve the following inequality

$$\sqrt{\frac{y+z}{x}}+\sqrt{\frac{z+x}{y}}+\sqrt{\frac{x+y}{z}} \geq \sqrt{\frac{16(x+y+z)^3}{3(x+y)(y+z)(z+x)}}.$$

Let $a=y+z, b=z+x$, $c=x+y$, and $s=x+y+z=\frac{a+b+c}{2}$. Observe that a, b, c determine triangle ABC with semiperimeter s, area F, and circumradius R. Using our notations we can rewrite our inequality

$$\sqrt{\frac{a}{s-a}}+\sqrt{\frac{b}{s-b}}+\sqrt{\frac{c}{s-c}} \geq \sqrt{\frac{16s^3}{3abc}} \iff$$

$$\sum_{cyc}\sqrt{\frac{(s-b)(s-c)}{bc}} \geq \frac{4}{\sqrt{3}}\cdot\frac{Fs}{abc}=\frac{1}{\sqrt{3}}\cdot\frac{s}{R}.$$

We know that $\frac{s}{R}=\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ and

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}, \sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}, \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Our inequality is equivalent to

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} \geq \frac{4}{\sqrt{3}}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Denote by $\alpha=\frac{\pi-A}{2}, \beta=\frac{\pi-B}{2}, \gamma=\frac{\pi-C}{2}$. Observe that $\alpha+\beta+\gamma=\pi$ and $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right)$. Consider now an acute-angled triangle $A'B'C'$ with $A'=\alpha, B'=\beta, C'=\gamma$ with the same notations a, b, c, s, R, r, for the lengths of sides, the semiperimeter, the circumradius, and the inradius, respectively. Our inequality can be rewritten as

$$\cos\alpha+\cos\beta+\cos\gamma \geq \frac{4}{\sqrt{3}}\sin\alpha\sin\beta\sin\gamma.$$

Using the identity $\cos\alpha+\cos\beta+\cos\gamma=\frac{R+r}{R}$ and Euler's Inequality $R\geq 2r$ we get $\cos\alpha+\cos\beta+\cos\gamma\geq\frac{3r}{R}$. Also we know $\sin\alpha\sin\beta\sin\gamma=\frac{abc}{8R^3}=\frac{4Rrs}{8R^3}=\frac{rs}{2R^2}$. Thus, it suffices to prove

$$\frac{4}{\sqrt{3}}\cdot\frac{rs}{2R^2}\leq\frac{3r}{R}\text{ or }2s\leq 3\sqrt{3}R,$$

that is clear from the famous fact $9R^2\geq a^2+b^2+c^2\geq\dfrac{(a+b+c)^2}{3}=\dfrac{4s^2}{3}$.

Second solution by Kee-Wai Lau, Hong Kong, China

Our inequality is homogeneous, therefore we can assume that $a+b+c=1$. Let us rewrite it in the following form

$$\frac{b+c}{\sqrt{a}}\sqrt{(c+a)(a+b)}+\frac{c+a}{\sqrt{b}}\sqrt{(a+b)(b+c)}+\frac{a+b}{\sqrt{c}}\sqrt{(b+c)(c+a)}\geq\frac{4\sqrt{3}}{3}.$$

We have

$$\frac{b+c}{\sqrt{a}}\sqrt{(c+a)(a+b)}=\left(\frac{1}{\sqrt{a}}-\sqrt{a}\right)\sqrt{(c+a)(a+b)}=\sqrt{1+\frac{bc}{a}}-\sqrt{a}\sqrt{a+bc}.$$

Using similar expressions for $\frac{c+a}{\sqrt{b}}\sqrt{(a+b)(b+c)}$ and $\frac{a+b}{\sqrt{c}}\sqrt{(b+c)(c+a)}$ we see that the left hand side is equal to S_1-S_2, where

$$S_1=\sqrt{1+\frac{ab}{c}}+\sqrt{1+\frac{bc}{a}}+\sqrt{1+\frac{ca}{b}}$$

and

$$S_2=\sqrt{a}\sqrt{a+bc}+\sqrt{b}\sqrt{b+ac}+\sqrt{c}\sqrt{c+ab}.$$

From the AM-GM inequality we have

$$ab+bc+ca\leq\frac{1}{3},\ a^2+b^2+c^2\geq\frac{1}{3},\ \frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq 1.$$

Let us prove that $S_1\geq 2\sqrt{3}$. Using the AM-GM inequality we have

$$\left(\frac{S_1}{3}\right)^6\geq\left(1+\frac{ab}{c}\right)\left(1+\frac{bc}{a}\right)\left(1+\frac{ca}{b}\right)=1+a^2+b^2+c^2+\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}+abc=$$

$$=1+\frac{1}{9}(a+b+c)^2+\frac{8}{9}(a^2+b^2+c^2)+\frac{26}{27}\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)+$$

$$+ab\left(\frac{1}{3\sqrt{3}c}-\sqrt{\frac{c}{3}}\right)^2+bc\left(\frac{1}{3\sqrt{3}a}-\sqrt{\frac{a}{3}}\right)^2+ac\left(\frac{1}{3\sqrt{3}b}-\sqrt{\frac{b}{3}}\right)^2\geq$$

$$\geq 1+\frac{1}{9}+\left(\frac{8}{9}\right)\left(\frac{1}{3}\right)+\frac{26}{27}=\frac{64}{27}.$$

From the Cauchy-Schwarz inequality we have

$$S_2\leq\sqrt{a+b+c}\cdot\sqrt{(a+bc)+(b+ca)+(c+ab)}=$$

$$=\sqrt{1+ab+bc+ca}\leq\sqrt{1+\frac{1}{3}}=\frac{2\sqrt{3}}{3}.$$

Thus $S_1-S_2\geq 2\sqrt{3}-\frac{2\sqrt{3}}{3}=\frac{4\sqrt{3}}{3}$ and we are done.

Third solution by Pham Huu Duc, Ballajura, Australia

Using Holder Inequality we get

$$\left(\sum_{\text{cyc}}\sqrt{\frac{b+c}{a}}\right)^2\sum_{\text{cyc}}a(b+c)^2\geq 8(a+b+c)^3.$$

It suffices to prove that

$$3(a+b)(b+c)(c+a)\geq 2\sum_{\text{cyc}}a(b+c)^2.$$

The inequality is equivalent to

$$3\left(\sum_{cyc}a(b^2+c^2)+2abc\right)\geq 2\left(\sum_{cyc}a(b^2+c^2)+6abc\right)$$

or

$$\sum_{cyc}a(b^2+c^2)\geq 6abc,$$

which is true from the AM-GM inequality.

Also solved by Jingjun Han, High School Affiliated to Fudan University, China; Orif Ibrogimov, SamSU, Samarkand, Uzbekistan; Diyora Salimova, Lyceum N1, Samarkand, Uzbekistan; Paolo Perfetti, Mathematical Department of "Tor Vergata" University, Roma, Italy; Daniel Campos Salas, Costa Rica; Magkos Athanasios, Kozani, Greece; Vardan Verdiyan, Yerevan, Armenia; Tigran Sloyan, Yerevan, Armenia

O44. Let $ABCD$ be a cyclic quadrilateral and let O be the intersection of its diagonals. Denote by I_{ab} and I_{cd} the centers of incircles of triangles OAB and OCD, respectively. Prove that the perpendiculars from O, I_{ab}, I_{cd} to lines AD, BD, AC, respectively, are concurrent.

Proposed by Mihai Miculita, Oradea, Romania

Solution by Vardan Verdiyan, Yerevan, Armenia

Let $I_{cd}H_1$, OH_2 and $I_{ab}H_3$ be the perpendiculars from I_{cd}, O, I_{ab} to lines DB, BC, AC, respectively.

Denote by M_1 the intersection point of lines OH_2 and $I_{cd}H_1$ and by M_2 the intersection point of lines OH_2 and $I_{ab}H_3$.

Note that if $OM_1 = OM_2$ then the lines $I_{cd}H_1, OH_2$ and $I_{ab}H_3$ are concurrent. Since $\angle H_1OM_1 = \angle H_2OB$ and $\Delta H_1OM_1, \Delta H_2OB$ are right-angled triangles $\Rightarrow$

$$\Delta H_1OM_1 \sim \Delta H_2OB, \Rightarrow OM_1 = OB \cdot \frac{OH_1}{OH_2}.$$

Analogously, $OM_2 = OC \cdot \frac{OH_3}{OH_2}$. It suffices to prove that

$$OC \cdot \frac{OH_3}{OH_2} = OB \cdot \frac{OH_1}{OH_2} \text{ or } OC \cdot OH_3 = OB \cdot OH_1.$$

Because $I_{cd}H_1 \perp OD$ we have H_1 is the point of tangency of the incircle of triangle COD with the line OD. Thus $OH_1 = \frac{CO+DO-CD}{2}$. Similarly we get $OH_3 = \frac{AO+BO-AB}{2}$. It is enough to prove that

$$OC \cdot \frac{AO+BO-AB}{2} = OB \cdot \frac{CO+DO-CD}{2} \iff$$

$$OC(AO-AB) = OB(DO-CD).$$

Because $ABCD$ is cyclic quadrilateral we have $\Delta AOB \sim \Delta DOC$. Therefore

$$\frac{OB}{OC} = \frac{AO}{DO} = \frac{AB}{CD} = \frac{AO-AB}{DO-CD},$$

which completes the proof.

Also solved by Andrea Munaro, Italy; Tigran Sloyan, Yerevan, Armenia; David E. Narvaez, Universidad Tecnologica de Panama

O45. Consider a positive real number t. A grasshopper has a finite number of pointwise nests. It can add new nests as follows: from two nests A and B it can jump to point C with $\frac{\overrightarrow{AC}}{\overrightarrow{AB}} = t$, and make C its nest. Prove that there are points in the plane that cannot be made nests.

Proposed by Iurie Boreico, Moldova

Solution by Iurie Boreico, Moldova

Let $x_1, x_2, \ldots, x_n$ be the affixes of the initial nests of the grasshopper. The new nest C produced from the nests A, B with affixes a, b has affix $b+k(b-a)$. It is therefore straightforward by induction to show that each new next may be represented as $P(k, x_1, x_2, \ldots, x_n)$ where P is an integer polynomial in $n+1$ variables. It is known that the set of integer polynomials in $n+1$ variables is countable. Thus the set of all possible nests is also at most countable. As C is uncountable, there are infinitely many points that cannot be made nests.

O46. Let O and I be the circumcenter and the incenter of triangle ABC, respectively. Denote by D the intersection of the incircle of ABC with BC and by E and F the intersections of AI and AO with the circumcircle of ABC, respectively. Let S be the intersection of FI and ED, M the intersection of SC and BE, and N the intersection of AC and BF. Prove that M, I and N are collinear.

Proposed by Pohoata Cosmin, Bucharest, Romania

Solution by Son Ta Hong, Ha Noi University of Education, Vietnam

Denote by D' the intersection of AE and BC. Let R, r be the circumradius and the inradius of triangle ABC. Without loss of generality suppose $AB < AC$, we have

$\angle ID'D = \angle D'AC + \angle D'CA = \angle EAB + \angle AEB = 180 - \angle ABE = \angle EFA.$

Thus $90^\circ - \angle ID'D = 90^\circ - \angle EFA$ and $\angle EID = \angle FAI \quad (1).$

On the otherhand, looking at the power of a point I with respect to the circumcircle and using Euler's theorem we have

$$p(I) = IA \cdot IE = 2R \cdot r = AF \cdot ID \Longrightarrow \frac{IA}{AF} = \frac{ID}{IE} \quad (2)$$

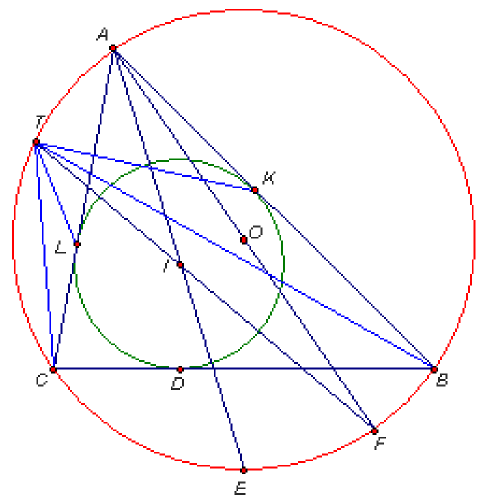
From (1) and (2) we obtain

$$\triangle AIF \sim \triangle IDE, \ \Rightarrow \angle AFS = \angle AFI = \angle IED = \angle AES, \ \Rightarrow S \in (O).$$

Applying Pascal's theorem to the hexagon $ASCEFB$ we get M, I and N are collinear.

Second solution by Tigran Sloyan, Yerevan, Armenia

At first we prove that S lies on the circumcircle of triangle ABC. Let T be the intersection point of FI with the circumcircle and L, K be the points of tangency of the incircle with the sides AC and AB, respectively.



Then we have

$$\angle ITA = \angle FTA = 90^\circ = \angle ILA = \angle IKA.$$

It follows that points T, L, K and A lie on the circle with diameter AI. Thus $\angle TLA = \angle TKA$ which means that $\angle TLC = \angle TKB$. Also $\angle TCL = \angle TCA = \angle TBA = \angle TBK$ and therefore triangles TCL and TBK are similar. Hence

$$\frac{CT}{BT} = \frac{CL}{BK} = \frac{CD}{BD}.$$

We get that TD is the angle bisector of $\angle CTB$, and therefore it passes through E. Thus both ED and FI pass through T, $\Rightarrow S \equiv T$.

To complete the proof we consider the self-intersecting inscribed hexagon $AEBFTC$. Using Pascal's theorem we obtain that the intersection points of the pairs of opposite sides are collinear. That is the intersection point of AE and FT (point I),the intersection point of EB and TC (point M) and the intersection point of BF and CA (point N) are collinear, as desired.

O47. Consider the Fibonacci sequence $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Prove that

$$\sum_{i=0}^{n} \frac{(-1)^{n-i} F_i}{n+1-i} \binom{n}{i} = \begin{cases} \dfrac{2F_{n+1}}{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Proposed by Gabriel Alexander Reyes, San Salvador, El Salvador

First solution by David E. Narvaez, Universidad Tecnologica de Panama

Observe that

$$\frac{1}{n+1-i}\binom{n}{i} = \frac{1}{n+1}\binom{n+1}{i}$$

thus

$$\sum_{i=0}^{n} \frac{(-1)^{n-i} F_i}{n+1-i}\binom{n}{i} = \sum_{i=0}^{n} \frac{(-1)^{n-i} F_i}{n+1}\binom{n+1}{i} = \frac{1}{n+1}\sum_{i=0}^{n} (-1)^{n-i} F_i \binom{n+1}{i}$$

and we are left to prove that

$$\sum_{i=0}^{n} (-1)^{n-i} F_i \binom{n+1}{i} = \begin{cases} 2F_{n+1}, & \text{if } n \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

But is easy to prove that $F_i = \frac{\phi^i - (1-\phi)^i}{\sqrt{5}}$, where $\phi = \frac{\sqrt{5}+1}{2}$. Thus

$$\sum_{i=0}^{n} (-1)^{n-i} F_i \binom{n}{i} = \sum_{i=0}^{n} (-1)^{n-i} \frac{\phi^i - (1-\phi)^i}{\sqrt{5}} \binom{n}{i} =$$

$$= \frac{1}{\sqrt{5}} \left(\sum_{i=0}^{n} \binom{n+1}{i} (-1)^{n-i} \phi^i - \sum_{i=0}^{n} \binom{n+1}{i} (-1)^{n-i} (1-\phi)^i \right) =$$

$$= \frac{1}{\sqrt{5}} \left[-(-1+\phi)^{n+1} + \phi^{n+1} - \left(-(-1+(1-\phi))^{n+1} + (1-\phi)^n \right) \right]$$

$$\sum_{i=0}^{n} (-1)^{n-i} F_i \binom{n}{i} = \frac{1}{\sqrt{5}} \left[- \left(-(1-\phi)^{n+1} + \phi^{n+1} + (-\phi)^{n+1} + (1-\phi)^n \right) \right]$$

Clearly, this last expression takes values of 0 if n is even and $2F_i$ if n is odd, and we are done.

Second solution by Tigran Sloyan, Yerevan, Armenia

Note that

$$\frac{1}{n+1-i}C_n^i = \frac{1}{n+1-i}\cdot\frac{n!}{i!(n-i)!} = \frac{1}{n+1}\cdot\frac{(n+1)!}{i!((n+1)-i)!} = \frac{1}{n+1}C_{n+1}^i.$$

Then

$$\sum_{i=0}^{n}\frac{(-1)^{n-i}F_i}{n+1-i}C_n^i = \frac{1}{n+1}\sum_{i=0}^{n}(-1)^{n-i}F_iC_{n+1}^i.$$

From Binet's formula for Fibonacci numbers we have that for any positive integer i

$$F_i = \frac{1}{\sqrt{5}}\left((\frac{\sqrt{5}+1}{2})^i - (\frac{1-\sqrt{5}}{2})^i\right).$$

We denote $\frac{1+\sqrt{5}}{2} = x$ and $\frac{1-\sqrt{5}}{2} = y$ then $F_i = \frac{1}{\sqrt{5}}(x^i - y^i)$. Thus

$$\frac{1}{n+1}\sum_{i=0}^{n}(-1)^{n-i}F_iC_{n+1}^i = \frac{1}{\sqrt{5}(n+1)}(\sum_{i=0}^{n}(-1)^{n-i}x^iC_{n+1}^i - \sum_{i=0}^{n}(-1)^{n-i}y^iC_{n+1}^i) =$$

$$= \frac{1}{\sqrt{5}(n+1)}(\sum_{i=0}^{n}(-1)^{(n+1)-i}y^iC_{n+1}^i - \sum_{i=0}^{n}(-1)^{(n+1)-i}x^iC_{n+1}^i)$$

We have that $(a+b)^m = \sum_{i=0}^{m} a^ib^{m-i}C_m^i$, therefore

$$\sum_{i=0}^{n}(-1)^{(n+1)-i}y^iC_{n+1}^i = (y-1)^{n+1} - y^{n+1} = (-x)^{n+1} - y^{n+1}$$

and

$$\sum_{i=0}^{n}(-1)^{(n+1)-i}x^iC_{n+1}^i = (x-1)^{n+1} - x^{n+1} = (-y)^{n+1} - x^{n+1}$$

therefore

$$\frac{1}{n+1}\sum_{i=0}^{n}(-1)^{n-i}F_iC_{n+1}^i = \frac{1}{n+1}(\frac{1}{\sqrt{5}}(x^{n+1}-y^{n+1})+(-1)^{n+1}\frac{1}{\sqrt{5}}(x^{n+1}-y^{n+1})).$$

Hence $\frac{1}{n+1}\sum_{i=0}^{n}(-1)^{n-i}F_iC_{n+1}^i = \frac{F_{n+1}(1+(-1)^{n+1})}{n+1} = \begin{array}{cc} \frac{2F_{n+1}}{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{array}$

Also solved by Jose Hernandez Santiago, UTM, Oaxaca, Mexico; Arkady Alt, San Jose, California, USA; Daniel Campos Salas, Costa Rica

O48. Let $f \in \mathbb{Z}[X]$ be a monic irreducible polynomial of degree n whose zeros $x_1, x_2, ..., x_n$ are all real numbers. Let $S_k = x_1^{2k} + x_2^{2k} + ... + x_n^{2k}$. Prove that there exist a universal constant $c > 0$, such that

$$S_1 \cdot S_2 \cdot ... \cdot S_{n-1} \geq c \cdot \frac{e^{2n}}{n^2}$$

holds for all n.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution by Iurie Boreico, Moldova

Let $f(x) = x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n = (x - x_1)(x - x_2)...(x - x_n)$. Clearly $|a_n| = |x_1 \cdot x_2 \cdot ... \cdot x_n| \geq 1$. Using the AM-GM inequality we get

$$S_k = x_1^{2k} + x_2^{2k} + ... + x_n^{2k} \geq n|x_1 \cdot x_2 \cdot ... \cdot x_n|^{\frac{2k}{n}} \geq n.$$

Thus we have

$$S_1 \cdot S_2 \cdot ... \cdot S_{n-1} \cdot \frac{n^2}{e^{2n}} \geq \frac{n^{n+1}}{e^{2n}} \geq \frac{1}{e^2},$$

for $n \geq 1$.