Solutions for Mathematical Reflections 1(2007)

Juniors

J37. Let $a_1, a_2 ..., a_{2n+1}$ be distinct positive integers not exceeding $3n+1$. Prove that among them there are two such that

$$a_i - a_j = m, \text{ for all } m \in \{1, 2, ..., n\}.$$

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by Jose Alejandro Samper, Colegio Helvetia de Bogota, Colombia

Let $m \in \{1, 2, ..., n\}$. Color the numbers from 1 to $3n+1$ with m colors so that i and j have the same color if an only if $i \equiv j \pmod{m}$. The number of numbers of the same color is at most $\lceil \frac{3n+1}{m} \rceil$. If more than half of the numbers of the same color are chosen, there are two that differ by m, because the numbers of the same color are all the numbers that have the same residue modulo m.

We have to choose $2n+1$ different numbers from 1 to $3n+1$. By the Pigeonhole principle, there are at least $\frac{2n+1}{m}$ numbers having the same color. This means that the problem is equivalent to:
$2 \cdot \frac{2n+1}{m} = \frac{4n+2}{m} > \lceil \frac{3n+1}{m} \rceil$, because there exists a color i, such that the number of chosen numbers of color i is greater than half of the total number of numbers of color i.

Using the fact that $n+1 > m$ we get

$$4n+2 > 3n+1+m \geq m\lfloor \frac{3n+1}{m} \rfloor + m.$$

Therefore

$$\frac{4n+2}{m} > \lfloor \frac{3n+1}{m} \rfloor + 1 \geq \lceil \frac{3n+1}{m} \rceil.$$

This completes the proof.

J38. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{a^2+bc}{(a+b)(a+c)}+\frac{b^2+ca}{(b+a)(b+c)}+\frac{c^2+ab}{(c+a)(c+b)}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

Solution by Paolo Perfetti, Roma, Italy

The inequality is equivalent to

$$\frac{a^3+b^3+c^3+3abc-ab(a+b)-ac(a+c)-bc(b+c)}{(a+b)(a+c)(b+c)} \geq 0$$

which is in turn equivalent to

$$a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b) \geq 0.$$

This is the well–known Schur's inequality and we are done.

Also solved by Vishal Lama, Southern Utah University, USA; Vardan Verdiyan, Yerevan, Armenia; Son Ta Hong, Ha Noi University of Education, Vietnam; Jose Alejandro Samper, Colegio Helvetia de Bogota, Colombia; Courtis G. Chryssostomos, Larissa, Greece; Pham Huu Duc, Australia; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vo Quoc Ba Can, Can Tho University, Vietnam; Vicente Vicario Garcia, University of Huelva, Spain

J39. Evaluate the product

$$(\sqrt{3}+\tan 1^\circ)(\sqrt{3}+\tan 2^\circ)\ldots(\sqrt{3}+\tan 29^\circ).$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

First solution by Courtis G. Chryssostomos, Larissa, Greece

Let $A=(\sqrt{3}+\tan 1^\circ)(\sqrt{3}+\tan 2^\circ)...(\sqrt{3}+\tan 29^\circ)$. We have

$$\sqrt{3}+\tan 1^\circ=\tan 60^\circ+\tan 1^\circ=\frac{\sin 60^\circ\cos 1^\circ+\cos 60^\circ\sin 1^\circ}{\cos 60^\circ\cdot\cos 1^\circ}=\frac{\sin 61^\circ}{\cos 60^\circ\cdot\cos 1^\circ}.$$

Analogously we obtain

$$\sqrt{3}+\tan 2^\circ=\frac{\sin 62^\circ}{\cos 60^\circ\cdot\cos 2^\circ},\quad\ldots,\sqrt{3}+\tan 29^\circ=\frac{\sin 89^\circ}{\cos 60^\circ\cdot\cos 29^\circ}.$$

Thus

$$A=\frac{\sin 61^\circ}{\cos 60^\circ\cdot\cos 1^\circ}\cdot\frac{\sin 62^\circ}{\cos 60^\circ\cdot\cos 2^\circ}\cdot\ldots\cdot\frac{\sin 89^\circ}{\cos 60^\circ\cdot\cos 29^\circ}=\left(\frac{1}{\cos 60^\circ}\right)^{29}=2^{29}.$$

Second solution by Jose Alejandro Samper, Colegio Helvetia de Bogota, Colombia

We will use the following identities:

$$\tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\cdot\tan b},\quad\tan 30^\circ=\frac{1}{\sqrt{3}},\quad\tan 15^\circ=2-\sqrt{3}$$

We obtain

$$\frac{1}{\sqrt{3}}=\tan 30^\circ=\frac{\tan k+\tan(30-k)}{1-\tan k\cdot\tan(30-k)}$$

$$\Leftrightarrow\sqrt{3}(\tan k+\tan(30-k))=1-\tan k\cdot\tan(30-k).$$

Then

$$\prod_{k=1}^{29}(\sqrt{3}+\tan k)=(\sqrt{3}+\tan 15^\circ)\prod_{k=1}^{14}(\sqrt{3}+\tan k)(\sqrt{3}+\tan(30-k))=$$

$$=\left(\sqrt{3}+(2-\sqrt{3})\right)\prod_{k=1}^{14}\left(3+\sqrt{3}(\tan k+\tan(30-k))+\tan k\tan(30-k)\right)=$$

$$= 2\prod_{k=1}^{14}\Big(3+\sqrt{3}(\tan k+\tan(30-k))+1-\sqrt{3}(\tan k+\tan(30-k)\Big) =$$

$$= 2\cdot\prod_{k=1}^{14} 4 = 2^{29}, \text{ and we are done.}$$

Also solved by Jose Hernandez Santiago, Oaxaca, Mexico; Vardan Verdiyan, Yerevan, Armenia; Vishal Lama, Southern Utah University, USA; Daniel Campos Salas, Costa Rica; Gabriel Alexander Reyes, San Salvador, El Salvador; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Son Ta Hong, Ha Noi University of Education, Vietnam; Vo Quoc Ba Can, Can Tho University, Vietnam; Lee Ju-Hyeong, Suwon, Republic of Korea; Vicente Vicario Garcia, University of Huelva, Spain

J40. A 5×6 rectangle is cut into eight rectangles with integral dimensions and whose sides are parallel to the ones of the initial rectangle. Prove that among them there are two congruent rectangles.

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by Ivan Borsenco, University of Texas at Dallas

Let us assume that all the rectangles obtained by cutting are different. The table below shows rectangles with the least possible areas.

Area 1 : 1×1
Area 2 : 1×2
Area 3 : 1×3
Area 4 : 1×4, 2×2
Area 5 : 1×5
Area 6 : 1×6, 2×3

It is not difficult to see that if we choose 8 different rectangles with the least possible areas, the sum of their areas is $1+2+3+4+4+5+6+6 = 31 > 30$. Thus it is not possible to obtain 8 different rectangles.

J41. Let a, b, c be positive real numbers such that $a+b+c+1=4abc$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3 \geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}.$$

Proposed by Daniel Campos Salas, Costa Rica

First solution by Vardan Verdiyan, Yerevan, Armenia

By the AM-GM inequality,

$$4\sqrt[4]{abc} \leq a+b+c+1=4abc \Rightarrow abc \geq 1.$$

Thus $a+b+c+abc \geq a+b+c+1=4abc \Rightarrow a+b+c \geq 3abc$.

Again from the AM-GM inequality

$$(ab+bc+ca)^2 \geq 3abc(a+b+c) \Rightarrow (ab+bc+ca)^2 \geq (3abc)^2.$$

It follows that

$$ab+bc+ca \geq 3abc \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3.$$

For the second part of the inequality assume $3\sqrt{abc} < \sqrt{a}+\sqrt{b}+\sqrt{c}$. Using Schur's third degree inequality we get

$$(\sum \sqrt{a})^3+3(\sum \sqrt{a}) > (\sum \sqrt{a})^3+9\sqrt{abc} \geq 4(\sum \sqrt{a})(\sum \sqrt{bc}) \;\Rightarrow$$

$$(\sum \sqrt{a})^2+3 > 4\sum \sqrt{bc}, \;\Leftrightarrow\; a+b+c+3 > 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}).$$

From the AM-GM inequality, $a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca} \;\Rightarrow$

$$3(a+b+c+3)+2(a+b+c) > 8(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \Leftrightarrow 9(\sum a+1) > 4(\sum \sqrt{a})^2$$

$$\Leftrightarrow\; 9abc > (\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \Leftrightarrow 3\sqrt{abc} > \sqrt{a}+\sqrt{b}+\sqrt{c}, \text{ contradiction.}$$

Thus

$$3 \geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}.$$

Second solution by Gabriel Alexander Reyes, San Salvador, El Salvador

The idea is to perform some substitutions in order to transform each part of our inequality. First we write the constraint $a+b+c+1=4abc$ as follows:

$$4=\frac{a+b+c+1}{abc}=\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}+\frac{1}{abc}$$

So we may set $x = 1/a,\ y = 1/b,\ z = 1/c$. We get a new constraint $xy + yz + zx + xyz = 4$, and now we have to show that $x + y + z \ge 3 \ge \sqrt{xy} + \sqrt{yz} + \sqrt{zx}$.

Let us prove that $x + y + z \ge 3$. The constraint $xy + yz + zx + xyz = 4$ can be written as

$$\frac{x}{2} \cdot \frac{y}{2} + \frac{y}{2} \cdot \frac{z}{2} + \frac{z}{2} \cdot \frac{x}{2} + 2 \cdot \frac{x}{2} \cdot \frac{y}{2} \cdot \frac{z}{2} = 1.$$

Because it has the form $pq + qr + rp + 2pqr = 1$, this enables us to perform the substitution

$$x = \frac{2p}{q+r},\ y = \frac{2q}{r+p},\ z = \frac{2r}{p+q}$$

Now our inequality becomes

$$2\left(\frac{p}{q+r} + \frac{q}{r+p} + \frac{r}{p+q}\right) \ge 3$$

or

$$\frac{p}{q+r} + \frac{q}{r+p} + \frac{r}{p+q} \ge \frac{3}{2}$$

which is Nesbitt's inequality.

In order to deal with the second inequality, $\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le 3$, we use another substitution. Observe that the constraint has the equivalent form

$$(\sqrt{xy})^2 + (\sqrt{yz})^2 + (\sqrt{zx})^2 + \sqrt{xy} \cdot \sqrt{yz} \cdot \sqrt{zx} = 4.$$

This allows us to substitute

$$\sqrt{yz} = 2\cos A,\ \sqrt{zx} = 2\cos B,\ \sqrt{xy} = 2\cos C$$

where A, B, C are the angles of an acute triangle. Solving for x, y, z yields

$$x = \frac{2\cos B\cos C}{\cos A},\ \frac{2\cos C\cos A}{\cos B},\ \frac{2\cos A\cos B}{\cos C}$$

Hence the inequality $\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le 3$ is equivalent to

$$\cos A + \cos B + \cos C \le \frac{3}{2},$$

which is well-known. This completes our proof.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let us define $u = \frac{1}{a}, v = \frac{1}{b}, w = \frac{1}{c}$, and let us denote the arithmetic, geometric and quadratic means of u, v, w by A, G, Q, respectively. Then, the condition given in the problem may be written as

$$4 = vw + wu + uv + uvw = \frac{9A^2 - 3Q^2}{2} + G^3,$$

while the first inequality is equivalent to $A \geq 1$. But since $Q \geq A \geq G$, with equalities if and only if $u = v = w$ we have

$$4 = \frac{9A^2}{2} - \frac{3Q^2}{2} + G^3 \leq 3A^2 + A^3,$$

$$0 \leq A^3 + 3A^2 - 4 = (A-1)(A+2)^2 .$$

Thus $A \geq 1$, with equality if and only if $u = v = w$, i.e., if and only if $a = b = c$.

Let us now define $x = \frac{1}{\sqrt{bc}}, y = \frac{1}{\sqrt{ca}}, z = \frac{1}{\sqrt{ab}}$, and $s = x + y + z$. Then, $a = \frac{x}{yz}, b = \frac{y}{zx}, c = \frac{z}{xy}$. The second inequality is equivalent to $s \leq 3$, and the condition given in the problem may be written as

$$\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} + 1 = \frac{4}{xyz};$$

$$x^2 + y^2 + z^2 + xyz = 4.$$

From this last equation, it is obvious that x, y, z are all less than 2, while

$$(s-2)^2 = 8 - 4s + 2(xy + yz + zx) - xyz = (2-x)(2-y)(2-z) .$$

Since $2-x, 2-y, 2-z$ are all positive, we may apply the AM-GM inequality, obtaining

$$(s-2)^2 \leq \left(\frac{6-s}{3}\right)^3,$$

$$0 \geq (s-6)^3 + 27(s-2)^2 = s^3 + 9s^2 - 108 = (s-3)(s+6)^2 ,$$

with equality if and only if $x = y = z$. Therefore, $s \leq 3$ with equality if and only if $a = b = c$.

Also solved by Vo Quoc Ba Can, Can Tho University, Vietnam; Paolo Perfetti, Roma, Italy; Daniel Campos Salas, Costa Rica

J42. Find all triples (m, n, p) of positive integers such that $m+n+p = 2008$ and the system of equations

$$\frac{x}{y} + \frac{y}{x} = m,\ \frac{y}{z} + \frac{z}{y} = n,\ \frac{z}{x} + \frac{x}{z} = p$$

has at least one solution in nonzero real numbers.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Dr. Titu Andreescu, University of Texas at Dallas

We have

$$mnp = \left(\frac{x}{y} + \frac{y}{x}\right)\left(\frac{y}{z} + \frac{z}{y}\right)\left(\frac{z}{x} + \frac{x}{z}\right) = \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{y^2}{z^2} + \frac{z^2}{y^2} + \frac{z^2}{x^2} + \frac{x^2}{z^2} =$$

$$= \left(\frac{x}{y} + \frac{y}{x}\right)^2 + \left(\frac{y}{z} + \frac{z}{y}\right)^2 + \left(\frac{z}{x} + \frac{x}{z}\right)^2 - 4 = m^2 + n^2 + p^2 - 4. \quad (1)$$

Note it cam be rewritten as $p(mn-p) = m^2+n^2-4$, that gives us $mn \geq p$ or $(m+2)(n+2) \geq p+2$. Analogously, $(n+2)(p+2) \geq m+2$ and $(m+2)(p+2) \geq p+2$. Equation (1) is equivalent to

$$mnp + 2(mn + np + pm) = (m + n + p)^2 - 4.$$

Adding $4(m + n + p) + 8$ to both sides, we obtain

$$(m + 2)(n + 2)(p + 2) = (m + n + p + 2)^2 = 2010^2,$$

where m, n, p are integers greater than or equal to 2.

Observe that $(m + 2)(n + 2)(p + 2) = 2^2 \cdot 3^2 \cdot 5^2 \cdot 67^2$. Suppose $d|m + 2$ and $d|n + 2$, then $d^2|(m + 2)(n + 2)(p + 2)$. Therefore $d^2|(m + n + p + 2)^2$ or $d|(m + 2 + n + 2 + p - 2)$ implying $d|p - 2$. Observe that 67^2 cannot divide $m + 2, n + 2, p + 2$. Assume without loss of generality $67^2|p + 2$, then $30^2 = 2^2 \cdot 3^2 \cdot 5^2 \geq (m + 2)(n + 2) \geq (p + 2) = 67^2$, contradiction. It follows that $67|m+2$ and $67|n+2$. From the observation above we get $67|p-2$, where $p + 2 \leq 2^2 \cdot 3^2 \cdot 5^2 = 900it$. We have $p + 2$ can be represented as $67k + 4$ and is less than 900. A quick case analysis shows that the only solution is $p = 2$. Therefore $x = y$, $m = n$ and the desired triples are $(m, n, p) = (1003, 1003, 2)$ and their permutations. The system of equations clearly has solutions in real numbers:

$$(x, y, z) = (2r, 2r, r(1003 \pm \sqrt{1003^2 - 1})).$$

Seniors

S37. Let x, y, z be real numbers such that

$$\cos x + \cos y + \cos z = 0,$$

and

$$\cos 3x + \cos 3y + \cos 3z = 0.$$

Prove that

$$\cos 2x \cdot \cos 2y \cdot \cos 2z \leq 0.$$

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

First solution by Lee Ju-Hyeong, Suwon, Republic of Korea

Because $\cos 3a = 4\cos^3 a - 3\cos a$ and $\cos x + \cos y + \cos z = 0$,

$$\begin{aligned}\cos 3x + \cos 3y + \cos 3z &= 4(\cos^3 x + \cos^3 y + \cos^3 z) - 3(\cos x + \cos y + \cos z) \\ &= 4(\cos^3 x + \cos^3 y + \cos^3 z)\end{aligned}.$$

Then since $\cos 3x + \cos 3y + \cos 3z = 0$,

$$\cos^3 x + \cos^3 y + \cos^3 z = 0.$$

Hence

$$\cos^3 x + \cos^3 y + \cos^3 z = 3\cos x \cos y \cos z = 0.$$

Thus

$$\cos x = 0 \quad \text{or} \quad \cos y = 0 \quad \text{or} \quad \cos z = 0.$$

Without loss of generality, assume $\cos x = 0$. Then

$$\cos y = -\cos z.$$

Hence

$$\begin{aligned}\cos 2x \cos 2y \cos 2z &= (2\cos^2 x - 1)(2\cos^2 y - 1)(2\cos^z - 1) \\ &= -(2\cos^2 z - 1)^2 \leq 0.\end{aligned}.$$

The problem is solved.

Second solution by José Gibergans-Báguena and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain

From the well known trigonometric identity $\cos 3A = 4\cos^2 A - 3\cos A$, we have

$$\cos 3x + \cos 3y + \cos 3z = 4\cos^3 x - 3\cos x + 4\cos^3 y - 3\cos y + 4\cos^3 z - 3\cos z$$

$$= 4(\cos^3 x + \cos^3 y + \cos^3 z) - 3(\cos x + \cos y + \cos z)$$

or equivalently

$$\cos^3 x + \cos^3 y + \cos^3 z = 0.$$

From $\cos x + \cos y + \cos z = 0$ we have $\cos x = -(\cos y + \cos z)$ and replacing it into the previous expression, we get

$$\cos^2 y \cos z + \cos y \cos^2 z = 0 \Leftrightarrow \cos y \cos z(\cos y + \cos z) = 0.$$

Now, we discuss the following cases:

(1) If $\cos y = 0$, then $y = \pi/2 + 2k\pi$ $(k \in \mathbb{Z})$ and from $\cos x = -\cos z$ we get $x = (2k+1)\pi - z$ $(k \in \mathbb{Z})$. Therefore,

$$\cos 2x \cdot \cos 2y \cdot \cos 2z = -\cos^2 2z \leq 0.$$

Note that we get the same conclusion when $\cos z = 0$.

(2) If $\cos y = -\cos z$ then $y = (2k+1)\pi - z$ $(k \in \mathbb{Z})$ and from $\cos x = 0$ we have $x = \pi/2 + 2k\pi$ $(k \in \mathbb{Z})$. Therefore,

$$\cos 2x \cdot \cos 2y \cdot \cos 2z = -\cos^2 2z \leq 0.$$

This completes the proof.

Also solved by Vicente Vicario Garcia, University of Huelva, Spain; Vo Quoc Ba Can, Can Tho University, Vietnam; Daniel Campos Salas, Costa Rica; Gabriel Alexander Reyes, San Salvador, El Salvador; Vardan Verdiyan, Yerevan, Armenia

S38. Prove that for each positive integer n, there is a positive integer m such that

$$(1+\sqrt{2})^n = \sqrt{m} + \sqrt{m+1}.$$

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

It may be shown by induction that, for each positive integer n, we may find positive integers x_n, y_n such that

$$\left(1+\sqrt{2}\right)^n = x_n + y_n\sqrt{2},$$

where $x_1 = y_1 = 1$, and if positive integers x_n, y_n can be found, then

$$\left(1+\sqrt{2}\right)^{n+1} = \left(x_n + y_n\sqrt{2}\right)\cdot\left(1+\sqrt{2}\right) = (x_n + 2y_n) + (x_n + y_n)\sqrt{2},$$

$$x_{n+1} = x_n + 2y_n,$$

$$y_{n+1} = x_n + y_n.$$

Therefore

$$\left(1+\sqrt{2}\right)^n = \sqrt{x_n^2} + \sqrt{2y_n^2}.$$

Moreover, we may show, also by induction, that

$$2y_n^2 - x_n^2 = (-1)^{n+1}.$$

This result is true for $n = 1$, and being true for n, then

$$2y_{n+1}^2 - x_{n+1}^2 = 2x_n^2 + 4x_ny_n + 2y_n^2 - x_n^2 - 4x_ny_n - 4y_n^2 = -\left(2y_n^2 - x_n^2\right) = (-1)^{n+2}.$$

Therefore, we may choose, for all odd positive integers n,

$$m = x_n^2 = 2y_n^2 - 1,$$

and for all even positive integers n,

$$m = 2y_n^2 = x_n^2 - 1.$$

Although the proof is complete, we may go further and give general relations for m; it is not difficult to prove by induction that, for all positive integer n,

$$x_n = \frac{\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n}{2},$$

$$y_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}.$$

Hence for all positive integer n,

$$m = \frac{\left(1+\sqrt{2}\right)^{2n} + \left(1-\sqrt{2}\right)^{2n} - 2}{4}.$$

Second solution by Vicente Vicario Garcia, University of Huelva, Spain

Let us prove the statement for $n = 2j$, in case $n = 2j + 1$ solution is analogous. Using the binomial theorem

$$(1+\sqrt{2})^n = \sum_{k=0}^{n} \binom{n}{k} (\sqrt{2})^k = 1 + \binom{n}{1}\sqrt{2} + ... + \binom{n}{n-1}(\sqrt{2})^{n-1} + (\sqrt{2})^n =$$

$$= \left[\binom{n}{0} + 2\binom{n}{2} + ... + 2^j\binom{n}{2j}\right] + \left[\binom{n}{1} + 2\binom{n}{3} + ... + 2^{j-1}\binom{n}{2j-1}\right]\sqrt{2} =$$

$$= \sqrt{\left[\binom{n}{0} + 2\binom{n}{2} + ... + 2^j\binom{n}{2j}\right]^2} + \sqrt{2\left[\binom{n}{1} + 2\binom{n}{3} + ... + 2^{j-1}\binom{n}{2j-1}\right]^2}$$

$$= \sqrt{A+1} + \sqrt{B}.$$

We will prove that $A = B$. Observe that

$$(1+\sqrt{2})^n + (1-\sqrt{2})^n = 2\left[\binom{n}{0} + 2\binom{n}{2} + ... + 2^j\binom{n}{2j}\right].$$

$$(1+\sqrt{2})^n - (1-\sqrt{2})^n = 2\left[\binom{n}{1} + 2\binom{n}{3} + ... + 2^{j-1}\binom{n}{2j-1}\right].$$

The proof reduces to the following:

$$\left[\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}\right]^2 - 1 = \left[\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2}\right]^2 \Leftrightarrow$$

$$\Leftrightarrow \left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]^2 - 4 = \left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right]^2 \Leftrightarrow$$

$$\Leftrightarrow 2(-1)^n - 4 = -2(-1)^n \Leftrightarrow 4(-1)^n = 4.$$

Third solution by Gabriel Alexander Reyes, San Salvador, El Salvador

We use some algebraic number theory. Consider the quadratic field $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2},\ a,\ b \in \mathbb{Q}\}$. Here the norm function N is defined as $N(a+b\sqrt{2}) = a^2 - 2b^2$. We know that $1+\sqrt{2}$ is a unit, because $N(1+\sqrt{2}) = = 1^2 - 2 \cdot 1^2 = -1$. But the set of units is closed under multiplication; hence $(1+\sqrt{2})^n$ is a unit for every integer $n \geq 1$. If $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$, for integers A_n and B_n, the latter implies that $N[(1+\sqrt{2})^n] = \pm 1$, that is, $A_n^2 - 2B_n^2 = \pm 1$. Now we are done, because $(1+\sqrt{2})^n = A_n + B_n\sqrt{2} = \sqrt{A_n^2} + \sqrt{2B_n^2}$, and we have showed that A_n^2 and $2B_n^2$ differ by 1.

Fourth solution is another solution received from Gabriel Alexander Reyes, San Salvador, El Salvador

Recall the following theorem from theory of Pell's equations:

Theorem. Let D be a square-free positive integer. If the equation $x^2 - Dy^2 = -1$ is solvable in positive integers, and $(p_0,\, q_0)$ is its minimal solution, then all the positive integer solutions $(p_n,\, q_n)$ of the equation $|x^2 - Dy^2| = 1$ are given by $(p_0 + q_0\sqrt{D})^n = p_n + q_n\sqrt{D}$. In addition, we have $p_n^2 - Dq_n^2 = (-1)^n$.

If $D = 2$, the minimal solution of $x^2 - 2y^2 = -1$ is $x = y = 1$. Hence all the positive solutions $(p_n,\, q_n)$ of the equation $x^2 - 2y^2 = \pm 1$ satisfy $(1 + \sqrt{2})^n = p_n + q_n\sqrt{2}$. The claim immediately follows after observing that $(1 + \sqrt{2})^n = p_n + q_n\sqrt{2} = \sqrt{p_n^2} + \sqrt{2q_n^2}$. Furthermore, from $p_n^2 - 2q_n^2 = (-1)^n$ we infer that, if n is even, we can take $m = 2q_n^2$, and if n is odd, $m = p_n^2$.

Remark: This result is stronger than the previous solution. Indeed, we have proved that every unit of $\mathbb{Q}[\sqrt{2}]$ is of the form $\pm(1 + \sqrt{2})^n$, where n is a integer. Observe that n can take negative values, since the reciprocal of a unit is a unit too.

S39. Let a be a positive integer and let

$$A = \left\{\sqrt{a}, \sqrt[3]{a}, \sqrt[4]{a}, \ldots\right\}.$$

Prove that for every positive integer n the set A contains n consecutive terms of a geometric sequence, but it does not contain a geometric sequence with infinitely many terms.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The elements of A are all the numbers of the form $a^{\frac{1}{m}}$, where m takes all integer values greater than or equal to 2.

Let us consider the following set of n numbers in A:

$$\left\{a^{\frac{1}{n!}}, a^{\frac{2}{n!}}, a^{\frac{3}{n!}}, \ldots, a^{\frac{n}{n!}}\right\}.$$

Clearly they all belong to A, since $1, 2, 3, ..., n$, all divide $n!$. Furthermore, the result of dividing each number in the set by the previous one is $a^{\frac{1}{n!}}$, so they are consecutive terms of a geometric sequence. For each positive integer n a subset of A formed by n consecutive terms of a geometric sequence has been found.

Let us consider the set

$$B = \left\{\frac{\ln a}{2}, \frac{\ln a}{3}, \frac{\ln a}{4}, \ldots\right\}.$$

Clearly the n-th element of set B is the natural logarithm of the n-th element of set A. Therefore, a subset of A forms a geometric sequence if and only if the corresponding subset in B forms an arithmetic progression. Or, the second part of the problem is equivalent to showing that no infinite arithmetic progression may be found in B.

If $a = 1$, all elements in A are equal to 1, and all elements in B are zero. An infinite geometric sequence with ratio 1 may be found in A, and an infinite arithmetic progression with common difference 0 may be found in B. Let us now show that no infinite arithmetic progression may be found in B when $a \neq 1$.

The set B is bound by $\frac{\ln a}{2}$ and 1, the former being the upper bound and the latter the lower bound if $a > 1$, vice versa otherwise. Furthermore, all elements in B are distinct when $\ln a \neq 0$. Therefore, if an infinite geometric sequence may be found in A, an infinite, bound, arithmetic sequence will be found in B, which is absurd, since all arithmetic sequences of nonzero difference are not bound.

S40. Let f and g be irreducible polynomials with rational coefficients and let a and b be complex numbers such that $f(a) = g(b) = 0$. Prove that if $a+b$ is a rational number, then f and g have the same degree.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

Solution by Iurie Boreico, Moldova

Consider $h(x) \in \mathbb{Q}[x]$ such that $h(x) = g(a+b-x)$. Clearly $h(a) = f(a) = 0$, and because $f(x)$ is irreducible, $f|h$. Thus $\deg f(x) \leq \deg h(x) = \deg g(x)$. Analogously we can prove that $\deg g(x) \leq \deg f(x)$. It follows that f and g have the same degree.

S41. Prove that for any positive real numbers a, b and c,

$$\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}} \geq \sqrt{6 \cdot \frac{a+b+c}{\sqrt[3]{abc}}}.$$

Proposed by Pham Huu Duc, Ballajura, Australia

First solution by Tigran Sloyan, Yerevan, Armenia

We can assume that $abc = 1$ and rewrite the inequality in the following form

$$\sqrt{bc(b+c)}+\sqrt{ac(a+c)}+\sqrt{ab(a+b)} \geq \sqrt{6(a+b+c)}.$$

Squaring both sides we obtain

$$bc(b+c)+ac(a+c)+ab(a+b)+2\left(\sum \sqrt{c^2(ab+ac)(ab+bc)}\right) \geq 6(a+b+c).$$

From the inequality $\sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)} \geq x_1y_1+x_2y_2$ we can say that

$$\sum \sqrt{c^2(ab+ac)(ab+bc)} \geq \sum c(ab+\sqrt{abc^2}) = \sum c(ab+\sqrt{c})$$

$$\Rightarrow \sum \sqrt{c^2(ab+ac)(ab+bc)} \geq 3+\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}.$$

Therefore it is sufficient to prove that

$$bc(b+c)+ac(a+c)+ab(a+b)+6+2(\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}) \geq 6(a+b+c).$$

From the AM-GM inequality for six positive real numbers we have

$$\sqrt{c^3}+\sqrt{c^3}+ac^2+bc^2+1+1 \geq 6\sqrt[6]{c^7ab} = 6c$$

$$\Rightarrow 2\sqrt{c^3}+c^2(a+b)+2 \geq 6c.$$

Similarly we can prove that

$$2\sqrt{b^3}+b^2(a+c)+2 \geq 6b$$

$$2\sqrt{a^3}+a^2(b+c)+2 \geq 6a.$$

Finally, adding up the last three inequalities we get the desired one.

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let s, A, G respectively denote the sum, the arithmetic, and geometric means of a, b, c. Dividing both sides of the proposed inequality by $\sqrt{s}$, we find it to be equivalent to:

$$\sqrt{\frac{1}{a}-\frac{1}{s}}+\sqrt{\frac{1}{b}-\frac{1}{s}}+\sqrt{\frac{1}{c}-\frac{1}{s}} \geq \sqrt{\frac{6}{G}}.$$

Squaring both sides and multiplying throughout by G allows us to write the proposed inequality in the following equivalent form:

$$\frac{ab+bc+ca}{G^2}-\frac{G}{A}+\frac{2G}{3A}\cdot\left(\sqrt{1+3\frac{A\cdot c^2}{G^3}}+\sqrt{1+3\frac{A\cdot b^2}{G^3}}+\sqrt{1+3\frac{A\cdot a^2}{G^3}}\right) \geq 6,$$

where it has been used that

$$\left(\frac{1}{a}-\frac{1}{s}\right)\cdot\left(\frac{1}{b}-\frac{1}{s}\right)=\frac{s-a-b}{sab}+\frac{1}{s^2}=\frac{c^2}{sG^3}+\frac{1}{s^2}=\frac{1}{9A^2}\cdot\left(1+\frac{3A\cdot c^2}{G^3}\right),$$

and similarly for the other two cross-products.

Now, using the inequalities between arithmetic and quadratic means, and between arithmetic and geometric means, allows us to find:

$$\frac{1}{\sqrt{4}}\sqrt{1+3\frac{A\cdot c^2}{G^3}} \geq \frac{1}{4}\left(1+3\frac{c}{G}\sqrt{\frac{A}{G}}\right) \geq \frac{1}{4}\left(1+3\frac{c}{G}\right),$$

with equality if and only if $A\cdot c^2 = G^3$ and simultaneously $A = G$, i.e., if and only if $a = b = c$. Or,

$$\sqrt{1+3\frac{A\cdot c^2}{G^3}}+\sqrt{1+3\frac{A\cdot b^2}{G^3}}+\sqrt{1+3\frac{A\cdot a^2}{G^3}} \geq \frac{3}{2}+\frac{3s}{2G}=\frac{3}{2}+\frac{9A}{2G},$$

with equality if and only if $a = b = c$. Therefore, in order to complete the proof, it is sufficient to show that

$$\frac{ab+bc+ca}{G^2}+3 \geq 6,$$

which boils down to

$$\frac{ab+bc+ca}{3} \geq G^2.$$

But this is always true, as a result of the AM-GM inequality applied to ab, bc, ca. Or, the proof is complete, and the equality holds if and only if $a = b = c$.

Third solution by Ho Phu Thai, Da Nang, Vietnam

Squaring both sides yields:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+$$

$$2\left(\sqrt{\frac{(a+b)(a+c)}{bc}}+\sqrt{\frac{(b+c)(b+a)}{ca}}+\sqrt{\frac{(c+a)(c+b)}{ab}}\right)\geq\frac{6(a+b+c)}{\sqrt[3]{abc}}.$$

Using the well-known inequality $\sqrt{(p+q)(p+r)}\geq p+\sqrt{qr}$, we will prove that

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+2\left(\frac{a}{\sqrt{bc}}+\frac{b}{\sqrt{ca}}+\frac{c}{\sqrt{ab}}\right)+6\geq\frac{6(a+b+c)}{\sqrt[3]{abc}}.$$

Using Chebyshev's inequality we have

$$\frac{a}{\sqrt{bc}}+\frac{b}{\sqrt{ca}}+\frac{c}{\sqrt{ab}}\geq\frac{a+b+c}{3}\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)\geq\frac{a+b+c}{\sqrt[3]{abc}}.$$

Our inequality becomes

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+6\geq\frac{4(a+b+c)}{\sqrt[3]{abc}}.$$

From the substitution $a=x^3,b=y^3,c=z^3$ and expansion we get

$$x^6(y^3+z^3)+y^6(z^3+x^3)+z^6(x^3+y^3)+6x^3y^3z^3\geq 4(x^5y^2z^2+y^5z^2x^2+z^5x^2y^2).$$

$$\Leftrightarrow\sum_{cyc}z^2\left[z^4x+z^4y+x^4y+xy^4-x^2y^2z\right](x-y)^2\geq 0.$$

We just need to prove that for all positive numbers x,y,z:

$$z^4x+z^4y+x^4y+xy^4\geq x^2y^2z,$$

which is a consequence of the AM-GM inequality:

$$z^4x+z^4y+3\cdot\frac{1}{3}x^4y+3\cdot\frac{1}{3}xy^4\geq 8\sqrt[8]{\frac{1}{3^6}x^6y^{16}z^8}>x^2y^2z.$$

The proof is complete. Equality holds if and only if $a=b=c$.

Also solved by Daniel Campos Salas, Costa Rica; Vo Quoc Ba Can, Can Tho University, Vietnam; Vardan Verdiyan, Yerevan, Armenia

S42. Prove that in any triangle there exist a pair (M_1, M_2) of isogonal conjugates such that $OM_1 \cdot OM_2 > OI^2$, where O and I are the circumcenter and the incenter, respectively.

Proposed by Ivan Borsenco, University of Texas at Dallas

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let ABC be the triangle. We will consider two cases:

1) ABC is equilateral. The isogonal conjugate of $O = I$ is clearly itself, so the isogonal conjugate of any point M_1 other than $O = I$ inside the triangle is not O. Therefore, choosing $M_1 \neq O$ inside triangle ABC results in OM_1 and OM_2 being both positive, and the inequality is true.

2) ABC is not equilateral. Let a, b, c be the lengths of sides BC, CA, AB, respectively, and let us assume without loss of generality that $a \leq b \leq c$. Let us call M the point where the incircle touches side BC. It is well known that $BM = r \cot \frac{B}{2}$ and $CM = r \cot \frac{C}{2}$, where r is the inradius of ABC. Writing the power of point M with respect to the circumcircle of ABC as $BM \cdot CM$, and calling R the circumradius of ABC, results in

$$OM^2 = R^2 - BM \cdot CM = R^2 - r^2 \cot \frac{B}{2} \cot \frac{C}{2}.$$

Let us take any point M_1 inside ABC, on AM, arbitrarily close to M. Lines BM_1 and CM_1 are then arbitrarily close to BC, or their reflections about the angle bisectors of B and C, respectively, are arbitrarily close to BA and CA, or the isogonal conjugate of M_1 is arbitrarily close to A. Therefore, isogonal conjugates M_1, M_2 may be chosen such that OM_2 is arbitrarily close to $OA = R$, and OM_1 is arbitrarily close to OM. It suffices therefore to prove that

$$OM > \frac{OI^2}{R} = R - 2r,$$

or, squaring both sides of the resulting inequality,

$$R^2 - 4Rr + 4r^2 < R^2 - r^2 \cot \frac{B}{2} \cot \frac{C}{2},$$

$$R > r + \frac{r}{4} \cot \frac{B}{2} \cot \frac{C}{2}.$$

But since we have assumed that $C > \frac{\pi}{3}$ and $B \geq A$,

$$\cot \frac{B}{2} \cot \frac{C}{2} = 1 + \frac{\sin \frac{A}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}} \leq 1 + \frac{1}{\sin \frac{C}{2}} < 1 + \frac{1}{\sin \frac{\pi}{6}} = 3,$$

all that needs to be proved is $R > \frac{7r}{4}$, and this is always true, since $R > 2r$ for non-equilateral triangles.

Undergraduate

U37. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with f' continuous, such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1.$$

Prove that there exists $c \in (0,1)$ for which $f'(c) = 6$.

Proposed by Cezar Lupu, University of Bucharest, Romania

Solution by Li Zhou, Polk Community College, USA

Let $p(x) = 6x - 2$. Then $\int_0^1 p(x)dx = \int_0^1 xp(x)dx = 1$. Thus we cannot have $f(x) - g(x)$ entirely positive or entirely negative on $(0,1)$.

Suppose that $\{x \in (0,1) : f(x) = p(x)\} = \{a\}$. If $f(x) > p(x)$ for all $x \in (0,a)$, then $f(x) < p(x)$ for all $x \in (a,1)$

$$\int_0^1 xf(x) - 1 = \int_0^a x(f(x) - p(x))dx + \int_a^1 x(f(x) - p(x))dx <$$

$$a\int_0^a (f(x)-p(x))dx+a\int_a^1 (f(x)-p(x))dx = a\left(\int_0^1 f(x)dx - \int_0^1 p(x)dx\right) = 0.$$

Similarly, if $f(x) < p(x)$ for all $x \in (0,a)$, then $f(x) > p(x)$ for all $x \in (a,1)$ and $\int_0^1 xf(x)dx > 1$.

Therefore, there are a and b in $(0,1)$, $a < b$, such that $f(a) = p(a)$ and $f(b) = p(b)$. By the Mean Value Theorem, there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{p(b) - p(a)}{b - a} = 6.$$

U38. Let $n > 1$ be an odd positive integer and let x be a real number. Set

$$t = \sum_{k=1}^{n-1} \arctan\left(\frac{\cos\left(\frac{2\pi k}{n}\right) - x}{\sin\left(\frac{2\pi k}{n}\right)}\right).$$

Compute the value of $\tan t$ in terms of n and x.

Proposed by Alex Anderson, New Trier High School, Winnetka, IL

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

For each integer k such that $1 \le k \le \frac{n-1}{2}$, we find that

$$\frac{\cos\left(\frac{2\pi(n-k)}{n}\right) - x}{\sin\left(\frac{2\pi(n-k)}{n}\right)} = \frac{\cos\left(\frac{2\pi k}{n}\right) - x}{-\sin\left(\frac{2\pi k}{n}\right)} = -\frac{\cos\left(\frac{2\pi k}{n}\right) - x}{\sin\left(\frac{2\pi k}{n}\right)}.$$

Or, the angles

$$\arctan\left(\frac{\cos\left(\frac{2\pi k}{n}\right) - x}{\sin\left(\frac{2\pi k}{n}\right)}\right)$$

and

$$\arctan\left(\frac{\cos\left(\frac{2\pi(n-k)}{n}\right) - x}{\sin\left(\frac{2\pi(n-k)}{n}\right)}\right)$$

add up to a multiple of π. Since t is then the sum of $\frac{n-1}{2}$ such angles, t is an integer multiple of π, hence $\tan t = 0$, for each x and each positive odd integer n.

U39. Prove that $\lim\limits_{n\to\infty} \dfrac{1}{\ln n} \sum\limits_{k=1}^{n} \dfrac{\{\ln k\}}{k} = \dfrac{1}{2}$,

where $\{x\}$ denotes the fractional part of x.

Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania

Solution by Vicente Vicario Garcia, University of Huelva, Spain

We use the famous Cauchy's integral test for the convergence of numerical infinite series. If $f(x) : [1, \infty) \to \mathbb{R}$ is a decreasing function such that $\lim_{x\to\infty} f(x) = 0$, then

$$0 \le \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx - D \le f(n),$$

with $D = \lim d_n = \lim(s_n - t_n)$ (D is a constant) and $s_n = \sum_{k=1}^{n} f(k)$, $t_n = \int_1^n f(x)dx$. Thus

$$\sum_{k=1}^{n} \frac{\{\ln k\}}{k} = \sum_{k=1}^{n} \frac{\ln k - [lnk]}{k} = \sum_{k=1}^{n} \frac{\ln k}{k} - \sum_{k=1}^{n} \frac{[\ln k]}{k}.$$

By Cauchy's integral test we obtain

$$\sum_{k=1}^{n} \frac{\ln k}{k} = \frac{1}{2} \ln^2 n + B + O\left(\frac{\ln n}{n}\right), \text{ B is constant and } \int_1^n \frac{\ln x}{x} dx = \frac{1}{2} \ln^2 n. \quad (1)$$

Again using Cauchy's test we get

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + O(\frac{1}{n}), \text{ where } \gamma \text{ is Euler-Mascheroni constant.} \quad (2)$$

Now we use (2) to evaluate

$$\sum_{k=[e^j]+1}^{[e^{j+1}]-1} \frac{[\ln k]}{k} = j\left(\frac{1}{[e^j]+1} + \frac{1}{[e^j]+2} + ... + \frac{1}{[e^{j+1}]-1}\right) \sim j(\ln e^{j+1} - \ln e^j) = j,$$

where $\sim$ denotes asymptotic equivalent formulas. Then if $e^{j+1} = n \to j = -1 + \ln n$,

$$\sum_{k=1}^{n} \frac{[\ln k]}{k} \sim 1+2+...+j = \frac{j(j+1)}{2} = \frac{\ln n(\ln n - 1)}{2} + O(\ln n) = \frac{\ln^2 n}{2} - \frac{\ln n}{2} + O(\ln n).$$

Finally we have

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{\{\ln k\}}{k} = \lim_{n\to\infty} \frac{\left(\frac{\ln^2 n}{2} + A + O\left(\frac{\ln n}{n}\right) - \frac{\ln^2 n}{2} + \frac{\ln n}{2} + O(\ln n)\right)}{\ln n} = \frac{1}{2}.$$

U40. Show that $\mathrm{GL}_4(\mathbb{Q})$ has no element of order 7.

Proposed by Jean-Charles Mathieux, Dakar University, Sénégal

Solution by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Suppose A is such a matrix. Observe that if a is an eigenvalue of A, then $a^7 = 1$ and $a^6 + ... + a + 1$ is nonzero. Indeed, otherwise the irreducible polynomial $X^6 + ... + X + 1$ (over the field of rationals) would have a common root with the characteristic polynomial of A, therefore it would divide the latter. This is impossible, looking at the degrees. Thus each eigenvalue of A is 1. It follows that $A = I_4$. Because A is diagonalisable over the field of complex numbers, we obtain the desired contradiction.

U41. Let k be a positive integer and let α be a real number greater than 1. A number is called a k-prime if it is the product of at most k (not necessarily distinct) primes. Let $p(r)$ be the probability that a random integer x contains r k-prime divisors $d_1 < d_2 < \ldots < d_r$ such that $d_r < \alpha d_1$. Prove that $\lim_{r\to\infty} p_r = 0$.

Proposed by Iurie Boreico, Moldova

Solution by Iurie Boreico, Moldova

We use induction on k. The base case $k = 0$ is clear.

First of all let us prove some basic facts.

a) A graph having $\binom{n+k-2}{k-1}$ edges contains either a K_n or a $\overline{K_k}$. This is proven by induction on $n + k$. If $n + k = 2$, then a graph with at least one vertex satisfies the condition, and the same holds if at least one of n, k is 0. Now assume $n+k = m$ and for $n+k < m$ we have already proven the problem.

Take a vertex A and let S_1 be the set of its neighbors, S_2 be the set of its non-neighbors. Then $|S_1| + |S_2| \geq \binom{n+k-2}{k-1} - 1 = \binom{n+k-2}{k-1} + \binom{n+k-2}{k-2}$, therefore $|S_1| \geq \binom{n+k-2}{k-1}$ or $S_1 \geq \binom{n+k-2}{k-2}$. In the first case, by the induction hypothesis, S_1 contains either a K_{n-1} or a $\overline{K_k}$. If it contains a $\overline{K_k}$ we are done and so we are if it contains a K_{n-1} because we can add A to it. In the second case, by the induction hypothesis, S_2 contains either a K_n or a $\overline{K_{k-1}}$. Again, if it contains a K_n we are done, and if it contains a $\overline{K_{k-1}}$, then we add A to it and complete the inductive step.

b) Set q_r be the probability that a number x is divisible by r pairwise coprime k-prime numbers. Then $\lim_{r\to\infty} q_r = \lim_{r\to\infty} p_r$.

To do this, it suffices to prove that the density of numbers x divisible by R pairwise coprime k-prime numbers so that no r of these divisors are pairwise coprime is 0. According to a) we can select a R such that if from these R divisors no r are mutually coprime then at least $km + 1$ are mutually not coprime. Now take one of these numbers. It is divisible by at most k prime numbers so out of the $km + 1$ at least $m + 1$ are divisible by the same prime divisor p. These numbers divisible by p constitute at least m $k - 1$-prime numbers dividing the number x. But the density of such numbers x tends to zero according to the induction hypothesis.

c)We thus need to show that $\lim_{r\to\infty} q_r = 0$.

We prove that the interval $(n; \alpha n)$ contains at most $C(\frac{n \ln\ln n^k}{\ln n})$ k-prime numbers. This is done by induction on k. The base case $k = 1$ is clear.

Now the induction step. A k-prime number M between n and αn satisfies the condition that its smallest divisor p is at most $\sqrt[k]{n}$.

Then the number $\frac{M}{p}$ is a $k-1$-prime number between $\frac{n}{p}$ and $\alpha\frac{n}{p}$. By the induction hypothesis there are at most $C\frac{\frac{n}{p}(\ln\ln\frac{n}{p}^{k-1})}{\ln\frac{n}{p}} < C_1\frac{1}{p}\frac{n\ln\ln n^{k-1}}{\ln n}$ such numbers. So by summing over all $p < \sqrt[k]{n}$ as $\sum_{p<\sqrt[k]{n}} < \ln\ln n$ we conclude the total number is at most $C_1(\frac{n\ln\ln n^k}{\ln n})$ and the induction step is proven.

d) We now show that $\lim_{r\to\infty} q_r = 0$. Consider $A = \sqrt{\alpha}$. Let's take a x having $2r$ pairwise coprime k-prime divisors in an interval $(n;\alpha n)$. It is easy to see x has r pairwise coprime k-prime divisors in an interval $(A^k;A^{k+1})$. Say that it has divisors $d_1, d_2, \ldots, d_r$. The probability that x is divisible by all of them is $\frac{1}{d_1 d_2 \ldots d_r} < \frac{1}{A^{kr}}$ as $d_1, d_2, \ldots, d_r$ are pairwise coprime. Finally $d_1, d_2, \ldots, d_r$ can be chosen in less than $\binom{t_k}{r} < \frac{t_k^r}{r!}$ where t_k is the number of k-prime numbers between A^k and A^{k+1}. But according to c) $t_k < C\frac{A^k \ln ka^k}{ka}$ where $a = \ln A$. So $\frac{t_k^r}{r!} < \frac{C^r e^r A^{kr}\ln ka^k}{k^r a^r r^r}$ as $r! \sim (\frac{r}{e})^r$. It is clear that $\ln x^k < C'x$ for some x therefore $\frac{t_k^r}{r!} < C'\frac{(CE)^r A^{kr}}{r^r k^{r-1} a^r}$. Now multiplying by the probability which is less than $\frac{1}{A^{kr}}$ and summing by all k we get

$$\lim_{r\to\infty} q_r < \lim_{r\to\infty}\sum_{k=1}^{\infty}\frac{1}{A^{kr}}\cdot C'\frac{(CE)^r A^{kr}}{r^r k^{r-1} a^r}.$$

Now $\sum_{k=1}^{\infty}\frac{1}{k^{r-1}a^r}$ clearly is convergent to some C_1 when $r\to\infty$ while $\frac{C'(CE)^r}{r^r} < \frac{1}{r}$ for sufficiently big r. Therefore this sum is at most $C_1\frac{1}{r}$ for sufficiently big r and tends to 0, as desired.

U42. Let $A_1, ..., A_n, B_1, ..., B_n$ be points in a plane such that $B_iA_1 \cdot B_iA_2 \cdot ... \cdot B_iA_n \leq A_jB_i$ for all i and j. Prove that

$$\prod_{1 \leq i < j \leq n} A_iA_j \cdot B_iB_j \leq n^{\frac{n}{2}},$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Recall the expression of the Cauchy determinant

$$\det\left(\frac{1}{a_i + b_j}\right) = \frac{\prod_{1 \leq i < j \leq n}(b_j - b_i)(a_j - a_i)}{\prod_{i,j}(a_i + b_j)}.$$

Using Hadamard's inequality, we deduce that

$$(\prod_{1 \leq i < j \leq n} |b_j - b_i||a_j - a_i|)^2 \leq (\prod_{i,j} |a_i + b_j|^2)^2 \cdot \prod_j \left(\frac{1}{|b_j + a_1|^2} + ... + \frac{1}{|b_j + a_n|^2}\right).$$

However, the last quantity is at most n^n, because each factor can be rewritten as

$$|b_j + a_2|^2 ... |b_j + a_n|^2 + ... + |b_j + a_1|^2 ... |b_j + a_{n-1}|^2 \leq n.$$

From here the conclusion follows, by interpreting A_i and B_j as points of affixes a_i and $-b_j$.

Olympiad

O37. Let a, b, c, d be nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 4$. Prove that

$$\sqrt{2}(4-ab-bc-cd-da) \geq (\sqrt{2}+1)(4-a-b-c-d).$$

Proposed by Vasile Cartoaje, University of Ploiesti, Romania

First solution by Daniel Campos Salas, Costa Rica

Let $s = a+b+c+d$. From the Cauchy-Schwarz inequality we have

$$4 = \sqrt{4(a^2+b^2+c^2+d^2)} \geq s, \tag{1}$$

and since a, b, c, d are nonnegative reals numbers,

$$s \geq \sqrt{a^2+b^2+c^2+d^2} = 2. \tag{2}$$

Applying the AM-GM inequality it follows that

$$\sqrt{2}(4-ab-bc-cd-da) = \sqrt{2}(4-(a+b)(c+d)) \geq \sqrt{2}\left(4-\frac{s^2}{4}\right).$$

Note that

$$\sqrt{2}\left(4-\frac{s^2}{4}\right) - (\sqrt{2}+1)(4-s) = -\frac{s^2\sqrt{2}}{4} + (\sqrt{2}+1)s - 4 =$$

$$= \frac{\sqrt{2}}{4}\left(-s^2+(4+2\sqrt{2})s - 8\sqrt{2}\right) = \frac{\sqrt{2}}{4}(4-s)(s-2\sqrt{2}). \tag{3}$$

From (1) and (3) it follows that the inequality holds for $s \geq 2\sqrt{2}$. Suppose that $s \leq 2\sqrt{2}$. Then

$$\sqrt{2}(4-ab-bc-cd-da) \geq \sqrt{2}(4-ab-bc-cd-da-ac-bd) = \frac{\sqrt{2}}{2}(12-s^2).$$

Hence

$$\frac{\sqrt{2}}{2}(12-s^2) - (\sqrt{2}+1)(4-s) = -\frac{s^2\sqrt{2}}{2} + (\sqrt{2}+1)s + (2\sqrt{2}-4) =$$

$$= \frac{\sqrt{2}}{2}\left(-s^2+(2+\sqrt{2})s+(4-4\sqrt{2})\right) = \frac{\sqrt{2}}{2}(2\sqrt{2}-s)(s-2+2\sqrt{2}). \tag{4}$$

From (2) and (4) it follows that the inequality also holds for $s \leq 2\sqrt{2}$, and this completes the proof.

Second solution by Vardan Verdiyan, Yerevan, Armenia

Because $ab+bc+cd+da=(a+c)(b+d)$, our inequality can be rewritten as

$$\sqrt{2}(4-(a+c)(b+d)) \geq (\sqrt{2}+1)(4-(a+c)-(b+d)) \Leftrightarrow$$
$$(\sqrt{2}+1)(a+b+c+d) \geq \sqrt{2}(a+c)(b+d)+4.$$

Denote $x=a+c$ and $y=b+d$. From the condition of the problem we have $x^2+y^2 \geq a^2+b^2+c^2+d^2=4 \;\Rightarrow\; (x+y)^2 \geq 4+2xy$. Therefore $x+y \geq 2$. We can rewrite our inequality

$$(\sqrt{2}+1)(x+y) \geq \sqrt{2}xy+4 \;\Leftrightarrow\; (2+\sqrt{2})(x+y) \geq 2xy+4\sqrt{2} \;\Leftrightarrow$$
$$4+(2+\sqrt{2})(x+y)-4\sqrt{2} \geq 4+2xy.$$

Thus it is enough to prove

$$4+(2+\sqrt{2})(x+y)-4\sqrt{2} \geq (x+y)^2.$$

Denote $x+y=k$. We have to prove

$$k^2-(2+\sqrt{2})k+4(\sqrt{2}-1) \;\Leftrightarrow$$
$$\frac{2+\sqrt{2}-\sqrt{22-12\sqrt{2}}}{2} \leq k \leq \frac{2+\sqrt{2}+\sqrt{22-12\sqrt{2}}}{2}=2\sqrt{2}.$$

It follows that our inequality is true if $x+y \leq 2\sqrt{2}$.

Assume that $x+y>2\sqrt{2}$. Observe that $xy<4$ as $4=a^2+b^2+c^2+d^2 \geq (a+c)(b+d)=xy$. Because $x+y \geq 2\sqrt{xy}$, it suffices to prove that

$$(2+\sqrt{2})2\sqrt{xy} \geq 2xy+4\sqrt{2} \;\Leftrightarrow xy-(2+\sqrt{2})\sqrt{xy}+2\sqrt{2} \leq 0.$$

This is true when $\sqrt{2}=\frac{2+\sqrt{2}-\sqrt{6-4\sqrt{2}}}{2} \leq \sqrt{xy} \leq \frac{2+\sqrt{2}+\sqrt{6-4\sqrt{2}}}{2}=2$.

But if $\sqrt{xy}<\sqrt{2}$, then $(2+\sqrt{2})(x+y)>4+4\sqrt{2}>2xy+4\sqrt{2}$ and we are done.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let us call $s=a+b+c+d$, and let us find first the maximum and minimum values of s. Since the quadratic mean of a,b,c,d is 1, the inequality between the arithmetic and quadratic means ensures that the maximum value of the sum is 4, achieved if and only if $a=b=c=d=1$. Furthermore, since ab,ac,ad,bc,bd,cd are all non-negative,

$$s^2 \geq a^2+b^2+c^2+d^2=4,$$

with equality if and only if $ab = ac = ad = bc = bd = cd = 0$, or if and only if one of a, b, c, d equals 2, and the rest are 0. Therefore,

$$2 \leq s \leq 4.$$

Let us now consider two different cases, $2 \leq s < 2\sqrt{2}$ and $2\sqrt{2} \leq s \leq 4$. In the first case,

$$s^2 - \left(2 + \sqrt{2}\right) s + 4\sqrt{2} - 4 = \left(s - 2\sqrt{2}\right) \left(s - 2 + \sqrt{2}\right) < 0 \leq 2ac + 2bd.$$

In the second case,

$$4 + 2ac + 2bd = a^2 + b^2 + c^2 + d^2 + 2ac + 2bd = (a + c)^2 + (b + d)^2 \geq \frac{s^2}{2},$$

with equality if and only if $a + c = b + d$, and where the arithmetic-quadratic inequality has been used. Now, since

$$s^2 - \left(2 + \sqrt{2}\right) s + 4\sqrt{2} = \frac{s^2}{2} + \frac{(s - 4)\left(s - 2\sqrt{2}\right)}{2} \leq \frac{s^2}{2},$$

we have

$$s^2 - \left(2 + \sqrt{2}\right) s + 4\sqrt{2} - 4 \leq 2ac + 2bd,$$

with equality if and only if $s = 4$ or $s = 2\sqrt{2}$.

Finally, in either case,

$$ab + bc + cd + da = \frac{s^2 - 4 - 2ac - 2bd}{2} \leq \frac{1 + \sqrt{2}}{\sqrt{2}} s - \frac{4}{\sqrt{2}},$$

$$\sqrt{2}\,(4 - ab - bc - cd - da) \geq\ 4\sqrt{2} + 4 - \left(1 + \sqrt{2}\right) s = \left(1 + \sqrt{2}\right)(4 - s)\,,$$

Equality happens in two cases:

1) if and only if $s = 4$, or if and only if $a = b = c = d = 1$

2) if and only if $s = 2\sqrt{2}$ and simultaneously $a + c = b + d$, or if and only if $a + c = b + d = \sqrt{2}$. In this case, since

$$(a + c)^2 + (b + d)^2 = 4 = a^2 + b^2 + c^2 + d^2,$$

it follows that $ac = bd = 0$, or one of a, c equals $\sqrt{2}$ and the other one is zero, and likewise with b, d.

O38. Let w_1 be a circle smaller than and internally tangent to the circle w_2 at T. A tangent to w_1 (at T'), intersects w_2 at A and B. If A, T', and B are fixed, what is the locus of T.

Proposed by Alex Anderson, New Trier High School, Winnetka, IL

First solution by Gabriel Alexander Reyes, San Salvador, El Salvador

We need to prove the following lemma:

Lemma. TT' is the angle bisector of $\angle ATB$.

Proof. This problem appeared at the British Olympiad (first round, 1993). Let A' and B' be the second point of intersection of AT and BT with w_1, respectively. Because T is the center of similarity of circles w_1 and w_2, we know that AB and $A'B'$ are parallel. Then $\angle AT'A = \angle T'A'B$. But $\angle AT'A = \angle A'TT'$, because AT' is tangent to w_1. On the other hand, $\angle T'A'B' = \angle T'TB'$, given that $ATB'T'$ is a cyclic quadrilateral. Hence $\angle A'TT' = \angle T'TB'$, so TT' is the angle bisector of $\angle A'TB'$, as claimed.

Applying now the angle bisector theorem in the triangle ATB yields $AT/TB = AT'/T'B$. Upon observing that the ratio on the right hand side is fixed, observe that the condition of the Apollonius circle is satisfied. Therefore the locus of T is a circle, namely the Apollonius circle corresponding to the points A, B and the ratio $AT'/T'B$. If $AT'/T'B = 1$, we know this circle degenerates into the perpendicular bisector of AB.

Second solution is another solution received from Gabriel Alexander Reyes, San Salvador, El Salvador

Perform an inversion with center T'. The line AB remains fixed, w_1, which is tangent to AB at T', becomes a line ℓ parallel to AB, and w_2 becomes a circle through A and B, tangent to ℓ at T. But the latter implies, by symmetry, that T belongs to the perpendicular bisector ℓ' of AB. Then the locus of the inverse of T is a line orthogonal to AB. If ℓ' passes through T', this line is its own inverse, so the locus of T is the perpendicular bisector of AB (of course, this is the case when T' is the midpoint of AB). Otherwise, the locus of T is a circle through T', whose center lies on AB. This happens to be an Apollonius circle, as we noted above.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Let P be the point where the tangents to w_1 at T and T' meet. Clearly, $PT = PT'$. Because PT' intersects w_2 at A and B, and PT is tangent to w_2 at T, it follows that $PA \cdot PB = PT^2 = PT'^2$. Or, given A, B, T', point P is fixed for any T, and T is on the circle with center P that passes through T'. Using Stewart's theorem on cevian TA in triangle PTB,

$$AT^2 = \frac{PT^2 \cdot AB + BT^2 \cdot PA - PA \cdot AB \cdot PB}{PB} = \frac{BT^2 \cdot PA}{PB};$$

$$\frac{AT^2}{BT^2} = \frac{PA}{PB} = \frac{PA^2}{PT'^2} = \frac{PT'^2}{PB^2}.$$

Therefore, the ratio $\frac{AT}{BT}$ is independent on T, and the circle is the Apollonius circle such that any point T on it satisfies

$$\frac{AT}{BT} = \frac{AT'}{BT'}.$$

Let us show that for any given point on the circle, except for T' and its diametrally opposite point, there exist circles w_1 and w_2 such that the given point is a valid location of T.

Given any point on the Apollonius circle described above, because $PT = PT'$, a circle tangent to PT at T and to PT' at T' exists. Its center is the angle bisector of $\angle TPT'$. Denote this circle will be w_1. Let us w_2 be the circumcircle of triangle ATB. Because $PT^2 = PA \cdot PB$, PT is a tangent to w_2 at T, which means that w_1 and w_2 are tangent at T. Because T' is on side AB of triangle ATB, w_1 is partially inside w_2. But since both these circles are tangent, so w_1 is completely inside w_2, since two tangent circles may only be equal, exterior to each other, or one interior to the other. Thus, for any point T in the Apollonius circle described, we have found circles w_1 and w_2 satisfying the conditions of the problem. In the case of T' and its diametrally opposite point T'', circle w_1 becomes point T' and line AB, respectively, w_2 becoming line AB in both cases.

Note finally that, when T' is the midpoint of AB, the Apollonius circle becomes the perpendicular bisector of segment AB and circles w_1 and w_2 have their centers on this bisector, the former having TT' as a diameter, the latter being, as always, the circumcircle of ATB.

O39. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+2bc}}+\frac{b}{\sqrt{b^2+2ca}}+\frac{c}{\sqrt{c^2+2ab}}\leq\frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

Proposed by Ho Phu Thai, Da Nang, Vietnam

First solution by Daniel Campos Salas, Costa Rica

Since the inequality is symmetric with respect a, b, c we can assume without loss of generality that $a \geq b \geq c$. Then $(a-c)(b-c) \geq 0$, which implies

$$\frac{c}{\sqrt{c^2+2ab}}\leq\frac{c}{\sqrt{ab+bc+ca}}.$$

It is not difficult to prove that

$$\frac{a}{\sqrt{a^2+2bc}}+\frac{b}{\sqrt{b^2+2ca}}\leq\frac{a+b}{\sqrt{ab+bc+ca}}.$$

Note that this inequality is equivalent to

$$\frac{a(\sqrt{a^2+2bc}-\sqrt{ab+bc+ca})}{\sqrt{a^2+2bc}}\geq\frac{b(\sqrt{ab+bc+ca}-\sqrt{b^2+2ca})}{\sqrt{b^2+2ca}}. \tag{1}$$

It is easy to verify that

$$\sqrt{a^2+2bc}-\sqrt{ab+bc+ca}, \sqrt{ab+bc+ca}-\sqrt{b^2+2ca}\geq 0,$$

given $a \geq b \geq c$. Let us prove that

$$\frac{a}{\sqrt{a^2+2bc}}\geq\frac{b}{\sqrt{b^2+2ac}} \tag{2}$$

is equivalent to $a^3c \geq b^3c$, which is clearly true.
Let us also prove that

$$\sqrt{a^2+2bc}-\sqrt{ab+bc+ca}\geq\sqrt{ab+bc+ca}-\sqrt{b^2+2ca}. \tag{3}$$

This inequality is equivalent to

$$\sqrt{a^2+2bc}+\sqrt{b^2+2ca}\geq 2\sqrt{ab+bc+ca},$$

which (after squaring both sides and cancelling some terms) becomes

$$a^2+b^2+2\sqrt{(a^2+2bc)(b^2+2ca)}\geq 4ab+2bc+2ca.$$

Because $a^2+b^2 \geq 2ab$, it is enough to prove that

$$\sqrt{(a^2+2bc)(b^2+2ca)} \geq ab+bc+ca,$$

which is equivalent to

$$c\big(2a^3+2b^3+2abc-a^2c-b^2c-2a^2b-2ab^2\big)= c(2a+2b-c)(a-b)^2 \geq 0.$$

Multiplying (2) and (3) we prove that (1) holds and the conclusion follows.

Second solution by Pham Huu Duc, Australia

By the Cauchy-Schwarz inequality,

$$\left(\sum \frac{a}{\sqrt{a^2+2bc}}\right)^2 \leq \sum a \sum \frac{a}{a^2+2bc}$$

It remains to show that

$$\sum \frac{a}{a^2+2bc} \leq \frac{a+b+c}{ab+bc+ca}$$

or, equivalently,

$$\sum \frac{a(ab+bc+ca)}{a^2+2bc} \leq a+b+c.$$

We have

$$\sum \left(a - \frac{a(ab+bc+ca)}{a^2+2bc}\right) = \sum \frac{a(a-b)(a-c)}{a^2+2bc}$$

Without loss of generality, assume $a \geq b \geq c$. Then

$$\frac{c(c-a)(c-b)}{c^2+2ab} \geq 0$$

and

$$\frac{a(a-b)(a-c)}{a^2+2bc} + \frac{b(b-c)(b-a)}{b^2+2ca} = \frac{c(a-b)^2[3ab+2a(a-c)+2b(b-c)]}{(a^2+2bc)(b^2+2ca)} \geq 0$$

and the conclusion follows.

Also solved by Anuj Kumar, New Delhi, India; Tigran Sloyan, Yerevan, Armenia; Vardan Verdiyan, Yerevan, Armenia; Vo Quoc Ba Can, Can Tho University, Vietnam

O40. In the AwesomeMath summer camp there are 80 boys and 40 girls. It has been noticed that any two boys have an even number of acquaintances among the girls and exactly 19 boys know an odd number of girls. Prove that one can choose a group of at least 50 boys such that any girl is acquainted to an even number of boys from this group.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Solution by Gabriel Dospinescu, Ecole Normale Superieure, Paris

Let us consider the matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1, & \text{if } B_i \text{ knows } F_j \\ 0, & \text{otherwise.} \end{cases}$$

Here $B_1, B_2, \ldots, B_{80}$ are the boys and $F_1, F_2, \ldots, F_{40}$ are the girls. Now, consider the matrix $T = A \cdot {}^tA$. Observe that all the elements of the matrix T, except those from the main diagonal, are even (because $t_{ij} = \sum_{k=1}^{40} a_{ik}a_{jk}$ is the number of common acquaintances among the girls of the boys B_i, B_j). Each element on the main diagonal of T is precisely the number of girls known by the corresponding boy. If we consider the matrix T in $(\mathbb{Z}_2, +, \cdot)$, it will be diagonal, with exactly 19 nonzero elements on its main diagonal. From now on, we will work only in $\mathbb{Z}/2\mathbb{Z}$. We have seen that $\text{rank}(T) = 19$. Using Sylvester's inequality, we have

$$19 = \text{rank}(T) \geq \text{rank}(A) + \text{rank}\left({}^tA\right) - 40 = 2\text{rank}\left({}^tA\right) - 40,$$

hence $r = \text{rank}\left({}^tA\right) \leq 29$. Let us consider now the linear system in $(\mathbb{Z}_2, +, \cdot)$:

$$\begin{cases} a_{11}x_1 + a_{21}x_2 + \cdots + a_{80,1}x_{80} = 0 \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{80,2}x_{80} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1,40}x_1 + a_{2,40}x_2 + \cdots + a_{80,40}x_{80} = 0. \end{cases}$$

The set of solutions of this system is a vector space of dimension $80 - r \geq 51$. That is why we can choose a solution $(x_1, x_2, \ldots, x_{80})$ of the system, having at least 50 components equal to $\widehat{1}$. Finally, consider the set $S = \{i \in \{1, 2, \ldots, 80\} |\ x_i = \widehat{1}\}$. We have proved that $|S| \geq 29$ and also $\sum_{j \in S} a_{ji} = 0$ for all $i = 1, 2, ..., 40$. But observe that $\sum_{j \in S} a_{ji}$ is the number of boys B_k with $k \in S$ such that B_k knows F_i. Thus if we choose the group of those boys B_k with $k \in S$, then each girl is known by an even number of boys from this group and the problem is solved.

O41. Prove the following identity:

$$\sum_{0\le a\le\sqrt{n}} \left\lfloor \sqrt{n-a^2} \right\rfloor = \sum_{0\le i\le \frac{n}{2}} (-1)^i \left\lfloor \frac{n}{2i+1} \right\rfloor.$$

Proposed by Ashay Burungale, India

Solution by Ashay Burungale, India

Observe that $\lfloor\sqrt{n-a^2}\rfloor$ is the number of positive integers b with $a^2+b^2 \le n$. Therefore the left hand side of the identity is the number of $a \ge 0$, $b \ge 1$ with $a^2+b^2 \le n$. If $f(k)$ is the number of pairs (a,b) then we use the classical result

$$f(k) = \sum_{d|k,\ d\ odd} (-1)^{(d-1)/2}.$$

Thus the left hand side equals to $\sum_{k=1}^{n} f(k)$. But

$$\sum_{k=1}^{n} f(k) = \sum_{0\le i\le \frac{n}{2}} (-1)^i \left\lfloor \frac{n}{2i+1} \right\rfloor,$$

because the term $(-1)^{(d-1)/2}$ occurs exactly for those k that satisfy $d|k$ and there are $\left\lfloor \frac{n}{d} \right\rfloor$ of those.

O42. Let $a_1, a_2, ..., a_5$ be positive real numbers such that

$$a_1a_2...a_5 = a_1(1+a_2) + a_2(1+a_3) + ... + a_5(1+a_1) + 2.$$

Find the minimal value of $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \dfrac{1}{a_4} + \dfrac{1}{a_5}$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris

First solution by Vo Quoc Ba Can, Can Tho University, Vietnam

First, we will prove that for all positive real numbers x, y, z, t, u, the following inequality holds

$$(x+y+z+t+u)^3 \geq 25(xyz + yzt + ztu + tux + uxy)$$

Indeed, without loss of generality, assume $x = \min\{x, y, z, t, u\}$. Setting $y = x+b, z = x+c, t = x+d, u = x+e$ then $b, c, d, e \geq 0$, we have

$$LHS - RHS = 5Ax + (b+c+d+e)^3 - 25cd(b+e),$$

where

$$\begin{aligned} A &= 3(b+c+d+e)^2 - 5be - 10bc - 5bd - 10cd - 5ce - 10de \\ &= \frac{1}{12}(6b+d+e-4c)^2 + \frac{5}{84}(7d-4c-5e)^2 + \frac{5}{28}(2c-e)^2 + \frac{5}{4}e^2 \geq 0. \end{aligned}$$

By the AM - GM inequality, we have

$$25cd(b+e) \leq 27cd(b+e) \leq (c+d+(b+e))^3 = (b+c+d+e)^3.$$

The inequality is proved. Now, using this inequality with $x = \frac{1}{a_1}, y = \frac{1}{a_2}, z = \frac{1}{a_3}, t = \frac{1}{a_4}, u = \frac{1}{a_5}$, we obtain

$$P^3 \geq \frac{25(a_1a_2 + a_2a_3 + \ldots + a_5a_1)}{a_1a_2\ldots a_5}$$

On the other hand, by the MacLaurin and AM - GM inequalities, we have

$$\frac{125(a_1+a_2+\ldots+a_5)}{a_1a_2\ldots a_5} \leq P^4, \qquad \frac{3125}{a_1a_2\ldots a_5} \leq P^5.$$

Hence

$$1 = \frac{a_1+a_2+\ldots+a_5}{a_1a_2\ldots a_5} + \frac{a_1a_2+a_2a_3+\ldots+a_5a_1}{a_1a_2\ldots a_5} + \frac{2}{a_1a_2\ldots a_5} \leq \frac{P^4}{125} + \frac{P^3}{25} + \frac{2P^5}{3125}$$

or

$$\frac{(2P-5)(P^4+15P^3+100P^2+250P+625)}{3125} \geq 0.$$

From this, we have $P \geq \frac{5}{2}$. Now, letting $a_1 = a_2 = \ldots = a_5 = 2$, we get $P = \frac{5}{2}$, therefore

$$\min P = \frac{5}{2}.$$

Second solution by Ho Phu Thai, Da Nang, Vietnam

Denote the variables by a, b, c, d, e. From the hypothesis, $abcde \geq 32$ and

$$1 = \frac{1}{abcd} + \frac{1}{bcde} + \frac{1}{cdea} + \frac{1}{deab} + \frac{1}{eabc} + \frac{1}{abc} + \frac{1}{bcd} + \frac{1}{cde} + \frac{1}{dea} + \frac{1}{eab} + \frac{2}{abcde}.$$

Let us prove the following inequalities:

$$(x+y+z+t+w)^4 \geq 125(xyzt + yztw + ztwx + twxy + wxyz) \Leftrightarrow$$

$$x^4+y^4+z^4+t^4+w^4+7\sum_{20\text{terms}} x^3y+12\sum_{30\text{terms}} x^2yz+24\sum_{24\text{terms}} xyzt \geq 125\sum_{24\text{terms}} xyzt \Leftrightarrow$$

$$x^4+y^4+z^4+t^4+w^4+7\sum_{20\text{terms}} x^3y+12\sum_{30\text{terms}} x^2yz \geq 101\sum_{24\text{terms}} xyzt.$$

This is a consequence of Muirhead's theorem. We will now prove that

$$(x+y+z+t+w)^3 \geq 25(xyz + yzt + ztw + twx + wxy).$$

Normalizing $x+y+z+t+w = 5$, we show that $xyz+yzt+ztw+twx+wxy \leq 5$ by Lagrange multipliers. Define the function $F = xyz + yzt + ztw + twx + wxy + \lambda(x+y+z+t+w-5)$. Solving the system

$$\frac{dF}{dx} = \frac{dF}{dy} = \frac{dF}{dz} = \frac{dF}{dt} = \frac{dF}{dw} = 0,$$

we get $x = y = z = t = w = 1$ and $\lambda = -3$. The condition $x+y+z+t+w = 5$ implies $\frac{dF}{dx} + \frac{dF}{dy} + \frac{dF}{dz} + \frac{dF}{dt} + \frac{dF}{dw} = 0$. Observing that $d^2x = d^2y = d^2z = d^2t = d^2w = 0$, we have

$$dF = \sum_{cyc}(xydz+yzdx+zxdy), d^2F = \frac{1}{2}\sum_{sym} dxdy \leq \frac{12}{5}(dx+dy+dz+dt+dw)^2 = 0.$$

Hence F attains its maximum when $x = y = z = t = w = 1$. Applying the two inequalities, we get

$$1 \leq \frac{1}{125}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)^4 + \frac{1}{25}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)^3 + \frac{1}{16}.$$

This implies $\frac{1}{a} + \frac{1}{b} + \cdots + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \leq \frac{5}{2}$. Equality holds if and only if $a = b = c = d = e = 2$.