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RENORMALIZED HIGHER POWERS OF WHITE NOISE (RHPWN) AND CONFORMAL FIELD THEORY

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The Virasoro–Zamolodchikov $*$-Lie algebra w_∞ has been widely studied in string theory and in conformal field theory, motivated by the attempts of developing a satisfactory theory of quantization of gravity. The renormalized higher powers of quantum white noise (RHPWN) $*$-Lie algebra has been recently investigated in quantum probability, motivated by the attempts to develop a nonlinear generalization of stochastic and white noise analysis. We prove that, after introducing a new renormalization technique, the RHPWN Lie algebra includes a second quantization of the w_∞ algebra. Arguments discussed at the end of this note suggest the conjecture that this inclusion is in fact an identification.

Keywords: Renormalized powers of quantum white noise; Virasoro algebra; w_∞-algebra; conformal field theory.

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1. Introduction

Here we recall the basic definitions that pertain to the renormalized higher powers of quantum white noise (RHPWN) algebra. We will use the notations of Ref. 2 which contains the proofs of all the results recalled in this section. The standard Boson white noise $*$-Lie algebra is defined by the commutation relations

$$[b_t, b_s^\dagger] = \delta(t-s)\cdot 1\,,$$
$$[b_t^\dagger, b_s^\dagger] = [b_t, b_s] = 0\,,$$
$$(b_s^\dagger)^\dagger = b_s\,, \qquad 1^\dagger = 1\,,$$

where 1 (often omitted from the notations) denotes the central element and all the identities are meant in the operator distribution sense described in Ref. 1.

The formal extension of the above commutation relations to the associative $*$-algebra generated by b_t, $b_s^\dagger$, 1 leads to the identities:

$$[b_t^{\dagger^n} b_t^k, b_s^{\dagger^N} b_s^K] = \varepsilon_{k,0}\varepsilon_{N,0} \sum_{L\geq 1} \binom{k}{L} N^{(L)} b_t^{\dagger^n} b_s^{\dagger^{N-L}} b_t^{k-L} b_s^K \delta^L(t-s)$$
$$- \varepsilon_{K,0}\varepsilon_{n,0} \sum_{L\geq 1} \binom{K}{L} n^{(L)} b_s^{\dagger^N} b_t^{\dagger^{n-L}} b_s^{K-L} b_t^k \delta^L(t-s)\,, \quad (1.1)$$

where $\forall\ n,\ k,\ N,\ K \in \mathbb{N}\cup 0$

$$\binom{K}{L} := \frac{K!}{L!(K-L)!} \quad \text{(binomial coefficient)}; \binom{K}{L} = 0, \text{if } K < L$$
$$\varepsilon_{n,k} := 1 - \delta_{n,k} \quad \text{(Kronecker's delta)}$$
$$n^{(L)} := n(n-1)\cdots(n-L+1)\,; \qquad n^{(0)} = 1; n^{(L)} = 0, \text{if } n < L\,.$$

The right-hand side of the above identity is ill defined because of the powers $\delta^L(t-s)$ of the δ-function. Any procedure to give a meaning to these powers will be called a renormalization rule. In this note we will use the following renormalization rule whose motivations are discussed in Ref. 4:

$$\delta^l(t-s) = \delta(s)\delta(t-s)\,, \qquad l = 2,3,4,\ldots\,. \quad (1.2)$$

The right-hand side of (1.2) is well defined as a convolution of distributions. Using this, (1.1) can be rewritten in the form:

$$[b_t^{\dagger^n} b_t^k, b_s^{\dagger^N} b_s^K] = \varepsilon_{k,0}\varepsilon_{N,0}\Bigg(kN b_t^{\dagger^n} b_s^{\dagger^{N-1}} b_t^{k-1} b_s^K \delta(t-s)$$
$$+ \sum_{L\geq 2} \binom{k}{L} N^{(L)} b_t^{\dagger^n} b_s^{\dagger^{N-L}} b_t^{k-L} b_s^K \delta(s)\delta(t-s)\Bigg)$$
$$- \varepsilon_{K,0}\varepsilon_{n,0}\Bigg(Kn b_s^{\dagger^N} b_t^{\dagger^{n-1}} b_s^{K-1} b_t^k \delta(t-s)$$
$$+ \sum_{L\geq 2} \binom{K}{L} n^{(L)} b_s^{\dagger^N} b_t^{\dagger^{n-L}} b_s^{K-L} b_t^k \delta(s)\delta(t-s)\Bigg)\,. \quad (1.3)$$

Introducing test functions and the associated smeared fields

$$B_k^n(f) := \int_{\mathbb{R}^d} f(t) b_t^{\dagger^n} b_t^k dt ,$$

the commutation relations (1.3) become:

$$\begin{aligned}[][B_k^n(g), B_K^N(f)] &= (\varepsilon_{k,0}\varepsilon_{N,0} kN - \varepsilon_{K,0}\varepsilon_{n,0} Kn) B_{K+k-1}^{N+n-1}(gf) \\ &\quad + \sum_{L=2}^{(K\wedge n)\vee(k\wedge N)} \theta_L(n,k;N,K) g(0) f(0) b_0^{\dagger^{N+n-L}} b_0^{K+k-L} \end{aligned} \tag{1.4}$$

$$\theta_L(n,k;N,K) := \varepsilon_{k,0}\varepsilon_{N,0} \binom{k}{L} N^{(L)} - \varepsilon_{K,0}\varepsilon_{n,0} \binom{K}{L} n^{(L)} \tag{1.5}$$

which still contain the ill-defined symbols $b_0^{\dagger^{N+n-L}}$, b_0^{K+k-L}. However, if the test function space is chosen so that

$$f(0) = g(0) = 0 , \tag{1.6}$$

then the singular term in (1.4) vanishes and the commutation relations (1.4) become:

$$[B_k^n(g), B_K^N(f)]_R := (kN - Kn) B_{k+K-1}^{n+N-1}(gf) \tag{1.7}$$

which no longer include ill-defined objects. The symbol $[\cdot,\cdot]_R$ denotes these renormalized commutation relations.

A direct calculation shows that the commutation relations (1.7) define, on the family of symbols $B_k^n(f)$, a structure of $*$-Lie algebra with involution

$$B_k^n(f)^* := B_n^k(\bar{f}) .$$

From the commutation relations (1.7) it is clear that, fixing a subset $I \subseteq \mathbb{R}^d$, not containing 0, and the test function

$$\chi_I(s) = \begin{cases} 1 , & s \in I \\ 0 , & s \notin I \end{cases} \tag{1.8}$$

the commutation relations (1.7) restricted to the (self-adjoint) family

$$\{B_k^n := B_k^n(\chi_I) : n,k \in \mathbb{N} \cup 0, n+k \geq 3\} \tag{1.9}$$

give

$$[B_k^n, B_K^N]_R := (kN - Kn) B_{k+K-1}^{n+N-1} . \tag{1.10}$$

The arguments in Ref. 4 then suggest the natural interpretation of the $*$-Lie-algebra, defined by the relations (1.9), (1.10), as the 1-*mode algebra* of the RHPWN and, conversely, the interpretation of the RHPWN $*$-Lie-algebra as a current algebra of its 1-mode version.

Now recall the following definition (see Refs. 6–11):

Definition 1. The w_∞-$*$-Lie-algebra is the infinite dimensional Lie algebra spanned by the generators $\hat{B}^n_k$, where $n \in \mathbb{N}$, $n \geq 2$ and $k \in \mathbb{Z}$, with commutation relations:

$$[\hat{B}^n_k, \hat{B}^N_K]_{w_\infty} = (k(N-1) - K(n-1))\hat{B}^{n+N-2}_{k+K} \tag{1.11}$$

and involution

$$(\hat{B}^n_k)^* = \hat{B}^n_{-k}\,. \tag{1.12}$$

Remark 1. The w_∞-$*$-Lie-algebra, whose elements are interpreted as area-preserving diffeomorphisms of 2-manifolds, contains as a sub-Lie-algebra (not as a $*$-sub-algebra) the (centerless) Virasoro (or Witt) algebra with commutation relations

$$[\hat{B}^2_k(g), \hat{B}^2_K(f)]_V := (k-K)\hat{B}^2_{k+K}(gf)\,.$$

Both w_∞ and a quantum deformation of it, denoted W_∞ and defined as a (non-unique) large N limit of Zamolodchikov's W_N algebra,[8] have been studied extensively[6,7,9–11] in connection to two-dimensional conformal field theory and quantum gravity.

The striking similarity between the commutation relations (1.11) and (1.10) suggests that the two algebras are deeply related. The following theorem shows that the current algebra, over $\mathbb{R}^d$, of the w_∞-$*$-Lie-algebra can be realized in terms of the renormalized powers of white noise. The converse of this statement is intuitively obvious at the level of formal white noise operators, but a precise statement about this topic will be discussed elsewhere.

Theorem 1. *Let $\mathcal{S}_0$ be the test function space of complex valued* (*right-continuous*) *step functions on $\mathbb{R}^d$ assuming a finite number of values and vanishing at zero*, *and let the powers of the δ-function be renormalized by the prescription*

$$\delta^l(t-s) = \delta(s)\delta(t-s)\,, \qquad l = 2, 3, \ldots \tag{1.13}$$

(*cf. Ref.* 4). *Then the white noise operators*

$$\hat{B}^n_k(f) := \int_{\mathbb{R}^d} f(t) e^{\frac{k}{2}(b_t - b^\dagger_t)} \left(\frac{b_t + b^\dagger_t}{2}\right)^{n-1} e^{\frac{k}{2}(b_t - b^\dagger_t)} dt; n \in \mathbb{N}, n \geq 2, k \in \mathbb{Z} \tag{1.14}$$

satisfy the relations (1.11) *and* (1.12) *of the w_∞-Lie algebra.*

Remark 2. The integral on the right-hand side of (1.14) is meant in the sense that one expands the exponential series, applies the commutation relations (1.1) to bring the resulting expression to normal order, introduces the renormalization prescription (1.13), integrates the resulting expressions after multiplication by a test function and interprets the result as a quadratic form on the exponential vectors.

Proof. The relation (1.12) is obvious, thus we will only prove (1.11). To this goal notice that the left-hand side of (1.11) is equal to:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t)f(s)\Bigg[e^{\frac{k}{2}(b_t-b_t^\dagger)}\left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(b_t-b_t^\dagger)},$$

$$e^{\frac{K}{2}(b_s-b_s^\dagger)}\left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1} e^{\frac{K}{2}(b_s-b_s^\dagger)}\Bigg]dt\,ds$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t)f(s)e^{\frac{k}{2}(b_t-b_t^\dagger)}\left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(b_t-b_t^\dagger)}$$

$$\times\, e^{\frac{K}{2}(b_s-b_s^\dagger)}\left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1} e^{\frac{K}{2}(b_s-b_s^\dagger)}dt\,ds$$

$$- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t)f(s)e^{\frac{K}{2}(b_s-b_s^\dagger)}\left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1} e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

$$\times\, e^{\frac{k}{2}(b_t-b_t^\dagger)}\left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(b_t-b_t^\dagger)}dt\,ds\,.$$

Since $[b_t-b_t^\dagger, b_s-b_s^\dagger]=0$, this is equal to:

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t)f(s)e^{\frac{k}{2}(b_t-b_t^\dagger)}\left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1} e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

$$\times\, e^{\frac{k}{2}(b_t-b_t^\dagger)}\left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1} e^{\frac{K}{2}(b_s-b_s^\dagger)}dt\,ds$$

$$- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t)f(s)e^{\frac{K}{2}(b_s-b_s^\dagger)}\left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1} e^{\frac{k}{2}(b_t-b_t^\dagger)}$$

$$\times\, e^{\frac{K}{2}(b_s-b_s^\dagger)}\left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(b_t-b_t^\dagger)}dt\,ds$$

$$= \frac{1}{2^{n+N-2}}\Bigg\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t)f(s)e^{\frac{k}{2}(b_t-b_t^\dagger)}(b_t+b_t^\dagger)^{n-1}e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

$$\times\, e^{\frac{k}{2}(b_t-b_t^\dagger)}(b_s+b_s^\dagger)^{N-1}e^{\frac{K}{2}(b_s-b_s^\dagger)}dt\,ds$$

$$- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t)f(s)e^{\frac{K}{2}(b_s-b_s^\dagger)}(b_s+b_s^\dagger)^{N-1}e^{\frac{k}{2}(b_t-b_t^\dagger)}$$

$$\times\, e^{\frac{K}{2}(b_s-b_s^\dagger)}(b_t+b_t^\dagger)^{n-1}e^{\frac{k}{2}(b_t-b_t^\dagger)}dt\,ds\Bigg\}\,.$$

From the formal expression (1.1) of the CCR we deduce the identities:

$$e^{\frac{K}{2}(b_s-b_s^\dagger)}(b_t+b_t^\dagger)^{n-1}$$
$$=\sum_{m=0}^{n-1}\binom{n-1}{m}(b_t+b_t^\dagger)^m K^{n-1-m}\delta^{n-1-m}(t-s)e^{\frac{K}{2}(b_s-b_s^\dagger)}\,,$$

$$e^{\frac{k}{2}(b_t-b_t^\dagger)}(b_s+b_s^\dagger)^{N-1}$$
$$=\sum_{m=0}^{N-1}\binom{N-1}{m}(b_s+b_s^\dagger)^m k^{N-1-m}\delta^{N-1-m}(t-s)e^{\frac{k}{2}(b_t-b_t^\dagger)}\,,$$

$$(b_t+b_t^\dagger)^{n-1}e^{\frac{K}{2}(b_s-b_s^\dagger)}$$
$$=e^{\frac{K}{2}(b_s-b_s^\dagger)}\sum_{m=0}^{n-1}\binom{n-1}{m}(b_t+b_t^\dagger)^m(-1)^{n-1-m}K^{n-1-m}\delta^{n-1-m}(t-s)\,,$$

$$(b_s+b_s^\dagger)^{N-1}e^{\frac{k}{2}(b_t-b_t^\dagger)}$$
$$=e^{\frac{k}{2}(b_t-b_t^\dagger)}\sum_{m=0}^{N-1}\binom{N-1}{m}(b_s+b_s^\dagger)^m(-1)^{N-1-m}k^{N-1-m}\delta^{N-1-m}(t-s)\,.$$

These identities imply that:

$$\begin{aligned}
&[\hat{B}_k^n(g),\hat{B}_K^N(f)]\\
&\quad=\frac{1}{2^{n+N-2}}\Bigg\{\sum_{m_1=0}^{n-1}\sum_{m_2=0}^{N-1}\binom{n-1}{m_1}\binom{N-1}{m_2}(-1)^{n-1-m_1}K^{n-1-m_1}k^{N-1-m_2}\\
&\qquad\times\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}g(t)f(s)e^{\frac{k}{2}(b_t-b_t^\dagger)}e^{\frac{K}{2}(b_s-b_s^\dagger)}(b_t+b_t^\dagger)^{m_1}(b_s+b_s^\dagger)^{m_2}e^{\frac{k}{2}(b_t-b_t^\dagger)}e^{\frac{K}{2}(b_s-b_s^\dagger)}\\
&\qquad\times\delta^{n-1-m_1+N-1-m_2}(t-s)dt\,ds\\
&\qquad-\sum_{m_3=0}^{N-1}\sum_{m_4=0}^{n-1}\binom{N-1}{m_3}\binom{n-1}{m_4}(-1)^{N-1-m_3}k^{N-1-m_3}K^{n-1-m_4}\\
&\qquad\times\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}g(t)f(s)e^{\frac{K}{2}(b_s-b_s^\dagger)}e^{\frac{k}{2}(b_t-b_t^\dagger)}(b_s+b_s^\dagger)^{m_3}(b_t+b_t^\dagger)^{m_4}e^{\frac{K}{2}(b_s-b_s^\dagger)}e^{\frac{k}{2}(b_t-b_t^\dagger)}\\
&\qquad\times\delta^{N-1-m_3+n-1-m_4}(t-s)dt\,ds\Bigg\}\,.
\end{aligned}$$

The term $(m_1=n-1,\, m_2=N-1)$ cancels out with $(m_3=N-1,\, m_4=n-1)$.

The renormalization prescription (1.13) and the choice of test functions vanishing at zero imply that

$$\sum_{m_1=0}^{n-3} \sum_{m_2=0}^{N-3} (\cdots) = \sum_{m_3=0}^{N-3} \sum_{m_4=0}^{n-3} (\cdots) = 0 \,.$$

Therefore, after the renormalization prescription (1.13), the only surviving terms are those corresponding to the pairs:

$$\begin{aligned} &(m_1 = n-1, m_2 = N-2)\,, \qquad &&(m_3 = N-1, m_4 = n-2)\,, \\ &(m_1 = n-2, m_2 = N-1)\,, \qquad &&(m_3 = N-2, m_4 = n-1) \end{aligned}$$

and we obtain:

$$\begin{aligned} &[\hat{B}^n_k(g), \hat{B}^N_K(f)] \\ &\quad = \frac{1}{2^{n+N-2}}((N-1)k - (n-1)K - (n-1)K + (N-1)k) \\ &\qquad \times \int_{\mathbb{R}^d} g(t)f(t)e^{\frac{k+K}{2}(b_t - b_t^\dagger)}(b_t + b_t^\dagger)^{n+N-3} e^{\frac{k+K}{2}(b_t - b_t^\dagger)} dt \\ &\quad = \frac{2}{2^{n+N-2}}((N-1)k - (n-1)K) \\ &\qquad \times \int_{\mathbb{R}^d} g(t)f(t)e^{\frac{k+K}{2}(b_t - b_t^\dagger)}(b_t + b_t^\dagger)^{n+N-3} e^{\frac{k+K}{2}(b_t - b_t^\dagger)} dt \\ &\quad = \frac{1}{2^{n+N-3}}((N-1)k - (n-1)K) \\ &\qquad \times \int_{\mathbb{R}^d} g(t)f(t)e^{\frac{k+K}{2}(b_t - b_t^\dagger)}(b_t + b_t^\dagger)^{n+N-3} e^{\frac{k+K}{2}(b_t - b_t^\dagger)} dt \\ &\quad = (k(N-1) - K(n-1))\hat{B}^{n+N-2}_{k+K}(gf)\,. \end{aligned}$$

□

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