PROBLEMS

11257. *Proposed by Raimond Struble, Santa Monica, CA.* Let $\langle z_n \rangle$ be a sequence of complex numbers, and let $s_n = \sum_{k=1}^{n} z_k$. Suppose that all s_n are nonzero.
(a) Given that s_n does not tend to zero, show that $\sum_{n=1}^{\infty} z_n/s_n$ converges if and only if $\lim_{n\to\infty} s_n$ exists.
(b) Show that if s_n tends to a limit s, and $s - s_n$ is never zero, then $\sum_{k=1}^{\infty} z_n/(s - s_{n-1})$ diverges.

11258. *Proposed by Manuel Kauers, Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria.* Let F_n denote the nth Fibonacci number, and let i denote $\sqrt{-1}$. Prove that

$$\sum_{k=0}^{\infty} \frac{F_{3^k} - 2F_{1+3^k}}{F_{3^k} + i\,F_{2\cdot 3^k}} = i + \tfrac{1}{2}\left(1 - \sqrt{5}\right).$$

11259. *Proposed by Nobuhisa Abe, NBU Attached Senior High School, Saiki, Japan.* For integers n greater than 2, let

$$f(n) = \sum_{j=1}^{n-2} 2^j \sum_{S} \prod_{k\in S} k,$$

where the sum is over all j-element subsets S of the set $\{1, \ldots, n-1\}$. Show that $4(2n-1)! + (f(n))^2$ is never the square of an integer.

11260. *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.* Find those nonnegative values of α and β for which

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\alpha + k\log k}{\beta + (k+1)\log(k+1)}$$

converges. For those values of α and β, evaluate the sum.

11261. *Proposed by Isaac Sofair, Fredericksburg, VA.* A triangle of area 1 has vertices A_1, A_2, and A_3. The sides A_2A_3, A_3A_1, and A_1A_2 subtend angles of measure α_1, α_2, and α_3, respectively, at an internal point P. The triangle has angles at A_1, A_2, and A_3 of measure a_1, a_2, and a_3 respectively. The extensions of A_1P, A_2P, and A_3P to their opposite sides meet those sides at B_1, B_2, and B_3, respectively.

For $k = 1..3$, let $T_k = \sin a_k \sin\alpha_k \sin(\alpha_k - a_k)$. For an even permutation p of $(1,2,3)$, let $S_p = (\sin a_{p_1} \sin\alpha_{p_2} \sin\alpha_{p_3} + \sin\alpha_{p_1} \sin a_{p_2} \sin a_{p_3})$, and let S be the product of S_p over all three such permutations. Prove that the area of triangle $B_1B_2B_3$ is $2T_1T_2T_3/S$.

11262. *Proposed by Ashay Burungale, Satara, Maharashtra, India.* In a certain town of population $2n+1$, one knows those to whom one is known. For any set A of n citizens, there is some person amongst the other $n+1$ who knows everyone in A. Show that some citizen of the town knows all the others.

11263. *Proposed by Gregory Keselman, Oak Park, MI, and formerly of Lvov Polytechnic Institute, Ukraine.* Show that when n is a positive integer and a is real,

$$\sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k a^k \binom{n-k}{k} = \begin{cases} \frac{(1+\sqrt{1-4a})^{n+1} - (1-\sqrt{1-4a})^{n+1}}{2^{n+1}\sqrt{1-4a}}, & \text{if } a < 1/4; \\ (n+1)/2^n, & \text{if } a = 1/4; \\ a^{n/2}\left(\cos(n\beta) + \sin(n\beta)/\sqrt{4a-1}\right), & \text{if } a > 1/4. \end{cases}$$

Here, β denotes $\arcsin\sqrt{1 - 1/(4a)}$.