PROBLEMS

11215. *Proposed by Shmuel Rosset, Tel Aviv University, Tel Aviv, Israel.* A car moves along the real line from $x = 0$ at $t = 0$ to $x = 1$ at $t = 1$, with differentiable position function $x(t)$ and differentiable velocity function $v(t) = x'(t)$. The car begins and ends the trip at a standstill; that is, $v = 0$ at both the beginning and the end of the trip. Let L be the maximum velocity attained during the trip. Prove that at some time between the beginning and end of the trip $|v'| > L^2/(L-1)$.

11216. *Proposed by Ted Chinburg, University of Pennsylvania, Philadelphia, PA, and Shahriar Shahriari, Pomona College, Claremont, CA.* Let K be a field, and let G be an ordered Abelian group. The *support* Supp(a) of a formal sum $a = \sum_\gamma a_\gamma t^\gamma$ with coefficients a_γ in K and exponents γ in G is the set $\{\gamma \in G \mid a_\gamma \neq 0\}$. The *generalized power series ring* $K((G^{\leq 0}))$ is the set of all formal sums a for which Supp(a) is a well-ordered subset of the nonpositive elements of G. Addition and multiplication in $K((G^{\leq 0}))$ are defined in the same way they are for ordinary power series. Show that $K((G^{\leq 0}))$ is Noetherian if and only if either $G = \{0\}$ or G is order isomorphic to $\mathbb{Z}$ with the usual ordering. (An ordered Abelian group is an Abelian group G with a total order $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ for all a, b, and c in G.)

11217. *Proposed by Michel Bataille, Rouen, France.* For n a positive integer let S_n denote the set of all numbers of the form

$$\frac{x^n}{y^{n-1}(y-1)(1-x)} + \frac{y^n}{z^{n-1}(z-1)(1-y)} + \frac{z^n}{x^{n-1}(x-1)(1-z)}$$

such that x, y, and z are positive numbers, each different from 1, with $xyz = 1$. Show that S_n is bounded below, and find the greatest lower bound of S_n in terms of n.

11218. *Proposed by Gary Gordon, Lafayette College, Easton, PA.* Consider the following algorithm, which takes as input a positive integer n and proceeds by *rounds*, listing in each round certain positive integers between 1 and n inclusive, ultimately producing as output a positive integer $f(n)$, the last number to be listed. In the 0th round, list 1. In the first round, list, in increasing order, all primes less than n. In the second round, list in increasing order all numbers that have not yet been listed and are of the form $2p$, where p is prime. Continue in this fashion, listing numbers of the form $3p$, $4p$, and so on until all numbers between 1 and n have been listed. Thus $f(10) = 8$ because the list eventually reaches the state $(1, 2, 3, 5, 7, 4, 6, 10, 9, 8)$, while $f(20) = 16$ and $f(30) = 27$.
(a) Find $f(2006)$.
(b) Describe the range of f.
(c) Find $\underline{\lim}_{n\to\infty} f(n)/n$ and $\overline{\lim}_{n\to\infty} f(n)/n$.

11219. *Proposed by R. A. Strubel, Santa Monica, CA.* Prove that when n is a positive integer and s is a real number greater than 1

$$1 + n(\zeta(s) - 1) \leq \sum_{k=0}^{\infty} \left(\frac{n}{n+k}\right)^s \leq n\zeta(s).$$

11220. *Proposed by David Beckwith, Sag Harbor, NY.* Show that when n is a positive integer

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{2n-2r}{n-1} = 0.$$

11221. *Proposed by Paolo Perfetti, University "Tor Vergata," Rome, Italy.* Give an example of a function g from $\mathbb{R}$ into $\mathbb{R}$ such that g is differentiable everywhere, g' is differentiable on one dense subset of $\mathbb{R}$, and g' is discontinuous on another dense subset of $\mathbb{R}$.