Snapshots on quantum probability

Luigi Accardi
Centro Vito Volterra, Università di Roma Tor Vergata,
00133 Roma, Italy
accardi@volterra.mat.uniroma2.it, http://volterra.mat.uniroma2.it
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Abstract

As suggested by the title, the goal of the present talk is to describe some casual photographs of different parts of quantum probability without pretense of completeness. I will choose three topics which, in my opinion, efficiently illustrate the fruitful interplay between mathematics and physics which has characterized the development of quantum probability in the past thirty years: (i) the description of a recent experiment which has brought to a conclusion the long standing debate about possible non local effects as necessary consequences of the basic principles of quantum mechanics; (ii) the notion of interacting Fock space, which emerged from quantum electrodynamics without dipole approximation and turned out to be a fruitful tool in such disparate fields as orthogonal polynomials, asymptotics of graphs, quantum structure of classical probability measures, exclusion statistics, . . . ; (iii) the square (and higher powers) of white noise and its relation to renormalization theory and infinitely divisible processes.

1 Introduction

The following philosophical thumb rule will be the guideline for the exposition in the present section:

If I cannot solve a difficult problem, maybe I am not good enough, but if I cannot solve an apparently simple problem, maybe there is something deep behind this apparent simplicity.

Following this guideline I will use two simple problems to illustrate the basic thesis of Quantum probability as it begun to emerge in the late 70's of the past century:

Probability theory must go beyond the classical (Kolmogorov) model!

This is a strong statement and therefore, to be taken into consideration, it needs strong arguments in its support.

How to prove such a statement? The answer is: by means of the **statistical invariants**. To explain what they are an analogy with geometry might be useful.

It is generally recognized that the starting point of non euclidean geometry is Gauss' *theorema egregium* (the excellent theorem) which gives a criterium to distinguish between flat and curved surfaces in terms of experimentally measurable parameters. Gauss was well aware of the importance of this fact as shown by his historical attempt to experimentally measure a geological triangle to estimate the local curvature of the earth.

Just as physicists measure geometrical quantities such as areas, angles, ..., they can also measure statistical quantities such as (conditional) probabilities, correlations, ..., .

Can one combine some of these measurements into "statistical invariants" which allow to distinguish between Kolmogorovian and non–Kolmogorovian models of the laws of chance just as the "geometrical invariants" allow to distinguish between Euclidean and non–Euclidean models of space?

The following two simple but important examples show that this is indeed possible.

The statistical invariant of the two slit experiment: a quantum probabilistic analysis

We follow Feynman ([Feyn51] or [FeLeSa66]) in the exposition of this experiment.

A source emits microscopic particles which can pass a first screen through two slits, labeled 1 and 2, and are collected on a second screen. Denote X an arbitrary region on this second screen.

The experimentally measurable statistical data are the probabilities:

$P(X)$ = probability of hitting the region X of the screen

$P(X|1)$ = probability of hitting X when the slit 2 is closed

$P(X|2)$ = probability of hitting X when the slit 1 is closed

Definition 1 *The statistical data $P(X)$, $P(X|1)$, $P(X|2)$, admit a Kolmogorov model if there exists a probability space $(\Omega, \mathcal{F}, \mu)$ and events, still denoted X, 1 and 2 and identified to subsets of Ω, such that:*

$$P(X) = \mu(X) \tag{1}$$

$$P(X|1) = \frac{\mu(X \cap 1)}{\mu(1)} \tag{2}$$

$$P(X|2) = \frac{\mu(X \cap 2)}{\mu(2)} \tag{3}$$

Remark The above definition means simply that in the classical, Kolmogorovian model the conditional probabilities $P(X|1)$ and $P(X|2)$ must be given by the usual Bayes' formula.

The crucial fact to be noted here is that the statistical data $P(X)$, $P(X|1)$, $P(X|2)$, have a well defined experimental meaning independently of any mathematical model (just as the angles in Gauss' triangle). Therefore the existence of a Kolmogorov model for them cannot be postulated, but it must be proved. Exactly as one cannot postulate the existence of an Euclidean triangle determined by three empirically measured angles, but one must use Gauss' criterium to check if such a triangle exists or not.

Proposition 1 *The statistical data $P(X)$, $P(X|1)$, $P(X|2)$ admit a Kolmogorov model if and only if the given probabilities satisfy the constraints*

$$0 < \frac{P(X) - P(X|2)}{P(X|1) - P(X|2)} < 1 \tag{4}$$

Proof. One solves the system (1), (2), (3), in the unknowns $\mu(X \cap 1), \mu(1), \mu(X \cap 2), \mu(2)$, and one imposes the condition that the solutions are in $(0, 1)$.

Remark If one changes the statistical data in (4), i.e. the probabilities, so that the constraint (4) is still fulfilled, then the property of admitting a Kolmogorovian model will remain unaltered.

In this sense we say that the relation (4) is a statistical invariant for the set of statistical data $P(X)$, $P(X|1)$, $P(X|2)$ with respect to the Kolmogorovian model.

We can see that, if the empirical data $P(X)$, $P(X|1)$, $P(X|2)$ have been obtained in three different experiments on three different samples, then condition (4) is by no means a tautology on the relative frequencies: its validity is an experimental fact. If it holds, then a Kolmogorov model exists and the identity

$$P(X) = \mu(1)P(X|1) + \mu(2)P(X|2) \tag{5}$$

must be satisfied. The experiment discussed by Feynman (and before him by Bohr, Einstein, Heisenberg, ..., shows that in some cases (5) is not satisfied so that a Kolmogorov model for these data cannot exist.

Summing up: the 2–slit experiment, whose conceptual implications were first discussed in the Solvay conference in 1928, provides the first example of a simple set of statistical data, coming from different but related experiments, which do not admit a Kolmogorovian model.

The statistical invariant of the EPR type experiments: a quantum probabilistic analysis

In 1935 Einstein, Podolsky and Rosen (EPR) published a paper [EPR35] which pointed out some difficulties in the interpretation of quantum mechanics.

In 1964 Bell proved an inequality [Bell64], related to the EPR type experiments, which can be experimentally checked.

The discovery, in [Ac81a], that **both the 2–slit experiment and the Bell inequality are two different aspects of the same mathematical phenomenon** was the starting point of quantum probability.

More precisely: both the 2–slit experiment and Bell's inequality are necessary conditions for the existence of a Kolmogorovian model for a set of statistical data coming from different and mutually incompatible experiments: conditional probabilities in the 2–slit case; correlations in the Bell case.

In probabilistic language Bell's inequality can be formulated as follows:

Theorem 1 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and let* A, B, C *be any three random variables defined on* Ω *and taking values in the interval* $[-1, 1]$. *Then the following (Bell) inequality holds:*

$$|E(AB) - E(BC)| \leq 1 - E(AC) \tag{6}$$

Proof Simple exercise.

The physical implications of Bell's inequality can be formulated in the language of quantum probability as follows:

Corollary 1 *There exist triples* a, b, c *of unit vectors in* $\mathbf{R}^3$ *for which it is not possible to find six* $\{-1, +1\}$*–valued random variables* S_x^j $(x = a, b, c\,;\ j = 1, 2)$ *on the same probability space* $(\Omega, \mathcal{F}, P)$ *whose correlations are given by:*

$$E(S_x^1 \cdot S_y^2) = -x \cdot y \qquad ; \qquad x, y = a, b, c \tag{7}$$

(where, for $x, y \in \mathbf{R}^3$, $x \cdot y$ *denotes the Euclidean scalar product.)*

Proof Choose

$$a = 0 \qquad ; \qquad b = \frac{\pi}{4} \qquad ; \qquad c = \frac{\pi}{2}$$

and show that

$$A := S_a^1 \qquad ; \qquad B := S_b^2 \qquad ; \qquad C := S_c^1$$

violate Bell's inequality. Then use the fact that (7) implies:

$$S_c^1 = -S_c^2$$

The situation is as simple as in the case of the 2–slit experiment and, just as in that case there is a huge literature on Bell's inequality.

A mathematician, unaware of the 75 years of heated debate on the foundations of quantum mechanics, might ask: **Why such a huge literature**?

The fact is that the simple arguments above were believed, for several decades, to challange some of the basic pillars of contemporary physics, such as the principle of reality or the principle of locality.

In fact, according to Bell, the inequality (6) is necessarily implied by the locality principle: no physical interaction can propagate with a velocity higher than the speed of light in the vacuum.

But, according to experiments, the EPR correlations violate Bell's inequality.

Conclusion: if Bell's idea, that his inequality (6) is a necessary consequence of the locality principle, is correct, then quantum mechanics is non–local.

But the locality principle, is one of the pillars of relativity theory. Therefore if Bell's idea is correct, then QM contradicts relativity.

Thus we are faced with the following dramatic question:

Is it true that the two basic theories of contemporary physics
– the theory of relativity
– quantum mechanics
are mutually contradictory?

In the past 30 years the answer to this question, accepted by the majority of physicists has been:

yes they are mutually contradictory. Moreover this can be proved by theory and confirmed by experiment

But a minority of physicists and mathematicians still had some doubts and insisted to look for a way out of this apparent contradiction. The story of this pursuit is long and tormented and will not be discussed here. The final result can be synthetized in the bizarre statement:

Mathematics saves physics by an experiment!

More precisely, the paper [AcImRe01] describes an experiment which realizes a local, deterministic, classical, macroscopic, dynamical system reproducing the EPR correlations, hence violating Bell's inequality.

This experiment is based on a mathematical model of a new type of dynamical systems (adaptive dynamical systems) which will probably be very fruitful not only in mathematics and physics, but also in biological, economical and social sciences where adaptive dynamical systems are the rule rather than the exception.

This experiment proves that arguments such as Bell's inequality cannot be invoked to assert an hypothetical contradiction between the two basic theories of contemporary physics (relativity and quantum mechanics) for the simple reason that **it is (mathematically and experimentally) not true that locality implies Bell's inequality**.

This brings quantum physics out of what K. Popper defined: "... The great quantum muddle ... ".

The experiment of [AcImRe01], and more generally the theory of adaptive dynamical systems of which this paper is only a first example, also provides an intuitive answer to the following question: **how can non Kolmogorovian models of physical systems naturally arise within classical physics and classical probability**? In other words: how can we build non Kolmogorovian models of physical systems using classical probability?

This answer is based on the "chameleon effect" which ca be formulated as follows:

the dynamics of a system may depend on the observables we want to measure (or, more generally, on the local environment).

For example: the color of a chameleon is as "pre–determined" as the color of a ball in a ballot box but everybody understands that the word "pre–determination" has a different meaning in the two cases.

In the case of a ball in a ballot box you measure what was there, independently of the environment (passive property).

In the case of a chameleon you measure the **response** to an environment (adaptive property). In other words:

The symbol of classical probability is the color of a ball in a box (passive property: environment independent).

The symbol of quantum probability is the color of a chameleon (adaptive property: response to an environment).

This result has also some practical implications which make a bridge between mathematical philosophy and industrial applications because it gave rise to a new line of research which could be called *non–Kolmogorovian stochastic simulation* and which generalizes usual stochastic simulation in the sense that it allows to reproduce quantum entanglement by classical computer, a task which was considered impossible up to a few years ago.

In particular it is now possible to set up a competition between quantum cryptography based on quantum optics and quantum cryptography based on the chameleon effect. The group in Volterra Center is now actively persuing this goal in collaboration with an italian industry and we hope to arrive soon (say within one year from now) to the production of a software which can be effectively sold on the market.

Coming back to the foundations of probability theory, the conclusions which can be drawn from the above described experiments are the following:

there exist sets of statistical data empirically obtainable from simple real experiments which cannot be described within a single classical probability space but which can be described within a single quantum probability space.

So you have a choice:

(i) either you use many classical probability spaces to describe these data

(ii) or you use a single quantum probability space

But (i) is a bad choice for many reasons, the most important of which is that, to realize it, you must introduce a huge quantity of spurious, non observable, information (e.g., in a quantum mechanical context, all the joint probabilities of incompatible observables)

This also explains the difference between the term "Non commutative probablity", which refers to a purely mathematical generalization of the classical probabilistic formalism, and "Quantum probability" which underlines the experimental necessity to go beyond the Kolmogorov model.

In fact, from a purely mathematical point of view, every branch of mathematics **can** be made non commutative.

However probability theory **must** be made non commutative in the sense that, in thia case, noncommutativity is not an arbitrary choice but an experimental necessity expressing the seesnce of a physical situation. The precise sense of this statement, translated into a set of physically meaningful axioms, is explained in the papers [Ac95a], [Ac82c] where the reader will find the answer to several natural questions, which have accompanied the development of quantum theory since its birth, such as for example: Why just the Hilbert space model? Why the Hilbert space should be complex? Why the superposition principle? Why the evolution of probability should be described by the Schrödinger equation? Where do the canonical commutation relations come from? ...

The general picture which emerges from the above considerations suggests an analogy between the main thesis of general relativity (the interaction with masses determines the laws of space) and the main thesis of quantum probability (the interaction with the environment determines the laws of chance).

According to relativity theory the geometry of space with masses tends to be non Euclidean.

According to quantum probability the statistics of responses (i.e. of adaptive systems) tends to be non Kolmogorovian.

2 The Interacting Fock Functor

It is known that the Fock functor establishes an isomorphism between the category of Gaussian processes and the category of Fock Spaces.

In view of this the following natural question arises:

does there exist a functor realizing, for arbitrary probability measures, what the Fock functor realizes for Gaussian measures?

The answer is: yes, the Interacting Fock Functor realizes this goal.

Interacting Fock Spaces: intuitive idea

An Interacting Fock Space (IFS) is any Hilbert space $\mathcal{H}$ with the following properties:

(i) $\mathcal{H}$ is the orthogonal sum of "n–particle sub–spaces:

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

(ii) there exist a vector space $\mathcal{H}_0$ and a linear map

$$A^+ : f \in \mathcal{H}_0 \to A_f^+$$

from $\mathcal{H}_0$ to densely defined operators, called creation operators,

$$A_f^+ : \mathcal{H}_n \to \mathcal{H}_{n+1}$$

(iii1) there is a unit vector $\Phi \in \mathcal{H}$ such that

$$\mathcal{H}_0 \equiv \mathbf{C} \cdot \Phi$$

(iii2) For each $n \in \mathbf{N}$ and $f_1, \ldots, f_n, f_{n+1} \in \mathcal{H}_0$

$$A^+_{f_n} \cdot \ldots \cdot A^+_{f_1} \Phi \in Dom(A^+_{f_{n+1}})$$

(iii3) For each $n \in \mathbf{N}$ the vectors

$$\mathcal{N}_n := \{A^+_{f_n} \cdot \ldots \cdot A^+_{f_1} \Phi \ : \ f_1, \ldots, f_n \in \mathcal{H}_0\}$$

are total in $\mathcal{H}_n$

(iv) the adjoints of the creation operators, called annihilation operators, are defined on the (dense) sub–space

$$\mathcal{N} := \bigcup_{n \geq 0} \mathcal{N}_n$$

Interacting Fock Spaces were introduced in the paper [AcLu92b] because they natuarlly arise in quantum electrodynamics when one drops the usual dipole approximation (cf. [AcLuVo02] for more details). The first systematic discussion of this notion from an axiomatic point of view is in [AcLuVo97b], but the definition used here, which is more suitable for the aplications to the theory of orthogonal polynomials we are going to discuss, is taken from [AcSk98].

The connection between one–mode interacting Fock Spaces and orthogonal polynomials, associated to an arbitrary probability measure on the real line with moments of any order, was first established in the paper [AcBo97].

In order to understand the intuitive idea behind this result let us recall the Jacobi recurrence formula for orthogonal polynomials, associated to any probability measure μ on $\mathbf{R}$:

$$xP^{(n)}(x) = P^{(n+1)}(x) + \alpha_n P^{(n)}(x) + \omega_n P^{(n-1)}(x) \ , \tag{8}$$

where $n \in \mathbf{N}$ and $P^{(-1)}(x) := 0$. If we define the (creation and number) operators by:

$$a^+ P^{(n)} := P^{(n+1)} \tag{9}$$

$$NP^{(n)} := nP^{(n)} \tag{10}$$

Then (20) becomes equivalent to

$$X = a^{(+)} + a + \alpha_N \tag{11}$$

where α_N is defined by the spectral theorem ($\alpha_N P^{(n)} := \alpha_n P^{(n)}$).

This is the quantum decomposition of an arbitrary real valued classical random variable (with moments of any order).

Example In the usual Fock space, with vacuum distribution and with the usual creation and annihilation operators, the standard Gaussian random variable can be represented as

$$X = a^{(+)} + a \tag{12}$$

and the standard Poisson random variable intensity α can be represented as

$$X = a^{(+)} + a + \alpha a^{(+)} a \tag{13}$$

In the following we will see that there is a deep reason why both the standard Gaussian and the standard Poisson random variable can be represented in the same Fock space.

connections with works by Berezansky

Algebra implies statistics !

Another interesting fall out of the interacting Fock space is a new, purely probabilistic, approach to the commutation relations which allowed a natural generalization of the Heisenberg commutation relations to an arbitrary probability measure.

The question: "where do the canonical commutation relations come from?" was, as mentioned in the first section of this paper, one of the old problems in the foundations of quantum theory. The answer suggested by the theory of interacting Fock spaces is very simple: from classical probability theory! The intuitive idea on which this answer is based is very simple: in any one–mode interacting Fock space both $aa^{(+)}$ and $a^{(+)}a$ leave the n–particle space invariant. Therefore also their difference, i.e. the commutator $aa^{(+)} - a^{(+)}a$ leaves each n–particle space invariant and, in one–mode case, this is possible if and only if this commutator is a function of the number operator, i.e. if, for some function T, one has:

$$[a, a^{(+)}] = aa^{(+)} - a^{(+)}a = T(N) \tag{14}$$

More precisely:

Theorem 2 *Every classical probability measure on* $\mathbf{R}$ *with moments of any order is canonically associated to a commutation relation in an interacting Fock space. This commutation relation, together with the Fock prescription*

$$a\Phi = 0 \tag{15}$$

uniquely determines all the moments of the given probability measure (hence the measure itself, in all cases in which the moment problem admits a unique solution).

The explicit formula of $T(N)$ can be easily calculated in terms of the Jacobi parameters (i.e. the coefficients appearing in the Jacobi relation (20). The following table, taken from the joint paper with H.H. Kuo and A. Stan [ACKUST], shows the result of this calculation in the case of some well known probability measures.

Measure	Polynomials	Jacobi parameters
Gaussian $N(0,\sigma^2)$	Hermite $H_n(x;\sigma^2)$ $= (-\sigma^2)^n e^{x^2/2\sigma^2} \partial_x^n e^{-x^2/2\sigma^2}$	$\alpha_n = 0$ $\omega_n = \sigma^2 n$ $(\lambda_n = \sigma^{2n} n!)$
Poisson Poi (a)	Charlier $C_n(x;a) =$ $(-1)^n a^{-x} \Gamma(x+1) \Delta^n \left[\frac{a^x}{\Gamma(x-n+1)}\right]$	$\alpha_n = n + a$ $\omega_n = an$ $(\lambda_n = a^n n!)$
Gamma $\Gamma(\alpha)$, $(\alpha > -1)$ $\frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x}$, $x > 0$	Laguerre $\mathcal{L}_n^{(\alpha)}(x)$ $= (-1)^n x^{-\alpha} e^x \partial_x^n [x^{n+\alpha} e^{-x}]$	$\alpha_n = 2n + 1 + \alpha$ $\omega_n = n(n+\alpha)$ $(\lambda_n = n!(n+\alpha)\cdots(1+\alpha))$
Uniform on $[-1,1]$	Legendre $\tilde{L}_n(x) = \frac{1}{2^n(2n-1)!!} \partial_x^n [(x^2-1)^n]$	$\alpha_n = 0$ $\omega_n = \frac{n^2}{(2n+1)(2n-1)}$ $\left(\lambda_n = \frac{(n!)^2}{[(2n-1)!!]^2(2n+1)}\right)$
Arcsine $\frac{1}{\pi\sqrt{1-x^2}}$, $\lvert x\rvert < 1$	Chebyshev (1st kind) $\tilde{T}_0(x) = 1$ $\tilde{T}_n(x) = \frac{1}{2^{n-1}} \cos(n \cos^{-1} x)$, $n \geq 1$	$\alpha_n = 0$ $\omega_n = \begin{cases} \frac{1}{2}, & n = 1 \\ \frac{1}{4}, & n \geq 2 \end{cases}$ $\left(\lambda_n = \frac{1}{2^{2n-1}}\right)$
Semicircle $\frac{2}{\pi}\sqrt{1-x^2}$, $\lvert x\rvert < 1$	Chehyshev (2nd kind) $\tilde{U}_n(x) = \frac{1}{2^n} \frac{\sin[(n+1)\cos^{-1} x]}{\sin[\cos^{-1} x]}$	$\alpha_n = 0$ $\omega_n = \frac{1}{4}$ $\left(\lambda_n = \frac{1}{4^n}\right)$
$\frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta+1)}{\Gamma\left(\beta+\frac{1}{2}\right)} (1-x^2)^{\beta-\frac{1}{2}}$ $\lvert x\rvert < 1$, $\beta > -\frac{1}{2}$	Gegenbauer $\tilde{G}_n^{(\beta)}(x) = C_n^{(\beta)} (1-x^2)^{\frac{1}{2}-\beta} \partial_x^n [(1-x^2)^{n+\beta-\frac{1}{2}}]$ $C_n^{(\beta)} = \frac{(-1)^n 2^n \Gamma(2\beta+n)}{\Gamma(2\beta+2n)}$	$\alpha_n = 0$ $\omega_n = \frac{n(n+2\beta-1)}{4(n+\beta)(n+\beta-1)}$

$[a^-, a^+]e_n$	Coherent vector	Generating function
$\sigma^2 I$	$e^{\frac{zx}{\sigma^2} - \frac{z^2}{2\sigma^2}}$	$e^{tx-\frac{1}{2}\sigma^2 t^2} = \sum_{n=0}^{\infty} \frac{H_n(x;\sigma^2)}{n!} t^n$
aI	$e^{-z}\left(1+\frac{z}{a}\right)^x$	$e^{-at}(1+t)^x = \sum_{n=0}^{\infty} \frac{C_n(x;a)}{n!} t^n$
$(2n+\alpha+1)e_n$	$\sum_{n=0}^{\infty} \frac{\mathcal{L}_n^{(\alpha)}(x)}{n!(n+\alpha)\cdots(1+\alpha)} z^n$	$(1+t)^{-\alpha-1} e^{\frac{tx}{1+t}}$ $= \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_n^{(\alpha)}(x)$
$-\frac{1}{(2n+3)(2n+1)(2n-1)} e_n$	$\sum_{n=0}^{\infty} \frac{((2n-1)!!)^2(2n+1)}{(n!)^2} \tilde{L}_n(x) z^n$	$\frac{1}{\sqrt{1-2tx+t^2}}$ $= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!} \tilde{L}_n(x) t^n$
$\begin{cases} \frac{1}{2} e_0, & n=0 \\ -\frac{1}{4} e_1, & n=1 \\ 0, & n \geq 2 \end{cases}$	$\frac{1-2xz}{1-4xz+4z^2}$	$\frac{4-t^2}{4-4tx+t^2}$ $= \sum_{n=0}^{\infty} \tilde{T}_n(x) t^n$
$\begin{cases} \frac{1}{4} e_0, \ n=0 \\ 0, \ n \geq 1 \end{cases}$	$\frac{1}{1-4xz+4z^2}$	$\frac{4}{4-4tx+t^2} = \sum_{n=0}^{\infty} \tilde{U}_n(x) t^n$
$\frac{\beta^2-\beta}{2(n+1+\beta)(n+\beta)(n-1+\beta)}$	not in close form	$\frac{1}{(1-2tx+t^2)^\beta}$ $= \sum_{n=0}^{\infty} \frac{2^n \Gamma(\beta+n)}{\Gamma(\beta) n!} \tilde{G}_n^{(\beta)}(x) t^n$

The problem to extend the above results from one–mode interacting Fock spaces (corresponding to measures on the real line) to general interacting Fock spaces (corresponding to measures on multi (possibly infinite dimensional) spaces) was solved in the paper [AcNh01] in the finite dimensional case and in the paper [AcKuSt04] in the infinite dimensional case

The general programme to which this line of research is inspired can be synthetized in a single sentence: **algebra implies statistics**.

This means that the main goal of this programme is to realize, just like in the Gaussian and Poisson case, a complete coding of all the statistical information of a probability measure (e.g. mixed moments) into algebraic properties of the creation and annihilation operators, appearing in its quantum decomposition, such as, for example, the commutation relations (14) or the property of "killing the vacuum" (15).

3 The renormalized square of white noise and Meixner classification theorem

As an illustration of the slogan: **algebra implies statistics**, I will now discuss the quantum probabilistic solution of the old problem to give a meaning to the square of classical white noise.

The quantum decomposition of a classical random variable is functorial, hence it can be easily transported from single random variables to processes. For example the quantum decomposition of classical white noise is

$$w_t = b_t^+ + b_t \tag{16}$$

where b_t^+, b_t is the quantum (Boson Fock) white noise, algebraically characterized by the properties

$$[b_t, b_s^+] = \delta(t-s) \tag{17}$$

$$b_t \Phi = 0 \tag{18}$$

Given the quantum decomposition (16) of classical white noise an intuitive formal candidate for the square of classical white noise is:

$$w_t^2 = (b_t^+ + b_t)^2$$

$$= b_t^{+2} + b_t^2 + b_t^+ b_t + b_t b_t^+$$

$$= b_t^{+2} + b_t^2 + 2b_t^+ b_t + \delta(0)$$

Here the presence of the meaningless symbol $\delta(0)$ reflects the fact that, since the white noise is a distribution, its square is ill defined. Mimicking a standard procedure in physics (renormalization) we subtract this infinity and find:

$$b_t^{+2} + b_t^2 + 2b_t^+ b_t$$

But this does not solve our problem because, since b_t^+, b_t are operator valued distributions, also expressions like b_t^{+2}, b_t^2 are meaningless. Thus even after additive renormalizion, w_t^2 remains meaningless !

However, if we take seriously the slogan: **algebra implies statistics**, we could think of another approach to the problem. That is, instead of trying to give a direct meaning to expressions like b_t^{+2}, b_t^2, we try to give a meaning to the commutator $[b_t^2, b_{t'}^{+2}]$ and then we realize a representation of these commutation relations in a Hilbert space with a unit vector Φ satisfying

$$b_t^2 \Phi = 0$$

since algebra implies statistics, if we find the good algebra for the square of white noise then we will also find the good statistics!

Because of (17) we expect that b_t^{+2}, b_t^2 should satisfy the commutation relations:

$$[b_t^2, b_{t'}^{+2}] = [b_t^2, b_{t'}^+]b_{t'}^+ + b_{t'}^+[b_t^2, b_{t'}^+]$$
$$= b_t[b_t, b_{t'}^+]b_{t'}^+ + [b_t, b_{t'}^+]b_t b_{t'}^+ + b_{t'}^+, b_t[b_t, b_{t'}^+] + b_{t'}^+[b_t, b_{t'}^+]b_t$$
$$= \delta(t-t')b_t b_{t'}^+ + \delta(t-t')b_t b_{t'}^+ + b_{t'}^+ b_t \delta(t-t') + \delta(t-t')b_{t'}^+ b_t$$
$$= 2\delta(t-t')b_t b_{t'}^+ + 2\delta(t-t')b_{t'}^+ b_t = 2\delta(t-t')^2 + 4\delta(t-t')b_{t'}^+ b_t$$

where the singularity now appears in the emergence of the factor $\delta(t-t')^2$. To handle this singularity a new idea was proposed in [AcLuVo99]: **multiplicative renormalizion** . This amounts to use the following known formula from distribution theory:

Theorem 3 *$\delta(t)^2 = c\delta(t)$ where the constant $c \in \mathbf{c}$ is arbitrary.*

In order to define the renormalized commutation relations of the square of WN, let us introduce, as usual, the "smeared" fields:

$$b_\varphi^+ = \int dt \varphi(t) b_t^2 \quad ; \qquad b_\varphi = (b_\varphi^+)^+$$

$$n_\varphi = \int dt \varphi(t) b_t^+ b_t$$

With these notations we find the renormalized commutation relations:

$$[b_\varphi, b_\psi^+] = \gamma\langle\varphi, \psi\rangle + n_{\overline{\varphi}\psi}$$
$$[n_\varphi, b_\psi] = -2b_{\overline{\varphi}\psi}$$
$$[n_\varphi, b_\psi^+] = 2b_{\varphi\psi}^+$$
$$(b_\varphi^+)^+ = b_\varphi \quad ; \qquad n_\varphi^+ = n_{\overline{\varphi}}$$

Thus, to complete the definition of the Fock representation, we need only the prescription:

$$b_\varphi \Phi = 0$$

It is easy to verify that, if the Fock representation exists, then it is unique. The problem thus is: Does the Fock representation exist?

The answer, discovered in [AcLuVo99] is: Yes if the constant $c \in \mathbf{c}$ is > 0.

It was then proved in [AcFrSk00] that the algebra of the RSWN is isomorphic to a representation of the algebra of currents over (a central extension of) the Lie algebra $sl(2, \mathbf{R})$. It is known that this has 3 generators

$$B^- \qquad B^+ \qquad M$$

(the central extension corresponds to a multiple of the identity) and relations

$$[B^-, B^+] = M$$

$$[M, B^\pm] = \pm 2B^\pm$$

An application of a general result due to Schürmann, on the classification theorem for independent increment processes on *–bi–algebras [Schü93], allows to obtain a full classification of an important subclass of representations of this current algebra as well as a concrete realization of them on some explicitly constructed Fock space.

4 The five simplest classes of Levy processes: the Meixner processes

Meixner, in his classical paper [Meix34], considered the following problem: find all sequences of polynomials $P^{(n)}(x)$ ($n \in \mathbf{N}$), in one real variable x with the following properties:

(i) the leading coefficient of each $P^{(n)}(x)$ is 1

(ii) for each $n \in \mathbf{N}$, $P^{(n)}(x)$ is the n–th orthogonal polynomial with respect to some probability measure μ on $\mathbf{R}$.

(iii) there exist functions $f(z)$ and $\Psi(z)$ such that

$$G(x, z) := \exp(x\Psi(z))f(z) = \sum_{n=0}^{\infty} \frac{P^{(n)}(x)}{n!} z^n \tag{19}$$

If a probability measure μ on $\mathbf{R}$, which solves the Meixner problem, exists, then it is uniquely determined by two sequences:

– (a_n) real number

– (b_n) positive numbers

which are the coefficients of the Jacobi recurrence formula

$$xP^{(n)}(x) = P^{(n+1)}(x) + a_n P^{(n)}(x) + b_n P^{(n-1)}(x) \ , \tag{20}$$

$$n \in \mathbf{N} \ , \qquad P^{(1)}(x) := 0 \tag{21}$$

Meixner showed that the sequences corresponding to the solution of his problem are completely determined by two parameters λ and k through the equations

$$k = \frac{b_n}{n} - \frac{b_{n-1}}{n-1} \quad , \quad n \geq 2$$

$$\lambda = a_n - a_{n-1} \quad (\text{equivalently } a_n = \lambda n) \quad , \quad n \in \mathbf{N}$$

ii He then proved that these equation admit exactly 5 solutions, namely:

– 2 solutions for $k = 0$

– 3 solutions for $k \neq 0$

I) The 1–st Meixner class

If $k = 0$ and $\lambda = 0$, then

$$a_n = 0 \qquad , \qquad b_n = n$$

and the $P^{(n)}$ are the Hermite polynomials so that μ is the standard Gaussian distribution on $\mathbf{R}$.

II) The 2–d Meixner class

If $k = 0$ and $\lambda \neq 0$, then

$$a_n = \lambda n \qquad , \qquad b_n = n$$

and the $P^{(n)}$ are the Charlier polynomials so that μ is the centered Poisson distribution on $\mathbf{R}$ with parameter λ.

If $k \neq 0$ then

$$b_n = kn + \frac{n}{n-1} b_{n-1} =$$

$$= kn + \frac{n}{n-1} b_{n-2} = kn + \frac{n}{n-1}(k(n-1) + \frac{n-1}{n-2} b_{n-2} =$$

$$= \ldots = kn^2$$

Setting for simplicity of notations $k = 1$ and introducing two quantities α and β through the equation

$$1 + \lambda z + z^2 = (1 - \alpha z)(1 - \beta z) \ . \tag{22}$$

we distinguish the three following cases:

III) The 3–d Meixner class

$|\lambda| = 2$

In this case the $P^{(n)}(x)$ are the Laguerre polynomials and μ is a centered gamma distribution which is a compound Poisson measure.

(IV) The 4–th Meixner class

$|\lambda| > 2$

In this case the $P^{(n)}(x)$ are the Meixner polynomials (of the first kind), which are orthogonal with respect to a centered Pascal (negative binomial) distribution i.e., up to reparametrization, a distribution of the form (**??**).

(V) The 5–d Meixner class

$|\lambda| < 2$, so that $\alpha \neq \beta$, both complex conjugate.

In this case the $P^{(n)}(x)$ are the Meixner polynomials of the second kind, (or Meixner–Pollaczek polynomials). These are orthogonal with respect to a measure μ obtained by centering a probability measure of the form $C \exp(ax)|\Gamma(1 + imx)|^2 dx$, where $a \in \mathbf{R}$, $m > 0$, and C is the normalizing constant.

The measures of the 3–d Meixner class do not possess the chaotic decomposition property but, as shown by Nualart and Schoutens [NUSCH00], see also the recent book [SCHOU00], they enjoy a generalization of this property obtained by adding to the original process (X_t) its associated "power jump processes"

$$X_t^{(i)} := \sum_{0<s\leq t} (\Delta X_s)^{(i)} \qquad , \quad i \geq 2$$

The measures in the 4–th Meixner class, i.e. the Lévy processes, corresponding to the Pascal measures, were introduced in [BruRo91] in the context of optimal selection strategies based on relative ranks, when the total number of options is unknown.

The measures in the 5–th Meixner class are called Meixner measures in the papers [Grig99], [LYTVa], [Grig01] and they are a sub–class of Grigelionis' "generalized z–distributions" In fact, in [Grig00c] the term "Meixner distribution" is used for the class of probability measures on $\mathbf{R}$ whose characteristic function (Fourier transform of the probability density) has the form

$$\hat{f}(z) = \left(\frac{cos(\beta/2)}{cosh((\alpha z - i\beta)/2)}\right)^{2\delta}$$

with $z \in \mathbf{R}, -\pi < \beta < \pi, \delta > 0, \mu \in \mathbf{R}$. We refer to [Grig01] for several interesting properties of these distributions and explicit formulae related to them.

In [SchTeu98] it was proved that the measures in this class correspond to Levy processes and their connection with the Meixner–Pollaczek polynomials was established. In particular, in [Grig00c] the Meixner process was proposed as a model for risky assets and an analogue of the Black and Sholes formula was established for them.

The infinite dimensional and multidimensional analogues of orthogonal polynomials associated to a given measure have been widely studied both in the Gaussian ([BEKO], [HiKuPoStr93], [Kuo96]) and in the Poisson case ([CHIHA], [KOKUOL]).

The programme to extend this analysis to more general probability measures was developed by Berezansky, who introduced in this connection the notion of Jacobi field of operators, and his school ([BELILY], [LYTV], [BEREZa], [BEREZb]).

To write a stochastic equation as a white noise equation is a simple exercise, but the resulting equation is not Hamiltonian, so its connection with physics is not clear. The converse problem: given an Hamiltonian white noise equation, find the associated stochastic equation, is nontrivial and its solution was suggested by the stochastic limit of quantum theory. It involves a new type of normal order - causal normal order - and a purely white noise formulation of Ito table. This, in its turn, opens the way to a nonlinear extension of classical Ito calculus which turns out to be strictly related with a new approach to renormalization theory and to the theory of current representations of classical and infinite dimensional Lie algebras.

5 Bibliography

[AcLuVo02] Accardi L., Y.G. Lu, I. Volovich: Quantum Theory and its Stochastic Limit. Springer Verlag (2002) Japanese translation, Tokyo–Springer (2002)

[Ac81a] Accardi L.: Topics in quantum probability, Phys. Rep. 77 (1981) 169-192

[AcImRe01] L.Accardi, Kentaro Imafuku, Massimo Regoli: On the physical meaning of the EPR–chameleon experiment, Infinite dimensional analysis, quantum probability and related topics, 5 N. 1 (2002) 1–20 quant-ph/0112067 Volterra Preprint December (2001) N. 494

[Ac95a] L. Accardi: Can mathematics help solving the interpretational problems of quantum theory? Il Nuovo Cimento 110 B (1995) 685–721 cf. also: Mesoscopic Physics and fundamental problems in quantum mechanics, C.Di Castro, F.Guerra, G. Jona–Lasinio (1995) Volterra Preprint N. 203 (1995)

[Ac82c] Accardi L.: Some trends and problems in quantum probability, In: Quantum probability and applications to the quantum theory of irreversible processes, L. Accardi, A. Frigerio and V. Gorini (eds.) Springer LNM N. 1055 (1984) 1-19

[AcSk98] L. Accardi and M. Skeide: Interacting Fock space versus full Fock module, Volterra Preprint N. 328 (1998)

[AcLu92b] Accardi L., Lu Y.G.: The Wigner Semi–circle Law in Quantum Electro Dynamics, Commun. Math. Phys. 180 (1996) 605–632 Volterra preprint N.126 (1992)

[AcLuVo97b] Accardi L., Lu Y.G., I. Volovich The QED Hilbert module and Interacting Fock spaces. Publications of IIAS (Kyoto) (1997)

[AcBo97] Luigi Accardi, Marek Bożejko: Interacting Fock spaces and Gaussianization of probability measures Infinite dimensional analysis, quantum probability and related topics, 2, N. 4 (1998) Volterra Preprint (1997) N. 321

[AcKuSt04] Accardi L., Kuo H.–H., Stan A.: Orthogonal polynomials and interacting Fock spaces, Infinite Dimensional Anal. Quantum Probab. Related Topics (IDA-QP) 7 (4) (2004) 485–505

[AcLuVo99] Accardi, L., Lu, Y.G., Volovich, I.V.: A white noise approach to classical and quantum stochastic calculus, Volterra Preprint N.375 July (1999), Volterra–CIRM School: WN approach to classical and quantum Ito calculus, Trento july (1999)

[AcFrSk00] L. Accardi, U. Franz, M. Skeide: Renormalized squares of white noise and other non–gaussian noises as Lévy processes on real Lie algebras, Comm. Math. Phys. (2002) Preprint Volterra, N. 423 (2000)

[Acc00] Accardi L.: The white noise approach to stochastic calculus: motivations, applications, developments, in: Colloque SMT 2000, p. 1–18 Plenary Lecture given at: Colloque Société Mathématique de Tunisie, Sousse, 27–30 mars 2000

[AcNh01] Accardi L., Nhani M.: Interacting Fock Spaces and Orthogonal Polynomials in several variables, in: The Crossroad of Non-Commutativity, Infinite-Dimensionality and Obata, Hora, Matsui (eds.) World Scientific (2002) 192–205 Preprint Volterra, N. 523 (2002)

[Bell64] Bell J.S.: On the Einstein Podolsky Rosen paradox. Physics (Long Island City, N.Y.) 1 (1964) 195-200

[BEREZb98] Yu.M. Berezansky: Commutative Jacobi fields in Fock space, J. Func. Anal. 138 (1998) 163–190.

[BEREZa97] Yu.M. Berezansky: On the direct and inverse spectral problem for Jacobi fields, Algebra i Analys (in russian) 9 (6) (1997), english transl. St. Petersburg Math. J. 9 (6) (1997).

[BEKO94] Yu.M. Berezansky, Yu.G. Kondratiev: Spectral Methods in infinite dimensional analysis, Kluwer (1994).

[BELILY95] Yu.M. Berezansky, V.O. Livinsy,, E.W. Lytvynov: A generalization of Gaussian white noise analysis, Methods Func. Anal. and Topol. 1 (1995) 28–55.

[BruRo91] F.T. Bruss, L.C. Rogers: Pascal processes and their characterization, Stochastic Process. Appl. 37 (1991) 331–338.

[EPR35] Einstein A., Podolsky B., Rosen N. Can quantum mechanical description of reality be considered complete ? Phys. Rev. 47 (1935) 777-780

[Feyn51] Feynmann R.P. The concept of probability in quantum mechanics. in :Proc. II-d Berkeley Sym on Math. Stat. and Prob. ; Univ. of California Press , Berkeley, Los Angeles (1951) 533-541

[FeLeSa66] Feynmann R.P., Leighton R.P., Sands M.: Lectures on Physics, III, Addison–Wesley (1966)

[Grig01] B. Grigelionis: Generalized z–distributions and related stochastic processes, Lithuanian Math. J. 41 (2001) 303–319.

[Grig99] B. Grigelionis: Processes of Meixner type, Lithuanian Math. J. 39 (1999) 33–41.

[Grig00c] B. Grigelionis: On generalized z–diffusions, Preprint Vilnius (2000).

[LYTV02a] Eugene Lytvynov: Polynomials of Meixner's type in infinite dimensions Jacobi fields and orthogonality measures, arXiv:math.CA/0203026 v5, 27 May (2002).

[LYTV02b] Eugene Lytvynov: Orthogonal decompositions for Levy processes with an application to the gamma, Pascal and Meixner processes, arXiv:math.PR/0204087 v5, 27 May (2002).

[LYTV95c] E. Lytvynov: Multiple Wiener integrals and non–Gaussian white noises: a Jacobi field approach, Methods Func. Anal. and Topol. 1 (1995) 61–85.

[Meix34] Meixner J.: Orthogonale Polynomsysteme mit einen besonderen Gestalt der erzeugenden Funktion, J.Lond. Math. Soc. 9 (1934) 6–13

[NUSCH00] D. Nualart, W. Schoutens: Chaotic and predictable representations for Lévy processes, Stochastic Process. Appl. 90 (2000) 109–122.

[SCHOU00] W. Schoutens: Stochastic Processes and Orthogonal Polynomials, Lecture Notes in Statist., Vol. 146, Springer–Verlag, New York (2000).

[SchTeu98] W. Schoutens, J.L. Teugels: Lévy processes, polynomials and martingales, Commun. Statisti.–Stochastic Models 14 (1998) 335–349.

[Schü93] Schürmann, M.: White noise on bialgebras. Springer Lect Notes Math: 1544 (1993)