Proposed problem for Mathematical Reflections

Dear Ivan, dear Editor,
I would like to propose the following problem with solution.

Give an example of a real differentiable function f defined on $(0,1)$ such that $\sup\limits_{x\in E}|f'(x)| = M < +\infty$, E is a dense subset of the domain (say $\overline{E} \supset (0,1)$) and $|f|$ nowhere differentiable on $(0,1)$. Otherwise prove that such a function does not exist.

Comment: If one asks E infinite but not having any limit point in $(0,1)$, the function does exist (Amer. Math. Monthly Vol.75, No.6 (Jun.–Jul., 1968), 688–689)

Answer The function does not exist.

Let's suppose that the function f exists. Of course $f(x_0) = 0$ if $x_0 \in E$ because otherwise we would have $|f|$ differentiable at x_0. The differentiability of f at x_0 means $|f(x)| = |f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)| \le M|x - x_0| + o(x - x_0)$. The density of E in $(0,1)$ means

$$\forall\, x \in (0,1)\ \forall\, \varepsilon > 0\ \exists\, x_\varepsilon \in E\colon\ |x - x_\varepsilon| < \varepsilon$$

This implies that for any $x \in (0,1)$, for any $\varepsilon > 0$, there exists $x_0 \in E$ such that $|(x-x_0)+o(x - x_0)| < 2\varepsilon$ namely $f(x) = 0$ for any $x \in (0,1)$ contradicting the non differentiability of $|f|$.

Of course, feel free to change the statement if accepted.

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Best regards
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