PROBLEMS

11467. *Proposed by Xiang Qian Chang, Massachusetts College of Pharmacy and Health Sciences, Boston, MA.* Find in closed form the determinant of the $n \times n$ matrix with entries $a_{i,j}$ given by

$$a_{i,j} = \begin{cases} \sum_{k=0}^{i-1}(j-k)^2 & \text{if } i \le j; \\ \sum_{k=1}^{j} k^2 + \sum_{k=0}^{i-j-1}(n-k)^2 & \text{if } i > j. \end{cases}$$

11468. *Proposed by Cosmin Pohoata, Tudor Vianu National College of Informatics, Bucharest, Romania.* Let $A_1A_2A_3$ be a triangle, let $\mathcal{H}$ be a dilation mapping of the plane, and let $\mathcal{R}$ be a right angle rotation of the plane. Let P_1, P_2, and P_3 be the images under $\mathcal{H} \circ \mathcal{R}$ of A_1, A_2, and A_3, respectively, and suppose that P_1, P_2, and P_3 lie inside or on the boundary of $A_1A_2A_3$.

Let H_i for $i \in \{1, 2, 3\}$ be the foot of the perpendicular from P_i to the side of $A_1A_2A_3$ opposite A_i. Generalize the Erdős–Mordell inequality: show that

$$P_1A_1 + P_2A_2 + P_3A_3 \ge P_1H_2 + P_1H_3 + P_2H_3 + P_2H_1 + P_3H_1 + P_3H_2,$$

with equality if and only if $A_1A_2A_3$ is equilateral and each P_i is equal to the circumcenter of $A_1A_2A_3$.

11469. *Proposed by Slavko Simic, Mathematics Institute SANU, Belgrade, Serbia.* Let $\langle x_i \rangle$ be a sequence of positive numbers, and let $\langle p_i \rangle$ be a sequence of nonnegative numbers summing to 1. Let

$$A = \sum_{i=1}^{\infty} p_i x_i, \quad H = \left(\sum_{i=1}^{\infty} p_i / x_i \right)^{-1}.$$

Show that if s and t are nonnegative numbers such that $s \le \sqrt{x_i} \le s + t$ for all $i \ge 1$, then $H \le A \le t^2 + H$.

11470. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let $ABCDEF$ be a hexagon inscribed in a circle. Let M, N, and P be the midpoints of the line segments BC, DE, and FA, respectively, and similarly let Q, R, and S be the midpoints of AD, BE, and CF. Show that if both MNP and QRS are equilateral, then the segments AB, CD, and EF have equal lengths.

11471. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let A be an $r \times r$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. For $n \ge 0$, let $a(n)$ be the trace of A^n. Let $H(n)$ be the $r \times r$ Hankel matrix with (i, j) entry $a(i + j + n - 2)$. Show that

$$\lim_{n\to\infty} |\det H(n)|^{1/n} = \prod_{k=1}^{r} |\lambda_k|.$$

11472. *Proposed by Mahdi Makhul, Shahrood University of Technology, Shahrood, Iran.* Let t be a nonnegative integer, and let f be a $4t + 3$ times continuously differentiable function on $\mathbb{R}$. Show that there is a number a such that at $x = a$,

$$\prod_{k=0}^{4t+3} \frac{d^k f(x)}{dx^k} \ge 0.$$

11473. *Proposed by Paolo Perfetti, Mathematics Dept., University "Tor Vergata Roma," Rome, Italy.* Let α and β be real numbers such that $-1 < \alpha + \beta < 1$ and such that, for all integers $k \geq 2$,

$$-(2k)\log(2k) \neq \alpha, \qquad (2k+1)\log(2k+1) \neq \alpha,$$

$$1 + (2k+1)\log(2k+1) \neq \beta, \qquad -1 - (2k+2)\log(2k+2) \neq \beta.$$

Let

$$T = \lim_{N\to\infty} \sum_{n=2}^{N} \prod_{k=2}^{n} \frac{\alpha + (-1)^k \cdot k\log(k)}{\beta + (-1)^{k+1}(1 + (k+1)\log(k+1))},$$

$$U = \lim_{N\to\infty} \sum_{n=2}^{N} ((n+1)\log(n+1)) \prod_{k=2}^{n} \frac{\alpha + (-1)^k \cdot k\log(k)}{\beta + (-1)^{k+1}(1 + (k+1)\log(k+1))}.$$

(**a**) Show that the limits defining T and U exist.
(**b**) Show that if, moreover, $|\alpha| < 1/2$ and $\beta = -\alpha$, then $T = -2U$.