

Alberto Pettorossi

Techniques for Searching,
Parsing, and Matching

Second Edition

ARACNE

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Preface

These lecture notes present some basic techniques for: (i) exploring search spaces, (ii) parsing context-free languages, and (iii) matching patterns in strings. These techniques are taught in a course on Automata, Languages, and Translators at the University of Roma “Tor Vergata”. We assume that the reader is familiar with the basic notions of Automata Theory and Formal Languages. These notions can be found in many books such as [9, 10, 21].

Some of the algorithms we have presented in these notes are written in Java 1.5 and some others in Prolog. For the Java language the reader may refer to the Java Tutorial at <http://java.sun.com/docs/books/tutorial/>. (Recall that this Java version allows the use of parametrized types, also called *generics*.) All Java programs have been compiled using the Java compiler 1.5.0_13 running under Mac OS X 10.4.11 Darwin 8.11.1.

For the Prolog language the reader may refer to [5]. The Prolog language incorporates a backtracking mechanism which is very often required for exploring search spaces and solving parsing and matching problems.

Acknowledgements

I would like to thank my colleagues at the Department of Informatics, Systems, and Production of the University of Roma “Tor Vergata”. I am also grateful to my students and my co-workers Lorenzo Clemente, Corrado Di Pietro, Fulvio Forni, Fabio Lecca, Maurizio Proietti, and Valerio Senni for their support and encouragement.

Many thanks to the Aracne Publishing Company for its helpful cooperation.
Roma, September 2008

In the second edition we have corrected a few mistakes and we have improved Chapter 5 and Chapter 9.
Roma, September 2009

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CHAPTER 1

Preliminary Definitions on Languages and Grammars

In this chapter we recall some preliminary definitions of formal languages. Let us consider a countable set V of elements, called *symbols*. The set V is also called the *alphabet*.

1.1. Free Monoids and Languages

The *free monoid over* (or *generated by*) the set V is the set V^* consisting of all finite sequences of symbols in V , that is,

$$V^* = \{v_1 \dots v_n \mid n \geq 0 \text{ and for each } 0 \leq i \leq n, v_i \in V\}.$$

The unary operation $*$ (pronounced ‘star’) is called *Kleene closure*, or **closure* (pronounced ‘star closure’). Sequences of symbols are also called *words* or *strings*. The *length* of a sequence $w = v_1 \dots v_n$, where for $i = 1, \dots, n, v_i \in V$, is n . The length of w is also denoted $|w|$. The sequence of length 0 is called the *empty sequence* (or *empty word* or *empty string*) and it is denoted by ε .

Given two sequences w_1 and w_2 in V^* their *concatenation*, denoted $w_1 \cdot w_2$ or $w_1 w_2$, is the sequence in V^* defined by recursion on the length of w_1 as follows:

$$\begin{aligned} w_2 & \text{ if } w_1 = \varepsilon \\ v_1((v_2 \dots v_n) \cdot w_2) & \text{ if } w_1 = v_1 v_2 \dots v_n, \text{ for some symbols } v_1 v_2 \dots v_n, \text{ with } n > 0. \end{aligned}$$

We have that the length of $w_1 \cdot w_2$ is the length of w_1 plus the length of w_2 . Concatenation is an associative binary operation on V^* whose neutral element is the empty sequence.

Any set of sequences which is a subset of V^* is called a *language* (or a *formal language*) over the alphabet V .

Given two languages A and B , their *concatenation*, denoted $A \cdot B$ or simply AB , is defined as follows:

$$A \cdot B = \{w_1 \cdot w_2 \mid w_1 \in A \text{ and } w_2 \in B\}.$$

Concatenation of languages is associative and its neutral element is the singleton $\{\varepsilon\}$. For reasons of simplicity, when A is a singleton, say $\{w\}$, the concatenation $A \cdot B$ will also be written as $w \cdot B$ or simply wB . Analogously for B , instead of A .

We have that: $V^* = V^0 \cup V^1 \cup V^2 \cup \dots \cup V^k \cup \dots$, where for each $i \geq 0, V^k$ is the set of all sequences of length k of symbols in V , that is, for each $k \geq 0$,

$$V^k = \{v_1 \dots v_k \mid \text{for } i = 1, \dots, k, v_i \in V\}.$$

Obviously, $V^0 = \{\varepsilon\}$, $V^1 = V$, and for $i, j \geq 0, V^i \cdot V^j = V^j \cdot V^i = V^{i+j}$. By V^+ we denote $V^* - \{\varepsilon\}$. The unary operation $^+$ (pronounced ‘plus’) is also called *positive closure* or *+closure* (pronounced ‘plus closure’).

The set $V^0 \cup V^1$ is also denoted by $V^{0,1}$.

Given an element a in a set V ,

- (i) a^* denotes the set of all *finite* sequence of zero or more a 's (thus, a^* is an abbreviation for $\{a\}^*$),
- (ii) a^+ denotes the set of all *finite* sequence of one or more a 's (thus, a^+ is an abbreviation for $\{a\}^+$),
- (iii) $a^{0,1}$ denotes the set $\{\varepsilon, a\}$ (thus, $a^{0,1}$ is an abbreviation for $\{a\}^{0,1}$), and
- (iv) a^ω denotes the *infinite* sequence made out of all a 's.

Given a word w , for any $k \geq 0$, the *prefix of w of length k* , denoted \underline{w}_k , is defined as follows:

$$\underline{w}_k = \begin{cases} w & \text{if } 0 \leq |w| \leq k \\ u & \text{if } w = uv \text{ for some } u, v \in V^*, \text{ and } |u| = k. \end{cases}$$

In particular, for any w , we have that: $\underline{w}_0 = \varepsilon$ and $\underline{w}_{|w|} = w$.

From now on, unless otherwise specified, when referring to an alphabet, we will assume that it is a *finite* set of symbols.

1.2. Formal Grammars

Let us begin by recalling the following notions. For other notions of automata theory and formal language theory the reader may refer to [10, 21].

DEFINITION 1.2.1. [Formal Grammar] A *formal grammar* (or a *type 0 grammar*, or a *grammar*, for short) is a 4-tuple $\langle V_T, V_N, P, S \rangle$, where:

- (i) V_T is a finite set of symbols, called *terminal symbols*,
- (ii) V_N is a finite set of symbols, called *nonterminal symbols* or *variables*, such that $V_T \cap V_N = \emptyset$,
- (iii) P is a *finite* set of pairs of strings, called *productions* (or *type 0 productions*), each pair $\langle \alpha, \beta \rangle$ being denoted by $\alpha \rightarrow \beta$, where $\alpha \in (V_T \cup V_N)^+$ and $\beta \in (V_T \cup V_N)^*$, and
- (iv) S is an element of V_N , called *axiom* or *start symbol*.

The set V_T is called the *terminal alphabet*. The set V_N is called the *nonterminal alphabet*. The set $V = V_T \cup V_N$ is called the *alphabet* of the grammar. In a production $\alpha \rightarrow \beta$, α is called the *left hand side* (*lhs*, for short) and β is called the *right hand side* (*rhs*, for short).

Given a grammar $G = \langle V_T, V_N, P, S \rangle$ we may define a set of elements in V_T^* , called the *language generated by G* , as we now indicate.

Let us first introduce the binary relation $\rightarrow_G \subseteq V^* \times V^*$ which is defined as follows: for every sequence $\alpha \in V^+$ and every sequence β, γ , and δ in V^* ,

$$\gamma\alpha\delta \rightarrow_G \gamma\beta\delta \text{ iff there exists a production } \alpha \rightarrow \beta \text{ in } P.$$

For any $k \geq 0$, the k -fold composition of the relation \rightarrow_G is denoted \rightarrow_G^k . Thus, for instance, for every sequence $\sigma_0 \in V^+$ and every sequence $\sigma_2 \in V^*$, we have that:

$$\sigma_0 \rightarrow_G^2 \sigma_2 \text{ iff there exists } \sigma_1 \in V^+ \text{ such that } \sigma_0 \rightarrow_G \sigma_1 \text{ and } \sigma_1 \rightarrow_G \sigma_2.$$

The transitive closure of \rightarrow_G is denoted \rightarrow_G^+ . The reflexive, transitive closure of \rightarrow_G is denoted \rightarrow_G^* . When the grammar G is understood from the context, we will feel

free to omit the subscript G , and instead of writing \rightarrow_G , \rightarrow_G^k , \rightarrow_G^+ , and \rightarrow_G^* , we will simply write \rightarrow , \rightarrow^k , \rightarrow^+ , and \rightarrow^* , respectively.

DEFINITION 1.2.2. [Language Generated by a Grammar] Given a grammar $G = \langle V_T, V_N, P, S \rangle$, the *language generated by G* , denoted $L(G)$, is the set

$$L(G) = \{w \mid w \in V_T^* \text{ and } S \rightarrow_G^* w\}.$$

The elements of the language $L(G)$ are said to be the *words* or the *strings* generated by the grammar G .

A language generated by a type 0 grammar is said to be a *type 0 language*. For productions, grammars, and languages, the term ‘unrestricted’ is often used instead of the term ‘type 0’.

DEFINITION 1.2.3. [Language Generated by a Nonterminal Symbol of a Grammar] Given a grammar $G = \langle V_T, V_N, P, S \rangle$, the language generated by the nonterminal $A \in V_N$, denoted $L_G(A)$, is the set

$$L_G(A) = \{w \mid w \in V_T^* \text{ and } A \rightarrow_G^* w\}.$$

We will write $L(A)$, instead of $L_G(A)$, when the grammar G is understood from the context.

DEFINITION 1.2.4. [Type 1, Context-Sensitive, Type 2, and Type 3 Production] (1) Given a grammar $G = \langle V_T, V_N, P, S \rangle$ we say that a production is of *type 1* iff (1.1) *either* it is of the form $\alpha \rightarrow \beta$, where $\alpha \in (V_T \cup V_N)^+$, $\beta \in (V_T \cup V_N)^+$, and $|\alpha| \leq |\beta|$, *or* it is $S \rightarrow \varepsilon$, and (1.2) if the production $S \rightarrow \varepsilon$ is in P then the axiom S does *not* occur on the right hand side of any production.

(cs) Given a grammar $\langle V_T, V_N, P, S \rangle$, we say that a production in P is *context-sensitive* iff (cs.1) *either* it is of the form $uAv \rightarrow uvw$, where $u, v \in V^*$, $A \in V_N$, and $w \in (V_T \cup V_N)^+$, *or* it is $S \rightarrow \varepsilon$, and (cs.2) if the production $S \rightarrow \varepsilon$ is in P then the axiom S does *not* occur on the right hand side of any production.

(2) Given a grammar $G = \langle V_T, V_N, P, S \rangle$ we say that a production in P is of *type 2* (or *context-free*) iff it is of the form $\alpha \rightarrow \beta$, where $\alpha \in V_N$ and $\beta \in V^*$.

(3) Given a grammar $G = \langle V_T, V_N, P, S \rangle$ we say that a production in P is of *type 3* (or *regular*, or *right linear*) iff it is of the form $\alpha \rightarrow \beta$, where $\alpha \in V_N$ and $\beta \in V_T V_N \cup V_T \cup \{\varepsilon\}$.

DEFINITION 1.2.5. [Type 1, Context-Sensitive, Type 2, and Type 3 Grammar and Language] A grammar is of type 1, context-sensitive, of type 2 (or context-free), and of type 3 (or regular, or right linear) iff all its productions are of type 1, context-sensitive, of type 2, and of type 3, respectively.

A type 1, context-sensitive, type 2 (or context-free), and type 3 (or regular or right linear) language is the language generated by a type 1, context-sensitive, type 2, and type 3 grammar, respectively.

One can show that for each type 1 grammar there exists an equivalent context-sensitive grammar, and vice versa. Thus, the notions of type 1 and context-sensitive

languages coincide. For this reason, instead of saying ‘a language of type 1’, we will also say ‘a context-sensitive language’ and vice versa.

One can also show that for each type 3 (or regular, or right linear) grammar there exists an equivalent grammar whose productions are all of the form $\alpha \rightarrow \beta$, where $\alpha \in V_N$ and $\beta \in V_N V_T \cup V_T \cup \{\varepsilon\}$. These grammars are said to be *left linear* grammars.

Given a grammar $G = \langle V_T, V_N, P, S \rangle$, an element of V^* is called a *sentential form* of G .

The following fact is an immediate consequence of the definitions.

FACT 1.2.6. Given a grammar $G = \langle V_T, V_N, P, S \rangle$ and a word $w \in V_T^*$, we have that w belongs to $L(G)$ iff there exists a sequence $\langle \alpha_1, \dots, \alpha_n \rangle$ of $n (> 1)$ sentential forms such that:

- (i) $\alpha_1 = S$,
- (ii) for every $i = 1, \dots, n-1$, there exist $\gamma, \delta \in V^*$ such that $\alpha_i = \gamma\alpha\delta$, $\alpha_{i+1} = \gamma\beta\delta$, and $\alpha \rightarrow \beta$ is a production in P , and
- (iii) $\alpha_n = w$.

Let us now introduce the following concepts.

DEFINITION 1.2.7. [Derivation of a Word and Derivation of a Sentential Form] Given a grammar $G = \langle V_T, V_N, P, S \rangle$ and a word $w \in V_T^*$ in $L(G)$, any sequence $\langle \alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n \rangle$ of $n (> 1)$ sentential forms satisfying Conditions (i), (ii), and (iii) of Fact 1.2.6 above, is called a *derivation of the word w* from S in the grammar G . A derivation $\langle S, \alpha_2, \dots, \alpha_{n-1}, w \rangle$ is also written as:

$$S \rightarrow \alpha_2 \rightarrow \dots \rightarrow \alpha_{n-1} \rightarrow w.$$

If there is a derivation from S to w , we have that $S \rightarrow^+ w$ and $S \rightarrow^* w$.

A *derivation of a sentential form $\varphi \in V^*$* from S in the grammar G is any sequence $\langle \alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n \rangle$ of $n (\geq 1)$ sentential forms such that Conditions (i) and (ii) of Fact 1.2.6 hold and $\alpha_n = \varphi$. That derivation is also written as:

$$S \rightarrow \alpha_2 \rightarrow \dots \rightarrow \alpha_{n-1} \rightarrow \varphi.$$

If there is a derivation from S to φ , we have that $S \rightarrow^* \varphi$.

DEFINITION 1.2.8. [Derivation Step] Given a derivation $\langle \alpha_1, \dots, \alpha_n \rangle$ of $n (\geq 1)$ sentential forms, for any $i = 1, \dots, n-1$, the pair $\langle \alpha_i, \alpha_{i+1} \rangle$ is called a *derivation step* from α_i to α_{i+1} (or a *rewriting step* from α_i to α_{i+1}). A derivation step $\langle \alpha_i, \alpha_{i+1} \rangle$ is also denoted by $\alpha_i \rightarrow \alpha_{i+1}$.

Given a sentential form $\gamma\alpha\delta$ for some $\gamma, \delta \in V^*$ and $\alpha \in V^+$, if we apply the production $\alpha \rightarrow \beta$, we perform the derivation step $\gamma\alpha\delta \rightarrow \gamma\beta\delta$.

Given a context-free grammar G and a word w , a derivation $\alpha_0 \rightarrow \alpha_1 \rightarrow \dots \rightarrow \alpha_n$ in G , where $\alpha_0 = S$ and $\alpha_n = w$, is said to be a *leftmost derivation* of w from S iff for $i = 0, \dots, n-1$, in the derivation step $\alpha_i \rightarrow \alpha_{i+1}$ the sentential form α_{i+1} is obtained from the sentential form α_i by replacing the leftmost nonterminal symbol occurring in α_i . A derivation step $\alpha_i \rightarrow \alpha_{i+1}$ in which we get the sentential form α_{i+1} by replacing the leftmost nonterminal symbol in α_i , is also denoted by $\alpha_i \rightarrow_{lm} \alpha_{i+1}$ (the subscript lm stands for *leftmost*).

Analogous definitions are assumed by replacing the word ‘leftmost’ by the word ‘rightmost’. In particular, a *rightmost derivation* is a derivation where at each step we replace the rightmost nonterminal symbol of the sentential form at hand. A rightmost derivation step is usually denoted by \rightarrow_{rm} (the subscript *rm* stands for *rightmost*).

EXAMPLE 1.2.9. Let us consider the grammar with axiom E and whose productions are:

$$\begin{aligned} E &\rightarrow E + T \\ E &\rightarrow T \\ T &\rightarrow T \times F \\ T &\rightarrow F \\ F &\rightarrow (E) \\ F &\rightarrow a \end{aligned}$$

Let us also consider the following three derivations $D1$, $D2$, and $D3$, where for each derivation step $\alpha_i \rightarrow \alpha_{i+1}$, we have underlined in the sentential form α_i the nonterminal symbol which is replaced in that step:

$$\begin{aligned} D1 : & \underline{E} \rightarrow_{lm} \underline{E} + T \rightarrow_{lm} \underline{T} + T \rightarrow_{lm} \underline{F} + T \rightarrow_{lm} a + \underline{T} \rightarrow_{lm} a + \underline{F} \rightarrow_{lm} a + a \\ D2 : & \underline{E} \rightarrow_{rm} E + \underline{T} \rightarrow_{rm} E + \underline{F} \rightarrow_{rm} \underline{E} + a \rightarrow_{rm} \underline{T} + a \rightarrow_{rm} \underline{F} + a \rightarrow_{rm} a + a \\ D3 : & \underline{E} \rightarrow_{lm} E + \underline{T} \rightarrow_{rm} \underline{E} + F \rightarrow_{lm} T + \underline{F} \rightarrow_{rm} \underline{T} + a \rightarrow_{lm} \underline{F} + a \rightarrow_{lm} a + a \end{aligned}$$

We have that:

derivation $D1$ is a leftmost derivation,

derivation $D2$ is a rightmost derivation, and

derivation $D3$ is neither a rightmost nor a leftmost derivation. \square

THEOREM 1.2.10. Given a context-free grammar G , for every word $w \in L(G)$ there exists a leftmost derivation of w and a rightmost derivation of w .

PROOF. By structural induction on the derivation tree of w . \square

For context-free grammars we can associate a *derivation tree*, also called a *parse tree*, with every derivation of a word w from the axiom S .

Given a context-free grammar $G = \langle V_T, V_N, P, S \rangle$ and a derivation of a word w from the axiom S , that is, a sequence $\alpha_1 \rightarrow \alpha_2 \rightarrow \dots \rightarrow \alpha_n$ of $n (> 1)$ sentential forms such that $\alpha_1 = S$ and $\alpha_n = w$ (see Definition 1.2.7 on page 12), the corresponding derivation tree T is constructed by applying the following two rules.

Derivation Tree Rule (1). The root of T is a node labeled by S .

Derivation Tree Rule (2). For $i = 1, \dots, n-1$, we consider in the given derivation

$$\alpha_1 \rightarrow \alpha_2 \rightarrow \dots \rightarrow \alpha_n$$

the i -th derivation step $\alpha_i \rightarrow \alpha_{i+1}$. Let us assume that in the i -th derivation step we have applied the production $A \rightarrow \beta$, where:

- (i) $A \in V_N$,
- (ii) $\beta = c_1 \dots c_k$, for some $k \geq 0$, and
- (iii) for $j = 1, \dots, k$, we have that $c_j \in V_N \cup V_T$.

Thus, $\alpha_i = \gamma A \delta$ and $\alpha_{i+1} = \gamma \beta \delta$ for some γ and δ in $(V_T \cup V_N)^*$. In the derivation tree constructed so far, we consider the leaf node, say ℓ_i , labeled by the symbol A which is replaced by β in that derivation step. That node ℓ_i is identified by the fact that γ is the left-to-right concatenation of the labels of the leaves to the left of ℓ_i in the derivation tree constructed so far.

If $k \geq 1$ then we generate k son nodes of the node ℓ_i and they will be labeled, from left to right, by c_1, \dots, c_k , respectively. (Obviously, after the generation of these k son nodes, the leaf node ℓ_i will no longer be a leaf node and will become an internal node of the new derivation tree.)

If $k = 0$ then we generate one son node of the node ℓ_i . The label of that new node will be the empty word ε .

We have that at the end of the application of Rule 2 the left-to-right concatenation of the labels of all the leaves of the resulting derivation tree T is the word w .

The word w is said to be the *yield* of the derivation tree T .

EXAMPLE 1.2.11. Let us consider the grammar with axiom S and whose productions are:

$$\begin{aligned} S &\rightarrow a A S \\ S &\rightarrow a \\ A &\rightarrow S b A \\ A &\rightarrow b a \\ A &\rightarrow S S \end{aligned}$$

Let us also consider the following derivation D of the word $a a b b a a$:

$$\begin{aligned} D: \quad \underline{S} &\rightarrow a \underline{A} S \rightarrow a \underline{S} b A S \rightarrow a a b \underline{A} S \rightarrow a a b b a \underline{S} \rightarrow a a b b a a \\ &\quad (1) \quad (2) \quad (3) \quad (4) \quad (5) \end{aligned}$$

where in each sentential form α_i we have underlined the nonterminal symbol which is replaced in the derivation step $\alpha_i \rightarrow \alpha_{i+1}$. The corresponding derivation tree is depicted in Figure 1.2.1. In the derivation D the numbers below the underlined nonterminal symbols denote the correspondence between the derivation steps and the nodes with the same number in the derivation tree of Figure 1.2.1. \square

Now let us introduce the following definitions.

DEFINITION 1.2.12. [**Unfold of a Context-Free Production**] Let us consider a context-free grammar $G = \langle V_T, V_N, P, S \rangle$. Let A and B be elements of V_N and $\alpha, \beta_1, \dots, \beta_n, \gamma$ be elements of $(V_T \cup V_N)^*$. Let $A \rightarrow \alpha B \gamma$ be a production in P , and $B \rightarrow \beta_1 \mid \dots \mid \beta_n$, for $n \geq 0$, be *all* the productions in P whose left hand side is B .

The *unfolding of B in $A \rightarrow \alpha B \gamma$ with respect to P* (or the *unfolding of B in $A \rightarrow \alpha B \gamma$ by using P* , or simply, the *unfolding of B in $A \rightarrow \alpha B \gamma$*) is the replacement of the production:

$$A \rightarrow \alpha B \gamma$$

by the n productions:

$$\begin{aligned} A &\rightarrow \alpha \beta_1 \gamma \\ &\dots \\ A &\rightarrow \alpha \beta_n \gamma. \end{aligned}$$

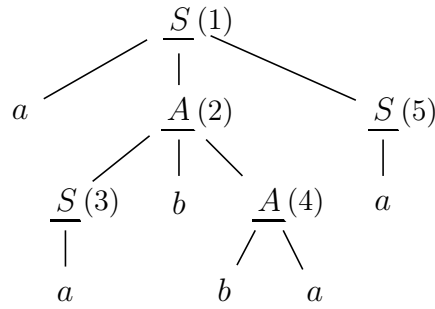


FIGURE 1.2.1. A derivation tree for the word $aabbaa$ and the grammar given in Example 1.2.11 on the facing page. This tree corresponds to the derivation $D: \underline{S} \rightarrow a\underline{A}S \rightarrow a\underline{S}bAS \rightarrow aab\underline{A}S \rightarrow aabba\underline{S} \rightarrow aabbaa$. The numbers, from 1 to 5, associated with the nonterminal symbols in the nodes of the tree, denote the correspondence between the nonterminal symbols and the five derivation steps of D .

DEFINITION 1.2.13. [Fold of a Context-Free Production] Let us consider a context-free grammar $G = \langle V_T, V_N, P, S \rangle$. Let A and B be elements of V_N and $\alpha, \beta_1, \dots, \beta_n, \gamma$ be elements of $(V_T \cup V_N)^*$. Let $A \rightarrow \alpha\beta_1\gamma \mid \dots \mid \alpha\beta_n\gamma$, for $n \geq 1$, be *some* productions in P whose left hand side is A , and $B \rightarrow \beta_1 \mid \dots \mid \beta_n$ be *all* the productions in P whose left hand side is B .

The *folding* of β_1, \dots, β_n in $A \rightarrow \alpha\beta_1\gamma \mid \dots \mid \alpha\beta_n\gamma$ with respect to P (or simply, the *folding* of β_1, \dots, β_n in $A \rightarrow \alpha\beta_1\gamma \mid \dots \mid \alpha\beta_n\gamma$) is the replacement of the n productions:

$$A \rightarrow \alpha\beta_1\gamma$$

...

$$A \rightarrow \alpha\beta_n\gamma$$

by the production:

$$A \rightarrow \alpha B\gamma.$$

DEFINITION 1.2.14. [Useful Symbol and Useless Symbol] Given a context-free grammar $G = \langle V_T, V_N, P, S \rangle$ a symbol $X \in V_T \cup V_N$ is *useful* iff $S \xrightarrow_G^* \alpha X \beta \xrightarrow_G^* w$ for some $\alpha, \beta \in (V_T \cup V_N)^*$ and $w \in V_T^*$. A symbol is *useless* iff it is not useful.

DEFINITION 1.2.15. [Reduced Grammar] A context-free grammar is said to be *reduced* if it has no useless symbols.

DEFINITION 1.2.16. [Nullable Symbol] Given a context-free grammar G , we say that a nonterminal symbol A is *nullable* iff $A \xrightarrow_G^* \varepsilon$.

DEFINITION 1.2.17. [Production for a Nonterminal Symbol] A production of the form $A \rightarrow \beta$, with $A \in V_N$ and $\beta \in (V_T \cup V_N)^*$, is said to be a *production for the nonterminal symbol* A .

DEFINITION 1.2.18. [Left Recursive Context-Free Production and Left Recursive Context-Free Grammar] Let us consider a context-free grammar $G =$

$\langle V_T, V_N, P, S \rangle$. We say that a production in P is *left recursive* if it is the form: $A \rightarrow A\alpha$ with $A \in V_N$ and $\alpha \in V^*$. A context-free grammar is said to be *left recursive* if there exists a nonterminal symbol A such that for some $\alpha, \beta, \gamma \in (V_T \cup V_N)^*$ we have that: (i) $S \rightarrow^* \alpha A \beta$, and (ii) $A \rightarrow^+ A \gamma$.

If a reduced, context-free grammar G has a left recursive production, then G is left recursive.

CHAPTER 2

Exploring Search Spaces

In this chapter we introduce the reader to some algorithms and techniques for exploring search spaces. In particular, we will consider:

- (i) the technique for exploring *linear search spaces* represented as arrays (or linear lists),
- (ii) the backtracking technique for exploring *general search spaces* represented as graphs, and
- (iii) the depth first and breadth first algorithms for exploring *tree-like search spaces* with the objective of finding leaves satisfying a given property.

The presentation of these algorithms can be viewed as an introduction to the parsing algorithms which we will consider in the following chapters. Indeed, we will see that every parsing algorithm performs the exploration of a search space. (Actually, every algorithm performs the exploration of a suitably defined search space.)

2.1. Exploring Linear Search Spaces

In this section we will indicate how to explore a linear search space represented as an array. As a typical example of this technique, now we present the following elementary program *Sum* for summing up the elements of a given array A of length $n \geq 1$.

We assume that the elements of the array are all integers. The array A can also be viewed as a finite function from the set $\{1, \dots, n\}$ to the set of integers.

Here is the Program *Sum*.

Program <i>Sum</i>	
$\{n \geq 1\}$: <i>precondition</i>
$i := 0; \text{ sum} := 0;$	establishing the invariant I .
$I \equiv \{\text{sum} = \sum_{j=1}^i A[j] \wedge i \leq n\}$: <i>invariant I</i>
while $i < n$ do $i := i + 1;$	working towards termination and
$\quad \text{sum} := \text{sum} + A[i]$	re-establishing the invariant I .
od	($I \wedge \neg(i < n)$) \rightarrow postcondition
$\{\text{sum} = \sum_{j=1}^n A[j]\}$: <i>postcondition</i>

Let us make the following comments of the structure of this program.

- (i) The program is made out of two initialization statements followed by a while-loop. As indicated by Kleene's Normal Form Theorem, the structure of this program is *universal*, in the sense that every Turing computable function can be computed via a unique while-loop (apart from a preliminary evaluation of a primitive recursive

function and then, at the end of the while-loop, a subsequent evaluation of a primitive recursive function).

(ii) The computation of the while-loop is characterized by the invariant I which holds before the execution of the while-loop, after every execution of the body of the while-loop, and at the end of the execution of the while-loop.

The reader may easily check that, indeed, the initialization statements establish the invariant I and, in our case, they do so in a trivial way because if $n \geq 1$, we have that $(0 = \sum_{j=1}^0 A[j]) \wedge 0 \leq n$.

In the body of the while-loop, once the value of i has been increased by 1, in order to re-establish the invariant we need to modify the value of the variable sum and to increase it by $A[i]$. Finally, when we exit from the while-loop and thus, it is no longer the case that $i < n$, the invariant should imply the postcondition $sum = \sum_{j=1}^n A[j]$, and indeed, this is the case.

(iii) Termination of the while-loop is guaranteed by the fact that at every execution of the body of the while-loop, i is increased by 1 and thus, sooner or later, it will be the case that $i < n$ does *not* hold and the execution of the while-loop terminates. Thus, as indicated by the usual approach to termination, there exists a function, namely $\lambda i. n-i$, which decreases at every execution of the body of the while-loop and it bounded from below by the value 0 because i cannot become larger than n .

Now, let us suppose that, instead of summing up all the elements of an array, we want to visit a given array A for finding out whether or not one of its elements satisfies a given property (or predicate) $prop : \Sigma \rightarrow \{true, false\}$, where Σ is the set from which the elements of the array A are taken. Thus, when viewing the array A of $n (\geq 1)$ elements as a finite function, we have that $A : \{1, \dots, n\} \rightarrow \Sigma$. In order to perform the visit of the array A we can use the following *Linear Search* program which is a simple variant of the *Sum* program on page 17.

$\{n \geq 1\}$ $i := 0; res := false;$ $I = \{res = \bigvee_{j=1}^i prop(A[j]) \wedge i \leq n\}$ $\mathbf{while} \ i < n \ \mathbf{do} \ i := i + 1;$ $\qquad \qquad \qquad res := res \vee A[i]$ $\qquad \qquad \qquad \mathbf{od}$ $\{res = \bigvee_{j=1}^n prop(A[j])\}$	Program for <i>Linear Search</i>
--	----------------------------------

Starting from this program *Linear Search*, now we derive a more efficient *Linear Search* program by applying some simple transformations. In particular, we may use the following properties:

- (i) $true \vee x = true$,
- (ii) $false \vee x = x$, and
- (iii) we may halt when we find an element $A[i]$ of A such that $prop(A[i])$ is *true*.

We derive the following program:

$$\begin{array}{l}
i := 0; \text{ res} := \text{false}; \\
\{ \text{false} = \bigvee_{j=1}^{i-1} \text{prop}(A[j]) \wedge (\text{if } i \geq 1 \text{ then } \text{res} = \text{prop}(A[i])) \wedge i \leq n \} \\
\mathbf{while} \ i < n \wedge \neg \text{res} \ \mathbf{do} \ i := i + 1; \\
\qquad \qquad \text{res} := \text{prop}(A[i]) \\
\mathbf{od}
\end{array}$$

By replacing i by a new identifier, also named i , which gets the value of the old i increased by 1, we derive the following program:

$$\begin{array}{l}
i := 1; \text{ res} := \text{false}; \\
\mathbf{while} \ i \leq n \wedge \neg \text{res} \ \mathbf{do} \ i := i + 1; \ \text{res} := \text{prop}(A[i-1]) \ \mathbf{od}
\end{array}$$

Now we can avoid the use of the index $i-1$ by anticipating, so to speak, the access to the array A . We swap the two instructions in the body of the while-loop and we get:

$$\begin{array}{l}
i := 1; \text{ res} := \text{false}; \\
\mathbf{while} \ i \leq n \wedge \neg \text{res} \ \mathbf{do} \ \text{res} := \text{prop}(A[i]); \ i := i + 1 \ \mathbf{od}
\end{array}$$

In order to avoid accessing the element $A[n+1]$ which is not defined, we may use the **and** operator and we get the following Program $P1$. Recall that ‘ a **and** b ’ is equivalent to ‘**if** a **then** b **else** false ’, and thus, b is not evaluated if a is false .

$$\begin{array}{l}
\{n \geq 1\} \qquad \qquad \qquad \text{Program } P1 \\
i := 1; \text{ res} := \text{false}; \\
\mathbf{while} \ i \leq n \ \mathbf{and} \ \neg \text{prop}(A[i]) \ \mathbf{do} \ i := i + 1 \ \mathbf{od}; \\
\mathbf{if} \ i \leq n \ \mathbf{then} \ \text{res} := \text{true} \\
\qquad \{ \text{res} = \bigvee_{j=1}^n \text{prop}(A[j]) \}
\end{array}$$

We can transform Program $P1$ by using conditional rewrite rules and recursion, and we get the following Program $P2$:

$$\begin{array}{l}
\{n \geq 1\} \qquad \qquad \qquad \text{Program } P2 \\
i := 1; \\
\text{search}(i): \left| \begin{array}{ll}
(1) \ i \leq n \wedge \neg \text{prop}(A[i]) & \rightarrow \text{search}(i+1) \\
(2) \ i \leq n \wedge \text{prop}(A[i]) & \rightarrow \text{res} := \text{true} \\
(3) \ i > n & \rightarrow \text{res} := \text{false}
\end{array} \right. \\
\{ \text{res} = \bigvee_{j=1}^n \text{prop}(A[j]) \}
\end{array}$$

We will not formally define here the operational semantics of the conditional rewrite rules which we have used in the above Program $P2$. It will suffice to say that, in order to perform $\text{search}(i)$, if it is the case that $i \leq n \wedge \neg \text{prop}(A[i])$ holds then Rule (1) tells us to perform $\text{search}(i+1)$, that is, to apply again one of the three rules either (1), or (2), or (3), having increased the value of i by one unit.

The equivalence between Program $P1$ and Program $P2$ is shown in the following Exercise 2.1.1.

EXERCISE 2.1.1. (i) Show that $P1$ terminates iff $P2$ terminates.

Hint. Both programs terminate iff $i > n \vee (\exists i \ 1 \leq i \leq n \wedge A[i] = true)$.

(ii) Show that if $P1$ terminates then ($P1$ sets res to the boolean value b iff $P2$ sets res to the same boolean value b).

Now we want to present a different kind of search of a linear search space. We call it *State Dependent Linear Search*. In this kind of search we want to visit an array, say A , whose elements are taken from a set Σ , for finding out whether or not one of its elements satisfies a given property (or predicate) $prop$, but this time the truth value of $prop$ depends on both: (i) a *state* q belonging to a set Q of states, and (ii) the element of the array which is currently visited.

The visit of the array A begins in the initial state $q_0 \in Q$ and any time a new element of the array is visited, we get to a new state which is computed by the so called *next-state function* $\delta : Q \times \Sigma \rightarrow Q$. The function δ depends on the previous state and the element of the array which is currently visited. Thus, we have:

- (i) an array A whose elements are taken from a set Σ ,
- (ii) an *initial state* $q_0 \in Q$,
- (iii) a *next-state function* $\delta : Q \times \Sigma \rightarrow Q$, and
- (iv) a property $prop : Q \times \Sigma \rightarrow \{true, false\}$.

We want a procedure which gives us the value *true* iff there exist a state q and an element a_i in the array A such that $prop(q, a_i) = true$.

Here is the program for the *State Dependent Linear Search*.

$\{n \geq 1\}$	Program for <i>State Dependent Linear Search</i>												
$i := 1;$ $q := q_0;$ $st-search(q, i):$ <table style="display: inline-table; vertical-align: middle; border-left: 1px solid black; border-right: 1px solid black; border-collapse: collapse;"> <tr> <td style="padding: 0 10px;">(1)</td> <td style="padding: 0 10px;">$i \leq n \wedge \neg prop(q, A[i])$</td> <td style="padding: 0 10px;">\rightarrow</td> <td style="padding: 0 10px;">$st-search(\delta(q, A[i]), i+1)$</td> </tr> <tr> <td style="padding: 0 10px;">(2)</td> <td style="padding: 0 10px;">$i \leq n \wedge prop(q, A[i])$</td> <td style="padding: 0 10px;">\rightarrow</td> <td style="padding: 0 10px;">$res := true$</td> </tr> <tr> <td style="padding: 0 10px;">(3)</td> <td style="padding: 0 10px;">$i > n$</td> <td style="padding: 0 10px;">\rightarrow</td> <td style="padding: 0 10px;">$res := false$</td> </tr> </table>		(1)	$i \leq n \wedge \neg prop(q, A[i])$	\rightarrow	$st-search(\delta(q, A[i]), i+1)$	(2)	$i \leq n \wedge prop(q, A[i])$	\rightarrow	$res := true$	(3)	$i > n$	\rightarrow	$res := false$
(1)	$i \leq n \wedge \neg prop(q, A[i])$	\rightarrow	$st-search(\delta(q, A[i]), i+1)$										
(2)	$i \leq n \wedge prop(q, A[i])$	\rightarrow	$res := true$										
(3)	$i > n$	\rightarrow	$res := false$										
$\{res = \bigvee_{j=1}^n prop(\delta(\dots \delta(\delta(q_0, A[1]), A[2]) \dots, A[j-1]), A[j])\}$													

By using the result of the above Exercise 2.1.1 which shows the equivalence between Program $P1$ and Program $P2$, we get the following program which is equivalent to the program for the *State Dependent Linear Search*:

$i := 1; q := q_0; res := false$ while $i \leq n$ and $\neg prop(q, A[i])$ do $q := \delta(q, A[i]); i := i+1$ od ; if $i \leq n$ then $res := true$
--

EXERCISE 2.1.2. Write a program to check whether or not in a given array there is an element which is equal to the sum of all the preceding elements.

Solution. Instantiating the above State Dependent Linear Search program, we get:

$i := 1; sum := 0; res := false$ while $i \leq n$ and $\neg (sum = A[i])$ do $sum := sum + A[i]; i := i+1$ od ; if $i \leq n$ then $res := true$
--

□

2.2. Backtracking Algorithms

Backtracking is a method for visiting all the nodes of a given tree starting from its root, when at each node we know:

- (i) the list of all the son nodes which are assumed to be ordered in a *linear order*, which is the left-to-right order of the elements of the list,
- (ii) a pointer p which identifies the node of that list which should be visited next, and
- (iii) the father node, if any.

When reasoning about our programs and also in our pictures below, we assume that the head of every list of nodes is ‘to the left’ of all the other elements of the list. When we visit a list of nodes, we first visit its head and then, recursively, we visit the tail of the list. In what follows we will also use the following terminology. Given a list of son nodes, (i) the head of the list is said to be *oldest son node*, and (ii) the node, if any, which is immediately to the right of the given node m in the list, is said to be the *immediately younger brother node* of m , or the *next node after m* .

The general idea of the backtracking algorithms can be explained as follows [6].

When the visit of a given node n of a tree is completed (and, thus, also the visits of all the descendant nodes of that node n are completed), the visit of the tree continues by performing the visit of the subtree, if any, which is rooted at the node which is the immediately younger brother node of n . If that immediately younger brother node of n does not exist, then the visit of the father node of the node n is completed.

Thus, by applying the backtracking technique, the visit of the tree search space is the *preorder visit*, that is, at each node of the tree we first visit that node and then, recursively, we make the preorder visit of each node in the list of the son nodes of that node. The son nodes of that list are visited according to the left-to-right order of the elements of the list.

The following program, called `Dispositions.java`, uses the backtracking technique, implemented as a recursion inside a do-while loop, for computing all the dispositions of length D whose elements are taken from the set $\{1, \dots, B\}$ of B elements. In particular, if $B = 2$, that program computes in increasing order the following sequence of tuples each of which is made out of D elements (thus, for instance, the first tuple has D occurrences of 1’s, the second tuple has $D - 1$ occurrences of 1’s and 1 occurrence of 2, and the last tuple has D occurrences of 2’s):

$$\begin{aligned} &\langle 1, 1, \dots, 1, 1 \rangle, \quad \langle 1, 1, \dots, 1, 2 \rangle, \quad \langle 1, 1, \dots, 2, 1 \rangle, \quad \langle 1, 1, \dots, 2, 2 \rangle, \quad \dots, \\ &\quad \dots, \\ &\langle 1, 2, \dots, 2, 2 \rangle, \quad \langle 2, 1, \dots, 1, 1 \rangle, \quad \dots, \quad \langle 2, 2, \dots, 1, 1 \rangle \quad \dots, \quad \langle 2, 2, \dots, 2, 2 \rangle \end{aligned}$$

The program `Dispositions.java` has the structure that we have indicated in Figure 2.2.1 on the following page (see also Figure 2.2.2 on page 23). In that program we have that:

- (i) the first local choice identifies the oldest son node of the given initial node,
- (ii) «to make the next local choice» is to increment the value of the *index of breadth* ib , that is, to move the pointer p one position to the right along the list of son nodes,
- (iii) «to go down the tree» is to increment by 1 the value of the *index of depth* id , that is, to begin the visit of a son node,

```

/* procedure generate() */
void generate();
begin
  select the first local choice;
  do
    ( $\alpha$ ): make the local choice;      ( $\beta$ ): down the tree (global value);
    if deepest level reached then do the leaf operation else generate()
    ( $\beta'$ ): up the tree (global value)  ( $\alpha'$ ): remove local choice;
    select the next local choice;
  while there is one more local choice to make;
end
/* main program */
id=0; generate();

```

FIGURE 2.2.1. Structure of a backtracking algorithm. The ‘local choice’ refers to the index of breadth ib and the ‘global value’ refers to the index of depth id .

(iv) «to go *up the tree*» is to decrement by 1 the value of id , that is, to go to the father node for beginning the visit of the immediately younger brother node, if any, and

(v) «to remove the local choice» is the step we have to make before making the next local choice (see the above Point (ii)).

By removing the local choice we complete the visit of the node, say n , which is our current local choice, and then, the next local choice, if any, can be made and the visit of the immediately younger brother node of n , if any, begins by executing one more time the body of the do-while loop.

For the index of breadth ib we have that: $1 \leq ib \leq B$, and for the index of depth id we have that: $0 \leq id \leq D$. The deepest level is reached when $id = D$.

Notice that «to make the local choice» does *not* increase the value of ib .

When the deepest level is reached, instead of going below a leaf node with $id = D$, we perform an operation associated with that leaf. That leaf operation, in general, depends on some global data structure which is manipulated while the visit of the tree progresses. In the case of the program `Dispositions.java` that global structure is the array `arrayDisp`.

The removal of the local choice, denoted by (α'), may not be necessary if the action of making the next local choice is performed *destructively*. This means that if the action of making a choice is done by assigning a value to a variable (as it is in our case, because we assign a value to the variable ib) then the action of making a new local choice automatically removes the old value of that variable (in our case the variable ib) and thus, automatically removes the local choice which was previously made. This is why in Figure 2.2.2 on the facing page there is no reference to the action (α').

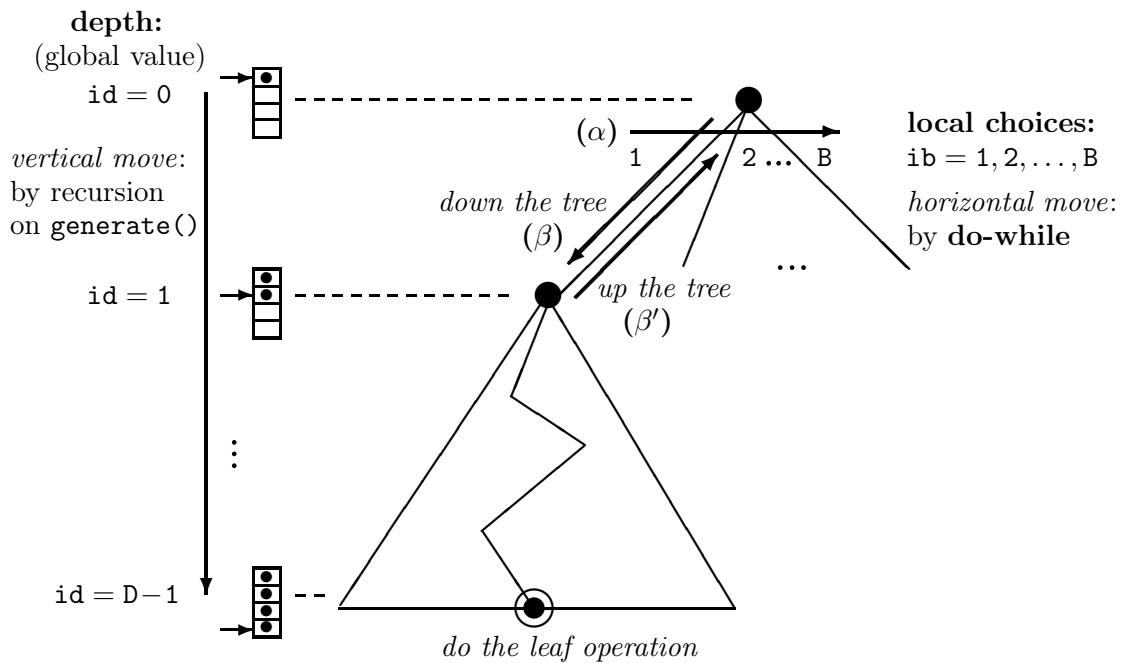


FIGURE 2.2.2. Visit of a tree of nodes by backtracking, starting from the topmost node. The dots (•) in the pictures of the array on the left represent the local choices which have been already made.

In the program `Dispositions.java` listed on page 24 we have that $D=3$ and $B=2$, and the dispositions of length 3 whose elements are taken from the set $\{1, \dots, B\}$, that is, $\{1, 2\}$, are the $8 (= B^D)$ sequences: 111, 112, 121, 122, 211, 212, 221, 222. When computing these sequences we get the tree of nodes depicted in Figure 2.2.3. The nodes of that tree are visited using the program `Dispositions.java`, starting from the root node a , in the following depth first order (which realizes the so called *preorder traversal* of the tree): $a, b, d, h, i, e, j, k, c, f, \ell, m, g, n, o$.

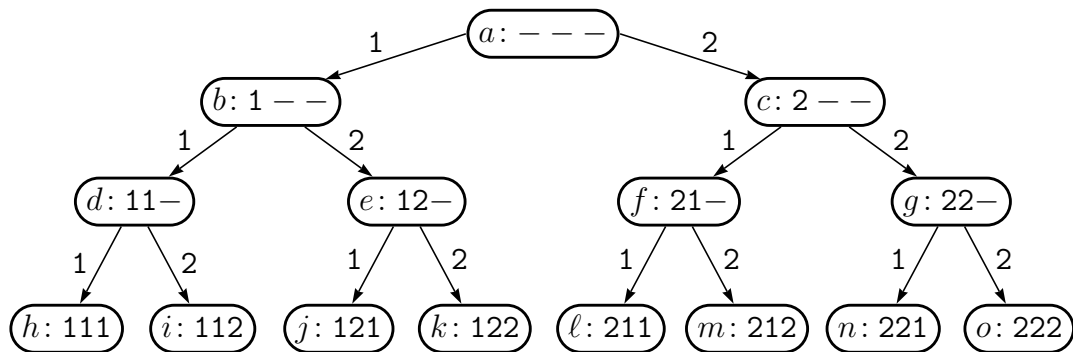


FIGURE 2.2.3. The tree of nodes when computing the 8 dispositions of length 3 of the elements 1 and 2, from 111 to 222. The dispositions are at the leaves.

```

/**
 * =====
 *                               Backtracking: DISPOSITIONS
 * Filename: "Dispositions.java"
 *
 * A disposition is a sequence of length D whose elements (not necessarily
 * distinct) are taken from a set of B elements.
 * =====
 */
public class Dispositions {
    private int id = 0;
    private final int D = 3;
    private final int B = 2;
    private int[] arrayDisp = new int[D]; // by default: initially the elements
                                         // of arrayDisp are all 0's
    private void printar(int[] array, int size) {
        for (int k=0; k<size; k++) {System.out.print(array[k]+" ");}
        System.out.println();
    }
    private void generate() {
        int ib=1;
        do {arrayDisp[id]=ib; id++;
            if (id==D) {printar(arrayDisp,D);} else generate();
            id--;
            ib++;}
        while (ib <= B);
    }
    public static void main(String[] args) {
        Dispositions disp = new Dispositions();
        disp.generate();
    }
}
/**
 * input:                               output:
 * -----
 * javac Dispositions.java
 * java Dispositions
 *
 *          1 1 1
 *          1 1 2
 *          1 2 1
 *          1 2 2
 *          2 1 1
 *          2 1 2
 *          2 2 1
 *          2 2 2
 * -----
 */

```

Now we present a program, called `Combinations.java`, for computing all the combinations of B (≥ 1) elements taken in groups of cardinality D , with $1 \leq D \leq B$. They are also called the *B-over-D combinations*. The number of B-over-D combinations is the value of the binomial coefficient $\binom{B}{D}$, that is, $\frac{B!}{(B-D)!D!}$.

The program `Combinations.java` is a variant of the program `Dispositions.java` (see page 24) in the sense that in this case the local choices that can be made, depend on some global information. Indeed, the computation of the B-over-D combinations can be viewed as the computation of the dispositions of length D whose elements are taken from the set $\{1, \dots, B\}$ with the following constraint:

« if an element has been already used for making a local choice at an ancestor node, then it cannot be used again for making a local choice at the current node. »

In order to recall all the choices made at the ancestor nodes, one can show that, since a combination of D elements is stored in the array `arrayComb` of D elements, it is enough to test whether or not `((id == 0) || (ib > arrayComb[id-1]))` holds. (We leave it to the reader the proof of this simple fact.) The identifier `ib`, with $1 \leq ib \leq B$, is the index of breadth and `id`, with $0 \leq id \leq D-1$, is the index of depth (see also Figure 2.2.2 on page 23). In our case, in order to recall all the choices made at the ancestor nodes, we could have also used a boolean vector.

```
/**
 * =====
 *                               Backtracking: COMBINATIONS
 * Filename: "Combinations.java"
 *
 * Combinations are dispositions, where the order of the elements
 * is NOT significant. In the dispositions the order is significant.
 *
 * The combinations of 'B over D' are the different SETS (not sequences) of
 * the B elements taken in groups of cardinality D.
 * The B elements are assumed to be: 1,...,B.
 * The combinations will be presented in lexicographic order, that is,
 * X_1,...,X_D comes before Y_1,...,Y_D iff there exists k, with 0<=k<D,
 * such that X_1 = Y_1, ..., X_k = Y_k, X_{k+1} < Y_{k+1}.
 *
 * In order to compute the combinations of 'B over D' we use a backtracking
 * algorithm which takes into account some global information to know
 * whether or not a given element can be used for making the current
 * local choice. That global information depends on previous local choices
 * made at the nodes which are ancestors of the node where the current
 * local choice is made.
 * =====
 */
public class Combinations {
    private int id = 0;
    private final int B = 4;           // B should be positive
    private final int D = 2;           // D should be positive and D <= B
    private int[] arrayComb = new int[D]; // by default: initially the elements
                                         // of arrayComb are all 0's

    private void printar(int[] array, int size) {
        for (int k=0; k<size; k++) {System.out.print(array[k]+" ");}
        System.out.println();
    }

    private void generate() {
        int ib=1;
        do {// ----- For id == 0 the first element is always ok.
            // ----- Combinations are generated in 'increasing order':
            //           this is due to the test (ib > arrayComb[id-1]) below.
            if ((id == 0) || (ib > arrayComb[id-1]))
                {arrayComb[id] = ib; id++;
                 if (id == D) {printar(arrayComb,D);} else generate();
                 id--;}
            ib++;}
        while (ib <= B);
    }
}
```

```

public static void main(String[] args) {
    Combinations comb = new Combinations();
    comb.generate();
}
}
/**
 * input:                output:
 * -----
 * javac Combinations.java
 * java  Combinations
 *                1 2
 *                1 3
 *                1 4
 *                2 3
 *                2 4
 *                3 4
 * -----
 */

```

Finally, we present a program, called `Queens.java`, for solving the n -queens problem [6]. This program implements a backtracking algorithm which can easily be understood because it is a variant of the algorithm we have presented in the program `Combinations.java` (see page 25). This variant is based on the fact that when making the local choices, we have to comply with the following constraint:

« when making any local choice, we have to determine whether or not the candidate local choice is *available*, that is, we have to determine whether or not the following global condition holds: the new queen to be placed on the board, is *not* under attack of any other queen already placed on the board. »

Attacks to a queen can come from a queen which is either in the same *column*, or in the same *up-diagonal*, or in the same *down-diagonal*. No attack to a queen can come from a queen on the same *row* simply because, by construction, every queen is placed on a different row (see Figure 2.2.4 on page 28). The columns taken by the queens on the board are identified by the value `false` in the array `col[0, ..., m-1]`. Analogously for the up-diagonals and the down-diagonals for which we use the boolean arrays `up[0, ..., 2(m-1)]` and `down[0, ..., 2(m-1)]`, respectively.

A new queen can be placed in column `k`, up-diagonal `u`, and down-diagonal `d` iff the corresponding elements of the arrays `col`, `up`, and `down`, that is, `col[k]`, `up[u]`, and `down[d]`, are all `true`. Once a queen is placed in that position, all those elements are set to `false`, meaning that that position is no longer available for a subsequent queen to be placed on the board. All the elements of the arrays `col`, `up`, and `down` are initialized to `true` meaning that all positions are available for the first queen.

In our program `Queens.java` which we now present, the number of queens is stored in the variables `nqueens` and `m`.

```

/**
 * =====
 *                               Backtracking: n QUEENS
 * * Filename: "Queens.java"
 * *
 * * From: E.W. Dijkstra: 'A Short Introduction to the Art of Programming'.
 * * EWD 316 August 1971. The command line for executing this program is:
 * * java Queens n (n should be at least 1)
 * * The queens are placed from top to bottom, one for each row.
 * * =====
 */

```



```

public class Queens {
    int nq = 0;          // index of queen to be placed in the current solution.
                        // See generate().
    int nqueens, nqueens1;
    int[] arrayQ;
    boolean[] col, up, down;

    // constructor of the class Queens
    Queens(int[] arrayQ, boolean[] col, boolean[] up,
           boolean[] down, int nqueens) {
        this.nqueens = nqueens;
        this.nqueens1 = nqueens-1;
        this.arrayQ = arrayQ;
        this.col = col;
        this.up = up;
        this.down = down;
    }
}
/**
 * -----
 * In Dijkstra: up is an array [-(nqueens-1)..nqueens-1] of booleans.
 * We have indicated the indexes he used in the comments below:// up[index]
 * We increased Dijkstra's indexes by nqueens1, which is nqueens-1.
 * -----
 */
private void printar(int[] arrayA, int size) {
    for (int k=0; k<size; k++) {System.out.print(arrayA[k]+" ");}
    System.out.println();}

private void generate()
{int h=0;
 do {if (!(col[h] || up[nq+h] || down[nq-h+nqueens1])) // up[nq-h]
     {arrayQ[nq]=h; col[h]=true; up[nq+h]=true; // up[nq-h]
     down[nq-h+nqueens1]=true; nq=nq+1;
     if (nq==nqueens) {printar(arrayQ,nqueens);} else generate();
     nq=nq-1;col[h]=false; up[nq+h]=false; // up[nq-h]
     down[nq-h+nqueens1]=false;};
  h=h+1;}
 while (h != nqueens);}

//-----
public static void main(String[] args) {

    int m=0; // m is the number of queens. See the field nqueens.
    try {m = Integer.parseInt(args[0]);
        if (m <= 0) {throw new Exception();}
    } catch (Exception e) {
        System.out.print("The number of queens should be >= 1.");
        System.out.println(" By default we assume 1 queen only.");
        m=1;
    }
    System.out.println("Number of queens = "+m);
    int[] arrayQ = new int[m]; // integer array [0..m-1]
    boolean[] col = new boolean[m]; // boolean array [0..m-1]
    boolean[] up = new boolean[2*m-1]; // boolean array [0..2(m-1)]
    boolean[] down = new boolean[2*m-1]; // boolean array [0..2(m-1)]

    Queens queens = new Queens(arrayQ, col, up, down, m);
    queens.generate();
}
}

```


2.3. Visiting Trees While Looking for Good Nodes

In this section we will present some algorithms for visiting trees with the objective of finding ‘good’ nodes, that is, nodes which satisfy a given property (or predicate) p . We assume that the trees are generated in a dynamic fashion, in the sense that any given tree is specified by: (i) its root node, and (ii) a function f which for each node of the tree, returns the list of its son nodes. Any such function f will be called a *tree-generating function*. We assume that the nodes of the trees we will consider, are of type α . We also assume that:

- (i) the predicate p is a function whose type is: $\alpha \rightarrow bool$, where $bool = \{true, false\}$, and
- (ii) the tree-generating function f is a function whose type is: $\alpha \rightarrow \alpha list$, where $\alpha list$ denotes the type of the lists whose elements are of type α .

In what follows the infix *append* function between two lists is denoted by $\langle \rangle$. Thus, $\langle \rangle$ has type: $(\alpha list) \times (\alpha list) \rightarrow (\alpha list)$.

2.3.1. Depth First Visit of Trees: Basic Version.

Now we present a function, called *existsev* (short for *exists eventually*), which given: (i) a predicate p , (ii) a tree-generating function f , and (iii) a list L of nodes, returns *true* if there exists a node n in L which is the root of the tree, say t_n , generated by the function f such that in t_n there exists a node m such that $p(m) = true$, otherwise it returns *false*.

The correctness of the algorithm we will give for evaluating the function *existsev* is based on the assumption that for any node n the tree which is rooted in n is finite. This finiteness assumption is required because given any node n , the tree rooted in n is generated and visited in a *depth first* manner (see the line marked with †).

If a node is the root of an infinite tree (because of the particular tree-generating function f which is given), then the evaluation of *existsev* may not terminate even if in that infinite tree there exists a node which satisfies p .

The *existsev* function is defined as follows. (In the first line we have given its *type* and in the other lines we have given its *definition*.)

```

existsev :  $(\alpha \rightarrow bool) \times (\alpha \rightarrow \alpha list) \times (\alpha list) \rightarrow bool$ 
existsev  $p f L =$  if  $L = []$  then false
                    else if  $p(hd(L))$  then true
                    else existsev  $p f (f(hd(L)) \langle \rangle tl(L))$       (†)

```

Let us consider a predicate p , a tree-generating function f , and a node n . Suppose that f generates a finite tree rooted in n . Then we have that there exists a node m in the tree rooted in n generated by f such that $p(m) = true$ iff $existsev(p, f, [n]) = true$. The inductive proof of this statement is left to the reader.

The following example illustrates the behaviour of the function *existsev* while performing the depth first visit of a tree of natural numbers (thus, in this example the type α is the type of the natural numbers.)

EXAMPLE 2.3.1. Let N denote the set of natural numbers. Let us consider the predicates $p : N \rightarrow \text{bool}$ and $\text{odd} : N \rightarrow \text{bool}$, and the tree-generating function f defined as follows:

$$\begin{aligned} p(x) &= x=25 \\ \text{odd}(0) &= \text{false} \\ \text{odd}(1) &= \text{true} \\ \text{odd}(n+1) &= \text{not}(\text{odd}(n)) \\ f(x) &= [x+4, x+5] \quad \text{if } x < 25 \text{ and } \text{odd}(x) \\ f(x) &= [x+5] \quad \text{if } x < 25 \text{ and } \text{not}(\text{odd}(n)) \\ f(x) &= [] \quad \text{otherwise} \end{aligned}$$

The search space is the tree of natural numbers depicted in Figure 2.3.1. In that figure we have also depicted the first six frontiers which are determined by the depth first visit of the tree, as we will explain below. The other frontiers are depicted on Figure 2.3.2 on the facing page.

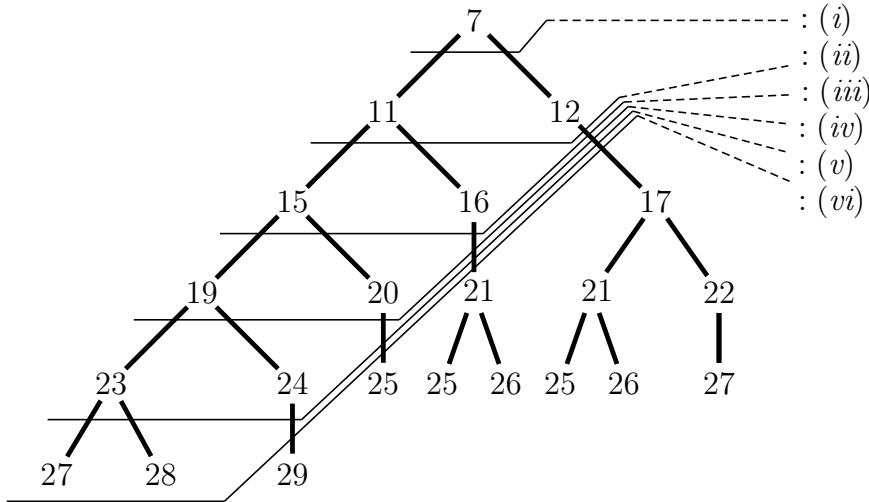


FIGURE 2.3.1. The tree generated from the initial node 7 by applying the tree-generating function f of Example 2.3.1. We have also depicted the first six frontiers of the depth first visit of the tree.

Every recursive call of the function existsev is of the form: $\text{existsev } p f l$, where the predicate p and the function f do not change, while the list l of natural numbers which is initially $[7]$ (because 7 is the root of the tree), changes as follows for each new recursive call (see also the frontiers depicted on Figure 2.3.1 and Figure 2.3.2 on the next page):

$$\begin{aligned} (i) : & [7] \\ (ii) : & [11, 12] \\ (iii) : & [15, 16, 12] \\ (iv) : & [19, 20, 16, 12] \\ (v) : & [23, 24, 20, 16, 12] \\ (vi) : & [27, 28, 24, 20, 16, 12] \end{aligned}$$

- (vii) : [28, 24, 20, 16, 12]
- (viii) : [24, 20, 16, 12]
- (ix) : [29, 20, 16, 12]
- (x) : [20, 16, 12]
- (xi) : [25, 16, 12]

Note that the head of the last list [25, 16, 12] is 25 and, since $p(25)$ is *true*, the execution of the function *existsev* stops. As the reader may verify, this sequence of eleven lists of nodes is the sequence of the *frontiers of the tree*, which are determined by the depth first visit of the tree with root 7. Indeed, in each successive call of the function *existsev*, the leftmost element of the list is replaced by its son nodes. In Figure 2.3.1 on the facing page we have depicted the first frontier [7] (the node 7 is the initial node), the second frontier [11, 12], the third frontier [15, 16, 12], . . . , until the sixth frontier which is [27, 28, 24, 20, 16, 12]. The other frontiers, from the seventh frontier which is [28, 24, 20, 16, 12], until the eleventh one which is [25, 16, 12], are shown in Figure 2.3.2. □

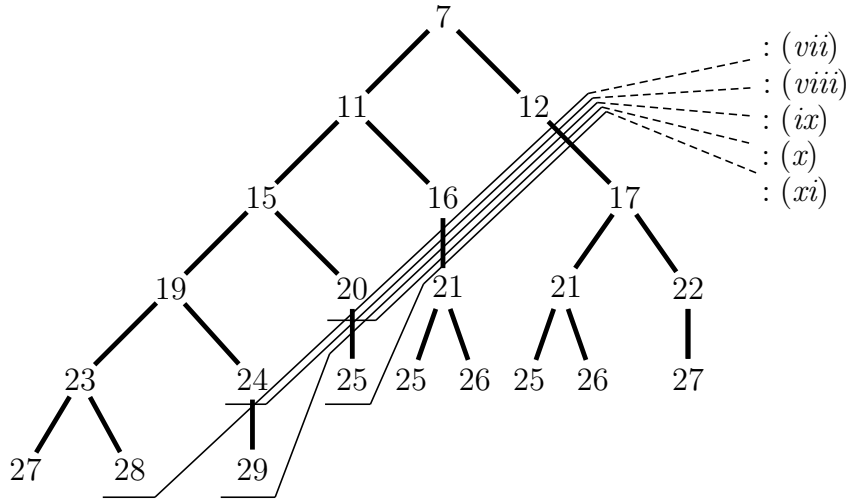


FIGURE 2.3.2. The frontiers, from the seventh one to the eleventh one, of the depth first visit of the tree generated from the initial node 7 by applying the tree-generating function f of Example 2.3.1 on the facing page.

2.3.2. Depth First Visit of Trees: Burstall's Version.

In this section we present a different algorithm for visiting trees in a depth first manner. It has been suggested to us by R. M. Burstall.

As in the previous Section 2.3.1, we are given a predicate p , a tree-generating function f , and a node n . Suppose that f generates a finite tree rooted in n . The following function $existsev1(p, f, n)$ returns *true* iff in the tree rooted in n and generated from n by the function f there exists a node m such that $p(m) = true$, otherwise it returns *false*.

As for the function *existsev* which is defined on page 29, the correctness of the *existsev1* function is based on the assumption that for any node n , the tree rooted

in n , is finite. This assumption is necessary because the tree rooted in n is generated and visited in a *depth first* manner (see the line marked with (††)).

Given a predicate p and a list L of nodes, the function $exists(p, L)$ returns *true* if in the list L there exists a node m such that $p(m)$ is *true*, otherwise it returns *false*.

The functions $existsev1$ and $exists$ are defined as follows.

```

existsev1 : ( $\alpha \rightarrow bool$ )  $\times$  ( $\alpha \rightarrow \alpha list$ )  $\times$   $\alpha \rightarrow bool$ 
existsev1  $p f x =$  if  $p(x)$  then true
                    else exists (existsev1  $p f$ )  $f(x)$            (††)

exists : ( $\alpha \rightarrow bool$ )  $\times$  ( $\alpha list$ )  $\rightarrow bool$ 
exists  $p L =$  if  $L = []$  then false
              else if  $p(hd(L))$  then true
              else exists  $p tl(L)$ 

```

Note that in the above function we use the partial application technique: the predicate which is the first argument of the function $exists$ in the line marked with (††), is the result of the application of the function $existsev1$ (which is of arity 3) to the predicate p and the function f . Thus, at line (††) the type of $existsev1 p f$ is $\alpha \rightarrow bool$, which is the type of a predicate on nodes, and the type of $f(x)$ is $\alpha list$, which is the type of a list of α 's.

NOTE 2.3.2. The functions $existsev$ and $existsev1$ are equivalent in the sense that for any given predicate p of type $\alpha \rightarrow bool$, any tree generating function f of type $\alpha \rightarrow \alpha list$, and any given initial node n of type α , we have that:

$$existsev p f [n] = existsev1 p f n. \quad \square$$

As we will see in more detail in Chapter 3, one can use this depth first visit algorithm for parsing strings (or words) generated by formal grammars. Here is an example where we consider the regular grammar G with axiom P and whose productions are:

$$P \rightarrow b \qquad P \rightarrow bQ \qquad Q \rightarrow a \qquad Q \rightarrow aQ$$

We want to check whether or not the string ba belongs to the language $L(G)$ generated by G . We can do so by constructing a tree whose root node is the pair $\langle P, ba \rangle$ of the axiom P and the *string-to-parse* ba . The nodes of that tree are constructed by applying at any node $\langle \sigma_1, \sigma_2 \rangle$, where σ_1 is a *sentential form* and σ_2 is a string of terminal symbols, either (i) the *chop* operation, or (ii) the *expand* operation, which we will describe below.

The construction of the tree finishes when a node of the form $\langle \varepsilon, \varepsilon \rangle$ is constructed, and this fact indicates that the string ba belongs to the language generated by G . Indeed, one can show that, in general, given a grammar H with axiom S , and a string σ , we have that $\sigma \in L(H)$ iff the node $\langle \varepsilon, \varepsilon \rangle$ belongs to the tree constructed from the root node $\langle S, \sigma \rangle$. Here is the description of the *chop* and *expand* operations.

(i) The *chop* operation is applied at a node of the form $\langle a\sigma_1, a\sigma_2 \rangle$, where a is a terminal symbol, and it constructs the single son node $\langle \sigma_1, \sigma_2 \rangle$. Thus, the *chop* operation consists in deleting the leftmost terminal symbol of both components of the pair in the node, if they are the same terminal symbol.

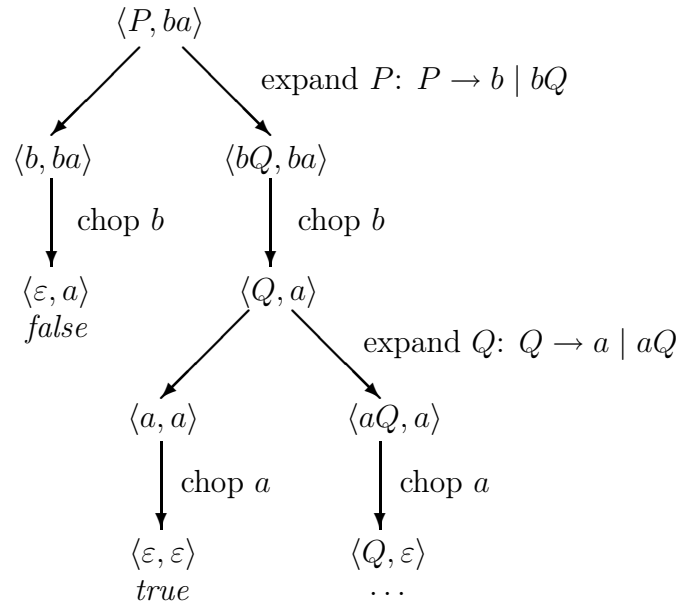


FIGURE 2.3.3. The tree of nodes which shows that the string ba is generated by the grammar G with axiom P and whose productions are: $P \rightarrow b \mid bQ$, $Q \rightarrow a \mid aQ$.

(ii) The *expand* operation is applied at a node of the form $\langle A\sigma_1, \sigma_2 \rangle$, where A is a nonterminal symbol, and it constructs the list of the son nodes of that node which is $[\langle \alpha_1\sigma_1, \sigma_2 \rangle, \dots, \langle \alpha_n\sigma_1, \sigma_2 \rangle]$, if $\alpha_1, \dots, \alpha_n$ are the right hand sides of all the productions of the given grammar whose left hand side is A . Thus, the *expand* operation is based on the replacement in all possible ways of the leftmost nonterminal symbol of the sentential form of the node. For instance, given the node $\langle P, ba \rangle$, the *expand* operation constructs the list $[\langle b, ba \rangle, \langle bQ, ba \rangle]$ of son nodes, because for the symbol P in the grammar G we have the two productions $P \rightarrow b$ and $P \rightarrow bQ$.

Figure 2.3.3 shows the tree with root $\langle P, ba \rangle$ which is generated by the *chop* and *expand* operations. The fact that in that tree there is the node $\langle \varepsilon, \varepsilon \rangle$ indicates that the string ba belongs to the language generated by G .

The construction of the tree which we have described above in terms of the *chop* and *expand* operations, can also be described in terms of a suitable tree-generating function f . We leave the definition of that tree-generating function to the reader. The predicate p which tells us when we have found the node $\langle \varepsilon, \varepsilon \rangle$, may be defined as follows: for any given node $\langle \sigma_1, \sigma_2 \rangle$, we have that $p(\langle \sigma_1, \sigma_2 \rangle) = \text{true}$ iff $\sigma_1 = \sigma_2 = \varepsilon$.

2.3.3. Breadth First Visit of Trees.

In this section we present a third algorithm for visiting trees. The visit is performed in a *breadth first* manner and thus, in order to ensure the correctness of this algorithm, we do not need the hypothesis that the tree generated from any given node by the given tree-generating function, is finite.

Given a function f with type $\alpha \rightarrow \alpha \text{list}$ and a list L of elements of type α , the function $\text{flatmap}(f, L)$ returns the concatenation of the lists produced by applying the function f to every element of L .

The function *exists* is the one we have presented on page 32.

The function *bf-existsev* (short for *breadth first exists eventually*) is very similar to the function *existsev* presented on page 29. They have the same type. However, in the case of the function *bf-existsev*, the tree is generated and visited in a breadth first manner (see the line marked with (†††)). Indeed, at each recursive call of the function *bf-existsev*, we construct the list of the son nodes of *all* nodes occurring in the list of the previous call of *bf-existsev*. Thus, the tree is generated level-by-level, by generating the nodes at distance $k+1$ from the root, only after the generation of all nodes at distance k , for any $k \geq 0$.

The functions *flatmap*, *exists*, and *bf-existsev* are defined as follows.

```

bf-existsev : ( $\alpha \rightarrow \text{bool}$ )  $\times$  ( $\alpha \rightarrow \alpha \text{ list}$ )  $\times$  ( $\alpha \text{ list}$ )  $\rightarrow \text{bool}$ 
bf-existsev p f L = if L = [] then false
                   else if exists p L then true
                   else bf-existsev p f (flatmap f L)           (†††)

exists : ( $\alpha \rightarrow \text{bool}$ )  $\times$  ( $\alpha \text{ list}$ )  $\rightarrow \text{bool}$ 
exists p L = if L = [] then false
              else if p(hd(L)) then true
              else exists p tl(L)

flatmap : ( $\alpha \rightarrow \alpha \text{ list}$ )  $\times$  ( $\alpha \text{ list}$ )  $\rightarrow$  ( $\alpha \text{ list}$ )
flatmap f L = if L = [] then []
               else (f(hd(L)) <> (flatmap f tl(L))

```

Note that the functions *existsev* and *bf-existsev* may not terminate if starting from a given node, the iterated application of the function *f* generates an infinite tree.

EXERCISE 2.3.3. Show that the following function definition of *existsev* is *not* correct:

```

existsev p f L = if exists p L then true
                  else existsev p f (f(hd(L)) <> tl(L))

```

Hint: Consider the case when $L = []$. □

EXERCISE 2.3.4. Show that the following function definition for *bf-existsev* is *not* correct:

```

bf-existsev p f L = if L = [] then false
                    else if p(hd(L)) then true
                    else bf-existsev p f (flatmap f L)

```

Hint: Some nodes are used for generating their son nodes, but for them we never test whether or not *p* holds. □

EXERCISE 2.3.5. Show that the following function definition for *bf-existsev* is *not* correct:

```

bf-existsev p f L = if exists p L then true
                    else bf-existsev p f (flatmap f L)

```

Hint: Consider the case when $L = []$. □

Chop-and-Expand Parsers for Context-Free Languages

In this chapter we illustrate the Chop-and-Expand parser for context-free languages in a functional language (similar to Standard ML [17]), an imperative language (Java 1.5), and a logic language (Prolog [5]). The idea of this parser is to perform a depth first visit of tree-like search space as indicated in Chapter 2. This parser works for context-free grammars which are *not left recursive*, that is, grammars in which no nonterminal symbol A exists such that: (i) $S \rightarrow^* \alpha A \beta$ for some $\alpha, \beta \in (V_T \cup V_N)^*$, and (ii) $A \rightarrow^+ A \gamma$ for some $\gamma \in (V_T \cup V_N)^*$ (see Definition 1.2.18 on page 15). If the given grammar is left recursive, the Chop-and Expand parser may not be able to explore via the depth first search the whole search space and, thus, parsing may not terminate.

The reader will find in the companion book [21] the description of two general algorithms which work for *any* context-free grammar (left recursive or not left recursive): (i) the Cocke-Younger-Kasami parser, and (ii) the Earley parser. We will not describe these algorithms here.

3.1. Chop-and-Expand Context-Free Parser in a Functional Language

As indicated in Section 2.3.2 on page 31, in order to solve a parsing problem in the case of context-free grammars, we have to explore a search space which is a tree, whose root node is made out of the pair $\langle S, \text{string-to-parse} \rangle$, where S is the axiom of the grammar and *string-to-parse* holds the input string to be parsed.

Below we present a program written in a functional language which performs the chop-and-expand parsing of a string generated by a context-free grammar $G = \langle V_T, V_N, P, S \rangle$. We use the recursive function *exists* which takes two arguments: (i) a list of *states*, also called *nodes*, and (ii) a context-free grammar which is not left recursive.

The list of states represents the current frontier of the tree which must be visited with the objective of finding in the tree a state of the form $\langle \varepsilon, \varepsilon \rangle$ (where ε is the empty string). The context-free grammar tells us how to construct the tree to be visited starting from its root, and indeed, by using that grammar we can construct, as we will see below, the son states of any given state by applying the function *expand*.

The portion of the tree which is strictly above the current frontier, has been already visited and no state of the form $\langle \varepsilon, \varepsilon \rangle$ exists in that portion of the tree.

Contrary to the function shown on page 32, the function *exists* presented here is a first order function (and *not* a higher-order function, because none of its arguments is a function or a predicate). This first order function *exists* has the advantage of allowing us to derive a chop-and-expand parser written in Java (see page 41) by a straightforward translation.

Now we indicate: (i) the conventions, (ii) the variables, and (iii) the data structures which are used in our first order function *exists*.

Any list of characters is written as a list and also as a string. For instance, $[b, c, c]$ is also written as bcc . Given a list ℓ , the head function $hd(\ell)$ and the tail function $tl(\ell)$ are defined as usual. Thus, we have that $hd(bcc) = b$ and $tl(bcc) = cc$. In particular, for any character c , the list $[c]$ made out of that character only, is also written as the string c . The empty string is denoted by ε . The type of the list of characters is called *char list*. It is also called *string*.

In what follows, when we write lists we will separate their members by either commas or blanks.

Pairs are denoted by using the angle brackets ' \langle ' and ' \rangle '. For instance, the pair of the strings b and cb is denoted by $\langle b, cb \rangle$.

The type of the lists of states is also called *state list*. The type of the lists of strings is also called *string list*. This type is also called *right-hand-sides* when the elements of the lists denote the right hand sides of all the productions for a given nonterminal symbol. For instance, given a context-free grammar whose productions for the nonterminal symbol P are $P \rightarrow b$ and $P \rightarrow aQQ$, the list $[b, aQQ]$ has type *right-hand-sides*.

Given a nonterminal symbol, say P , and the productions for it, say $P \rightarrow b$ and $P \rightarrow aQQ$, these productions can be grouped together and represented as the pair $\langle P, [b, aQQ] \rangle$ whose type is *symbol \times right-hand-sides*.

The type of the lists of *symbol \times right-hand-sides* pairs is called (*symbol \times right-hand-sides*) *list*. This type is also called *grammar* when the first components of the pairs of the list are all distinct symbols. For instance, given the context-free grammar whose productions are:

$$P \rightarrow b \quad P \rightarrow aQQ \quad Q \rightarrow \varepsilon \quad Q \rightarrow b \quad Q \rightarrow bP$$

it can be represented as the list of pairs $[\langle P, [b, aQQ] \rangle, \langle Q, [\varepsilon, b, bP] \rangle]$ which has type *grammar*. Each element in that list is a pair of a nonterminal symbol and the list of the right hand sides of all the productions for that nonterminal symbol.

In Table 1 on page 37 we show the syntactic categories of the identifiers occurring in the function *exists* and some examples of these syntactic categories. As usual, we write the terminal symbols and the nonterminal symbols using lower case and upper case letters, respectively.

The variable *sf* whose type is *char list* (also called *sentform* in Table 1), stores the value of a sentential form. A state, that is, a node of the tree, is a pair of a sentential form and a string of terminal symbols. We have that a descendant of a state $\langle \sigma_1, \sigma_2 \rangle$ is the state $\langle \varepsilon, \varepsilon \rangle$ iff the sentential form σ_1 generates the string σ_2 of terminal symbols.

The function *rhs* (short for *right hand sides*) takes in input a nonterminal symbol, that is, an element of type *char*, and produces in output the list of the right hand sides of the productions in the given grammar whose left hand side is the given nonterminal symbol. The type of the function *rhs* is: $char \rightarrow ((char\ list)\ list)$, which can also be written as: $char \rightarrow (string\ list)$ or as: $char \rightarrow right-hand-sides$.

Types of identifiers.	Examples. (a and b are terminals. P, Q, and R are nonterminals.)
• Sentential form (with terminal and nonterminal symbols). <i>sentform sf : char list</i>	PRb
• String (with terminal symbols only). <i>string s : char list</i>	bb
• State. <i>⟨sentform sf, string s⟩ : state</i>	⟨PRb, bb⟩
• List of states. <i>stl : state list</i>	[⟨PRb, bb⟩ ⟨bPQ, ab⟩]
• The right hand sides of the productions for the same nonterminal symbol. <i>rhs : char → (char list) list</i>	Given the productions $P \rightarrow b \mid aQQ$ we have that: $rhs(P) = [b, aQQ]$
• Grammar as a list of pairs. Each pair is made out of a nonterminal symbol and the list of all the right hand sides of the productions for that symbol. <i>gr : (symbol × right-hand-sides) list</i>	The grammar with the productions $P \rightarrow b \mid aQQ$ and $Q \rightarrow \varepsilon \mid b \mid bP$ is represented as: $[\langle P, [b, aQQ] \rangle \quad \langle Q, [\varepsilon, b, bP] \rangle]$

TABLE 1. Types and examples of the identifiers occurring in the function *exists*. The members of the lists of states and the members of the lists of productions are separated by blanks (not by commas).

Note that our representation of a context-free grammar as a list of $\langle symbol, right-hand-sides \rangle$ pairs makes it easy to evaluate the function *rhs*. Indeed, (i) no two pairs in the list which represents a grammar have the same nonterminal symbol as their first component, and (ii) for each element of the form $\langle A, Rs \rangle$ in the list which represents a grammar, we have that: $rhs(A) = Rs$.

Given a list of states which is the frontier of a tree to be visited, the function *expand* allows us to produce the new frontier of the expanded tree to be visited. This new frontier is obtained by replacing the *leftmost state* in the given list of states (that is, the head of the list of states) by its son states. These son states are generated according to the given context-free grammar by using the function *expand*, as we now indicate.

Let the frontier of the given tree be the state list $h : T$, where the state h is the pair of the form $\langle A\sigma, s \rangle$, for some nonterminal $A \in V_N$, string $\sigma \in (V_T \cup V_N)^*$, and string $s \in V_T^*$. Then the frontier of the new tree is:

$$[\langle \alpha_1\sigma, s \rangle, \dots, \langle \alpha_n\sigma, s \rangle] \ll T$$

where $A \rightarrow \alpha_1 \mid \dots \mid \alpha_n$ are *all* the productions of the given context-free grammar whose left hand side is A , and $\langle \rangle$ denotes the infix *append* operation on lists. The function *expand* has the following type:

$$\text{expand} : (\text{char list}) \text{ list} \times \text{char list} \times \text{char list} \rightarrow \text{state list}$$

that is, $\text{expand} : \text{right-hand-sides} \times \text{sentform} \times \text{string} \rightarrow \text{state list}$, and it is defined as follows

$$\text{expand}(\text{rhs}(A), \sigma, s) = [\langle \alpha_1 \sigma, s \rangle, \dots, \langle \alpha_n \sigma, s \rangle].$$

For instance, given the two productions $P \rightarrow \mathbf{b} \mid \mathbf{aQQ}$ and the state list *stl* which is $[\langle \mathbf{PRb}, \mathbf{bb} \rangle \quad \langle \mathbf{bPQ}, \mathbf{ab} \rangle]$, after the execution of the statement

$$\text{new-stl} = \text{expand}(\text{rhs}(P), \mathbf{Rb}, \mathbf{bb}) \langle \rangle \text{tl}(\text{stl})$$

we get that a new value for *new-stl* which is

$$[\langle \mathbf{bRb}, \mathbf{bb} \rangle \quad \langle \mathbf{aQQRb}, \mathbf{bb} \rangle \quad \langle \mathbf{bPQ}, \mathbf{ab} \rangle]$$

Here is the code of the function *exists* for the Chop-and-Expand parser for a context-free grammar which is not left recursive. *terminal(x)* and *nonterminal(x)* are predicates which return *true* iff x is a terminal or a nonterminal symbol, respectively. The function *exists* has the following type:

$$\text{exists} : \text{state list} \times \text{grammar} \rightarrow \text{bool}$$

where, as already mentioned, the type *grammar* is $(\text{symbol} \times \text{right-hand-sides}) \text{ list}$.

Chop-and-Expand Parser
<pre> exists stl gr = if stl=[] then false else let hd(stl)=⟨sf, s⟩ in if sf=ε and s=ε then true else if sf≠ε and nonterminal(hd(sf)) then (†) begin // EXPAND new-stl = expand(rhs(hd(sf)), tl(sf), s); exists new-stl gr end else if sf≠ε and s≠ε and terminal(hd(sf)) and hd(sf)=hd(s) then (‡) exists (⟨tl(sf), tl(s)⟩ : tl(stl)) gr // CHOP else exists tl(stl) gr </pre>

Note that in line (‡) above, the condition ‘and *terminal(hd(sf))*’ is not necessary. We have added it for reasons of symmetry with respect to the above line (†).

The initial call of the function *exists* is of the form:

$$\text{exists} [\langle S, \text{string-to-parse} \rangle] \text{ given-grammar}$$

where S is the axiom of the *given-grammar* and the first argument of the function *exists* is a list of states which consists of one state only, and that state is the pair $\langle S, \text{string-to-parse} \rangle$.

3.2. Chop-and-Expand Context-Free Parser in Java

In this section we list the two Java programs: `List.java` and `CFParser.java`, which realize the context-free parsing using the Chop-and-Expand algorithm illustrated in Section 2.3.2 on page 31 and Section 3.1 on page 35.

The C++ versions of these programs can be found in [19, page 69]. A similar version of the program `List.java` can be found (with the same name) in Section 7.4 on page 186.

```

/**
 * =====
 *          Generic Class List <T>      (class used for parsing)
 * Filename: "List.java"
 *
 * Every object of the class List<T> is a list of elements of type T.
 * The following methods are available:
 *
 *   cons(T d), head(), tail(), isSingleton(), isEmpty(), makeEmpty(),
 *   copy(), and listPrint().
 *
 * Also cloneCopy() is available if we uncomment line (***) below and
 * comment the previous line.
 * Note that tail() is destructive.
 * After a tail operation, if we need the original value of the list, we
 * should reconstruct it by 'consing' the head of the list.
 * =====
 */
import java.util.*;          // needed for using ArrayList<T> and Iterator

public class List<T> {      // it is ok if we do not use cloneCopy()

// public class List<T> implements Cloneable { // (***) In order to use
//     cloneCopy(), uncomment this line and comment the previous one.

    private ArrayList<T> list;

    public List() {          // constructor
        list = new ArrayList<T>();
    }

    public void cons(T datum) { // the head of the list is to the right,
        list.add(datum);      // not to the left.
    }

    public T head() {        // the head of the list is to the right.
        if (list.isEmpty()) // It is the last element: >-----+
            {System.out.println("Error: head of empty list!");};//      |
            T obj = list.get(list.size() - 1); // <-----+
        return obj;         // get(int i) is a method of ArrayList<T>
    }

    public void tail() {     // destructive tail() method
        if (list.isEmpty())
            {System.out.println("Error: tail of empty list!");};
        list.remove(list.size() - 1);
    }                        // size() is a method of ArrayList<T>
}

```

```

public boolean isSingleton() {
    return list.size() == 1;    // size() is a method of ArrayList<T>
}

public boolean isEmpty() {
    return list.isEmpty();    // isEmpty() is a method of ArrayList<T>
}

/**
 * ----- Examples of isEmpty() -----
 *
 * System.out.println(new List<Integer>().isEmpty());    prints true
 * System.out.println(new List().isEmpty());            prints true
 * List <Integer> v = new List();
 * System.out.println(v.isEmpty());                    prints true
 * System.out.println(v == null);                      prints false
 * -----
 */
public void makeEmpty() {
    list.clear();    // clear() is a method of ArrayList<T>
}

// ----- Printing a list -----
// We assume that an element of the list has a toString() method.
public void listPrint() {
    System.out.print("[ ");
    for (Iterator iter = list.iterator(); iter.hasNext(); ) {
        System.out.print((iter.next()).toString() + " ");
    };
    System.out.print("]");
}

// ----- Making a copy of a list without clone() -----
public List<T> copy() {
    List<T> copyList = new List<T>();
    for (Iterator<T> iter = list.iterator(); iter.hasNext(); ) {
        copyList.list.add(iter.next());
    };
    return copyList;
}

// ----- Making a copy of a list with clone() -----
public List<T> cloneCopy() {    // see (***) above
    try { List<T> copyList = (List<T>)super.clone();
        copyList.list = (ArrayList<T>)list.clone();
        return copyList;
    } catch (CloneNotSupportedException e) {
        throw new InternalError();
    }
}
}
}
/**
 * -----
 * If we have the program fragment:
 *
 * Integer k = new Integer(6);
 * List<Integer> la = new List<Integer>();
 * la.cons(k);

```

```

* System.out.print("\nla: "); la.listPrint();
* List<Integer> lb = new List<Integer>();
* la.cons(new Integer(4));
* la.cons(9); // Java 1.5: we can write: 9, instead of: new Integer(9)
* lb = la.cloneCopy();
* System.out.print("\nlb: "); lb.listPrint();
* System.out.print("\nla: "); la.listPrint();
*
* After compilation we get:
* Note: ./List.java uses unchecked or unsafe operations.
* Note: Recompile with -Xlint:unchecked for details.
*
* When running the program we get:
* la: [ 6 ]
* lb: [ 6 4 9 ]
* la: [ 6 4 9 ]
* -----
* Note: the head of the list is to the right.
* -----
*/
/**
* =====
*                CHOP-AND-EXPAND PARSER FOR CONTEXT FREE GRAMMARS
* Filename:"CFParser.java". Thanks to R. M. Burstall and E. W. Dijkstra.
*
* -----
*                --- TYPES ---
* List<char>  = String
* State      = <sentential_form, string>
* List<State> = list of states
* Rsides     = list of String : list of right hand sides
* SyRs       = <Sy,Rs> = <char:symbol, Rsides:Rs = its right hand sides>
* Gram       = List<SyRs> : list of SyRs pairs (type of a grammar)
* -----
*
* We use GENERIC TYPES to avoid the redefinition of the functions
* head(), tail(), cons(), isEmpty(), makeEmpty() for:
*
* - list of String (type: List<String>). It is the type of the list of all
*   right hand sides of a nonterminal. This type is also called: Rsides.
* - list of States (type: List<State>)
* - list of SyRs (type: List<SyRs>). It is the type of a grammar. It is
*   the type of the list of <nonterminal, list of its right hand sides>.
*   This type is also called: Gram.
* Redefinition of functions is necessary for printing and appending lists
* of different types.
* -----
* The input grammar is given as a string named 'gg' of type 'Gram',
* according to the following productions:
*
* character ::= 'a'..'d' | 'f'..'z' | 'A'..'Z'
*            ('e' is reserved: see below)
* charlist  ::= character | character charlist (that is, String)
* Rsides    ::= 'e' | charlist | charlist '|' Rsides
* SyRs      ::= char '>' Rsides
* grammar   ::= syRs | syRs ';' grammar
*
* The empty productions, also called epsilon-rules, are allowed.

```

```

* The character 'e' is reserved for denoting the right hand side of an
* empty production, that is, the empty charlist.
*
* For instance:
* gg="P>b|aQQ;Q>e|b|bP" stands for: P->b, P->aQQ, Q->e, Q->b, Q->bP
* In P>b|aQQ;Q>e|b|bP we have that:
*   \_/      aQQ is a string: "aQQ"
*   \___/    b|aQQ is a list of elements of type String, i.e., an Rside.
*           It is: ["b", "aQQ"]
*   \_____/ P>b|aQQ is a SyRs, i.e., a pair of a symbol and its
*           right hand sides. It is: <P, ["b", "aQQ"]>
* The axiom of the grammar is the nonterminal of the left hand side
* of the first production of the grammar.
*
* The string to parse is the string named 'stp', and
* 'stp' is either a non-empty sequence of characters from the set
* {'a',..., 'd', 'f',..., 'z'} or 'stp' is the empty sequence named 'e'.
*
* ----- ASSUMPTIONS ON THE INPUT -----
* (1) When writing the input grammar, the productions relative to the same
* nonterminal should be grouped together. For instance, we have to
* write "P>a|aQQ" (instead of "P>a;P>aQQ").
* (2) The input strings to parsed should be written according to the above
* rules we have stated for 'stp'.
*
* -----
* The grammar should not be left recursive, that is, it should not be the
* case that a nonterminal A exists such that A r ->* A s, for some strings
* r and s of terminal and/or nonterminal symbols of the grammar.
*
* -----
* The function 'parseString' is used both for:
* (1) getting one of the alternatives of the right hand sides of the
* productions, and
* (2) generating the list 'stringtoparse' of characters from the string
* 'stp'.
* In both cases, in fact, we look for a list of characters.
*
* -----
* The context-free parser works by exploring in a depth first fashion all
* possible words generated by the grammar and consistent with the
* character at hand in the string to parse.
*
* -----
* There is a global variable 'tracoon'. If it is set to 'true' then one
* can see the various steps through the execution of the program, and in
* particular, the sequence of the lists of states which are generated
* during the depth first visit of search space.
*
* -----
* After performing a tail() operation on a list we need to reconstruct
* the list which, otherwise, is modified (see statement marked by (**)).
* =====
*/

```

```
import java.util.*;
```

```
// -----
```



```

// string with an index pointing at the next available character.

class IndexString {
    public String string;
    public int index;

    public IndexString(String string) { // constructor
        this.string = string;
        this.index = 0;
    }

    public void back(){
        index--;
    }

    public char getCh() {
        char x;
        if (index < string.length()) { x = string.charAt(index); }
        else x = '$'; index++; return x;
    }
}
//-----
// state : <String:sentential_form, String:input_string>
class State {
    public String sentForm;
    public String inString;

    public State() {
        this.sentForm = "";
        this.inString = "";
    }

    // state : printing function
    // for reasons of readability the empty string is printed as "e"
    public void statePrint() {
        String sentForm1 = sentForm.equals("") ? "e" : sentForm;
        String inString1 = inString.equals("") ? "e" : inString;
        System.out.print("(" + sentForm1 + ", " + inString1 + ")");
    }

    // -----
    // List<State> : append function
    public static List<State> stAppend(List<State> list1,
                                       List<State> list2) {
        List<State> stl = new List<State>();
        if (list1.isEmpty()) {stl = list2.copy(); return stl;}
        else {State st = list1.head(); list1.tail();
              stl = stAppend(list1,list2); stl.cons(st);
              list1.cons(st); // reconstructing the list list1 (***)
              return stl;}
    }

    // -----
    // List<State> : printing functions
    public static void stlPrintNE(List<State> L) {
        if (!L.isEmpty())
            {State h = L.head(); h.statePrint(); L.tail();
              if (!L.isEmpty()) {System.out.print(" ");}; stlPrintNE(L);
              L.cons(h); //reconstructing the list L of states (***)
            };
    }
}

```

```

public static void stlPrint(List<State> L) {
    if (L.isEmpty()){System.out.print(" []");}
    else {System.out.print(" [""); stlPrintNE(L); System.out.print("]");}
}
}
// -----
// SyRs: <char:symb (left hand side), List<String>:rhss (right hand sides)>
class SyRs {
    public char symb;
    public List<String> rhss;

    public SyRs() {
        this.symb = '-';
        this.rhss = null;
    }

    // -----
    // List<String> = R sides:printing functions
    public static void rPrintNE(List<String> L) {
        if (!L.isEmpty()) {
            String h = L.head();
            if (h.equals("")) {System.out.print("e");}
            else {System.out.print(h);}
            L.tail();
            if (!L.isEmpty()) {System.out.print(" | ")};
            rPrintNE(L);
            L.cons(h);          // reconstructing the list L          (***)
        }
    }

    public static void rPrint(List<String> L) {
        if (L.isEmpty()) {System.out.print("<>");}
        else {System.out.print(" "); rPrintNE(L); System.out.print(" ");}
    }

    // -----
    // SyRs : printing function
    public static void SyRsPrint(SyRs r) {
        System.out.print(r.symb + " -> "); rPrint(r.rhss);
    }
}
// -----
// Gram : List<SyRs> (it is the type of a grammar)

class Gram {
    public List<SyRs> gg;

    public Gram() {
        this.gg = new List<SyRs>();
    }

    // -----
    // List<SyRs> = Gram: printing functions
    public static void printGramNE(List<SyRs> L) {
        if (!L.isEmpty())
            {SyRs h = L.head(); SyRs.SyRsPrint(h);
            System.out.print("\n"); L.tail();
            if (!L.isEmpty()) {System.out.print(" ");}; printGramNE(L);
            L.cons(h);
        }
    }
}

```

```

    public static void printGram(List<SyRs> L) {
        if (L.isEmpty()) {System.out.print("{ }");}
        else {System.out.print("{ "); printGramNE(L); System.out.print("}");}
    }
}

// -----
public class CFParse {
//-----

//          Making the grammar from the string gg
public static String parseString(IndexString f) {
    char x = f.getCh();
    String chl;
    if (('a'<=x && x<='z')||('A'<=x && x<='Z'))
        { chl = x + parseString(f); }
    else { chl = ""; };
    return chl;
}

// -----
public static List<String> parseRsides(IndexString f) {
    List<String> rs = new List<String>();
    String chl2 = parseString(f); f.back();
    // this is the case of the epsilon production
    if (chl2.equals("e")) {chl2 = "";}
    char x = f.getCh();
    // x can be either ',' or '$' or '|'. No other characters are possible.
    if ((x==',' || (x=='$')) {rs.cons(chl2); return rs; }
    else { // here we have: x=='|'
        rs = parseRsides(f); rs.cons(chl2); return rs; }
}

// -----
public static SyRs parseprod(IndexString f) {
    SyRs sRs1 = new SyRs(); sRs1.symb = f.getCh();
    char x = f.getCh();
    if (x!='>') {System.out.print("Error in production.\n");};
    List<String> rs1 = new List<String>();
    rs1 = parseRsides(f); sRs1.rhss=rs1; return sRs1;
}

// -----
public static List<SyRs> parseGram(IndexString f)
                                throws RuntimeException {
    SyRs sRs1 = parseprod(f); f.back();
    char x = f.getCh();
    // x can be either ',' or '$'. No other characters are possible.
    if (x=='$') { List<SyRs> gram1 = new List<SyRs>(); gram1.cons(sRs1);
        return gram1; }
    else if (x==',' ) {List<SyRs> gram2 = new List<SyRs>();
        gram2 = parseGram(f); gram2.cons(sRs1);
        return gram2; }
    else { throw new RuntimeException("Error parseGram!"); }
}

//-----

```

```

//          trace function

public static void trace(boolean traceon, String s, List<State> stl) {
    if (traceon)
        {System.out.print("\n" + s + ":\n"); State.stlPrint(stl);}
    }
//-----
// visiting the search space: constructing new lists of states

public static List<String> rightSides(char s, List<SyRs> gr)
    throws RuntimeException {
    if (gr.isEmpty())
        { throw new RuntimeException ("No production for "+ s +
            " in the grammar.\n");}
    else {SyRs sR = new SyRs(); sR = gr.head();
        if (sR.symb == s) { return sR.rhss; }
        else {List<String> rs = new List<String>();
            gr.tail(); rs = rightSides(s,gr);
            gr.cons(sR);          //reconstructing the grammar gr (***)
            return rs; }
        }
    }

//-----
//          final function

public static boolean isFinal(State st) {
    return (st.sentForm.equals("")) && (st.inString.equals(""));
}

//-----
//          chop function

public static void chop(State st) { // st : the state st is modified
    st.sentForm = st.sentForm.substring(1, st.sentForm.length());
    st.inString = st.inString.substring(1, st.inString.length());
}

//-----
//          expand function

public static List<State> expand(List<String> rs, String v, String w) {
    State st = new State();
    List<State> stl = new List<State>();
    if (rs.isEmpty()) { stl.makeEmpty(); return stl; }
    else {String cs = rs.head();
        rs.tail(); stl = expand(rs,v,w);
        rs.cons(cs);          // reconstructing the list rs (***)
        st.sentForm = cs.concat(v);
        st.inString = w;
        stl.cons(st); return stl; }
    }

//-----
//          exists function
//
// This code is directly derived from the code of the exists function
// in functional style.
//

```

```

private static boolean exists (boolean traceon, List<State> stl1,
                               List<SyRs> gr) {
    if (stl1.isEmpty()) { return false; }
    // depth first visit of the search space
    else {State st1 = new State(); st1 = stl1.head();
          String st1sentForm = st1.sentForm;
          String st1inString = st1.inString;
          if (isFinal(st1)) { return true; }
          else if (!(st1sentForm.equals("")) &&
                   ('A'<=st1sentForm.charAt(0) && (st1sentForm.charAt(0)<='Z')))
            {List<State> stl2 = new List<State>();
              List<String> rs = new List<String>();
              stl1.tail();
              rs = rightSides(st1sentForm.charAt(0),gr);
              // first character chopped
              st1sentForm = st1sentForm.substring(1, st1sentForm.length());
              stl2 = expand(rs,st1sentForm,st1inString);
              stl1 = State.stAppend(stl2,stl1);
              trace(traceon,"expand",stl1); // tracing
              return exists(traceon,stl1,gr);}
          else if (!(st1sentForm.equals("")) && !(st1inString.equals(""))
                   && (((('a'<=st1sentForm.charAt(0) && (st1sentForm.charAt(0)<='d')) ||
                          (('f'<=st1sentForm.charAt(0) && (st1sentForm.charAt(0)<='z'))))
                       && (st1sentForm.charAt(0) == st1inString.charAt(0)))
            {chop(st1); stl1.tail(); stl1.cons(st1);
              trace(traceon,"chop",stl1); // tracing
              return exists(traceon,stl1,gr);}
          else {stl1.tail();
                trace(traceon,"fail",stl1); // tracing
                return exists(traceon,stl1,gr);}
    }
}
//-----
// main program

public static void main(String[] args) {
System.out.print("1: " + new List().isEmpty());
System.out.print("2: " + new List<Integer>().isEmpty());

/** -----
 * gg : given-grammar
 * stp : string-to-parse
 * -----
 */

String gg ="P>b|aQQ;Q>e|b|bP"; // example 0 (with an empty production)
String stp="ab"; // true
boolean traceon = true; // variable for tracing the execution

//-----
// From the string gg to the List<SyRs> gr0.
// We first construct the IndexString f from the String gg.
// gr0 is the grammar encoded by the given string gg.
//-----

IndexString f = new IndexString(gg);
List<SyRs> gr0 = new List<SyRs>();

```

```

char axiom = gg.charAt(0);
if (('A'<=axiom) && (axiom<='Z'))
    {System.out.print("\nThe input grammar is ");
    System.out.print("\n\"e\" denotes the empty string in the rhs):\n");
    gr0 = parseGram(f);

/**
 * -----
 * In order to avoid parsing, the above statement 'gr0 = parseGram(f);' can
 * be replaced by the following statements (erase the *'s, please):
 *
 * SyRs Prs = new SyRs(); Prs.symb='P';           // P
 * String chl1P = "b";                           // b
 * String chl2P = "aQQ";                         // |aQQ
 * List<String> rsP = new List<String>();
 *          rsP.cons(chl2P); rsP.cons(chl1P);     // b|aQQ
 * Prs.rhss = rsP;                               // P>b|aQQ
 *
 * SyRs Qrs = new SyRs(); Qrs.symb='Q';         // Q
 * String chl1Q = "";                            // e
 * String chl2Q = "b";                          // |b
 * String chl3Q = "bP";                         // |bP
 * List<String> rsQ = new List<String>();
 *          rsQ.cons(chl3Q); rsQ.cons(chl2Q); rsQ.cons(chl1Q);
 *          // e|b|bP
 * Qrs.rhss = rsQ;
 *          //gr0 corresponding to gg="P>b|aQQ;Q>e|b|bP"
 * gr0.cons(Qrs); gr0.cons(Prs);
 * -----
 */
    Gram.printGram(gr0); }
else System.out.print("Not a grammar in input.");

System.out.print("\n\nThe axiom is " + axiom + ".\n");

State st = new State();
st.sentForm = axiom + ""; // from char to String (of length 1).
st.inString = stp;
List<State> stl = new List<State>();
stl.cons(st);

System.out.print("The initial list of states is:\n");
State.stlPrint(stl);
boolean ans = exists(traceon,stl,gr0);

System.out.print("\n\nThe following string of characters:\n ");
System.out.print(stp + "\nis ");
if (!ans) {System.out.print("NOT ");};
System.out.print("generated by the given grammar.\n");
} // end of main
}

/**
 * When printing the statelist the head is to the left.
 * input:   output:
 * -----
 * javac CFPARSER.java
 * java CFPARSER

```

```

*
*   The input grammar is ("e" denotes the empty string in the rhs):
*   { P -> b | aQQ
*     Q -> e | b | bP
*   }
*
*   The axiom is P.
*   The initial statelist is:
*   [(P, ab)]
*   expand:
*   [(b, ab) (aQQ, ab)]
*   fail:
*   [(aQQ, ab)]
*   chop:
*   [(QQ, b)
*   expand:
*   [(Q, b) (bQ, b) (bPQ, b)]
*   expand:
*   [(e, b) (b, b) (bP, b) (bQ, b) (bPQ, b)]
*   fail:
*   [(b, b) (bP, b) (bQ, b) (bPQ, b)]
*   chop:
*   [(e, e) (bP, b) (bQ, b) (bPQ, b)]
*
*   The following string of characters:
*   ab
*   is generated by the given grammar.
* -----
*/
/**
* Examples already prepared:
* String gg="P>b|aQQ;Q>e|b|bP"; // example 0 - with empty production
* String stp="a"; // true
* String stp="abaa"; // false
* String stp="aba"; // true
*
* String gg="P>a|aQ;Q>b|bP"; // example 1
* String stp="ab"; // true
* String stp="ababa"; // true
* String stp="aaba"; // false
*
* String gg="P>a|bQ;Q>b|aQ"; // example 2
* String stp="baab"; // true
* String stp="bbaaba"; // false
*
* String gg="S>aB|bA;A>bAA|a|aS;B>aBB|b|bS"; // same numbers of a's and b's
* String stp="bbabaaab"; // true
* String stp="babaaab"; // false
*
* String gg="S>a|aS;A>SbA|ba|SS"; // example 2.7 of Hopcroft-Ullman
* String stp="aabaaa"; // true
* String stp="aabba"; // false
* String stp="aabbaa"; // true
* -----
*/

```

3.3. Chop-and-Expand Context-Free Parser in a Logic Language

Here is a logic program which realizes a Chop-and-Expand Parser.

Chop-and-Expand Parser

1. $parse(G, [], []) \leftarrow$
2. $parse(G, [A|X], [A|Y]) \leftarrow terminal(A), parse(G, X, Y) \quad // \text{ CHOP}$
3. $parse(G, [A|X], Y) \leftarrow nonterminal(A), member(A \rightarrow B, G), \quad // \text{ EXPAND}$
 $\quad\quad\quad append(B, X, Z), parse(G, Z, Y)$
4. $member(A, [A|X]) \leftarrow$
5. $member(A, [B|X]) \leftarrow member(A, X)$
6. $append([], L, L) \leftarrow$
7. $append([A|X], Y, [A|Z]) \leftarrow append(X, Y, Z)$

Together with these seven clauses we also need the clauses which define the terminal symbols and the nonterminal symbols. Recall that in logic programming $[A|X]$ represents the list whose head is the element A and whose tail is the list X . In particular, if $[A|X] = [2, 3, 1, 1]$ then $A = 2$ and $X = [3, 1, 1]$. All constants in Prolog should begin with a lower case letter, so that, in particular, the axiom S is represented by the character s .

The first argument of $parse$ is a context-free grammar, the second argument is a list of terminal symbols and nonterminal symbols (that is, a sentential form), and the third argument is a word represented as a list of terminal symbols. We assume that context-free grammars are represented as lists of productions of the form $x \rightarrow y$, where x is a nonterminal symbol and y is a list of terminal and nonterminal symbols.

We have that $parse(G, [s], W)$ holds iff from the symbol s we can derive the word W using the grammar G , that is, W is a word of the language generated by G .

EXAMPLE 3.3.1. Let us consider the context-free grammar $G = \langle \{a, b\}, \{S\}, \{S \rightarrow aSb, S \rightarrow ab\}, S \rangle$. It is represented by the clauses:

```
terminal(a) ←
terminal(b) ←
nonterminal(s) ←
```

together with the list $[s \rightarrow [a, s, b], s \rightarrow [a, b]]$ which represents its productions. The left hand side of the first production is assumed to be the start symbol s . For this grammar G the query $\leftarrow parse([s \rightarrow [a, s, b], s \rightarrow [a, b]], [s], [a, a, b, b])$ evaluates to true, and this means that the word $aabb$ belongs to the language generated by G .

Note that in the clause $member(A, [B|X]) \leftarrow member(A, X)$, we do not require that A is different from B . Indeed with this clause, the query of the form $\leftarrow member(A, l)$, where A is a variable and l is a ground list, generates by backtracking all members of the list l with their multiplicity. \square

More examples of Prolog programs for parsing words generated by grammars can be found in Section 10.1 on page 245.

Parsers for Deterministic Context-Free Languages: LL(k) Parsers

In this chapter we will present the so called $LL(k)$ parsers for deterministic context-free languages. We assume that the reader is familiar with the basic notions of context-free languages and context-free grammars which can be found in classical books such as [9, 10]. In particular, we assume that the reader knows the notions of *unfolding* and *folding* of context-free productions we have given in Definitions 1.2.12 (on page 14) and 1.2.13 (on page 15).

Unless otherwise specified, V_T , V_N , and S denote respectively, the set of terminal symbols, the set of nonterminal symbols, and the start symbol of the grammars we will consider. By V we denote the set $V_T \cup V_N$.

4.1. Introduction to LL(k) Parsing

The $LL(k)$ parsers are algorithms for parsing the languages which are generated by the $LL(k)$ grammars (see Definition 4.1.1 below). In those grammars we assume the following hypotheses.

Hypotheses for $LL(k)$ Parsing, for $k \geq 0$.

- (i) No useless symbols occur in the productions.
- (ii) ε -productions may be present for the start symbol or other symbols of V_N .
- (iii) The symbol \$ does not belong to $V_T \cup V_N$.
- (iv) The input string in V_T^* to be parsed is terminated by \$.

We begin by giving the formal definition of an $LL(k)$ grammar, for any $k \geq 0$. Recall that, as indicated on page 10, given a string w and a natural number $k \geq 0$, the prefix \underline{w}_k of length k of the string w is defined as follows:

$$\underline{w}_k = \begin{cases} w & \text{if } |w| \leq k \\ u & \text{if } w = uv \text{ for some } u, v \in V^* \text{ and } |u| = k. \end{cases}$$

DEFINITION 4.1.1. [$LL(k)$ Grammar] For any $k \geq 0$, a context-free grammar (possibly with ε -productions) is an $LL(k)$ grammar if for any $x, y, z \in V_T^*$ and for any $\alpha, \beta, \gamma \in (V_N \cup V_T)^*$, we have that:

- if (1) $S \xrightarrow{*}_{lm} xA\alpha \rightarrow_{lm} x\beta\alpha \xrightarrow{*}_{lm} xy$ and
- (2) $S \xrightarrow{*}_{lm} xA\alpha \rightarrow_{lm} x\gamma\alpha \xrightarrow{*}_{lm} xz$ and
- (3) $\underline{y}_k = \underline{z}_k$

then $\beta = \gamma$.

Note that (1) and (2) are two leftmost derivations of two (possibly different) words xy and xz .

DEFINITION 4.1.2. [$LL(k)$ Language] For any $k \geq 0$, an $LL(k)$ language is a language such that there exists an $LL(k)$ grammar which generates it.

In what follows we will also need the following definition of a strong $LL(k)$ grammar.

DEFINITION 4.1.3. [Strong $LL(k)$ Grammar] For any $k \geq 0$, a context-free grammar (possibly with ε -productions) is a *strong* $LL(k)$ grammar if for any $x_1, x_2, y, z \in V_T^*$ and for any $\alpha_1, \alpha_2, \beta, \gamma \in (V_N \cup V_T)^*$ we have that:

$$\begin{aligned} \text{if } (1) \quad & S \xrightarrow{*}_{lm} x_1 A \alpha_1 \rightarrow_{lm} x_1 \beta \alpha_1 \xrightarrow{*}_{lm} x_1 y \quad \text{and} \\ (2) \quad & S \xrightarrow{*}_{lm} x_2 A \alpha_2 \rightarrow_{lm} x_2 \gamma \alpha_2 \xrightarrow{*}_{lm} x_2 z \quad \text{and} \\ (3) \quad & \underline{y}_k = \underline{z}_k \\ \text{then} \quad & \beta = \gamma. \end{aligned}$$

Note that (1) and (2) are two leftmost derivations of two (possibly different) words $x_1 y$ and $x_2 z$.

One can show the following fact (see also Fact 4.3.15 on page 73).

FACT 4.1.4. [$LL(0)$ Grammar and $LL(0)$ Language] Any $LL(0)$ grammar or strong $LL(0)$ grammar generates a language $L \subseteq V_T^*$ which is either the empty set of words or it consists of one word only.

As a consequence of this fact, Fact 4.2.11 on page 58, and Theorem 4.3.17 on page 73, we have that the parsing problem when we deal with $LL(0)$ grammars, or strong $LL(0)$ grammars, or $LL(0)$ languages, is trivial. Thus, unless otherwise specified, when referring to $LL(k)$ grammars (or languages) or strong $LL(k)$ grammars (or languages), we will assume that k is greater than or equal to 1.

An $LL(k)$ parser, also called a *k-predictive parsing algorithm*, is a deterministic pushdown automaton (see Figure 4.1.1 on the facing page) which once initialized, performs its moves according to a table T , called a *parsing table*, as we will indicate.

If the string to parse is the string w , the string given in input to the parsing pushdown automaton is $w\$$. The initial configuration of the stack is the string: $S\$$ and S is at the top of the stack. Recall that, unless otherwise specified, we assume that the stack is represented as a string with the top symbol on the left.

DEFINITION 4.1.5. [Words Accepted by $LL(k)$ Parsers] A word w is accepted by an $LL(k)$ parser, for $k \geq 1$, if the parser, starting from the configuration in which: (i) the input tape holds the word $w\$$, (ii) the head of the input tape (or input head) points at the leftmost symbol of w , and (iii) the stack holds $S\$$, being S the top of the stack, eventually reaches the configuration where: (i) the head of the input tape points at $\$$, and (ii) the top of the stack is $\$$ and no other symbols are on the stack.

Note that when presenting the $LL(k)$ parsers, some textbooks use different conventions. For instance, some authors do *not* use the symbol $\$$ at the bottom of the stack and do *not* add the symbol $\$$ at the right end of the input string.

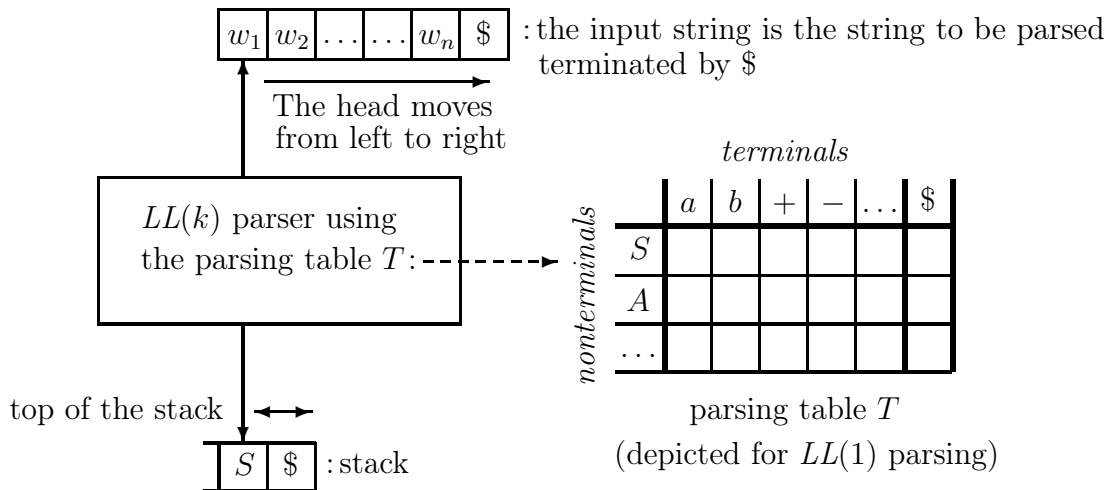


FIGURE 4.1.1. A deterministic pushdown automaton for $LL(k)$ parsing, with $k \geq 1$. The string to be parsed is $w_1 w_2 \dots w_n$. Initially, the stack has two symbols only: (i) S on top of the stack, and (ii) $\$$ at the bottom of the stack. The input string is the string to be parsed with the extra final symbol $\$$. We have depicted the parsing table T for the $LL(1)$ parsers. For the $LL(k)$ parsers, with $k > 1$, different tables should be used.

4.2. LL(1) Parsers

In this section we will study the $LL(1)$ parsers and the $LL(1)$ grammars. In these parsers the parsing automaton depicted in Figure 4.1.1 makes its moves according to a table T , called the *parsing table*, which is a matrix whose rows are labeled by the nonterminal symbols in V_N , and whose columns are labeled by the terminal symbols in V_T . In the table T there is also one extra column which is labeled by the symbol $\$$.

A move of the parsing automaton is either a *chop move* or an *expand move*. These moves are specified as follows (see also Figure 4.2.1 on the following page).

chop move:

if the input head is pointing at a terminal symbol, say a , and the same symbol a is at the top of the stack, then the input head is moved one cell to the right and the stack is popped;

expand move:

if the input head is pointing at a terminal symbol, say a , and the top of the stack is a nonterminal symbol, say A , then the stack is popped and a new string $\alpha_1 \alpha_2 \dots \alpha_n$, with $\alpha_i \in V$, for $i = 1, \dots, n$, is pushed onto the stack if the production $A \rightarrow \alpha_1 \alpha_2 \dots \alpha_n$ is at the entry (A, a) of the parsing table T (thus, after this move the new top symbol of the stack will be α_1).

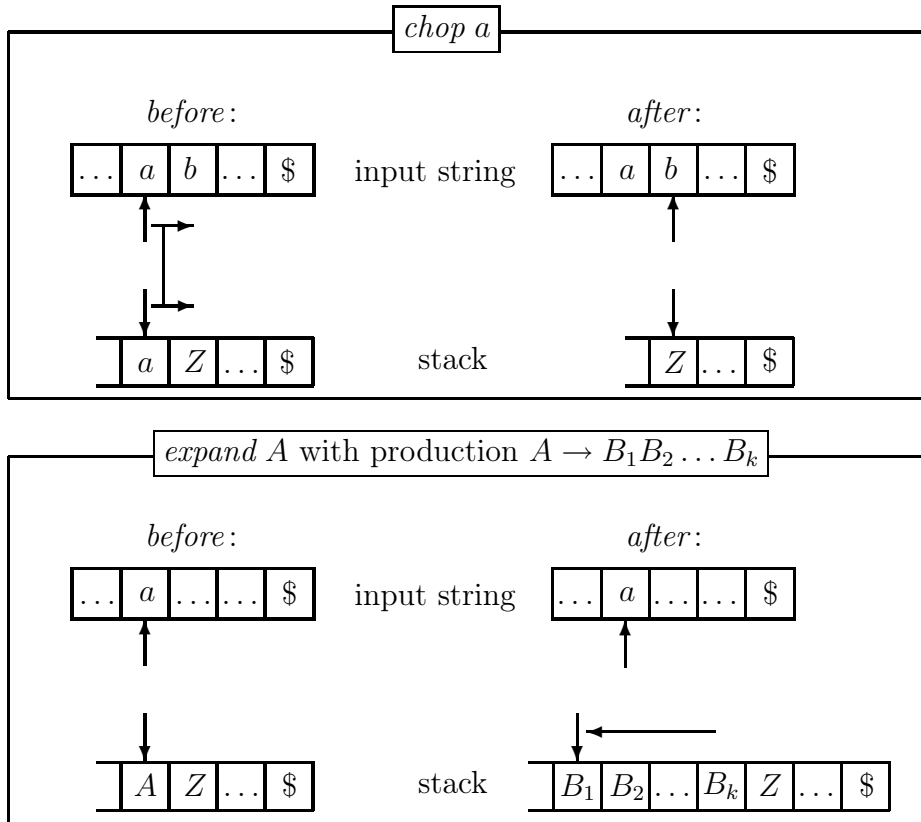


FIGURE 4.2.1. The *chop* move and the *expand* move of an $LL(1)$ parser. a and b are symbols in V_T and Z is a symbol in $V_T \cup V_N \cup \{\$\}$.

In order to explain how to construct the parsing table T of the parsing automaton for the language generated by an $LL(1)$ grammar, we need the following notions of $First_1$ and $Follow_1$ [2].

We first introduce the notion of $First_1(x)$.

For every $x \in V$, $First_1(x)$ is the set of the terminal symbols in V_T beginning any string β such that $x \rightarrow^* \beta$, with the stipulation that $First_1(x)$ should have the extra element ε iff either x is ε or x is a nullable nonterminal symbol (see Definition 1.2.16 on page 15).

We can compute the value of $First_1(x)$ as indicated by the following definition. Note that for every $x \in V$, the definition of $First_1(x)$ depends, in general, on the definition of $First_1(y)$ for all $y \in V$.

DEFINITION 4.2.1. [The Set $First_1(x)$ for Symbols] Let us consider a context-free grammar, possibly with ε -productions. $First_1$ is a function from $V \cup \{\varepsilon\}$ to the set of all subsets of $V_T \cup \{\varepsilon\}$. For every $x \in V \cup \{\varepsilon\}$, $First_1(x)$ is the *smallest* subset of $V_T \cup \{\varepsilon\}$ defined as follows:

- (i) $First_1(\varepsilon) = \{\varepsilon\}$.
- (ii) For $a \in V_T$, $First_1(a) = \{a\}$.
- (iii) For $A \in V_N$, $First_1(A)$ is a set of terminal symbols which is initialized to \emptyset and is modified according to the following rule.

For each production $A \rightarrow B_1 B_2 \dots B_{k-1} B_k$,

where: (1) $k \geq 0$, and (2) for $i = 1, \dots, k$, $B_i \in V_T \cup V_N$,

	add the elements of $First_1(B_1) - \{\varepsilon\}$ and
if $B_1 \rightarrow^* \varepsilon$	then add the elements of $First_1(B_2) - \{\varepsilon\}$ and
if $B_1 B_2 \rightarrow^* \varepsilon$	then add the elements of $First_1(B_3) - \{\varepsilon\}$ and
...	
if $B_1 B_2 \dots B_{k-1} \rightarrow^* \varepsilon$	then add the elements of $First_1(B_k) - \{\varepsilon\}$ and
if $B_1 B_2 \dots B_{k-1} B_k \rightarrow^* \varepsilon$	then add ε .

Note that if $B_1 B_2 \dots B_k \rightarrow^* \varepsilon$ then $A \rightarrow^* \varepsilon$ and $\varepsilon \in First_1(A)$. The following fact is a straightforward consequence of the definition of the function $First_1$.

FACT 4.2.2. For any $A \in V_N$, if $A \rightarrow^* \varepsilon$ then the empty string $\varepsilon \in First_1(A)$. In particular, if $A \rightarrow \varepsilon$ then $\varepsilon \in First_1(A)$.

In Definition 4.2.1 we have stipulated that $First_1(x)$ is the *smallest* subset of $V_T \cup \{\varepsilon\}$ satisfying Conditions (i), (ii), and (iii). The reason for that stipulation is that, since $First_1(x)$ may depend on itself (and, this happens, for instance, when in the given grammar there is a production of the form $A \rightarrow AB$), there could be more than one subset of $V_T \cup \{\varepsilon\}$ satisfying Conditions (i), (ii), and (iii).

We can extend the above definition of the function $First_1$ from elements in $V \cup \{\varepsilon\}$ to strings in V^* , by defining a new function, also called $First_1$, from V^* to the set of all subsets of $V_T \cup \{\varepsilon\}$ as follows.

DEFINITION 4.2.3. [**The Set $First_1(\alpha)$ for Strings**] Let us consider a context-free grammar, possibly with ε -productions. $First_1$ is a function from V^* to the set of all subsets of $V_T \cup \{\varepsilon\}$, such that for all $x \in V$ and $\alpha \in V^*$,

- (i) $First_1(\varepsilon) = \{\varepsilon\}$
- (ii) $First_1(x\alpha) = \text{if } x \rightarrow^* \varepsilon \text{ then } (First_1(x) - \{\varepsilon\}) \cup First_1(\alpha)$
else $First_1(x)$

where $First_1(x)$ on the right hand side is defined as in Definition 4.2.1 on the facing page.

FACT 4.2.4. For any string $\alpha \in V^*$, we have that $\varepsilon \in First_1(\alpha)$ iff $\alpha \rightarrow^* \varepsilon$.

Now we define the set $Follow_1(A)$ for any nonterminal A .

For any nonterminal $A \in V_N$, $Follow_1(A)$ is the smallest subset of $V_T \cup \{\$\}$ which includes every symbol occurring in a sentential form derived from $S\$$ immediately to the right of A , that is, for all $\alpha, \beta \in V^*$ and $b \in V_T \cup \{\$\}$,

if $S\$ \rightarrow^* \alpha A b \beta$ then $b \in Follow_1(A)$.

DEFINITION 4.2.5. [**The Set $Follow_1(A)$ for Nonterminals**] Let us consider a context-free grammar with axiom S , possibly with ε -productions. $Follow_1$ is a

function from V_N to the set of all subsets of $V_T \cup \{\$\}$ such that for each nonterminal $A \in V_N$, $Follow_1(A)$ is the *smallest* subset of $V_T \cup \{\$\}$ which satisfies the following rules.

- Rule (i). *If the nonterminal A is the axiom S then $\$ \in Follow_1(A)$;*
 Rule (ii). *If there is a production $B \rightarrow \alpha A \beta$ with $\alpha, \beta \in V^*$ and $\beta \rightarrow^* \varepsilon$ then $Follow_1(B) \subseteq Follow_1(A)$;*
 Rule (iii). *If there is a production $B \rightarrow \alpha A \beta$ with $\alpha, \beta \in V^*$, then $(First_1(\beta) - \{\varepsilon\}) \subseteq Follow_1(A)$.*

Note that for every $A \in V_N$, the definition of $Follow_1(A)$ depends, in general, on the definition of $Follow_1(B)$ for all $B \in V_N$, and the definition of $First_1(x)$ for all $x \in V$.

Note also that the premises of Rules (ii) and (iii) are *not* mutually exclusive, because the premise of Rule (iii) may hold independently of the fact that $\beta \rightarrow^* \varepsilon$ or $\beta \not\rightarrow^* \varepsilon$. If $\beta = \varepsilon$ then Rule (iii) is superfluous because its conclusion says that $\emptyset \subseteq Follow_1(A)$.

In Definition 4.2.5 we have stipulated that $Follow_1(x)$ is the *smallest* subset (and not *any* subset) of $V_T \cup \{\$\}$ satisfying Rules (i), (ii), and (iii). The reason for that stipulation is that, since $Follow_1(x)$ may depend on itself (and, this happens, for instance, when in the given grammar there is a production of the form $A \rightarrow BA$), there could be more than one subset of $V_T \cup \{\$\}$ satisfying Rules (i), (ii), and (iii).

Rule (i) of Definition 4.2.5 on the previous page is motivated by the fact that we have stipulated that the string given in input to the parsing automaton, is the string to be parsed followed by the symbol $\$$. We have that $Follow_1(S) = \{\$\}$ if S does *not* occur on the right hand side of any production.

In order to motivate Rule (ii) and, in particular, the inclusion $Follow_1(B) \subseteq Follow_1(A)$, let us consider the grammar with axiom S and the following productions:

$$S \rightarrow Bb \quad B \rightarrow A \quad A \rightarrow Aa \mid b$$

This grammar generates the language ba^*b . (Indeed, A and B generate the language ba^* .) For this grammar, since there is the production $B \rightarrow A$, we have that $Follow_1(B) = \{b\}$ and $Follow_1(A) = \{a, b\}$, as one can see from the derivation $S \rightarrow Bb \rightarrow Ab \rightarrow Aab \rightarrow bab$.

Note that, as a particular instance of Rule (iii), we have that if there is a production $B \rightarrow \alpha Ax\beta$ with $\alpha, \beta \in V^*$, $x \in V$, and $x \not\rightarrow^* \varepsilon$ then $First_1(x) \subseteq Follow_1(A)$. This is a consequence of Fact 4.2.4 on the preceding page.

The following fact is a straightforward consequence of the definition of the function $Follow_1$.

FACT 4.2.6. For any $A \in V_N$, the empty string $\varepsilon \notin Follow_1(A)$.

Note that in the theory of $LL(k)$ parsing we do *not* consider augmented grammars (see Definition 5.1.3 on page 76) and thus, in particular, we consider neither the new start symbol S' nor the extra production $S' \rightarrow S\$$.

In the case of an $LL(1)$ parser the table T is constructed as follows.

ALGORITHM 4.2.7. *Constructing the LL(1) parsing table for a given LL(1) grammar G .*

For each production $A \rightarrow \alpha$ of G with $A \in V_N$ and $\alpha \in (V_T \cup V_N)^*$,

Rule (1): if $a \in First_1(\alpha)$ then we place $A \rightarrow \alpha$ in row A and column a of the table T , and

Rule (2): if $\varepsilon \in First_1(\alpha)$ then for each $b \in Follow_1(A)$ we place $A \rightarrow \alpha$ in row A and column b of the table T . Note that, in particular, Rule (2) is applied if $\alpha = \varepsilon$, because in that case $\varepsilon \in First_1(\alpha)$.

One of the columns of the LL(1) parsing table is labeled by $\$$ because we stipulate that $\$ \in Follow_1(S)$, where as usual, S is the axiom of the grammar.

Note that given *any* context-free grammar G , we can construct a parsing table T by using this Algorithm 4.2.7. One can show that if an entry of the parsing table T is multiply defined (that is, more than one production must be placed in the same entry of the table T) then the given grammar G is *not* an LL(1) grammar.

Once the parsing table T has been constructed, it can be used by the deterministic pushdown automaton of Figure 4.1.1 on page 53 for parsing any given input string $w \$$. We have the following fact which we state without proof.

FACT 4.2.8. Let us consider the parsing automaton of Figure 4.1.1 on page 53 and let us assume that it is working using the table T . Initially, the stack of the automaton has the string $S \$$ (with S as top symbol and $\$$ on the bottom of the stack) and the input string is $w \$$.

- (i) If that automaton should use an entry of T without any production, then the input string w *does not belong* to the language generated by the given grammar G .
- (ii) If the input head of that automaton points at the symbol $\$$ in the input string and the top of the stack is $\$$ (in this case it could make a chop move of $\$$), then the input string w is accepted by the parsing automaton and w *belongs* to the language generated by the given grammar G .

We have the following theorem.

THEOREM 4.2.9. [**LL(1) Grammars and LL(1) Languages**] The LL(1) grammars are those which generate languages whose words are accepted by the deterministic pushdown automaton of Figure 4.1.1 on page 53, with the parsing table constructed as indicated by Algorithm 4.2.7.

The following result which we also state without proof, characterizes the LL(1) grammars and the strong LL(1) grammars.

THEOREM 4.2.10. [**LL(1) Grammars and Strong LL(1) Grammars**] (i) A context-free grammar G is an LL(1) grammar iff for every pair of distinct productions $A \rightarrow \alpha$ and $A \rightarrow \beta$ with $\alpha \neq \beta$, and for every string $w A \sigma$ such that $S \xrightarrow{*}_{lm} w A \sigma$ with $w \in V_T^*$ and $\sigma \in V^*$, we have that:

$$First_1(\alpha \sigma) \cap First_1(\beta \sigma) = \emptyset.$$

(ii) A context-free grammar G is a *strong LL(1)* grammar iff for every pair of distinct productions $A \rightarrow \alpha$ and $A \rightarrow \beta$ with $\alpha \neq \beta$, we have that:

$$\text{First}_1(\alpha \text{ Follow}_1(A)) \cap \text{First}_1(\beta \text{ Follow}_1(A)) = \emptyset.$$

As a consequence of this theorem we have the following fact.

FACT 4.2.11. [Equivalence of LL(1) Grammars and Strong LL(1) Grammars] The notions of an LL(1) grammar and a strong LL(1) grammar coincide.

EXAMPLE 4.2.12. [An LL(1) Grammar and Its LL(1) Parsing Table] Let us consider the grammar G with axiom S and whose productions are:

$$\begin{array}{l|l} S \rightarrow aAb & b \\ A \rightarrow a & bSA \end{array}$$

We have that:

$$\text{First}_1(aAb) = \{a\}, \quad \text{First}_1(b) = \{b\}, \quad \text{First}_1(a) = \{a\}, \quad \text{First}_1(bSA) = \{b\}.$$

The parsing table is:

	a	b	$\$$
S	$S \rightarrow aAb$	$S \rightarrow b$	
A	$A \rightarrow a$	$A \rightarrow bSA$	

In Figure 4.2.2 on the facing page we have depicted the sequence of the input string and stack configurations while parsing the string $abbab\$$. (That sequence from configuration (1) to configuration (10) is divided into three subsequences, each of them to be read from left to right.) The black triangle \blacktriangle indicates the symbols at hand in the input string and the top of the stack (as usual in our pictures of this section, in every stack configuration the top of the stack is the leftmost symbol).

In the last configuration of Figure 4.2.2 we have that both the top of the stack and the input head are pointing at the symbol $\$$. Thus, the given string $abbab$ belongs to the language generated by the grammar G . \square

EXAMPLE 4.2.13. [LL(1) Parsing of Arithmetic Expressions] Let us consider the context free grammar G with axiom E and the following productions:

$$\begin{array}{l|l} E \rightarrow E + T & T \\ T \rightarrow T \times F & F \\ F \rightarrow (E) & a \end{array}$$

This grammar is left recursive and thus, for all $k \geq 0$ it is not LL(k) (see [3, page 344]). If we want to use an LL(k) parsing algorithm we have to look for an equivalent grammar which is *not* left recursive. One such context free grammar is the following grammar \tilde{G} (with ε -productions):

$$\begin{array}{l|l} E \rightarrow T \tilde{E} & \tilde{E} \rightarrow \varepsilon \mid + T \tilde{E} \\ T \rightarrow F \tilde{T} & \tilde{T} \rightarrow \varepsilon \mid \times F \tilde{T} \\ F \rightarrow (E) & a \end{array}$$

The grammar \tilde{G} can be obtained from the grammar G by avoiding the left recursion and using ε -productions. The grammar \tilde{G} is an LL(1) grammar as shown by the

input string:	$abbab\$$ ▲		$abbab\$$ ▲		$abbab\$$ ▲		$abbab\$$ ▲
	(1)	$\xrightarrow{\text{expand } S}$	(2)	$\xrightarrow{\text{chop } a}$	(3)	$\xrightarrow{\text{expand } A}$	(4)
stack:	$S\$$ ▲		$aAb\$$ ▲		$Ab\$$ ▲		$bSAb\$$ ▲
		$\xrightarrow{\text{chop } b}$	(5)	$\xrightarrow{\text{expand } S}$	(6)	$\xrightarrow{\text{chop } b}$	(7)
			$SAb\$$ ▲		$bAb\$$ ▲		$Ab\$$ ▲
		$\xrightarrow{\text{expand } A}$	(8)	$\xrightarrow{\text{chop } a}$	(9)	$\xrightarrow{\text{chop } b}$	(10)
			$ab\$$ ▲		$b\$$ ▲		$\$$ ▲

FIGURE 4.2.2. $LL(1)$ parsing of the string $abbab\$$. The given grammar has axiom S and the following productions: $S \rightarrow aAb \mid b$, $A \rightarrow a \mid bSA$. The black triangle ▲ indicates the symbol at hand in the input string and the top of the stack (which is the leftmost symbol of the stack).

$LL(1)$ parsing table depicted in Figure 4.2.3 on the next page. In order to construct that parsing table, we have first to compute the $First_1$ sets and the $Follow_1$ sets. They are:

$$\begin{array}{lll}
 First_1(\varepsilon) = \{\varepsilon\} & & \\
 First_1(T\tilde{E}) = \{(\, a\} & First_1(+T\tilde{E}) = \{+\} & First_1(\tilde{E}) = \{\varepsilon, +\} \\
 First_1(F\tilde{T}) = \{(\, a\} & First_1(\times F\tilde{T}) = \{\times\} & First_1(\tilde{T}) = \{\varepsilon, \times\} \\
 First_1((E)) = \{(\} & First_1(a) = \{a\} & First_1(F) = \{(\, a\} \\
 Follow_1(E) = \{), \$\} & Follow_1(\tilde{E}) = \{), \$\} & Follow_1(F) = \{\times, +,), \$\} \\
 Follow_1(T) = \{+,), \$\} & Follow_1(\tilde{T}) = \{+,), \$\} &
 \end{array}$$

Figure 4.2.4 on the following page shows the sequence of input string and stack configurations while parsing the arithmetic expression a (thus, the input string is $a\$$) according to the grammar \tilde{G} . □

	a	$($	$)$	$+$	\times	$\$$
E	$E \rightarrow T\tilde{E}$	$E \rightarrow T\tilde{E}$				
\tilde{E}			$\tilde{E} \rightarrow \varepsilon$	$\tilde{E} \rightarrow +T\tilde{E}$		$\tilde{E} \rightarrow \varepsilon$
T	$T \rightarrow F\tilde{T}$	$T \rightarrow F\tilde{T}$				
\tilde{T}			$\tilde{T} \rightarrow \varepsilon$	$\tilde{T} \rightarrow \varepsilon$	$\tilde{T} \rightarrow \times F\tilde{T}$	$\tilde{T} \rightarrow \varepsilon$
F	$F \rightarrow a$	$F \rightarrow (E)$				

FIGURE 4.2.3. $LL(1)$ parsing table for the grammar with axiom E and productions: $E \rightarrow T\tilde{E}$, $\tilde{E} \rightarrow \varepsilon \mid +T\tilde{E}$, $T \rightarrow F\tilde{T}$, $\tilde{T} \rightarrow \varepsilon \mid \times F\tilde{T}$, $F \rightarrow (E) \mid a$.

input string:	$a \$$ ▲		$a \$$ ▲		$a \$$ ▲		$a \$$ ▲
	(1)	$\xrightarrow{\text{expand } E}$	(2)	$\xrightarrow{\text{expand } T}$	(3)	$\xrightarrow{\text{expand } F}$	(4)
stack:	$E \$$ ▲		$T\tilde{E} \$$ ▲		$F\tilde{T}\tilde{E} \$$ ▲		$a\tilde{T}\tilde{E} \$$ ▲
		$\xrightarrow{\text{chop } a}$	(5)	$\xrightarrow{\text{expand } \tilde{T}}$	(6)	$\xrightarrow{\text{expand } \tilde{E}}$	(7)
			$\tilde{T}\tilde{E} \$$ ▲		$\tilde{E} \$$ ▲		$\$$ ▲

FIGURE 4.2.4. $LL(1)$ parsing of the string $a \$$ according to the grammar \tilde{G} with axiom E and productions: $E \rightarrow T\tilde{E}$, $\tilde{E} \rightarrow \varepsilon \mid +T\tilde{E}$, $T \rightarrow F\tilde{T}$, $\tilde{T} \rightarrow \varepsilon \mid \times F\tilde{T}$, $F \rightarrow (E) \mid a$. The black triangle ▲ indicates the symbol at hand in the input string and the top of the stack (which is the leftmost symbol of the stack).

With reference to Example 4.2.13 above, note that the following grammar H , which is not left recursive and has no ε -productions, is *not* $LL(1)$:

$$\begin{array}{l}
 E \rightarrow T \quad \mid T\tilde{E} \qquad \qquad \tilde{E} \rightarrow +T \quad \mid +T\tilde{E} \\
 T \rightarrow F \quad \mid F\tilde{T} \qquad \qquad \tilde{T} \rightarrow \times F \quad \mid \times F\tilde{T} \\
 F \rightarrow (E) \quad \mid a
 \end{array}$$

This grammar H can be obtained from the given grammar G by eliminating the left recursion without using ε -productions (thus, grammar H is equivalent to grammars

G and \tilde{G}). To show that grammar H is not $LL(1)$, let us consider the following grammar which is a simplified version of H . This grammar has axiom T and the productions:

$$\begin{array}{l} T \rightarrow F \quad | \quad F\tilde{T} \\ F \rightarrow (T) \quad | \quad a \end{array} \qquad \begin{array}{l} \tilde{T} \rightarrow \times F \quad | \quad \times F\tilde{T} \end{array}$$

We have that:

$$\begin{array}{ll} First_1(F) = First_1(F\tilde{T}) = \{(\, a\} & First_1(\times F) = First_1(\times F\tilde{T}) = \{\times\} \\ First_1((T)) = \{(\, \} & First_1(a) = \{a\} \end{array}$$

We get the following parsing table (there is no need to compute the $Follow_1$ sets because ε does not occur in any of the $First_1$ sets):

	a	$($	$)$	\times	$\$$
T	$T \rightarrow F$ $T \rightarrow F\tilde{T}$	$T \rightarrow F$ $T \rightarrow F\tilde{T}$			
\tilde{T}				$T \rightarrow \times F$ $T \rightarrow \times F\tilde{T}$	
F	$F \rightarrow a$	$F \rightarrow (T)$			

Grammar H is not $LL(1)$ because in this table there exists an entry which has more than one production (actually, there are three entries). \square

EXAMPLE 4.2.14. [**A Grammar Which is Not $LL(1)$**] Let us consider the grammar G whose axiom is S and whose productions are:

$$\begin{array}{l} S \rightarrow \varepsilon \quad | \quad a b A \\ A \rightarrow S a a \quad | \quad b \end{array}$$

We have that:

$$\begin{array}{lll} First_1(\varepsilon) = \{\varepsilon\} & First_1(a b A) = \{a\} & First_1(S) = \{\varepsilon, a\} \\ First_1(S a a) = \{a\} & First_1(b) = \{b\} & \\ Follow_1(S) = \{\$, a\} & Follow_1(A) = \{\$, a\} & \end{array}$$

The parsing table is:

	a	b	$\$$
S	$S \rightarrow a b A$ $S \rightarrow \varepsilon$		$S \rightarrow \varepsilon$
A	$A \rightarrow S a a$	$A \rightarrow b$	$A \rightarrow S a a$

The given grammar is *not* $LL(1)$ because in this parsing table for the symbol S on the top of the stack and the input symbol a , there are two productions. \square

4.3. LL(k) Parsers (for $k \geq 1$)

In this section we consider the general case of $LL(k)$ parsers and $LL(k)$ grammars, with $k \geq 1$.

We first need the following definition which is a generalization of Definition 4.2.1 on page 54.

DEFINITION 4.3.1. [The Set $First_k(\alpha)$] Let us consider a context-free grammar, possibly with ε -productions. For $k \geq 0$, $First_k$ is a function from V^* to the set of all subsets of $V_T^0 \cup V_T^1 \cup V_T^2 \cup \dots \cup V_T^k$, that is, $\{\varepsilon\} \cup V_T \cup V_T^2 \cup \dots \cup V_T^k$. Given a string $\alpha \in V^*$, we have that:

$$First_k(\alpha) =_{def} \{w \mid (\alpha \rightarrow^* w \beta \text{ and } w \in V_T^k \text{ and } \beta \in V_T^*) \text{ or } (\alpha \rightarrow^* w \text{ and } w \in V_T^i \text{ for some } 0 \leq i < k)\}.$$

For any $\alpha \in V^*$ and $\beta \in V_T^*$, we have that: (i) if $\alpha \rightarrow^* \beta$ then $First_0(\alpha) = \{\varepsilon\}$, and (ii) for any $k \geq 1$, $\varepsilon \in First_k(\alpha)$ iff $\alpha \rightarrow^* \varepsilon$.

We will also need the following binary operation on languages.

DEFINITION 4.3.2. [k -bounded Concatenation] Given two languages L_1 and L_2 , their k -bounded concatenation, denoted \odot_k , for any $k \geq 1$, is defined as follows:

$$L_1 \odot_k L_2 = \{w \mid (|w| \leq k \text{ and } w \in L_1 \cdot L_2) \text{ or } (|w| = k \text{ and } \exists z, wz \in L_1 \cdot L_2)\}.$$

where the concatenation operation ' \cdot ' on languages is defined as usual (see Chapter 1).

For instance, given the languages $L_1 = \{\varepsilon, abb\}$ and $L_2 = \{b, bab\}$, we have that:

$$L_1 \cdot L_2 = \{b, bab, abbb, abbbab\} \text{ and}$$

$$L_1 \odot_2 L_2 = \{b, ba, ab\}.$$

Note that in $L_1 \odot_k L_2$ with $k \geq 1$, there may be words whose length is less than k .

Now we will describe how the $LL(k)$ parsing is performed, for any $k \geq 1$.

Supposed we are given a string v to be parsed and an $LL(k)$ grammar G . Suppose also that we have parsed a proper prefix p of the given string v , that is, $v = pu$ for some $u \in V_T^+$ and the input head is pointing at the leftmost symbol of u . We assume that the sentential form we have generated so far is: pAz , for some $z \in V^*$, that is, all symbols in p have been chopped and the stack is holding $Az\$$ with the top of the stack pointing at A .

In order to determine the production for expanding A , the parser uses a parsing table which is constructed as we will indicate, after the construction of some other tables, each of which is parameterized by: (i) a nonterminal symbol, and (ii) a language subset of $V_T^* \0,1 . A table named T and whose parameters are the nonterminal B and the language L , will be denoted by $T[B, L]$.

The table $T[B, L]$ can be represented as a matrix whose rows have three components (or columns):

- (i) the first one is a word w in $V_T^* \0,1 ,
- (ii) the second one is a production for B , and

(iii) the third one is a *list of sets of words* which depends on: (iii.1) the production for B which is the second component, and (iii.2) the language L . Each set in this list of sets of words is called a *local follow sets* for B .

The rows of table $T[B, L]$ are constructed by considering the productions for B . For each one of these productions, say $B \rightarrow \alpha$, we first compute the set of words:

$$\text{First}_k(\alpha) \odot_k L$$

and we then construct a row of the table $T[B, L]$ of the form:

w	$B \rightarrow \alpha$	M
-----	------------------------	-----

for each word w in $\text{First}_k(\alpha) \odot_k L$. In each of these rows M is a list of set of words such that:

- (i) if $\alpha \in V_T^*$ then $M = []$, and
- (ii) if α is of the form: $x_0 B_1 x_1 \dots B_m x_m$, with $m \geq 0$, and for $h=0, \dots, m$, $x_h \in V_T^*$, and for $i=1, \dots, m$, $B_i \in V_N$, then $M = [Y_1, \dots, Y_m]$, where for $i=1, \dots, m$,

$$Y_i = \text{First}_k(x_i B_{i+1} x_{i+1} \dots B_m x_m) \odot_k L.$$

The first table to be constructed is $T[S, \{\$\}]$.

When constructing the various tables, we maintain a set of tables to be constructed. Thus, initially, that set of tables to be constructed has exactly one element which is the table $T[S, \{\$\}]$.

At the end of the construction of a table, say $T[B, L]$, we update that set of tables to be constructed as follows. For each row of the table $T[B, L]$ of the form:

w	$B \rightarrow x_0 B_1 x_1 \dots B_m x_m$	$[Y_1, \dots, Y_m]$
-----	---	---------------------

where: (i) $w \in V_T^*$, (ii) $m \geq 1$, (iii) for $h=0, \dots, m$, $x_h \in V_T^*$, and (iv) for $i=1, \dots, m$, $B_i \in V_N$, we add to the set of tables to be constructed the m tables $T[B_i, Y_i]$, for $i=1, \dots, m$.

The process of constructing tables terminates when we have constructed all the tables which occur in the set of tables to be constructed.

At that point we can construct the parsing table for the $LL(k)$ parsing for $k \geq 1$. In that parsing table the rows are indexed by the parameters of the tables we have constructed (thus, the rows are as many as those tables), and the columns are indexed by the words in $V^0 \$ \cup V^1 \$ \cup \dots \cup V^{k-1} \$ \cup V^k$ (recall that $V^0 \$$ is equal to $\{\$\}$). The entry of the parsing table in row $[B_i, Y_i]$ and column w is the second component of the row of table $T[B_i, Y_i]$ whose first component is w .

Thus,

- (i) the index $[B_i, Y_i]$ of each row of the parsing table consists of the symbol B_i on the top of the stack and the string Y_i which is equal to $\text{First}_k(\sigma_i)$, where σ_i is the string of the symbols *below the top* of the stack, and
- (ii) the index w of each column of the parsing table is the string of at most k (≥ 1) symbols to the right of the input head (which is pointing at the leftmost symbol of w).

Now we will give an example of the construction of the tables in the case of an $LL(2)$ context-free grammar G . This example will clarify the rules we have given above for the general case of the $LL(k)$ parsing, for any $k \geq 1$.

EXAMPLE 4.3.3. Let us consider the following grammar G with axiom S and whose productions are:

$$S \rightarrow aAaa \mid bAba \qquad A \rightarrow b \mid \varepsilon$$

The construction of the $LL(2)$ parsing table will show that this grammar is an $LL(2)$ grammar.

• *Construction of the initial table $T[S, \{\$\}]$.*

For S we have the two productions: $S \rightarrow aAaa$ and $S \rightarrow bAba$. For each of them, say $S \rightarrow \alpha$, we have compute the set of words:

$$First_2(\alpha) \odot_2 \{\$\}.$$

Thus, we have:

$$First_2(aAaa) \odot_2 \{\$\} = \{ab, aa\} \odot_2 \{\$\} = \{ab, aa\}$$

$$First_2(bAba) \odot_2 \{\$\} = \{bb\} \odot_2 \{\$\} = \{bb\}$$

From these values we get the three rows of the table $T[S, \{\$\}]$. We have that:

$$T[S, \{\$\}]:$$

ab	$S \rightarrow aAaa$	$L1$
aa	$S \rightarrow aAaa$	$L2$
bb	$S \rightarrow bAba$	$L3$

where the lists $L1$, $L2$, and $L3$ of set of words are defined as follows:

$$L1 = [First_2(aa) \odot_2 \{\$\}] = [\{aa\}]$$

(Note that aa is the word in V_T^* which follows A in the right hand side of the production $S \rightarrow aAaa$, and there is only the nonterminal symbol A in $aAaa$).

$$L2 = [First_2(aa) \odot_2 \{\$\}] = [\{aa\}]$$

$$L3 = [First_2(ba) \odot_2 \{\$\}] = [\{ba\}]$$

(Note that ba is the word in V_T^* which follows A in the right hand side of the production $S \rightarrow bAba$ and there is only the nonterminal symbol A in $bAba$.)

Thus, we get the following table $T[S, \{\$\}]$:

$$T[S, \{\$\}]:$$

ab	$S \rightarrow aAaa$	$[\{aa\}]$
aa	$S \rightarrow aAaa$	$[\{aa\}]$
bb	$S \rightarrow bAba$	$[\{ba\}]$

Having constructed this table, we insert in the set of tables to be constructed the following two tables:

$$T[A, \{aa\}] \text{ and } T[A, \{ba\}].$$

Note that either the first row or the second row of table $T[S, \{\$\}]$ forces us to insert table $T[A, \{aa\}]$ in the set of tables to be constructed.

- *Construction of the table $T[A, \{aa\}]$.*

For A we have the two productions: $A \rightarrow b$ and $A \rightarrow \varepsilon$. Thus, we have that:

$$\begin{aligned} \text{First}_2(b) \odot_2 \{aa\} &= \{ba\} \\ \text{First}_2(\varepsilon) \odot_2 \{aa\} &= \{aa\} \end{aligned}$$

From these values we get the two rows of the table $T[A, \{aa\}]$. We have that:

$$T[A, \{aa\}]:$$

ba	$A \rightarrow b$	$M1$
aa	$A \rightarrow \varepsilon$	$M2$

where the lists $M1$ and $M2$ of sets of words are both empty because in the right hand sides of the productions $A \rightarrow b$ and $A \rightarrow \varepsilon$ there are no nonterminal symbols. Thus, we get the following table $T[A, \{aa\}]$:

$$T[A, \{aa\}]:$$

ba	$A \rightarrow b$	$[\]$
aa	$A \rightarrow \varepsilon$	$[\]$

After the construction of this table we do *not* add any new table to the set of tables to be constructed because in the right hand sides of the productions $A \rightarrow b$ and $A \rightarrow \varepsilon$ there are no nonterminal symbols.

- *Construction of the table $T[A, \{ba\}]$.*

For A we have the two productions: $A \rightarrow b$ and $A \rightarrow \varepsilon$. Thus, we have that:

$$\begin{aligned} \text{First}_2(b) \odot_2 \{ba\} &= \{bb\} \\ \text{First}_2(\varepsilon) \odot_2 \{ba\} &= \{ba\} \end{aligned}$$

From these values we get the two rows of the table $T[A, \{ba\}]$. We have that:

$$T[A, \{ba\}]:$$

bb	$A \rightarrow b$	$N1$
ba	$A \rightarrow \varepsilon$	$N2$

where the lists $N1$ and $N2$ of sets of words are both empty because in the right hand sides of the productions $A \rightarrow b$ and $A \rightarrow \varepsilon$ there are no nonterminal symbols. Thus, we get the following table $T[A, \{ba\}]$:

$$T[A, \{ba\}]:$$

bb	$A \rightarrow b$	$[\]$
ba	$A \rightarrow \varepsilon$	$[\]$

After the construction of this table we do *not* add any new table to the set of tables to be constructed because in the right hand sides of the productions $A \rightarrow b$ and $A \rightarrow \varepsilon$ there are no nonterminal symbols.

- *Construction of the parsing table.*

Having constructed the three tables $T[S, \{\$\}]$, $T[A, \{aa\}]$, and $T[A, \{ba\}]$, we can construct the $LL(2)$ parsing table. We have depicted it in Figure 4.3.1 on the following page.

In the following Figure 4.3.2 on the next page we have shown the sequence of the input string and stack configurations when parsing the string $bb a \$$. The symbol \blacktriangle

	aa	ab	ba	bb	$a\$$	$b\$$	$\$$
$[S, \{\$\}]$	$S \rightarrow aAaa$	$S \rightarrow aAaa$		$S \rightarrow bAba$			
$[A, \{aa\}]$	$A \rightarrow \varepsilon$		$A \rightarrow b$				
$[A, \{ba\}]$			$A \rightarrow \varepsilon$	$A \rightarrow b$			

FIGURE 4.3.1. The $LL(2)$ parsing table for the grammar with axiom S and productions: $S \rightarrow aAaa \mid bAba$ and $A \rightarrow b \mid \varepsilon$.

indicates the symbol at hand in the input string (that is, the position of the input head) and the top of the stack (as usual in this section, in every stack configuration the top of the stack is the leftmost symbol).

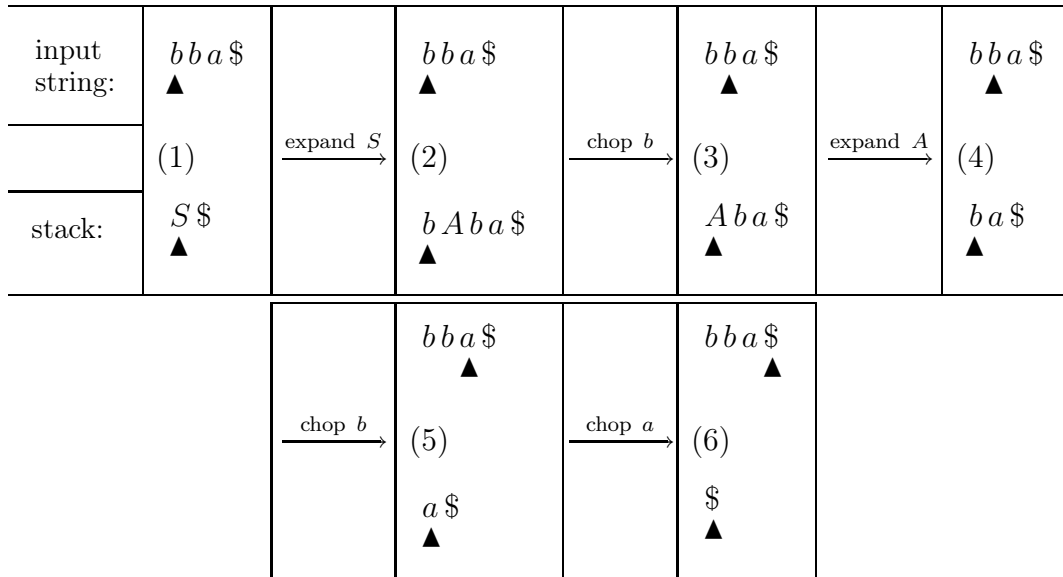


FIGURE 4.3.2. $LL(2)$ parsing of the string $bb a \$$. The given grammar with axiom S has the following productions: $S \rightarrow aAaa \mid bAba$, $A \rightarrow b \mid \varepsilon$. The symbol ▲ indicates the symbol at hand in the input string and the top of the stack (which is the leftmost symbol).

Note that the expansion of S (see configurations (1) and (2) in Figure 4.3.2) is done using the production $S \rightarrow bAba$ because: (i) below S on the stack there is $\$$, and (ii) the symbols in the position of the input head and at its right are bb (see the entry at row $[S, \{\$\}]$ and column bb of the parsing table depicted in Figure 4.3.1). The expansion of A is done using the production $A \rightarrow \varepsilon$ because: (i) below A on the stack there are the symbols ba , and (ii) the symbols in the position of the input head and at its right are ba (see the entry at row $[A, \{ba\}]$ and column ba of the parsing table depicted in Figure 4.3.1).

In the final configuration (6) in Figure 4.3.2 we have that both the input head and the top of the stack are pointing at the symbol $\$$. Thus, the given string $bb a$ belongs to the language generated by the grammar G . \square

We have the following property of the $LL(k)$ parsing, for any $k \geq 1$.

$LL(k)$ Parsing, for $k \geq 1$.

The production which has to be applied by the Chop-and-Expand parser for expanding the nonterminal A is uniquely determined by (see also Figure 4.3.3):

- (i) A itself,
- (ii) the leftmost k symbols of z (or z itself if $|z| < k$), and
- (iii) the leftmost k symbols of u (or u itself if $|u| < k$).

(The reader may want to look also at what we will state on page 68 with reference to strong $LL(k)$ grammars.)

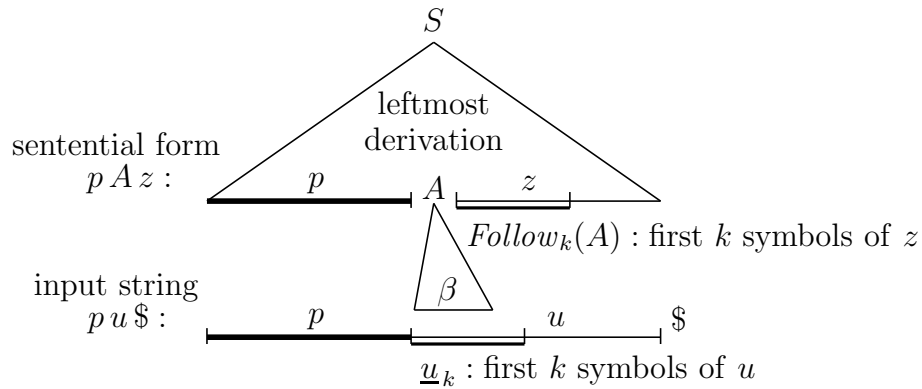


FIGURE 4.3.3. $LL(k)$ parsing, for $k \geq 1$. The derivation from S to pAz is a leftmost derivation.

Now in order to state Theorem 4.3.5 on the next page, we define for any nonterminal A and for any $k \geq 1$, the set $Follow_k(A)$.

DEFINITION 4.3.4. [The Set $Follow_k(A)$] Let us consider a context-free grammar with axiom S , possibly with ϵ -productions. For any $k \geq 1$, $Follow_k$ is a function from V_N to the set of all subsets of $V^*\0,1 . For any $A \in V_N$, $Follow_k(A)$ is the *smallest* set such that for any string $w \in (V^0\$ \cup V^1\$ \cup V^2\$ \cup \dots \cup V^{k-1}\$ \cup V^k)$,
 if $S\$ \rightarrow^* \alpha A \beta$ for some $\alpha \in V^*$ and $\beta \in V^*\$$ and $w \in First_k(\beta)$
 then $w \in Follow_k(A)$.

Thus, by definition, for every nonterminal $A \in V_N$, $Follow_k(A)$ is a set of words w in $V^0\$ \cup V^1\$ \cup V^2\$ \cup \dots \cup V^{k-1}\$ \cup V^k$ which occur in a sentential form derived from $S\$$ immediately to the right of A .

We have that for any $k \geq 1$ and $A \in V_N$, the empty string ϵ is *not* an element of $Follow_k(A)$.

Note that contrary to Definition 4.2.5 on page 55, where we have given an algorithm for the construction of the set $Follow_1(A)$, in the above Definition 4.3.4 we did *not* provide an algorithm that, for any $k \geq 1$, given the productions of an $LL(k)$ grammar, constructs the set $Follow_k(A)$.

The following result which we state without proof, characterizes the $LL(k)$ grammars and the strong $LL(k)$ grammars for any $k \geq 1$ [3, page 342].

THEOREM 4.3.5. [$LL(k)$ Grammars and Strong $LL(k)$ Grammars] (i) A context-free grammar G is an $LL(k)$ grammar iff for every pair of distinct productions $A \rightarrow \alpha$ and $A \rightarrow \beta$ with $\alpha \neq \beta$, and for every string $w A \sigma$ such that $S \xrightarrow{*}_{lm} w A \sigma$, we have that:

$$First_k(\alpha \sigma) \cap First_k(\beta \sigma) = \emptyset.$$

(ii) A context-free grammar G is a *strong* $LL(k)$ grammar iff for every pair of distinct productions $A \rightarrow \alpha$ and $A \rightarrow \beta$ with $\alpha \neq \beta$, we have that:

$$First_k(\alpha Follow_k(A)) \cap First_k(\beta Follow_k(A)) = \emptyset.$$

EXAMPLE 4.3.6. The grammar G with axiom S and the following productions:

$$S \rightarrow aAaa \mid bAba \qquad A \rightarrow b \mid \varepsilon$$

is an $LL(2)$ grammar as the above Example 4.3.3 shows, but it is *not* a strong $LL(2)$ grammar. Indeed, we have that:

$$Follow_2(A) = \{aa, ba\} \text{ and}$$

$$First_2(b Follow_2(A)) \cap First_2(\varepsilon Follow_2(A)) = \{ba, bb\} \cap \{aa, ba\} = \{ba\} \neq \emptyset.$$

We leave it to the reader to check the following facts which we state without proofs.

FACT 4.3.7. The rules we have given above for constructing the parsing table for $LL(1)$ parsing, can be obtained by instantiating for $k = 1$ the rules for $LL(k)$ parsing for $k \geq 1$.

FACT 4.3.8. [$LL(k)$ Languages Can Be Generated by Strong $LL(k)$ Grammars] For any $k \geq 0$, if a language L has an $LL(k)$ grammar then L has a strong $LL(k)$ grammar.

In view of this fact, if we consider the strong $LL(k)$ grammar which generates a given $LL(k)$ language L , then the language L can be parsed by using a Chop-and-Expand parser whose expansion steps satisfy the following property.

Strong $LL(k)$ Parsing, for $k \geq 1$.

The production which has to be applied by the Chop-and-Expand parser for expanding the nonterminal A is uniquely determined by:

- (i) A itself, and
- (ii) the first k symbols of the input string which are to the right of the last symbol which has been chopped (or the whole string to the right of that symbol, if there are less than k symbols to the right of it).

The reader may want to look also at what we stated on page 67, where the Chop-and-Expand $LL(k)$ parsing process has been described with respect to $LL(k)$ grammars, rather than strong $LL(k)$ grammars.

FACT 4.3.9. [From $LL(k)$ Grammars to $LL(k+1)$ Grammars in Greibach Normal Form] Let L be a language that can be generated by an $LL(k)$ grammar for some $k \geq 0$. Then: (i) $L - \{\varepsilon\}$ can be generated by an $LL(k+1)$ grammar in Greibach normal form, and (ii) L can be generated by an $LL(k+1)$ grammar G in Greibach normal form and we have to consider also the production $S \rightarrow \varepsilon$ iff $\varepsilon \in L$ [**3**, page 362].

In the following two examples the reader may see in action the techniques for producing from an $LL(1)$ grammar an equivalent $LL(2)$ grammar in Greibach normal form. These techniques constitute the basis for the proof of the above Fact 4.3.9. Without loss of generality, we assume that the given $LL(1)$ grammar has no unit productions, that is, productions of the form $A \rightarrow B$, for some $A, B \in V_N$ [**21**].

EXAMPLE 4.3.10. [From $LL(1)$ Grammars to $LL(2)$ Grammars in Greibach Normal Form: Example 1] Let us consider the $LL(1)$ grammar G with axiom S and the following productions:

$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow aA \mid \varepsilon \\ B &\rightarrow bA \mid \varepsilon \end{aligned}$$

This grammar G is, indeed, an $LL(1)$ grammar, as the reader may verify, because the $LL(1)$ parsing table is as follows:

	a	b	$\$$
S	$S \rightarrow AB$	$S \rightarrow AB$	$S \rightarrow AB$
A	$A \rightarrow aA$	$A \rightarrow \varepsilon$	$A \rightarrow \varepsilon$
B		$B \rightarrow bA$	$B \rightarrow \varepsilon$

This table is constructed after computing the following sets: $First_1(S) = \{a, b, \varepsilon\}$, $First_1(A) = \{a, \varepsilon\}$, $First_1(B) = \{b, \varepsilon\}$, $First_1(AB) = \{a, b, \varepsilon\}$, $Follow_1(S) = \{\$\}$, $Follow_1(A) = \{b, \$\}$, and $Follow_1(B) = \{\$\}$.

We want to derive a grammar in Greibach normal form which is equivalent to G . In that equivalent grammar we allow also the production $S \rightarrow \varepsilon$ if $\varepsilon \in L(G)$. In our case, indeed, we have the production $S \rightarrow \varepsilon$ because $\varepsilon \in L(G)$.

We start from the axiom S and we perform *leftmost unfolding* steps (that is, we unfold the leftmost nonterminals of the sentential forms), thereby producing new sentential forms from old ones. We stop these unfolding steps when we get: either

- (H1) ε , or
- (H2) a terminal symbol $a \in V_T$, or
- (H3) a string of the form $a\sigma_1 \dots \sigma_n$, for some $n \geq 1$, where $a \in V_T$, and for $i = 1, \dots, n$, we have that $\sigma_i \in V_T V_N^* \cup \{S\}$.

One can show that by doing so, we always construct a *finite* tree of sentential forms (see Figure 4.3.4 on the next page for the case of our grammar G). Every leaf of that tree allows us to generate, as we will indicate, a production of a grammar in Greibach normal form equivalent to G .

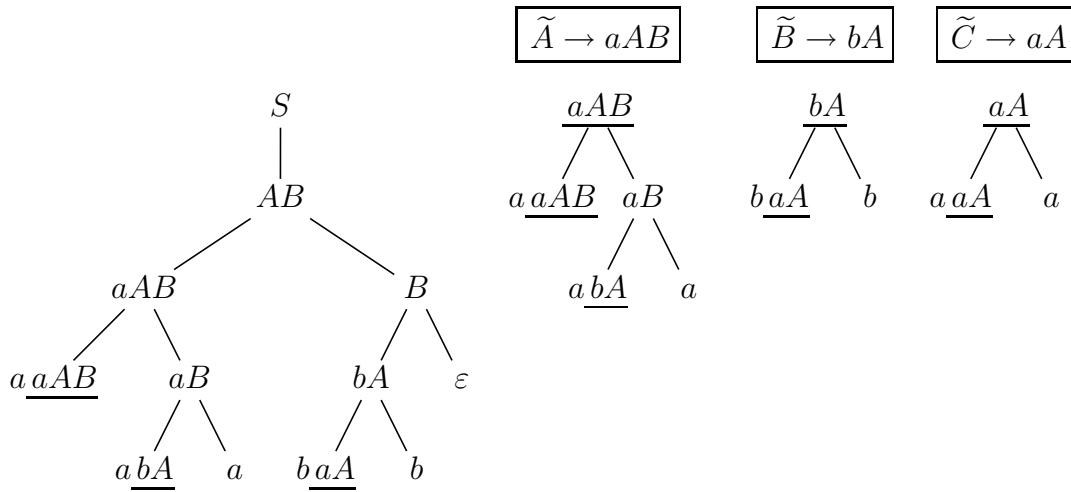


FIGURE 4.3.4. Trees of sentential forms obtained by unfolding the leftmost nonterminals. The given grammar has axiom S and productions: $S \rightarrow AB$, $A \rightarrow aA \mid \varepsilon$, $B \rightarrow bA \mid \varepsilon$. The nullable symbols are S , A , and B . We have indicated within boxes the productions which are generated by the roots of the trees of sentential forms whose root is *not* the axiom S .

Note that, since we perform only leftmost unfolding steps, the decomposition of the sentential form into a sequence of the form $a\sigma_1 \dots \sigma_n$ satisfying the Condition (H3) above, may not be unique. For instance, $abAS$ can be decomposed either into the two substrings: a and bAS , or into the three substrings: a , bA , and S .

Then we apply the following Tree Closure Rule.

Tree Closure Rule.

For each leaf where Condition (H3) holds, for $i = 1, \dots, n$, with $n \geq 1$, and for every σ_i which is *not* a terminal symbol and is different from S , we will construct a new tree of sentential forms whose root is σ_i and whose nodes are obtained from the root by performing leftmost unfolding steps until at each leaf one of the above two Conditions (H2) or (H3) becomes true.

(Note that Condition (H1) can never become true because $\sigma_i \in V_T V_N^*$.)

The leaves of these new trees we will construct, may determine the construction of more trees of sentential forms, because we have to apply the above *Tree Closure Rule* until all leaves of all trees of sentential forms are either:

- (i) ε , or
- (ii) a terminal symbol, or
- (iii) a string of the form $a\sigma_1 \dots \sigma_n$, for some $n \geq 1$, where $a \in V_T$, and for $i = 1, \dots, n$, each substring σ_i is the label of the root of a tree.

Having terminated the constructions of all these trees, we consider for each tree with root σ different from S , a new nonterminal A_σ and we introduce a production of the

form $A_\sigma \rightarrow \sigma$. We also introduce a production of the form $A_a \rightarrow a$, where A_a is a new nonterminal symbol, for each leaf of a tree which is made out of the terminal symbol a only.

We then perform all possible folding steps, both at the roots and at the leaves of the trees we have constructed, by using the productions of the form $A_\sigma \rightarrow \sigma$ or $A_a \rightarrow a$ we have introduced. In particular, a leaf with the sentential form $a\sigma_1 \dots \sigma_n$, with $n \geq 0$, is replaced by the leaf with the sentential form $aA_{\sigma_1} \dots A_{\sigma_n}$.

Finally, we get the desired productions of the grammar in Greibach normal form by considering for every root-to-leaf path in every tree, a production of the form $A \rightarrow \gamma$, where A is the label of the root and γ is the label of the leaf.

In the case of our grammar G above, starting from the trees of sentential forms depicted in Figure 4.3.4 on the facing page, we first introduce the following three productions (see the roots, different from S , of the trees of Figure 4.3.4):

$$\begin{aligned}\tilde{A} &\rightarrow aAB \\ \tilde{B} &\rightarrow bA \\ \tilde{C} &\rightarrow aA\end{aligned}$$

Then, by using these productions, we perform the folding steps at the leaves and at the roots of those trees (see the underlined substrings in Figure 4.3.4 on the preceding page). Finally, from the root-to-leaf paths of those trees, after the folding steps, we get the following productions in Greibach normal form (see also Figure 4.3.5):

$$\begin{aligned}S &\rightarrow a\tilde{A} \mid a\tilde{B} \mid a \mid b\tilde{C} \mid b \mid \varepsilon \\ \tilde{A} &\rightarrow a\tilde{A} \mid a\tilde{B} \mid a \\ \tilde{B} &\rightarrow b\tilde{C} \mid b \\ \tilde{C} &\rightarrow a\tilde{C} \mid a\end{aligned}$$

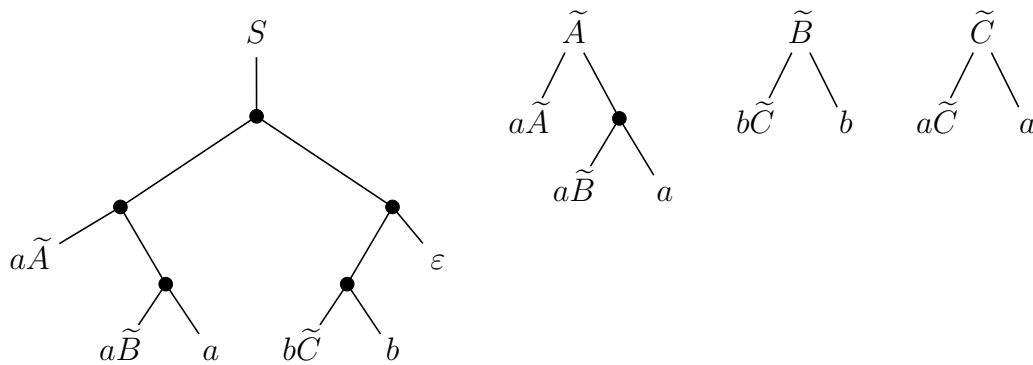


FIGURE 4.3.5. Root-to-leaf paths of the trees of Figure 4.3.4 on the preceding page after performing the folding steps.

Note that no unit productions are generated. We leave it to the reader to check that these productions belong to an $LL(2)$ grammar. \square

The following example is a bit more complex than the above Example 4.3.10 on page 69, in the sense that the sentential forms of some of the leaves will be viewed as

concatenations of the form $a\sigma_1 \dots \sigma_n$ with $n > 1$, while in Example 4.3.10 we always had $n = 1$.

EXAMPLE 4.3.11. [From $LL(1)$ Grammars to $LL(2)$ Grammars in Greibach Normal Form: Example 2] Let us consider the $LL(1)$ grammar G_1 with axiom T and the following productions:

$$\begin{aligned} T &\rightarrow FT' \\ T' &\rightarrow \varepsilon \quad | \quad \times FT' \\ F &\rightarrow (T) \quad | \quad a \end{aligned} \tag{G_1}$$

This grammar G_1 is an $LL(1)$ grammar which can be derived from the following left recursive grammar G_2 (which is *not* an $LL(1)$ grammar) with productions:

$$\begin{aligned} T &\rightarrow F \quad | \quad T \times F \\ F &\rightarrow (T) \quad | \quad a \end{aligned} \tag{G_2}$$

Starting from the axiom T of the grammar G_1 , by performing some unfolding steps by using the productions of the grammar G_1 itself, according to the rules described in the above Example 4.3.10 on page 69, we get the three trees of sentential forms depicted in Figure 4.3.6.

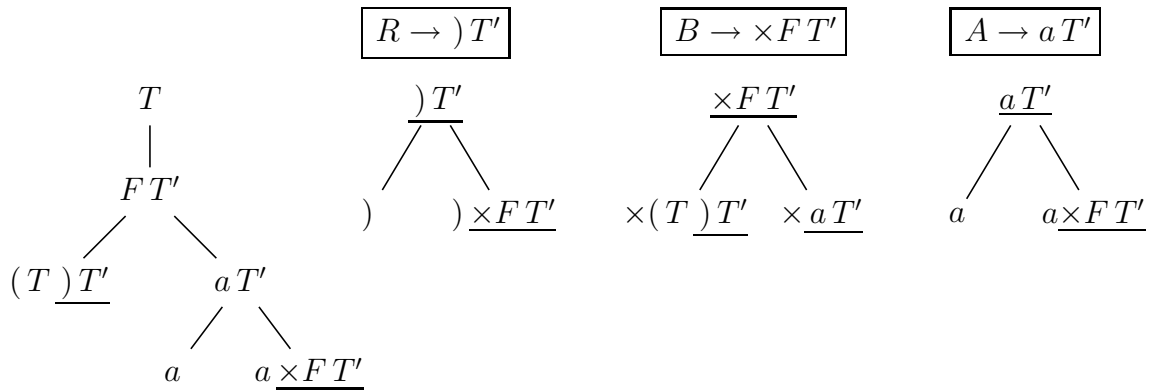


FIGURE 4.3.6. Trees of sentential forms obtained by unfolding the left-most nonterminals. The given grammar has axiom T and productions: $T \rightarrow FT'$, $T' \rightarrow \varepsilon \mid \times FT'$, $F \rightarrow (T) \mid a$. The only nullable symbol is T' . We have indicated within boxes the productions which are generated by the roots of trees of sentential forms whose root is not the axiom T .

Note that: (i) the string $\times(T)T'$ has been viewed as the concatenation $\times\sigma_1\sigma_2\sigma_3$ where $\sigma_1 = ($, $\sigma_2 = T$, and $\sigma_3 =)T'$, and (ii) the string $\times aT'$ has been viewed as the concatenation $\times\sigma_1$ where $\sigma_1 = aT'$.

Then we introduce the four productions:

$$\begin{aligned}
R &\rightarrow)T' \\
B &\rightarrow \times F T' \\
P &\rightarrow (\\
A &\rightarrow aT'
\end{aligned}$$

In particular, (i) the production $P \rightarrow ($ comes from $\sigma_1 = ($ in $\times(T)T' = \times\sigma_1\sigma_2\sigma_3$, and (ii) the production $A \rightarrow aT'$ comes from $\sigma_1 = aT'$ in $\times aT' = \times\sigma_1$.

Then if we perform the folding steps by using these productions, we get from the root-to-leaf paths of the trees of Figure 4.3.6 on the preceding page, the grammar G_3 which has the following productions in Greibach normal form:

$$\begin{aligned}
T &\rightarrow (TR \quad | \quad a \quad | \quad aB \\
R &\rightarrow) \quad | \quad)B \\
B &\rightarrow \times PTR \quad | \quad \times A \\
P &\rightarrow (\\
A &\rightarrow a \quad | \quad aB
\end{aligned} \tag{G_3}$$

No unit productions are generated, and since the axiom T is not nullable, we do not have the production $T \rightarrow \varepsilon$. We leave it to the reader to check that the grammar G_3 is an $LL(2)$ grammar (see, in particular, the productions for B). \square

FACT 4.3.12. For any $k \geq 0$, there are languages which are generated by an $LL(k+1)$ grammar and cannot be generated by any $LL(k)$ grammar.

FACT 4.3.13. [Decidability of Equivalence for $LL(k)$ grammars] For any $k \geq 0$, it is decidable whether or not two $LL(k)$ grammars are equivalent, that is, they generate the same language.

FACT 4.3.14. [Grammars in Greibach Normal Form and $LL(1)$ Grammars and Languages] If a grammar G is in Greibach normal form and for each nonterminal A and terminal a , there exists at most one production of the form $A \rightarrow a\alpha$ for some $\alpha \in V_N^*$ and the only ε -production allowed is $S \rightarrow \varepsilon$, then G is $LL(1)$.

Recall that in any $LL(1)$ grammar we allow ε -productions for *each* nonterminal symbol and, thus, the class of such grammars properly includes the class of the grammars in Greibach normal form mentioned in the above Fact 4.3.14.

As already mentioned, we have the following result.

FACT 4.3.15. [$LL(0)$ Languages] A language $L \in LL(0)$ iff $L = \emptyset$ or L is a singleton.

FACT 4.3.16. [4, page 689] [Grammars in Chomsky Normal Form and $LL(k)$ Grammars and Languages] For any $k \geq 0$, every $LL(k)$ language has an $LL(k+1)$ grammar in Chomsky normal form.

THEOREM 4.3.17. [Hierarchy of $LL(k)$ Grammars and Languages] For every $k \geq 0$, (i) the class of the $LL(k)$ grammars is properly contained in the class of the $LL(k+1)$ grammars, and (ii) the class of the $LL(k)$ languages is properly contained in the class of the $LL(k+1)$ languages.

FACT 4.3.18. [*LL(k) Languages Contained in LR(1) Languages*] For every $k \geq 0$, the class of the *LL(k)* languages is properly contained in the class of the *LR(1)* languages [22] (see also Section 5.8 on page 115).

Since, as we will state in the following Section 5.1, the class of the *LR(1)* languages coincides with the class of the deterministic context-free languages, we have the following consequence of Fact 4.3.18: every language which is generated by an *LL(k)* grammar, for some $k \geq 0$, is a deterministic context-free language, and there exists a proper subclass \mathcal{C} of the deterministic context-free languages such that for every language L in \mathcal{C} there is no $k \geq 0$ such that L can be generated by an *LL(k)* grammar.

FACT 4.3.19. [*LL(k) Languages Contained in LR(1) Languages: an Example*] The language $L = \{a^n b^n \mid n \geq 1\} \cup \{a^n c^n \mid n \geq 1\}$ is a deterministic context-free language (and, thus, an *LR(1)* language) and there is no $k \geq 0$ such that L is an *LL(k)* language [4, page 689].

One can show that the language $L = \{a^n b^n \mid n \geq 1\} \cup \{a^n c^n \mid n \geq 1\}$ is an *LR(1)* language by using the techniques we will present in the following Section 5.1.

The following fact is an immediate consequence of the Kleene Theorem (see, for instance, [21, page 42]).

FACT 4.3.20. [*Regular Languages and LL(1) Grammars*] Every regular language has an *LL(1)* grammar.

Parsers for Deterministic Context-Free Languages: LR(k) Parsers

In this chapter we will present the so called $LR(k)$ parsers for deterministic context-free languages.

5.1. Introduction to LR(k) Parsing

The $LR(k)$ parsers, for any $k \geq 0$, are algorithms for parsing the languages which are generated by the $LR(k)$ grammars which are defined as follows [3, pages 372-373].

DEFINITION 5.1.1. [$LR(k)$ Grammar. Version 1] For any $k \geq 0$, a context-free grammar $G = \langle V_T, V_N, P, S \rangle$, possibly with ε -productions and without any occurrence of the start symbol S on the right hand side of any production, is an $LR(k)$ grammar if for $\$ \notin V_T$, for every sentential form $\alpha\beta\gamma \in (V_T \cup V_N)^*$, there exists a rightmost derivation of the form:

$$S \$^k \xrightarrow{*}_{rm} \alpha A \gamma \$^k \xrightarrow{rm} \alpha \beta \gamma \k$

where: (i) the symbol $\$$ occurs only on the rightmost positions of every sentential form, (ii) $A \in V_N$, and (iii) the substring β which replaces A , can be determined by knowing the leftmost k symbols of $\gamma \k (see also Figure 5.4.9 on page 103).

Here is an equivalent definition of the $LR(k)$ grammars [3, page 372].

DEFINITION 5.1.2. [$LR(k)$ Grammar. Version 2] For any $k \geq 0$, a context-free grammar $G = \langle V_T, V_N, P, S \rangle$, possibly with ε -productions and without any occurrence of the start symbol S on the right hand side of any production, is an $LR(k)$ grammar if for $\$ \notin V_T$, for any $w, x, y \in V_T^*$, for any $\alpha, \beta, \gamma \in (V_T \cup V_N)^*$, and for any $A, B \in V_N$,

$$\text{if } (1) \ S \xrightarrow{*}_{rm} \alpha A w \xrightarrow{rm} \alpha \beta w \quad \text{and}$$

$$(2) \ S \xrightarrow{*}_{rm} \gamma B x \xrightarrow{rm} \alpha \beta y \quad \text{and}$$

$$(3) \ \underline{w}_k = \underline{y}_k$$

then $\alpha = \gamma$, $A = B$, and $x = y$. □

Note that in the case of the rightmost derivations in Conditions (1) and (2) of the above definition, the production which is applied in the last step is $A \rightarrow \beta$.

Note also that given a context-free grammar $G = \langle V_T, V_N, P, S \rangle$, we can avoid every occurrence of the start symbol S on the right hand side of the productions, by considering the new, equivalent grammar $G' = \langle V_T, V_N \cup \{S'\}, P \cup \{S' \rightarrow S\}, S' \rangle$, where S' is a new start symbol. It is easy to see that $L(G) = L(G')$ because the language generated by the symbol S in G is equal to the language generated by the symbol S' in G' .

In what follows we will introduce a few algorithms for parsing various subclasses of context-free languages. In particular, we will introduce an algorithm for parsing languages generated by:

- (i) $LR(0)$ grammars (see page 77),
- (ii) $SLR(1)$ grammars (see page 87),
- (iii) $LR(1)$ grammars (see page 90), and
- (iv) $LALR(1)$ grammars (see page 104).

Unless otherwise specified, in this section we assume that V_T, V_N , and S denote, respectively, the set of the terminal symbols, the set of nonterminal symbols, and the start symbol. As usual, we will use the symbol V to denote the set $V_T \cup V_N$.

When presenting the $LR(k)$ parsers, for any given context-free grammar $G = \langle V_T, V_N, P, S \rangle$, we consider its augmented version which is defined as follows.

DEFINITION 5.1.3. [Augmented Context-free Grammars] Given a context-free grammar $G = \langle V_T, V_N, P, S \rangle$ with start symbol S , its *augmented grammar*, call it $G' = \langle V_T \cup \{\$, V_N \cup \{S'\}, P', S' \rangle$, is the grammar obtained by considering: (i) the new start symbol $S' \notin V_N$, (ii) a new terminal symbol $\$ \notin V_T$, and (iii) the extra production $S' \rightarrow S \$$, that is, $P' = P \cup \{S' \rightarrow S \$\}$. In some cases we will assume that the extra production is $S' \rightarrow S$, instead of $S' \rightarrow S \$$. We will always indicate whether in the augmented grammar the extra production is $S' \rightarrow S \$$ or $S' \rightarrow S$ (see also Table 2 on page 116).

We have the following fact.

FACT 5.1.4. [Languages Generated by Grammars and Augmented Grammars] The language $L(G)$ generated by a context-free grammar G is equal to the language $L(G')$ generated by the augmented grammar G' if G' has the extra production $S' \rightarrow S$, while if G' has the extra production $S' \rightarrow S \$$ then $L(G') = L(G) \$$.

For the grammars we consider when presenting the $LR(k)$ parsers we assume the following hypotheses.

Hypotheses for $LR(k)$ Parsing, for $k \geq 0$.

- (i) The start symbol S (or S' in case of augmented grammars) does *not* occur on the right hand side of any production.
- (ii) No useless symbols occur in the productions.
- (iii) ε -productions may be present for the start symbol or any other symbols of V_N .
- (iv) The symbol $\$$ does not belong to the set $V_T \cup V_N$.
- (v) The input string to be parsed is terminated by the symbol $\$$ which can occur only at the right end of the input string.
- (vi) The symbol $\$$ does not occur on the right hand side of any production with the exception that it may occur in the production $S' \rightarrow S \$$, from the new start symbol S' of the augmented grammar.

5.2. LR(0) Parsers

In this section we will study the $LR(0)$ parsers and the $LR(0)$ grammars. In these parsers the parsing automaton we have depicted in Figure 5.2.1 on the following page, makes its moves according to a table T which is a matrix whose rows are labeled by the states of a finite automaton M we will define, and whose columns are labeled by the terminal symbols in V_T , $\$$, and the nonterminal symbols in V_N of the given $LR(0)$ grammar. The columns labeled by the elements in $V_T \cup \{\$\}$ constitute the so called *action part* of the table, while the columns labeled by the elements in V_N constitute the so called *goto part* of the table.

In the sequel we need the following definition.

DEFINITION 5.2.1. [Item and Complete Item] Given a context-free grammar $G = \langle V_T, V_N, P, S \rangle$, a production of G with an extra dot in any position in the right hand side, is called an *item* of G . Thus, a production whose right hand side has length n may give rise to $n+1$ items. In particular, a production of the form $A \rightarrow \varepsilon$, where A is a nonterminal symbol and ε is the empty string, may give rise to the following item only: $A \rightarrow \cdot$ (whose right hand side consists of the dot only).

An item is said to be *complete* if it is of the form $A \rightarrow \beta \cdot$, where $\beta \in (V_T \cup V_N)^*$, that is, the dot is at the rightmost position.

EXAMPLE 5.2.2. The context-free grammar G with axiom S and whose productions are:

1. $S \rightarrow SA$
2. $S \rightarrow A$
3. $A \rightarrow aSb$
4. $A \rightarrow ab$

gives rise to the following twelve items:

$$\begin{array}{cccc}
 S \rightarrow \cdot SA & S \rightarrow \cdot A & A \rightarrow \cdot aSb & A \rightarrow \cdot ab \\
 S \rightarrow S \cdot A & \boxed{S \rightarrow A \cdot} & A \rightarrow a \cdot Sb & A \rightarrow a \cdot b \\
 \boxed{S \rightarrow SA \cdot} & & A \rightarrow aS \cdot b & \boxed{A \rightarrow ab \cdot} \\
 & & \boxed{A \rightarrow aSb \cdot} &
 \end{array}$$

Among those items we have four complete items which are those within rectangles.

Now we present an algorithm in five steps which given an $LR(0)$ context-free grammar, constructs a table T which is needed for specifying the parsing automaton of Figure 5.2.1 on the next page. Then, we will explain how, once that table has been constructed, we can use that automaton for performing the $LR(0)$ parsing, that is, we will explain how that automaton can parse any given word and check whether or not it belongs to the language generated by the given $LR(0)$ context-free grammar.

In order to fix our ideas, each step of the algorithm will be illustrated using a running example which refers to the context-free grammar G of the above Example 5.2.2.

ALGORITHM 5.2.3. Constructing the $LR(0)$ Parsing Table.

We are given a context-free grammar $G = \langle V_T, V_N, P, S \rangle$ and we want to construct the $LR(0)$ parsing table for the automaton of Figure 5.2.1 on the next page.

Step (1). First we consider the augmented grammar $G' = \langle V_T \cup \{\$\}, V_N \cup \{S'\}, P', S' \rangle$, where the set P' of productions is $P \cup \{S' \rightarrow S \$\}$. As usual, we assume that $\$ \notin V_T$

$A \rightarrow \cdot$

Note also that in a big production: (i) the *leftmost node* is an item of the grammar G' of the form $A \rightarrow \cdot \alpha_1 \dots \alpha_n$ with $A \in V_N$ and for $i = 1, \dots, n$, we have that $\alpha_i \in V_T \cup V_N \cup \{\$\}$, and (ii) any other node is an item of the grammar G' with the dot in a different position, and (iii) the *rightmost node* is a complete item of the grammar G' .

In a big production the arc from a node m to a node n is labeled by the symbol in $V_T \cup V_N \cup \{\$\}$ which in the node m occurs on the right of the dot and in the node n occurs on the left of the dot.

Step (3). We consider the graph made out of all the big productions of the grammar G' we have constructed at Step (2). Then, for every nonterminal $A, B \in V_N \cup \{S'\}$ and strings $\alpha, \beta, \gamma \in (V_T \cup V_N \cup \{\$\})^*$, we add an arc labeled by ϵ from every node whose label is an item of the form $A \rightarrow \alpha \cdot B \beta$ to every node whose label is the item $B \rightarrow \cdot \gamma$. These ϵ -arcs may relate either nodes into two different big productions or nodes of the same big production.

In the case of our augmented grammar G' , at the end of Step (3) we get the graph of Figure 5.2.2.

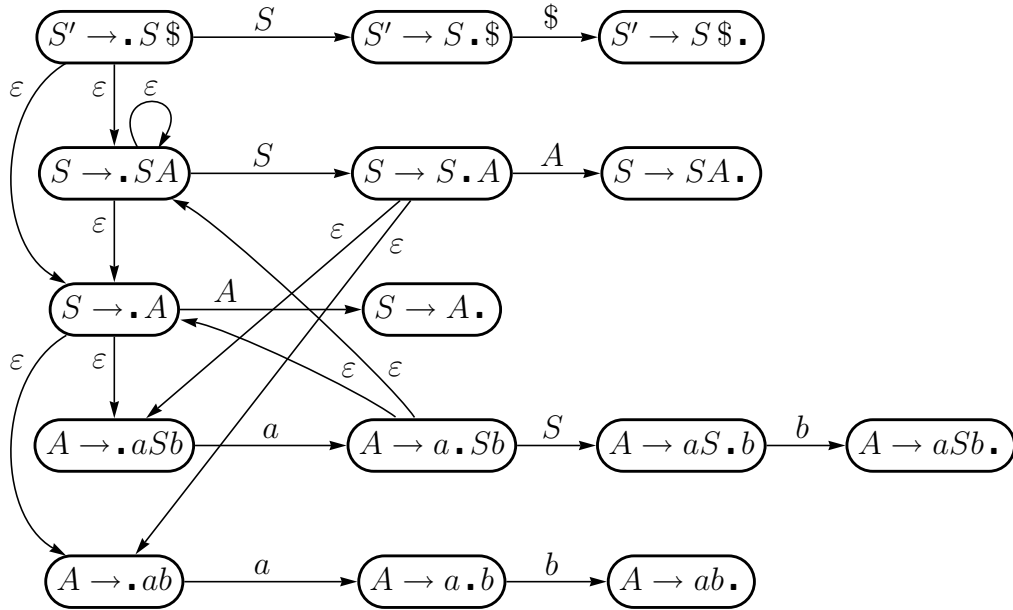


FIGURE 5.2.2. The graph of the big productions with the ϵ -arcs for the grammar G' with axiom S' and productions: 0. $S' \rightarrow S \$$, 1. $S \rightarrow S A$, 2. $S \rightarrow A$, 3. $A \rightarrow a S b$, 4. $A \rightarrow a b$. Each big production is depicted as a sequence of nodes which are horizontally aligned.

Step (4). By applying the Powerset Construction Procedure (see, for instance, [21]) to the graph of the big productions obtained at the end of Step (3), we get a finite automaton. Let that finite automaton be called M .

In the case of our augmented grammar G' , at the end of Step (4) we get the finite automaton of Figure 5.2.3.

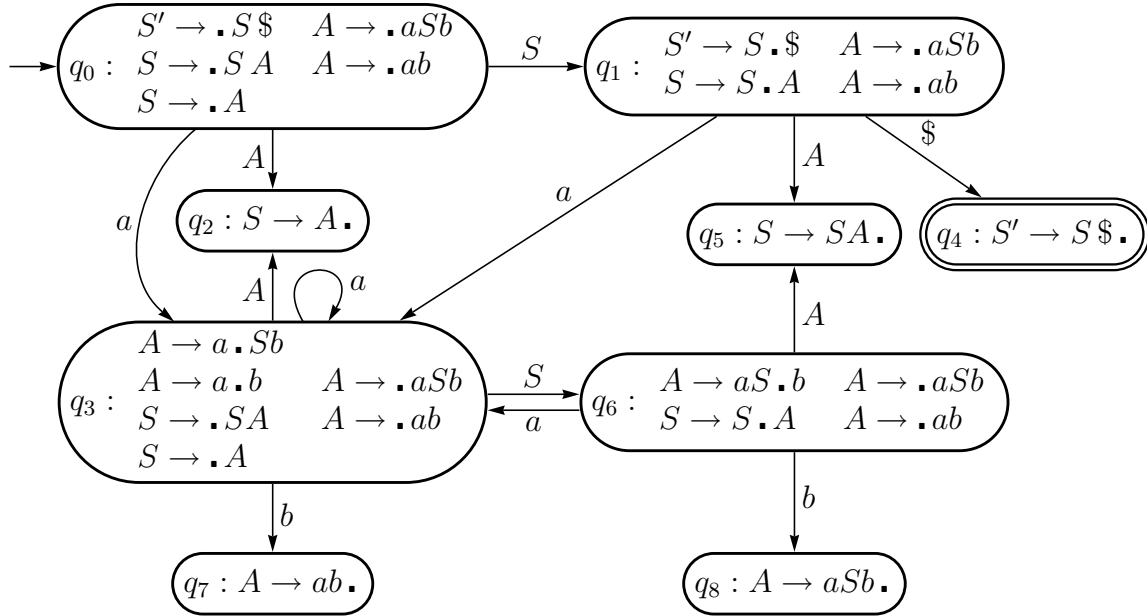


FIGURE 5.2.3. The finite automaton M for the $LR(0)$ parsing of the grammar G' with axiom S' and productions: 0. $S' \rightarrow S \$$, 1. $S \rightarrow SA$, 2. $S \rightarrow A$, 3. $A \rightarrow aSb$, 4. $A \rightarrow ab$. The label of each state is a set of items which is computed by the Powerset Construction Procedure from the graph of Figure 5.2.2 on the preceding page.

Recall that as a result of the Powerset Construction, each state of M is labeled by a set of items of the grammar G' . The initial state of M is the one which has in its label the item of the leftmost node of the big production associated with $S' \rightarrow S \$$, that is, the item $S' \rightarrow \cdot S \$$. The items in the states of the automaton M include those of the grammar G and the three extra items $S' \rightarrow \cdot S \$$, $S' \rightarrow S \cdot \$$, and $S' \rightarrow S \$ \cdot$, which are generated by the production $S' \rightarrow S \$$ of the augmented grammar G' .

The final state is the one which has in its label the item $S' \rightarrow S \$ \cdot$. (As usual we write double lines around the final states.) As indicated in Remark 5.2.4 on page 78, if the augmented grammar is equal to the given grammar with axiom S such that: (i) the only production for S is $S \rightarrow \alpha$, with $\alpha \in V^*$, and (ii) S does not occur on the right hand side of any production, then the initial state is the one which has in its label the item $S \rightarrow \cdot \alpha \$$ and the final state is the one which has in its label the item $S \rightarrow \alpha \$ \cdot$.

Step (5). The $LR(0)$ parsing table T is then constructed from the automaton M as follows. The meaning of the various entries of that table will be clarified when we will describe the behaviour of the parsing automaton of Figure 5.2.1 on page 78 which uses that parsing table.

(5.1) For each transition $\textcircled{q_i} \xrightarrow{a} \textcircled{q_j}$ where a is terminal symbol different from $\$$,

we insert in the element $T[q_i, a]$ which is in the *action* part of the table T , the entry $sh\ q_j$ (short for *shift* q_j). This means that the parsing automaton should take the input symbol a at hand, push it on the stack, and push the state q_j on the stack. Then the new input symbol at hand is the one to the right of the previous symbol a at hand. This action of the parsing automaton is called a *shift action*.

(5.2) For the transition $\textcircled{q_i} \xrightarrow{\$} \textcircled{\textcircled{q_j}}$

we insert in the element $T[q_i, \$]$ which is in the *action* part of the table T , the entry acc (short for *accept*). This means that the parsing automaton should accept the given input string. This action of the parsing automaton is called the *accept action*.

(5.3) For each transition $\textcircled{q_i} \xrightarrow{A} \textcircled{q_j}$ with the nonterminal symbol A ,

we insert in the element $T[q_i, A]$ which is in the *goto* part of the table T , the entry q_j . This means that the parsing automaton should push on the stack the state q_j (see the following Step (5.4)). This action of the parsing automaton is called a *goto action*.

(5.4) For each complete item $A \rightarrow \alpha_1 \dots \alpha_n \cdot$ for some $n \geq 0$, where the α_i 's are symbols in $V_T \cup V_N$, in the label of a state q_i ,

we insert in each element of row q_i in the *action* part of the table T , the entry $red\ p$ (short for *reduce production* p), where p is the number identifying the production $A \rightarrow \alpha_1 \dots \alpha_n$ of the grammar G' . This means that the parsing automaton should replace the string $\alpha_1 q_{\alpha_1} \dots \alpha_n q_{\alpha_n}$ which is on the stack (the top element being state q_{α_n}) by the nonterminal A . This action of the parsing automaton is called a *reduce action*.

In the case of our augmented grammar G' at the end of Step (5) we get the parsing table T of Figure 5.2.4 on the next page. The numbers which identify the productions of the grammar G' are necessary for identifying the *reduce* actions, and they are as follows:

- | | | | | |
|--------------------------|-----------------------|----------------------|------------------------|-----------------------|
| 0. $S' \rightarrow S \$$ | 1. $S \rightarrow SA$ | 2. $S \rightarrow A$ | 3. $A \rightarrow aSb$ | 4. $A \rightarrow ab$ |
|--------------------------|-----------------------|----------------------|------------------------|-----------------------|
-

Now let us explain how the parsing table T is used by the parsing automaton of Figure 5.2.1 on page 78 for performing the $LR(0)$ parsing. Initially, the stack has the initial state q_0 only. Then the parsing automaton reads the input string (which is terminated by the symbol $\$$), one symbol at a time from left to right, and it performs the actions as indicated by the table T . If the top of the stack is the state q and the input symbol is a , for any $a \in V_T \cup \{\$\}$, then the parsing automaton considers the entry $T[q, a]$. There are three cases.

Case (1): shift and goto. If $T[q, a] = sh\ q'$, then the parsing automaton pushes the symbol a from the input string on the stack, and then it pushes the state q' on the stack.

Case (2): accept. If $T[q, a] = acc$, then the parsing automaton accepts the whole input string.

	<u>action</u>			<u>goto</u>		
	<i>a</i>	<i>b</i>	<i>§</i>	<i>S</i>	<i>A</i>	
q_0	<i>sh</i> q_3			q_1	q_2	
q_1	<i>sh</i> q_3		<i>acc</i>		q_5	0. $S' \rightarrow S\ \$$
q_2	<i>red</i> 2	<i>red</i> 2	<i>red</i> 2			1. $S \rightarrow SA$
q_3	<i>sh</i> q_3	<i>sh</i> q_7		q_6	q_2	where: 2. $S \rightarrow A$
q_4	<i>red</i> 0	<i>red</i> 0	<i>red</i> 0			3. $A \rightarrow aSb$
q_5	<i>red</i> 1	<i>red</i> 1	<i>red</i> 1			4. $A \rightarrow ab$
q_6	<i>sh</i> q_3	<i>sh</i> q_8			q_5	
q_7	<i>red</i> 4	<i>red</i> 4	<i>red</i> 4			
q_8	<i>red</i> 3	<i>red</i> 3	<i>red</i> 3			

FIGURE 5.2.4. $LR(0)$ parsing table T for the grammar G' with axiom S' and productions: 0. $S' \rightarrow S\ \$$, 1. $S \rightarrow SA$, 2. $S \rightarrow A$, 3. $A \rightarrow aSb$, 4. $A \rightarrow ab$. It has been derived from the finite automaton of Figure 5.2.3 on page 80. The row for state q_4 can be eliminated, because in state q_1 , when $\$$ is read, the input string is accepted.

Case (3): reduce and goto. If $T[q, a] = \text{red } p$, where p is the number identifying the production $A \rightarrow \alpha_1 \dots \alpha_n$ (with $\alpha_i \in V_T \cup V_N \cup \{\ \$\}$, for $i = 1, \dots, n$) of the grammar G' , then the parsing automaton pops from the stack the whole string $\alpha_1 q_{\alpha_1} \dots \alpha_n q_{\alpha_n}$ (the top of the stack being the state q_{α_n}). This leaves a state, say \tilde{q} , on the top of the stack. Then the parsing automaton pushes A on the stack and then pushes also the state $T[\tilde{q}, A]$. Thus, after every *reduce* action the parsing automaton performs a *goto* action.

The behaviour of the parsing automaton we have described, satisfies the following property: every symbol α in $V_T \cup V_N \cup \{\ \$\}$ on the stack has above itself a state of the automaton M . We call that state the *cover state of the symbol* α .

Figure 5.2.5 on the facing page shows the behaviour of the automaton of Figure 5.2.1 on page 78 using the $LR(0)$ parsing table of Figure 5.2.4 on page 82, while parsing the input string $ababb\ \$$ which is read from left to right. The symbol \blacktriangle points at the input character at hand. The stack grows and shrinks at the right end, and the symbol \blacktriangle points at the top of the stack. In that figure time flows, so to speak, from top to bottom in the sense that, for any $i \geq 0$, the $\langle \text{stack}, \text{input} \rangle$ configuration at time $i+1$ is generated by the $\langle \text{stack}, \text{input} \rangle$ configuration at time i according to the actions described in the last column, which we named the *shift / reduce-goto* column.

In the literature, for every state q of the automaton M and for every symbol $a \in V_T \cup \{\ \$\}$, the entry $T[q, a]$ is also called '*action*(q, a)', and for every $A \in V_N$, the entry $T[q, A]$ is also called '*goto*(q, A)' (see also Figure 5.2.5 on the facing page).

With reference to Figure 5.2.5, the reader should note that the sequence of stack configurations shows *in reverse* the rightmost derivation of the input string when it

time	stack (the top is marked by ▲)	input (the symbol at hand is marked by ▲)	<i>shift / reduce-goto</i>
0	q_0 ▲	$a a b a b b \$$ ▲	$action(q_0, a) = sh\ q_3$
1	$q_0 a q_3$ ▲	$a a b a b b \$$ ▲	$action(q_3, a) = sh\ q_3$
2	$q_0 a q_3 a q_3$ ▲	$a a b a b b \$$ ▲	$action(q_3, b) = sh\ q_7$
3	$q_0 a q_3 a q_3 b q_7$ ▲	$a a b a b b \$$ ▲	$action(q_7, a) = red\ 4\ (A \rightarrow ab)$ $goto(q_3, A) = q_2$
4	$q_0 a q_3 A q_2$ ▲	$a a b a b b \$$ ▲	$action(q_2, a) = red\ 2\ (S \rightarrow A)$ $goto(q_3, S) = q_6$
5	$q_0 a q_3 S q_6$ ▲	$a a b a b b \$$ ▲	$action(q_6, a) = sh\ q_3$
6	$q_0 a q_3 S q_6 a q_3$ ▲	$a a b a b b \$$ ▲	$action(q_3, b) = sh\ q_7$
7	$q_0 a q_3 S q_6 a q_3 b q_7$ ▲	$a a b a b b \$$ ▲	$action(q_7, b) = red\ 4\ (A \rightarrow ab)$ $goto(q_6, A) = q_5$
8	$q_0 a q_3 S q_6 A q_5$ ▲	$a a b a b b \$$ ▲	$action(q_5, b) = red\ 1\ (S \rightarrow SA)$ $goto(q_3, S) = q_6$
9	$q_0 a q_3 S q_6$ ▲	$a a b a b b \$$ ▲	$action(q_6, b) = sh\ q_8$
10	$q_0 a q_3 S q_6 b q_8$ ▲	$a a b a b b \$$ ▲	$action(q_8, \$) = red\ 3\ (A \rightarrow aSb)$ $goto(q_0, A) = q_2$
11	$q_0 A q_2$ ▲	$a a b a b b \$$ ▲	$action(q_2, \$) = red\ 2\ (S \rightarrow A)$ $goto(q_0, S) = q_1$
12	$q_0 S q_1$ ▲	$a a b a b b \$$ ▲	$action(q_1, \$) = accept$

FIGURE 5.2.5. The sequence of configurations of the parsing automaton of Figure 5.2.1 on page 78 while parsing the string $a a b a b b \$$ using the $LR(0)$ parsing table of Figure 5.2.4 on the facing page for the grammar G' with axiom S' and productions: 0. $S' \rightarrow S \$$, 1. $S \rightarrow SA$, 2. $S \rightarrow A$, 3. $A \rightarrow aSb$, 4. $A \rightarrow ab$.

is accepted. In our case, in fact, by looking at the *reduce* action, we get the following rightmost derivation of $a a b a b b \$$ from the axiom S' :

$$\begin{aligned} S' &\xrightarrow{rm} (12) S \$ && \xrightarrow{rm} (11) A \$ && \xrightarrow{rm} (10) a S b \$ && \xrightarrow{rm} \\ &\xrightarrow{rm} (8) a S A b \$ && \xrightarrow{rm} (7) a S a b b \$ && \xrightarrow{rm} (4) a A a b b \$ && \xrightarrow{rm} \\ &\xrightarrow{rm} (3) a a b a b b \$. \end{aligned}$$

Since, according to our conventions production 0 of the grammar G' is $S' \rightarrow S \$$, and the symbol $\$$ occurs in this production only, we have that the entry *red* 0 is never present in the sequence of configurations of the parsing automaton of Figure 5.2.1 on page 78, such as those we have depicted in Figure 5.2.5 on the preceding page.

Moreover, in the $LR(0)$ parsing table we can always eliminate the row corresponding to the final state, because in that state, when $\$$ is read, the input string is accepted (see Step (5.2) of Algorithm 5.2.3 on page 77). For example, the row for state q_4 in Table 5.2.4 on page 82 can be eliminated. Therefore, in Step (5.4) of Algorithm 5.2.3 (see page 81) we could require that the state q_i be *not final* and we could eliminate from the $LR(0)$ parsing table the row corresponding to the final state.

Now let us make a few remarks on Algorithm 5.2.3 for constructing the $LR(0)$ parsing table.

REMARK 5.2.5. [*Reduce action is independent of the input symbol*] The *reduce* action does *not* depend on the input symbol at hand. For instance, in table T of Figure 5.2.4 on page 82 for the state q_2 we have that: $T[q_2, a] = T[q_2, b] = T[q_2, \$] = \text{red } 2$. Analogously, for the states q_4, q_5, q_7 , and q_8 . \square

REMARK 5.2.6. [*Multiple actions*] One can show that if during the construction of the table T we have to place more than one action in a given entry or, in particular, more than one complete item occurs as the label of a state of the finite automaton M , then the augmented grammar G' which has been derived from the given grammar G , is *not* $LR(0)$ [10]. This means that the language $L(G') = L(G) \$$ which is generated by the grammar G' cannot be parsed using the $LR(0)$ parsing algorithm.

If we have to place more than one action in a given entry of the table T , we say that there is a *conflict*. Depending on the actions simultaneously present in the same entry, a conflict can be either a *shift-shift conflict*, or a *shift-reduce conflict*, or *reduce-reduce conflict*. \square

REMARK 5.2.7. [*Undefined entry*] If during the parsing process of the input word $w \$$, the pushdown automaton of Figure 5.2.1 on page 78 when using the table T , finds an undefined entry, then $w \$$ is *not* in the $LR(0)$ language $L(G')$ generated by the augmented grammar G' and thus, the word w is *not* in the language generated by the given grammar G . \square

One can show that the following is an alternative definition of $LR(0)$ grammars. This definition is based on the construction of the finite automaton M of Step (4) of Algorithm 5.2.3 on page 77 [10, page 252].

DEFINITION 5.2.8. [$LR(0)$ Grammar and $LR(0)$ Language] A context-free grammar G is an $LR(0)$ grammar if

(i) the start symbol does *not* occur in the right hand side of any production, and
(ii) in every state of the automaton M of Step (4) of Algorithm 5.2.3 on page 77 if there is a complete item of G then in that state no other complete item occurs nor any item with a terminal symbol immediately to the right of the dot.

A language is an $LR(0)$ language if it is generated by an $LR(0)$ grammar.

Now we give an example of a grammar which is an $LR(0)$ grammar and it is *not* $LL(k)$ for any $k \geq 0$.

EXAMPLE 5.2.9. [**An $LR(0)$ Grammar**] Let us consider the grammar G with axiom S and whose productions are:

$$\begin{array}{l} S \rightarrow A \quad | \quad B \\ A \rightarrow aAb \quad | \quad 0 \\ B \rightarrow aBbb \quad | \quad 1 \end{array}$$

The language generated by this grammar is:

$$L = \{a^n 0 b^n \mid n \geq 0\} \cup \{a^n 1 b^{2n} \mid n \geq 0\}.$$

The grammar G is $LR(0)$ and it is *not* $LL(k)$ for any $k \geq 0$.

Now we will present a few results which we state without proof. First, we recall the following definitions (see [21, Section 3.3]).

DEFINITION 5.2.10. [**Prefix-Free Language**] A language L is said to be *prefix-free* (or to enjoy the *prefix property*) iff *no* word in L is a proper prefix of another word in L , that is, for every word $u \in L$, the word uv for $v \neq \varepsilon$ is *not* in L .

DEFINITION 5.2.11. [**Deterministic Pushdown Automaton**] A pushdown automaton $\langle Q, \Sigma, \Gamma, q_0, Z_0, F, \delta \rangle$ is said to be a *deterministic pushdown automaton* (or a *dpda*, for short) iff

- (i) $\forall q \in Q, \forall Z \in \Gamma$, if $\delta(q, \varepsilon, Z) \neq \{\}$ then $\forall a \in \Sigma, \delta(q, a, Z) = \{\}$ (that is, no other moves are allowed when an ε -move is allowed), and
- (ii) $\forall q \in Q, \forall Z \in \Gamma, \forall x \in \Sigma \cup \{\varepsilon\}$, $\delta(q, x, Z)$ is *either* $\{\}$ *or* a singleton (that is, if a move is allowed, then that move can be made in one way only, that is, there exists only one next configuration for the dpda).

DEFINITION 5.2.12. [**Deterministic Context-Free Language**] A context-free language L is said to be a *deterministic context-free language* iff it is accepted by a deterministic pushdown automaton *by final state*, that is, for each word $w \in L$, the deterministic pushdown automaton, starting from the input string $w\$$ and the initial state q_0 , eventually reaches a final state, having read all the symbols of $w\$$.

DEFINITION 5.2.13. [**Language Accepted by a Parsing Pushdown Automaton**] Let us consider the parsing automaton A of Figure 5.2.1 on page 78, with its parsing table T . That table specifies the moves of A for each symbol in input and each symbol on the top of the stack. An input word w is accepted by the automaton A if starting from the configuration where: (i) w is the string to be read with the extra symbol $\$$ at the right end (that is, initially the input head points at the leftmost symbol of w), and (ii) the initial state q_0 is on the stack, then A eventually performs the *accept* action. The *language accepted by a parsing pushdown automaton* A with table T is the set of input words accepted by A .

Let us consider an algorithm P which given a context-free grammar G , constructs its augmented grammar G' and the parsing table $T_{G'}$ for the parsing automaton A of Figure 5.2.1 on page 78. Depending on the augmented grammar G' , the algorithm P may or may not generate a table $T_{G'}$ such that the language accepted by the parsing automaton A using the parsing table $T_{G'}$, is equal to $L(G')$.

In particular, one can show that the augmented grammar G' is an $LR(0)$ grammar iff Algorithm 5.2.3 on page 77 produces a table such that the language accepted by the parsing automaton A is equal to $L(G')$ [10].

Now we define the language which is accepted by an algorithm for constructing parsing tables.

DEFINITION 5.2.14. [Language Accepted by an Algorithm for Constructing Parsing Tables] Let us consider an algorithm P that, given a context-free grammar G , constructs the augmented grammar G' and the parsing table $T_{G'}$ for the parsing automaton A of Figure 5.2.1 on page 78. We say that the *algorithm P accepts the language L* if L is the language accepted by the parsing automaton A when it uses the parsing table $T_{G'}$.

The following Theorems 5.2.15 and 5.2.16, which we state without proof, provide two alternative definitions of the class of the $LR(0)$ languages and relate the class of deterministic context-free languages to the class of the $LR(0)$ languages.

THEOREM 5.2.15. The class of the $LR(0)$ languages is the class of the context-free languages which are accepted by Algorithm 5.2.3 on page 77. (Thus, it is appropriate to call Algorithm 5.2.3 an algorithm for constructing $LR(0)$ parsing tables.)

THEOREM 5.2.16. [$LR(0)$ Languages and Deterministic Context-Free Languages] (i) The class of the $LR(0)$ languages is the class of deterministic context-free languages which enjoy the prefix property [10, pages 121 and 260]. (ii) A language $L \subseteq \Sigma^*$ is a deterministic context-free language iff $L\$$, with $\$$ not in V_T , can be generated by an $LR(0)$ grammar [10, page 260].

5.2.1. Avoiding the Powerset Construction.

Algorithm 5.2.3 on page 77 for constructing the $LR(0)$ parsing tables, can be replaced by the following algorithm which, given a context-free grammar, first constructs the corresponding augmented grammar G' , then constructs the deterministic finite automaton M , and finally constructs the $LR(0)$ parsing table.

After the construction of the augmented grammar G' , the construction of the automaton M is done by applying the following rules for generating: (i) states, (ii) sets of items in the labels of the states, and (iii) arcs.

Initialization Rule. The initial state q_0 of the finite automaton M has its set of items initialized by the item $S' \rightarrow \cdot S \$$.

Closure Rule (1). For any $A, B \in V_N$ and $\alpha, \beta, \gamma \in V^*$, if in the label of a state there is an item of the form $A \rightarrow \alpha \cdot B \beta$ then we add to that label *all* items of the grammar G' of the form: $B \rightarrow \cdot \gamma$.

Closure Rule (2). For any $A_1, \dots, A_n, B \in V_N$ and $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n \in V^*$, if all the items in the label of a state p with the dot immediately to the left of the nonterminal B are: $A_1 \rightarrow \alpha_1 \cdot B\beta_1, \dots, A_n \rightarrow \alpha_n \cdot B\beta_n$, for some $n > 0$, then we add to M a new state, call it q , such that: (i) there is an arc with label B from the state p to the state q , and (ii) the label of q includes $A_1 \rightarrow \alpha_1 B \cdot \beta_1, \dots, A_n \rightarrow \alpha_n B \cdot \beta_n$.

Closure Rule (3). For any $a \in V_T, A_1, \dots, A_n \in V_N$, and $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n \in V^*$, if all the items in the label of a state p with the dot immediately to the left of the terminal a are: $A_1 \rightarrow \alpha_1 \cdot a\beta_1, \dots, A_n \rightarrow \alpha_n \cdot a\beta_n$, for some $n > 0$, then we add to M a new state, call it q , such that: (i) there is an arc with label a from the state p to the state q , and (ii) the label of q includes $A_1 \rightarrow \alpha_1 a \cdot \beta_1, \dots, A_n \rightarrow \alpha_n a \cdot \beta_n$.

Those rules allow us to construct the deterministic finite automaton M without constructing the so called big productions and without applying the Powerset Construction Procedure. From M we then construct the $LR(0)$ parsing table T as indicated in Step (5) of Algorithm 5.2.3 on page 77. \square

5.3. SLR(1) Parsers

In this section we will describe a new algorithm for constructing the parsing table of the automaton of Figure 5.2.1 on page 78. This algorithm is a variant of Algorithm 5.2.3 on page 77 for constructing $LR(0)$ parsing tables. It provides the definition of a subclass of the deterministic context-free languages, called $SLR(1)$ (short for *Simple LR(1)*) languages, as specified by the following definition.

DEFINITION 5.3.1. [$SLR(1)$ Languages] A language is an $SLR(1)$ language iff it is accepted by the following Algorithm 5.3.2 (this notion of acceptance is given in Definition 5.2.14 on the preceding page).

Here is the algorithm for constructing $SLR(1)$ parsing tables.

ALGORITHM 5.3.2. *Constructing the $SLR(1)$ Parsing Table.*

Given a context-free grammar G , we first consider its augmented grammar G' . Then, in order to fill the entries of the $SLR(1)$ parsing table T for the parsing automaton of Figure 5.2.1 on page 78, we use the Algorithm 5.2.3 on page 77 for constructing $LR(0)$ parsing tables, with the following rule (5.4*), instead of rule (5.4) on page 81: (5.4*) for every state q of the deterministic finite automaton M ,

if there exists in q a complete item $A \rightarrow \alpha \cdot$, for some $A \in V_N$ and $\alpha \in V^*$,
then for every terminal symbol a of the set $Follow_1(A)$, we insert in the entry $T[q, a]$ of the parsing table the *reduce* action $red\ p$, where p identifies the production $A \rightarrow \alpha$ of the augmented grammar G' .

If by using this algorithm we fill some entries of the table T by either (i) a *shift* action (in the *action* columns), or (ii) a *reduce* action (in the *action* columns), or (iii) a *goto* action (in the *goto* columns), and no more than one action is placed in every entry of that table, then the augmented grammar G' is said to be an $SLR(1)$ grammar. Note that the parameter 1 in the qualifier ' $SLR(1)$ ' is due to the fact that in our algorithm we have used the set $Follow_1(A)$, rather than the set $Follow_k(A)$, for some $k > 1$.

Here is an example of an $SLR(1)$ grammar and the construction of corresponding parsing table according to Algorithm 5.3.2 on the preceding page.

EXAMPLE 5.3.3. [$SLR(1)$ Parsing: An Example] Let us consider the following augmented grammar:

0. $E' \rightarrow E \$$
1. $E \rightarrow E + T$
2. $E \rightarrow T$
3. $T \rightarrow T \times F$
4. $T \rightarrow F$
5. $F \rightarrow (E)$
6. $F \rightarrow a$

The finite automaton constructed at the end of Step (4) by Algorithm 5.2.3 on page 77 is shown in Figure 5.3.1 on the facing page.

The parsing table constructed from the finite automaton of Figure 5.3.1, is shown in Figure 5.3.2 on page 90. To construct that table we have first to compute the following $Follow_1$ sets:

$$Follow_1(E) = \{+,), \$\}, \quad Follow_1(T) = \{+, \times,), \$\}, \quad Follow_1(F) = \{+, \times,), \$\}$$

Note that, contrary to the $LR(0)$ parsing tables, in the $SLR(1)$ parsing tables we may have in the same row both a *reduce* action and a *shift* action (which, obviously, are placed in different columns). In particular, state q_9 shows that the given grammar is not an $LR(0)$ grammar. In state q_9 , by using one symbol of lookahead on the input string, we do the following actions:

(i) if the lookahead symbol is \times , then we perform a *shift* action by placing \times on the top of the stack, and we look for an F , because of the item $T \rightarrow T \cdot \times F$ in which the symbol \times is followed by F , and otherwise,

(ii) if the lookahead symbol is different from \times and belongs to $Follow_1(E)$, then we perform a *reduce* action by reducing production 1 which is: $E \rightarrow E+T$. Indeed, in state q_9 we have only the complete item $E \rightarrow E+T \cdot$, besides the item $T \rightarrow T \cdot \times F$.

Note that the row for state q_{12} can be eliminated because in state q_1 , when $\$$ is read, the input string is accepted and, thus, state q_{12} is never reached.

Now we give an example of a grammar which is *not* $SLR(1)$.

EXAMPLE 5.3.4. [A Grammars Which Is Not $SLR(1)$] Let us consider the grammar with axiom S and the following productions:

1. $S \rightarrow L = R$
2. $S \rightarrow R$
3. $L \rightarrow *R$
4. $L \rightarrow a$
5. $R \rightarrow L$

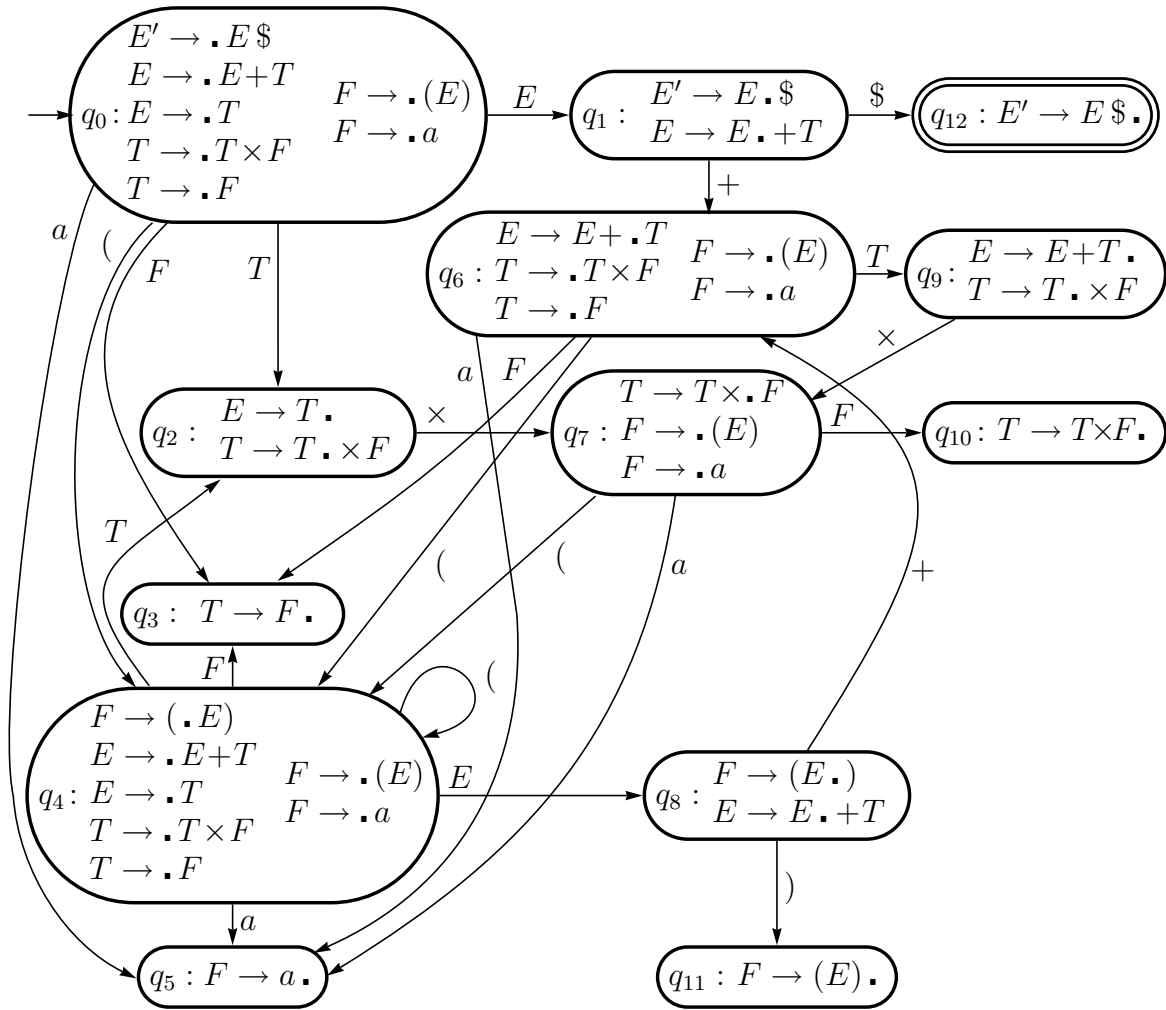


FIGURE 5.3.1. The finite automaton for the *SLR*(1) parsing of the grammar with axiom E' and productions: 0. $E' \rightarrow E\$$, 1. $E \rightarrow E+T$, 2. $E \rightarrow T$, 3. $T \rightarrow T \times F$, 4. $T \rightarrow F$, 5. $F \rightarrow (E)$, 6. $F \rightarrow a$. The label of each state is a set of items and it is generated by the Powerset Construction Procedure starting from the graph of the big productions whose construction is left to the reader (see Algorithm 5.2.3 on page 77 for constructing the *LR*(0) parsing tables).

Its augmented grammar has axiom S' and the extra production $S' \rightarrow S\$$. The language generated by this grammar is that of the so called *left-values* and *right-values* of assignments. In this language the string $*R$ denotes the content of the location R .

If we apply Algorithm 5.2.3 on page 77 for constructing the *LR*(0) parsing table, we get a state with items: $S \rightarrow L \cdot = R$ and $R \rightarrow L \cdot$ (see state q_3 in Figure 5.3.3 on page 91). We also have that $Follow_1(L) = \{=\}$ and $Follow_1(R) = \{=\}$.

Thus, in the *SLR*(1) parsing table for that state q_3 and terminal symbol '=', we should insert according to Algorithm 5.3.2 on page 87, both a *shift* action (because

	<u>action</u>						<u>goto</u>		
	<i>a</i>	+	×	()	\$	<i>E</i>	<i>T</i>	<i>F</i>
q_0	<i>sh</i> q_5			<i>sh</i> q_4			q_1	q_2	q_3
q_1		<i>sh</i> q_6				<i>acc</i>			
q_2		<i>red</i> 2	<i>sh</i> q_7		<i>red</i> 2	<i>red</i> 2			
q_3		<i>red</i> 4	<i>red</i> 4		<i>red</i> 4	<i>red</i> 4			
q_4	<i>sh</i> q_5			<i>sh</i> q_4			q_8	q_2	q_3
q_5		<i>red</i> 6	<i>red</i> 6		<i>red</i> 6	<i>red</i> 6			
q_6	<i>sh</i> q_5			<i>sh</i> q_4				q_9	q_3
q_7	<i>sh</i> q_5			<i>sh</i> q_4					q_{10}
q_8		<i>sh</i> q_6			<i>sh</i> q_{11}				
q_9		<i>red</i> 1	<i>sh</i> q_7		<i>red</i> 1	<i>red</i> 1			
q_{10}		<i>red</i> 3	<i>red</i> 3		<i>red</i> 3	<i>red</i> 3			
q_{11}		<i>red</i> 5	<i>red</i> 5		<i>red</i> 5	<i>red</i> 5			
q_{12}						<i>red</i> 0			

0. $E' \rightarrow E \$$
 1. $E \rightarrow E + T$
 2. $E \rightarrow T$
 3. $T \rightarrow T \times F$
 4. $T \rightarrow F$
 5. $F \rightarrow (E)$
 6. $F \rightarrow a$

where:

FIGURE 5.3.2. *SLR*(1) parsing table for the grammar with axiom E' and productions: 0. $E' \rightarrow E \$$, 1. $E \rightarrow E + T$, 2. $E \rightarrow T$, 3. $T \rightarrow T \times F$, 4. $T \rightarrow F$, 5. $F \rightarrow (E)$, 6. $F \rightarrow a$. It has been derived from the automaton of Figure 5.3.1 on the previous page. The row for state q_{12} can be eliminated because in state q_1 , when $\$$ is read, the input string is accepted.

of the symbol '=') and a *reduce* action (because $R \rightarrow L \cdot$ is a complete item and the symbol '=' is an element of $Follow_1(R)$). Then we would have a so called *shift-reduce conflict* and, thus, the given grammar is *not SLR*(1).

5.4. LR(1) Parsers

In this section we describe a new algorithm for constructing the parsing table of the automaton of Figure 5.2.1 on page 78. This algorithm is similar to Algorithm 5.2.3 on page 77 (which was used for constructing *LR*(0) parsing tables) and provides the definition of a subclass of the deterministic context-free languages, the so called *LR*(1) languages. Actually, one can show (see Theorem 5.4.12 on page 103) that every deterministic context-free language admits an *LR*(1) grammar which generates it.

ALGORITHM 5.4.1. Constructing the *LR*(1) Parsing Table.

Given a context-free grammar $G = \langle V_T, V_N, P, S \rangle$ we want to construct the *LR*(1) parsing table for the parsing automaton of Figure 5.2.1 on page 78.

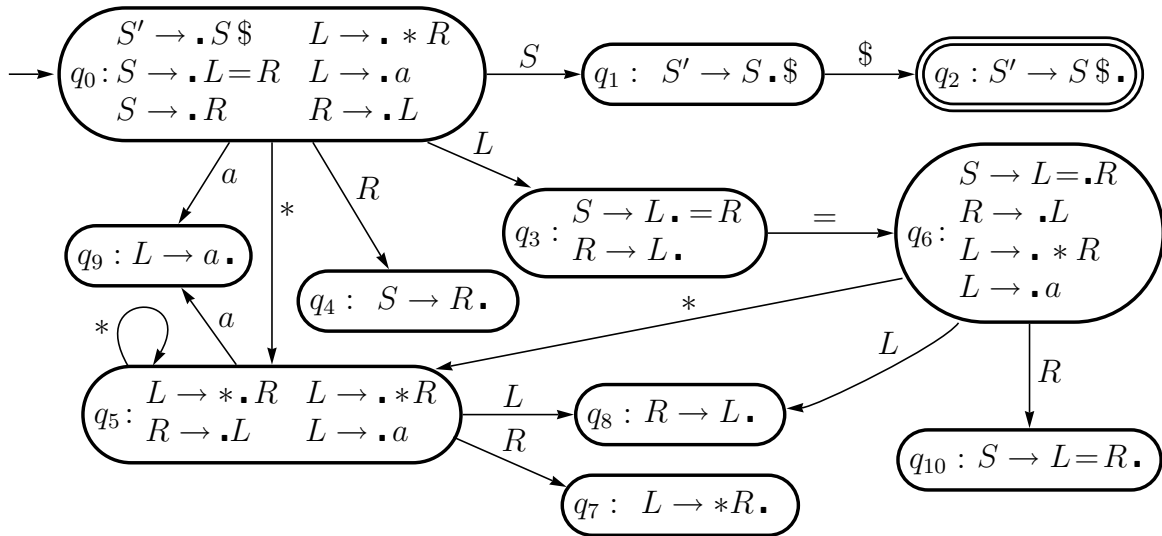


FIGURE 5.3.3. The finite automaton for the grammar with axiom S' and productions: 0. $S' \rightarrow S \$$, 1. $S \rightarrow L = R$, 2. $S \rightarrow R$, 3. $L \rightarrow * R$, 4. $L \rightarrow a$, 5. $R \rightarrow L$. The label of each state is generated by the Powerset Construction Procedure (see Algorithm 5.2.3 on page 77). State q_3 shows that this grammar is *not* an $SLR(1)$ grammar.

Step (1). First we consider the augmented grammar $G' = \langle V_T \cup \{\$, V_N \cup \{S'\}, P', S' \rangle$, where the set P' of productions is $P \cup \{S' \rightarrow S\}$. We assume that $\$ \notin V_T$ and $S' \notin V_N$. We have that the start symbol S' of the augmented grammar does not occur on the right hand side of any production of the augmented grammar.

For instance, in the case of the grammar G with axiom S and the three productions

1. $S \rightarrow C C$
2. $C \rightarrow c C$
3. $C \rightarrow d$

we consider the following augmented grammar G' :

0. $S' \rightarrow S$
1. $S \rightarrow C C$
2. $C \rightarrow c C$
3. $C \rightarrow d$

REMARK 5.4.2. Note that when the production for the start symbol S of the grammar G is *one only*, and S does *not* occur on the right hand side of any production of G , then we can take the augmented grammar G' to be G itself. \square

Step (2). We construct a set \mathcal{P} of *big productions with lookahead sets* as we now indicate. A big production with a lookahead set is a linear graph like the one described in Step (2) of Algorithm 5.2.3 on page 77 for constructing $LR(0)$ parsing tables, but in the label of each node of the graph there is also a set of symbols in $V_T \cup \{\$, called the lookahead set.$

The construction of the set \mathcal{P} of big productions with lookahead sets is done incrementally by applying to the following Rules (2.1) and (2.2).

Initialization Rule 2.1

Initially, the set \mathcal{P} consists of one big production only. It is the following one with lookahead set $\{\$\}$:

$$\boxed{S' \rightarrow \cdot S, \{\$\}} \xrightarrow{S} \boxed{S' \rightarrow S \cdot, \{\$\}}$$

If the only production for the axiom S is $S \rightarrow \alpha_1 \dots \alpha_n$, where $n \geq 0$ and for $i = 1, \dots, n$, $\alpha_i \in V_T \cup V_N$ and S does *not* occur on the right hand side of any production, and we take the augmented grammar G' to be G itself, then the set \mathcal{P} consists of the following big production with lookahead set $\{\$\}$:

$$\boxed{S \rightarrow \cdot \alpha_1 \alpha_2 \dots \alpha_n, \{\$\}} \xrightarrow{\alpha_1} \boxed{S \rightarrow \alpha_1 \cdot \alpha_2 \dots \alpha_n, \{\$\}} \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} \boxed{S \rightarrow \alpha_1 \alpha_2 \dots \alpha_n \cdot, \{\$\}}$$

Closure Rule 2.2

For every node in the graph constructed so far, of the form:

$\boxed{B \rightarrow \alpha \cdot A \beta, L}$ where $A, B \in V_N$ and $\alpha, \beta \in V^*$, and for every production $A \rightarrow \gamma$, with $\gamma = \gamma_1 \dots \gamma_n$, for $n \geq 0$, where the γ_i 's belong to $V_T \cup V_N$, in the augmented grammar G' , we add to the set \mathcal{P} the following big production:

$$\boxed{A \rightarrow \cdot \gamma_1 \gamma_2 \dots \gamma_n, L'} \xrightarrow{\gamma_1} \boxed{A \rightarrow \gamma_1 \cdot \gamma_2 \dots \gamma_n, L'} \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_n} \boxed{A \rightarrow \gamma_1 \gamma_2 \dots \gamma_n \cdot, L'}$$

whose lookahead set L' is defined as follows:

$$\begin{aligned} L' = & \text{if } \beta = \varepsilon && \text{then } L \\ & \text{if } \beta \neq \varepsilon \text{ and } \beta \rightarrow^* \varepsilon && \text{then } L \cup (\text{First}_1(\beta) - \{\varepsilon\}) \\ & \text{if } \beta \neq \varepsilon \text{ and } \beta \not\rightarrow^* \varepsilon && \text{then } \text{First}_1(\beta) \end{aligned}$$

Step (3). We consider the graph constructed at Step (2), made out of all the big productions with lookahead sets. Then, for every nonterminal $A, B \in V_N$ and strings $\alpha, \beta, \gamma \in (V_T \cup V_N)^*$, and for all lookahead sets L and L' , we add an arc labeled by ε from every node of the form:

$$\boxed{B \rightarrow \alpha \cdot A \beta, L} \text{ to every node of the form: } \boxed{A \rightarrow \cdot \gamma, L'}$$

Step (4). By applying the Powerset Construction Procedure (see, for instance, [21]) to the graph of the big productions with lookahead sets obtained at the end of Step (3), we get a finite automaton. Let that finite automaton be called M . The initial state of M is the one which has in its label the item $\boxed{S' \rightarrow \cdot S, \{\$\}}$. The final state of M

is the one which has in its label the item $\boxed{S' \rightarrow S \cdot, \{\$\}}$. Note that by the Powerset Construction the label of each state of M is a *set of items* of the grammar G' , each item being associated with its lookahead set.

In what follows, by abuse of language, we will feel free to say ‘an item’, instead of ‘an item with its lookahead set’.

If the only production for the axiom S is $S \rightarrow \alpha_1 \dots \alpha_n$, where $n \geq 0$ and for $i = 1, \dots, n$, $\alpha_i \in V_T \cup V_N$ and S does *not* occur on the right hand side of any

production, and we take the augmented grammar G' to be G itself, then the initial state of M is the one which has in its label the item $(S \rightarrow \cdot \alpha_1 \alpha_2 \dots \alpha_n, \{\$\})$ and the final state of M is the one which has in its label the item $(S \rightarrow \alpha_1 \alpha_2 \dots \alpha_n \cdot, \{\$\})$.

Step (5). The LR(1) parsing table T is then constructed from the automaton M as follows.

(5.1) For each transition $(q_i) \xrightarrow{a} (q_j)$ where a is a terminal symbol,

we insert in the element $T[q_i, a]$ which is in the *action* part of the table T , the entry *sh* q_j (short for *shift* q_j). This means that the parsing automaton of Figure 5.2.1 on page 78 should take the input symbol a at hand, push it on the stack, and push the state q_j on the stack. Then the new input symbol at hand is the one to the right of the previous symbol a at hand. This action of the parsing automaton is called a *shift action*.

(5.2) For the *final* state q_i with the item $(S' \rightarrow S \cdot, \{\$\})$,

we insert in the element $T[q_i, \$]$ which is in the *action* part of the table T , the entry *acc* (short for *accept*). This means that the parsing automaton of Figure 5.2.1 should accept the given input string. This action of the parsing automaton is called the *accept action*.

Recall that in the final state q_i , instead of the item $(S' \rightarrow S \cdot, \{\$\})$, we have the item $(S \rightarrow \alpha_1 \dots \alpha_n \cdot, \{\$\})$, for some $n \geq 0$ and the α_i 's in $V_T \cup V_N$, if according to

Remark 5.4.2 on page 91, we take the augmented grammar to be the given grammar itself, because in the given grammar: (i) there is *only one* production for the start symbol S , and (ii) S does *not* occur on the right hand side of any production.

(5.3) For each transition $(q_i) \xrightarrow{A} (q_j)$ with the nonterminal A ,

we insert in the element $T[q_i, A]$ which is in the *goto* part of the table T , the entry q_j . This means that the parsing automaton of Figure 5.2.1 should push on the stack the state q_j (see Point (5.4) below). This action of the parsing automaton is called a *goto action*.

(5.4) For each complete item $(A \rightarrow \alpha_1 \dots \alpha_n \cdot, L)$, for some $n \geq 0$, where the α_i 's are symbols in $V_T \cup V_N$, in the label of a state q_i which is *not* final, we insert for each $a \in L$, in row q_i and column a , in the *action* part of the table T , the entry *red* p (short for *reduce production* p), where p is the number identifying the production $A \rightarrow \alpha_1 \dots \alpha_n$ of the grammar G' . This means that the parsing automaton of Figure 5.2.1 should replace the string $\alpha_1 q_{\alpha_1} \dots \alpha_n q_{\alpha_n}$ which is on the stack (the top element being state q_{α_n}) by the nonterminal A . This action of the parsing automaton is called a *reduce action*.

Having constructed the $LR(1)$ parsing table, we can use the pushdown automaton of Figure 5.2.1 on page 78 with that table, for parsing a given input word w . Initially, the stack has the initial state q_0 only, and the input string is $w\$$. Then the pushdown automaton works by using the $LR(1)$ parsing table as it were an $LR(0)$ parsing table.

REMARK 5.4.3. [*Multiple actions*] As for $LR(0)$ grammars, also in the case of the $LR(1)$ grammars, if we have to place more than one action in an entry of the $LR(1)$ parsing table (that is, we have conflicts), then the language generated by the given grammar cannot be parsed by using a pushdown automaton which uses that table and the augmented grammar G' which has been derived from the given grammar G , is *not* $LR(1)$. \square

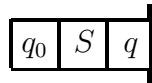
REMARK 5.4.4. [*Undefined entry*] If during the process of parsing a given input word $w\$$, the pushdown automaton requires an entry of the table, but that entry is empty, then the word w does *not* belong to the language generated by the augmented grammar G' and w does *not* belong to the language generated by the given grammar G . \square

REMARK 5.4.5. When applying Algorithm 5.4.1 on page 90, in any state, for any $n \geq 2$, we can replace n items of the form: $(A \rightarrow \gamma, L_1), \dots, (A \rightarrow \gamma, L_n)$, where $A \in V_N$ and $\gamma \in (V_T \cup V_N)^* \cup \{\cdot\} \cup (V_T \cup V_N)^*$ (that is, γ is a string of terminal or nonterminal symbols with a dot \cdot at the beginning, or in between, or at the end), by a unique item of the form: $(A \rightarrow \gamma, L_1 \cup \dots \cup L_n)$. Indeed, this replacement does not modify the outcome of that algorithm (see Step (5.4) on page 93). \square

Note that in the $LR(1)$ parsing table there is no column labeled by S' because no arc in the graph representing the finite automaton M constructed at the end of Step (4) of Algorithm 5.4.1 on page 90, has label S' .

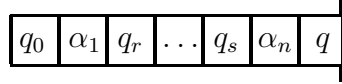
We leave it to the reader to check that, if the axiom of the augmented grammar is S' and the axiom of the given grammar is S , when the action *accept* is performed, then the pushdown automaton is in the final state q whose label has the item

$(S' \rightarrow S \cdot, \{\$\})$, the input symbol at hand is $\$$, and the stack is of the form (the top being the state q):



If, according to Remark 5.4.2 on page 91, we take the augmented grammar to be the given grammar itself, so that the only production for the axiom S is $S \rightarrow \alpha_1 \dots \alpha_n$, for some $n \geq 0$ and the α_i 's in $V_T \cup V_N$, then when the action *accept* is performed, the pushdown automaton is in the final state q whose label has the item

$(S \rightarrow \alpha_1 \dots \alpha_n \cdot, \{\$\})$, the input symbol at hand is $\$$, and the stack is of the form (the top being the state q):



Note that Algorithm 5.4.1 on page 90 for constructing $LR(1)$ parsing tables is analogous to Algorithm 5.2.3 on page 77 for constructing $LR(0)$ parsing tables and avoids the use of the Powerset Construction by using, instead, an Initialization Rule (see Rule 2.1 on page 92) and a Closure Rule (see Rule 2.2 on page 92).

The following fact states a property which is enjoyed by the lookahead sets of the items of the $LR(1)$ grammars.

FACT 5.4.6. [Lookahead Sets and $Follow_1$ Sets] For any item of the form $(B \rightarrow \alpha \cdot \beta, L)$, with $B \in (V_N \cup \{S'\})$ and $\alpha, \beta \in (V_T \cup V_N)^*$, we have that:

$$L \subseteq Follow_1(B).$$

PROOF. The proof is by cases.

Case (1). For the item $(S' \rightarrow \cdot S, \{\$\})$ where S' is the axiom of the augmented grammar and the production $S' \rightarrow S$ is the only production for S' , we get the thesis because $Follow_1(S') = \{\$\}$.

Case (2). For the item $(B \rightarrow \alpha \cdot A\beta, L)$ with $\beta \neq \varepsilon$ and $\beta \rightarrow^* \varepsilon$, we have to consider items of the form $(A \rightarrow \dots, L')$, where $L' = L \cup (First_1(\beta) - \{\varepsilon\})$. The thesis follows from the following Points (2.1) and (2.2).

Point (2.1). By induction hypothesis we have that: $L \subseteq Follow_1(B)$. We also have that $Follow_1(B) \subseteq Follow_1(A)$, by definition of $Follow_1$ if $\beta \neq \varepsilon$ and $\beta \rightarrow^* \varepsilon$ (see Definition 4.2.5 on page 55). Thus, $L \subseteq Follow_1(A)$.

Point (2.2). We have that: $(First_1(\beta) - \{\varepsilon\}) \subseteq Follow_1(A)$, by definition of $Follow_1$ if $\beta \neq \varepsilon$ and $\beta \rightarrow^* \varepsilon$.

Case (3). For the item $(B \rightarrow \alpha \cdot A\beta, L)$ with $\beta \neq \varepsilon$ and $\beta \not\rightarrow^* \varepsilon$, we have to consider items of the form $(A \rightarrow \dots, L')$, where $L' = First_1(\beta)$. The thesis follows because $First_1(\beta) - \{\varepsilon\} = First_1(\beta)$ (recall that, when $\beta \not\rightarrow^* \varepsilon$, we have that $\varepsilon \notin First_1(\beta)$) and $(First_1(\beta) - \{\varepsilon\}) \subseteq Follow_1(A)$ (by definition of $Follow_1$ if $\beta \neq \varepsilon$ and $\beta \not\rightarrow^* \varepsilon$). \square

One can show that the following is an alternative definition of $LR(1)$ grammars. The proof of this fact is based on Algorithm 5.4.1 on page 90 and can be found in [10, page 263].

DEFINITION 5.4.7. [$LR(1)$ Grammar and $LR(1)$ Language] A context-free grammar G is an $LR(1)$ grammar iff

- (i) the start symbol does *not* occur in the right hand side of any production, and
- (ii) if in the label of a state of the automaton M of Step (4) of Algorithm 5.4.1 for constructing the $LR(1)$ parsing tables, there is a complete item of the form

$(A \rightarrow \alpha \cdot, L)$ (recall that the label of a state is a set of items, each item being associ-

ated with its lookahead set), then (ii.1) for any other complete item $(B \rightarrow \beta \cdot, M)$ in the same label, $M \cap L = \emptyset$, and (ii.2) for any other incomplete item $(B \rightarrow \beta \cdot a \gamma, M)$, with $a \in V_T \cup \{\$\}$ in the same label, we have that $a \notin L$.

A language is an $LR(1)$ language if it is generated by an $LR(1)$ grammar.

Now we will present two examples of $LR(1)$ parsing.

EXAMPLE 5.4.8. [$LR(1)$ Parsing: Example 1] Let us consider the grammar G with axiom S and the following productions:

$$1. S \rightarrow CC \quad 2. C \rightarrow cC \quad 3. C \rightarrow d$$

The augmented grammar G' is:

$$0. S' \rightarrow S \quad 1. S \rightarrow CC \quad 2. C \rightarrow cC \quad 3. C \rightarrow d$$

Note that for the grammar G we can take the augmented grammar G' to be G itself, because there is only one production for S and S does not occur on the right hand side of any production in G .

The graph of the big productions with the lookahead sets is depicted in Figure 5.4.1 on the facing page.

Note the item $(S \rightarrow \cdot CC, \{\$\})$ in the second row from above, has an ε -arc both to the item $(C \rightarrow \cdot cC, \{c, d\})$ in the third row and to the item $(C \rightarrow \cdot d, \{c, d\})$ in the fifth row, both with lookahead set $\{c, d\}$, because $C \not\rightarrow^* \varepsilon$ and $First_1(C) = \{c, d\}$.

From the graph of Figure 5.4.1 the Powerset Construction Procedure constructs the finite automaton depicted in Figure 5.4.2 on page 98.

From that automaton we get the $LR(1)$ parsing table of Figure 5.4.3 on page 99. Figure 5.4.4 on page 99 shows the behaviour of the automaton of Figure 5.2.1 on page 78, while parsing the input string $dcd\$$ which is read from left to right. In Figure 5.4.4 we use the same conventions of Figure 5.2.5 on page 83 and, in particular, the symbol \blacktriangle points at the input character at hand. The stack grows and shrinks at the right end, and the symbol \blacktriangle points at the top of the stack. \square

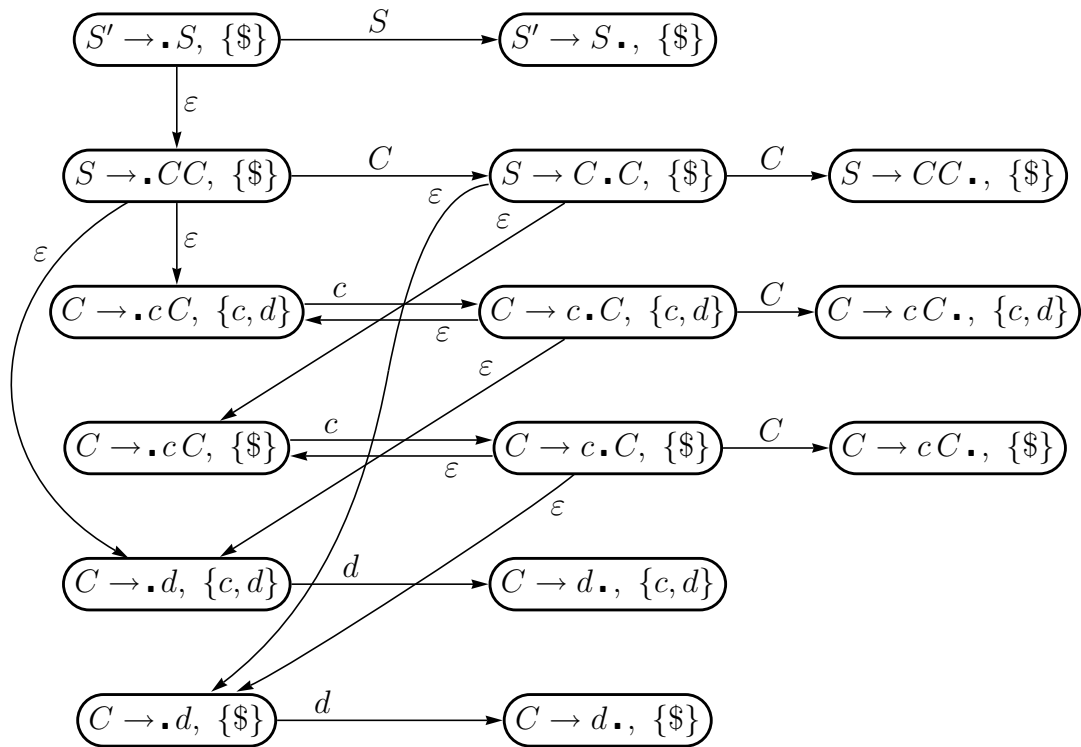


FIGURE 5.4.1. The graph of the big productions with lookahead sets and ε -arcs for the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow CC$, 2. $C \rightarrow cC$, 3. $C \rightarrow d$. Each big production with its lookahead set is depicted as a sequence of nodes which are horizontally aligned.

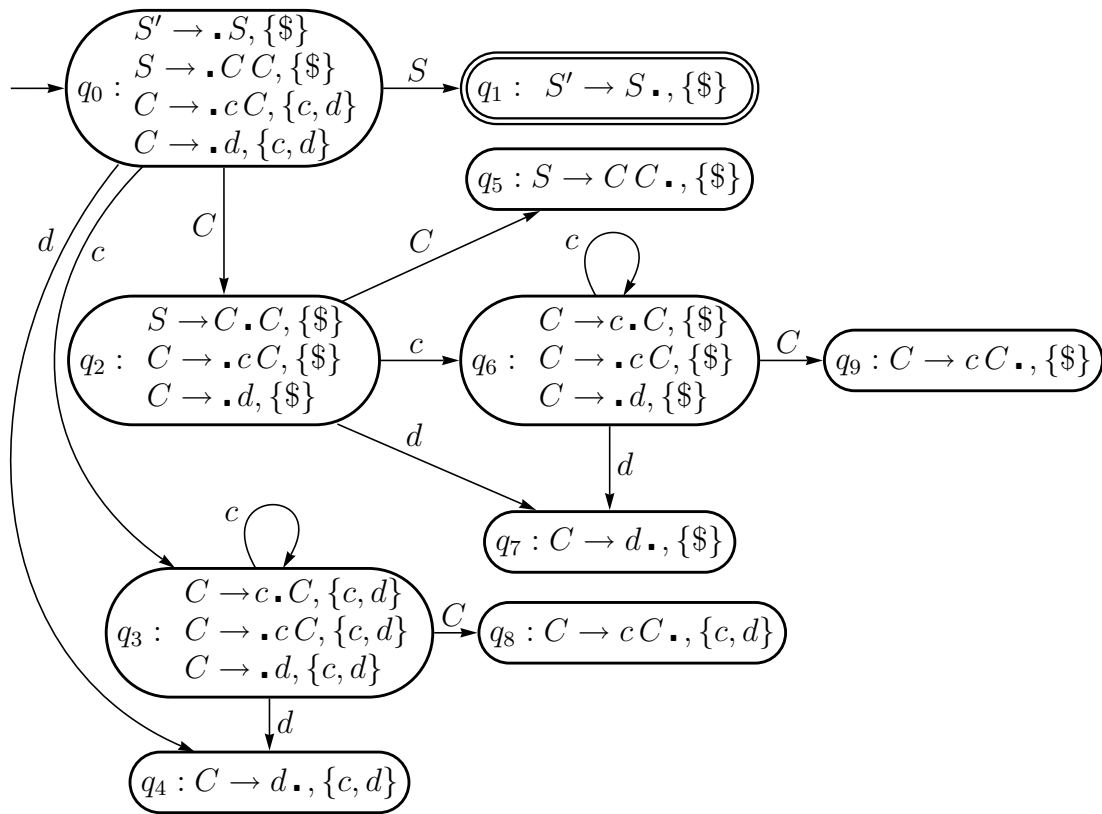


FIGURE 5.4.2. The finite automaton M for the $LR(1)$ parsing of the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow CC$, 2. $C \rightarrow cC$, 3. $C \rightarrow d$. The label of each state is a set of items which is computed by the Powerset Construction Procedure from the graph of Figure 5.4.1.

	<u>action</u>			<u>goto</u>	
	<i>c</i>	<i>d</i>	<i>⋄</i>	<i>S</i>	<i>C</i>
q_0	<i>sh</i> q_3	<i>sh</i> q_4		q_1	q_2
q_1			<i>acc</i>		
q_2	<i>sh</i> q_6	<i>sh</i> q_7			q_5
q_3	<i>sh</i> q_3	<i>sh</i> q_4			q_8
q_4	<i>red</i> 3	<i>red</i> 3			
q_5			<i>red</i> 1		
q_6	<i>sh</i> q_6	<i>sh</i> q_7			q_9
q_7			<i>red</i> 3		
q_8	<i>red</i> 2	<i>red</i> 2			
q_9			<i>red</i> 2		

where:

0. $S' \rightarrow S$
1. $S \rightarrow C C$
2. $C \rightarrow c C$
3. $C \rightarrow d$

FIGURE 5.4.3. LR(1) parsing table for the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow C C$, 2. $C \rightarrow c C$, 3. $C \rightarrow d$. It has been derived from the automaton of Figure 5.4.2 on page 98.

time	stack (the top is marked by \blacktriangle)	input (the symbol at hand is marked by \blacktriangle)	<i>shift / reduce-goto</i>
0	q_0 \blacktriangle	$d c d \text{⋄}$ \blacktriangle	$action(q_0, d) = sh\ q_4$
1	$q_0 d q_4$ \blacktriangle	$d c d \text{⋄}$ \blacktriangle	$action(q_4, c) = red\ 3\ (C \rightarrow d)$ $goto(q_0, C) = q_2$
2	$q_0 C q_2$ \blacktriangle	$d c d \text{⋄}$ \blacktriangle	$action(q_2, c) = sh\ q_6$
3	$q_0 C q_2 c q_6$ \blacktriangle	$d c d \text{⋄}$ \blacktriangle	$action(q_6, d) = sh\ q_7$
4	$q_0 C q_2 c q_6 d q_7$ \blacktriangle	$d c d \text{⋄}$ \blacktriangle	$action(q_7, \text{⋄}) = red\ 3\ (C \rightarrow d)$ $goto(q_6, C) = q_9$
5	$q_0 C q_2 c q_6 C q_9$ \blacktriangle	$d c d \text{⋄}$ \blacktriangle	$action(q_9, \text{⋄}) = red\ 2\ (C \rightarrow c C)$ $goto(q_2, C) = q_5$
6	$q_0 C q_2 C q_5$ \blacktriangle	$d c d \text{⋄}$ \blacktriangle	$action(q_5, \text{⋄}) = red\ 1\ (S \rightarrow C C)$ $goto(q_0, S) = q_1$
7	$q_0 S q_1$ \blacktriangle	$d c d \text{⋄}$ \blacktriangle	$action(q_1, \text{⋄}) = accept$

FIGURE 5.4.4. Sequence of configurations of the pushdown automaton of Figure 5.2.1 on page 78 while parsing the string $dcd\text{⋄}$ using the LR(1) parsing table of Figure 5.4.3, for the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow C C$, 2. $C \rightarrow c C$, 3. $C \rightarrow d$.

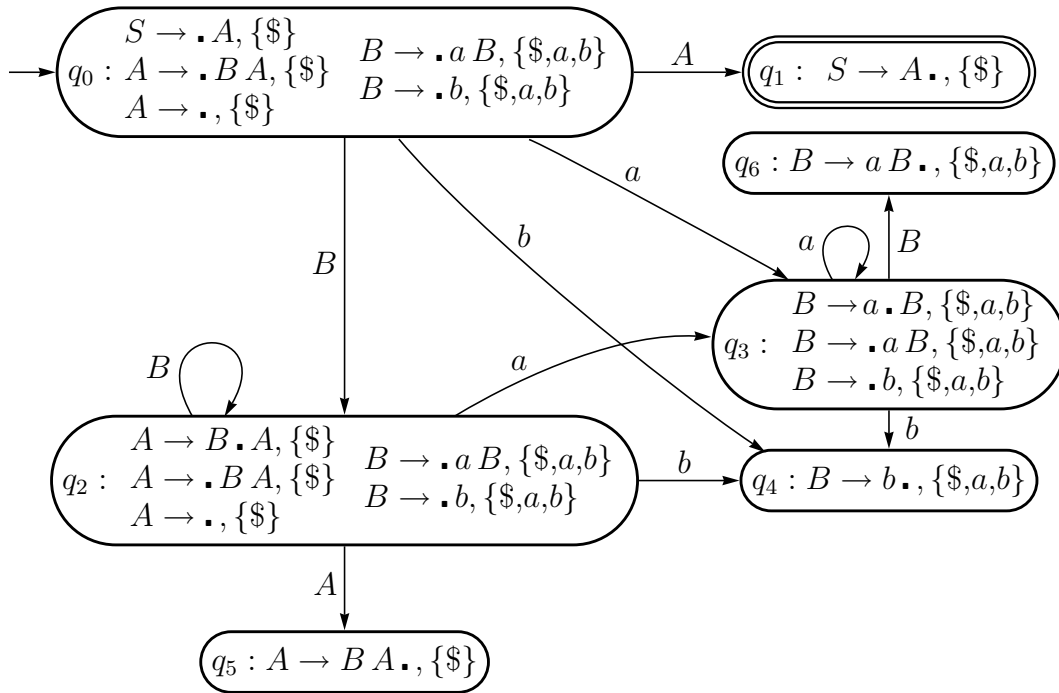


FIGURE 5.4.6. The finite automaton M for the $LR(1)$ parsing of the grammar G' with axiom S and productions: 0. $S \rightarrow A$, 1. $A \rightarrow BA$, 2. $A \rightarrow \varepsilon$, 3. $B \rightarrow aB$, 4. $B \rightarrow b$. The label of each state is a set of items which is computed by the Powerset Construction Procedure from the graph of Figure 5.4.5 on the preceding page.

	<u>action</u>			<u>goto</u>	
	a	b	$\$$	A	B
q_0	sh q_3	sh q_4	red 2	q_1	q_2
q_1			acc		
q_2	sh q_3	sh q_4	red 2	q_5	q_2
q_3	sh q_3	sh q_4			q_6
q_4	red 4	red 4	red 4		
q_5			red 1		
q_6	red 3	red 3	red 3		

where:

0. $S \rightarrow A$
1. $A \rightarrow BA$
2. $A \rightarrow \varepsilon$
3. $B \rightarrow aB$
4. $B \rightarrow b$

FIGURE 5.4.7. $LR(1)$ parsing table for the grammar G' with axiom S and productions: 0. $S \rightarrow A$, 1. $A \rightarrow BA$, 2. $A \rightarrow \varepsilon$, 3. $B \rightarrow aB$, 4. $B \rightarrow b$. It has been derived from the finite automaton of Figure 5.4.6.

time	stack (the top is marked by \blacktriangle)	input (the symbol at hand is marked by \blacktriangle)	<i>shift / reduce-goto</i>
0	q_0 \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_0, a) = sh\ q_3$
1	$q_0 a q_3$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_3, a) = sh\ q_3$
2	$q_0 a q_3 a q_3$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_3, b) = sh\ q_4$
3	$q_0 a q_3 a q_3 b q_4$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_4, b) = red\ 4\ (B \rightarrow b)$ $goto(q_3, B) = q_6$
4	$q_0 a q_3 a q_3 B q_6$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_6, b) = red\ 3\ (B \rightarrow a B)$ $goto(q_3, B) = q_6$
5	$q_0 a q_3 B q_6$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_6, b) = red\ 3\ (B \rightarrow a B)$ $goto(q_0, B) = q_2$
6	$q_0 B q_2$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_2, b) = sh\ q_4$
7	$q_0 B q_2 b q_4$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_4, \$) = red\ 4\ (B \rightarrow b)$ $goto(q_2, B) = q_2$
8	$q_0 B q_2 B q_2$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_2, \$) = red\ 2\ (A \rightarrow \varepsilon)$ $goto(q_2, A) = q_5$
9	$q_0 B q_2 B q_2 A q_5$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_5, \$) = red\ 1\ (A \rightarrow B A)$ $goto(q_2, A) = q_5$
10	$q_0 B q_2 A q_5$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_5, \$) = red\ 1\ (A \rightarrow B A)$ $goto(q_0, A) = q_1$
11	$q_0 A q_1$ \blacktriangle	$a a b b \$$ \blacktriangle	$action(q_1, \$) = accept$

FIGURE 5.4.8. Sequence of configurations of the pushdown automaton of Figure 5.2.1 on page 78 while parsing the string $a a b b \$$ using the $LR(1)$ parsing table of Figure 5.4.7 on the preceding page, for the grammar G' with axiom S and productions: 0. $S \rightarrow A$, 1. $A \rightarrow B A$, 2. $A \rightarrow \varepsilon$, 3. $B \rightarrow a B$ 4. $B \rightarrow b$.

Now we state the following results concerning the $LR(k)$ grammars and $LR(k)$ languages. For the proofs the interested reader may refer to [3, 4, 10].

THEOREM 5.4.10. [Hierarchy of $LR(k)$ Grammars] For any $k \geq 0$, the class of the $LR(k)$ grammars is properly contained in the class of the $LR(k + 1)$ grammars.

THEOREM 5.4.11. [Collapse of $LR(k)$ Languages for $k \geq 1$] (i) The class $LR(0)$ of languages is properly contained in the class of the $LR(1)$ languages. (ii) For any $k \geq 1$, the class of $LR(k)$ languages is equal to the class of $LR(k + 1)$ languages (see also Fact 5.8.10 on page 121).

THEOREM 5.4.12. [LR(1) Grammars, LR(1) Languages, and Deterministic Context-free Languages] (i) A language L is a deterministic context-free language iff there exists $k \geq 1$ such that L is generated by an $LR(k)$ grammar. (ii) A language L is a deterministic context-free language iff it is generated by an $LR(1)$ grammar. (Point (ii) is a consequence of Point (i) and Theorem 5.4.11.)

THEOREM 5.4.13. [LR(1) Grammars in Chomsky and Greibach Normal Form] (i) Every deterministic context-free language has an $LR(1)$ grammar in Chomsky normal form. (ii) Every deterministic context-free language has an $LR(1)$ grammar in Greibach normal form [4, page 708].

The following result refers to $LR(k)$ grammars and has been already mentioned with respect to $LR(k)$ languages (see Fact 4.3.18 on page 74). Reduced grammars have been introduced in Definition 1.2.15 on page 15.

THEOREM 5.4.14. [Reduced LL(k) Grammars Are Properly Included in Reduced LR(k) Grammars] For any $k \geq 0$, the class of reduced $LL(k)$ grammars is properly contained in the class of reduced $LR(k)$ grammars [23, page 248].

Let us close this section by recalling a property of the $LR(k)$ parsers when they make either a *shift* action or a *reduce-goto* action.

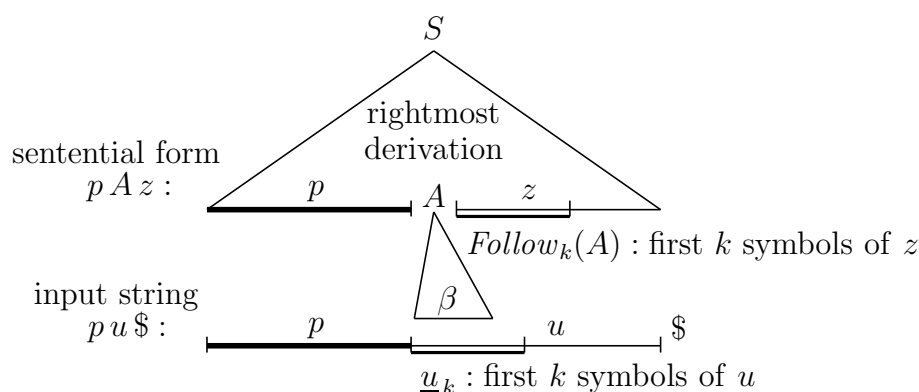
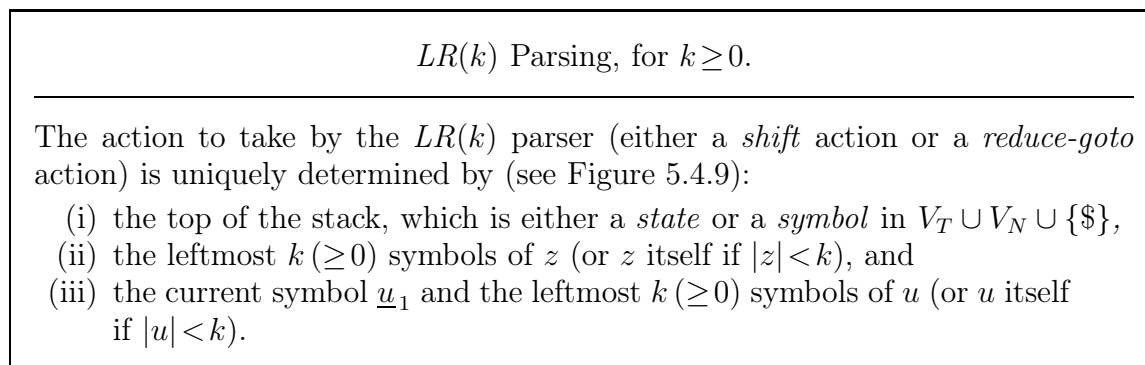


FIGURE 5.4.9. $LR(k)$ parsing, for $k \geq 0$. Note that, if $k=0$, the action of the $LR(0)$ parser is a function of the current symbol which is pointed by the input head, that is, the symbol \underline{u}_1 . The derivation from S to $p A z$ is a rightmost derivation.

5.5. LALR(1) Parsers

In this section we will study the class of *LALR*(1) grammars and we will present a technique for parsing the languages generated by those grammars. Basically, *LALR*(1) parsing is like *LR*(1) parsing, but it uses parsing tables which are derived by compaction from the *LR*(1) parsing tables. There are algorithms for directly constructing the tables for *LALR*(1) parsing without first constructing the tables for *LR*(1) parsing [2, page 242], but we do not consider them here.

The compaction of an *LR*(1) parsing table which makes it an *LALR*(1) parsing table is performed, as we now indicate, on the finite automaton from which the table is derived.

ALGORITHM 5.5.1. *Constructing the LALR(1) Parsing Table.*

Given a context-free grammar $G = \langle V_T, V_N, P, S \rangle$, where $V_T = \{a_1, \dots, a_r\}$ and $V_N = \{A_1, \dots, A_s\}$, we consider its augmented grammar G' with the new start symbol S' , and the finite automaton M which can be constructed from G' by applying Algorithm 5.4.1 on page 90 for constructing the *LR*(1) parsing tables.

Then, we replace *as long as possible* (and maybe no times) any subset $\{q_1, \dots, q_n\}$ of the states of M of the form

$$\left(\begin{array}{l} \text{item}_1, L_{11} \\ q_1 : \dots \\ \text{item}_m, L_{1m} \end{array} \right), \quad \dots, \quad \left(\begin{array}{l} \text{item}_1, L_{n1} \\ q_n : \dots \\ \text{item}_m, L_{nm} \end{array} \right)$$

(that is, any subset of states with the same first component) by the following single state $q_{1\dots n}$

$$\left(\begin{array}{l} \text{item}_1, L_{11} \cup \dots \cup L_{n1} \\ q_{1,\dots,n} : \dots \\ \text{item}_m, L_{1m} \cup \dots \cup L_{nm} \end{array} \right)$$

After each of these replacements which can be viewed as ‘fusions of states’, we readjust the arcs, that is, we replace, for all $k = 1, \dots, n$, (i) any arc *from* a state q_k by an arc *from* the new state $q_{1\dots n}$, and (ii) any arc *to* a state q_k by an arc *to* the new state $q_{1\dots n}$.

Each of the above replacements of states in the finite automaton M , corresponds in the *LR*(1) parsing table T to the replacement of a set $\{q_1, \dots, q_n\}$ of rows of the form

	<u>action</u>				<u>goto</u>		
	a_1	\dots	a_r	$\$$	A_1	\dots	A_s
$q_1 :$	$T[1, a_1]$	\dots	$T[1, a_r]$	$T[1, \$]$	$T[1, A_1]$	\dots	$T[1, A_s]$
\vdots	\vdots		\vdots	\vdots	\vdots		\vdots
$q_n :$	$T[n, a_1]$	\dots	$T[n, a_r]$	$T[n, \$]$	$T[n, A_1]$	\dots	$T[n, A_s]$

where some of the $T[i, j]$ entries may be absent, by the following single row $q_{1\dots n}$

	<i>action</i>				<i>goto</i>		
	a_1	\dots	a_r	$\$$	A_1	\dots	A_s
$q_{1\dots n} :$	$T[1, a_1],$	\dots	$T[1, a_r],$	$T[1, \$],$	$T[1, A_1],$	\dots	$T[1, A_s],$
	$\vdots,$	\dots	$\vdots,$	$\vdots,$	$\vdots,$	\dots	$\vdots,$
	$T[n, a_1]$	\dots	$T[n, a_r]$	$T[n, \$]$	$T[n, A_1]$	\dots	$T[n, A_s]$

where in each position of the row $q_{1\dots n}$ we have kept *one copy only* for each distinct entry, in the sense that for each column $j = a_1, \dots, a_r, \$, A_1, \dots, A_s$, and for each $i_1, i_2 = 1, \dots, n$, we have that $T[i_1, j] \neq T[i_2, j]$.

At the end of all these row replacements, if in the resulting parsing table T for each row i and for each column $j = a_1, \dots, a_r, \$, A_1, \dots, A_s$, there exists *at most one* element (that is, there are no conflicts), then the parsing table T is said to be an *LALR(1) parsing table*.

In this case, that is, in the case when there are no conflicts, the augmented grammar G' is an *LALR(1) grammar* and the language generated by G' is an *LALR(1) language*.

REMARK 5.5.2. [*Multiple actions*] As for *LR(1)* grammars, if we have to place more than one action in an entry of the *LALR(1)* parsing table, then there are conflicts and the language generated by the given grammar cannot be parsed by using a pushdown automaton which uses that table, and the augmented grammar G' which has been derived from the given grammar G , is *not* an *LALR(1) grammar*.

REMARK 5.5.3. [*Undefined entry*] If during the process of parsing a given input word $w\$$, the pushdown automaton requires an entry of the *LALR(1)* parsing table, but that entry is empty, then the word w does *not* belong to the language generated by the augmented grammar G' and w does *not* belong to the language generated by the given grammar G .

EXAMPLE 5.5.4. [*LALR(1) Parsing: Example 1*] In this example we construct the *LALR(1)* parsing table starting from the *LR(1)* parsing table of Figure 5.4.3 on page 99 relative to Example 5.4.8 on page 96. We have already constructed that *LR(1)* parsing table for the augmented grammar G' :

$$0. S' \rightarrow S \quad 1. S \rightarrow CC \quad 2. C \rightarrow cC \quad 3. C \rightarrow d$$

For *LALR(1)* parsing we have the finite automaton of Figure 5.5.1 on the next page, instead of the finite automaton of Figure 5.4.2 on page 98, and we have the *LALR(1)* parsing table of Figure 5.5.2 on page 107, instead of the *LR(1)* parsing table of Figure 5.4.3 on page 99.

The states of the finite automaton of Figure 5.4.2 on page 98 which have been fused are:

- (i) states q_3 and q_6 , fused into the state q_{36} ,
- (ii) states q_4 and q_7 , fused into the state q_{47} , and
- (iii) states q_8 and q_9 , fused into the state q_{89} . □

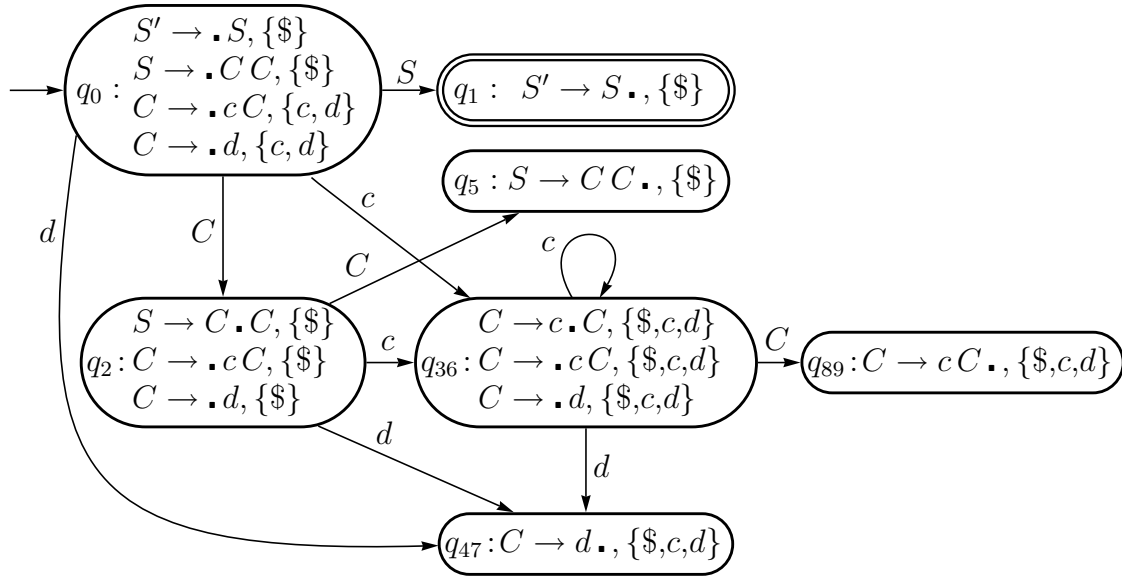


FIGURE 5.5.1. The finite automaton for the $LALR(1)$ parsing of the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow CC$, 2. $C \rightarrow cC$, 3. $C \rightarrow d$.

The fusion of the states which is performed by Algorithm 5.5.1 on page 104 does not create conflicts in the *goto* part of the parsing table because in that part the entries depend only on the first components of the $LR(1)$ items which are all equal in the states which are fused.

One can show [2, page 237] that when applying Algorithm 5.5.1 for constructing the $LALR(1)$ parsing tables, never *shift-reduce* conflicts may be produced, because *shift* actions depend on the first components of the $LR(1)$ items only. On the contrary, *reduce-reduce* conflicts may be produced.

One can also show [2, page 239–240] that for any input word which belongs to the language $L(G')$ generated by the augmented grammar G' , the $LALR(1)$ parser mimics the corresponding $LR(1)$ parser, in the sense that, with reference to Example 5.5.4 on page 105, it goes through the states q_{36} , q_{47} , and q_{89} iff the $LR(1)$ parser goes through the states (q_3 or q_6), (q_4 or q_7), and (q_8 or q_9), respectively.

One can show that for any input word w which does *not* belong to the language $L(G')$, the behaviour of the $LR(1)$ parser and the corresponding $LALR(1)$ parser, if there exists one, are similar, in the sense that after the $LR(1)$ parser has detected

	<u>action</u>			<u>goto</u>	
	<i>c</i>	<i>d</i>	<i>\$</i>	<i>S</i>	<i>C</i>
q_0	<i>sh</i> q_{36}	<i>sh</i> q_{47}		q_1	q_2
q_1			<i>acc</i>		
q_2	<i>sh</i> q_{36}	<i>sh</i> q_{47}			q_5
q_{36}	<i>sh</i> q_{36}	<i>sh</i> q_{47}			q_{89}
q_{47}	<i>red</i> 3	<i>red</i> 3	<i>red</i> 3		
q_5			<i>red</i> 1		
q_{89}	<i>red</i> 2	<i>red</i> 2	<i>red</i> 2		

where:

0. $S' \rightarrow S$
1. $S \rightarrow CC$
2. $C \rightarrow cC$
3. $C \rightarrow d$

FIGURE 5.5.2. *LALR*(1) parsing table for the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow CC$, 2. $C \rightarrow cC$, 3. $C \rightarrow d$. It has been derived from the automaton of Figure 5.5.1 on the facing page.

an error, the *LALR*(1) parser may do some extra *reduce* actions (but not extra *shift* actions) and then it will also detect an error.

EXAMPLE 5.5.5. [*LALR*(1) Parsing: Example 2] Let us consider the following grammar G :

$$1. S \rightarrow aA \quad 2. S \rightarrow bB \quad 3. A \rightarrow \varepsilon \quad 4. A \rightarrow cAd \quad 5. B \rightarrow \varepsilon$$

and its augmented version G' :

$$0. S' \rightarrow S \quad 1. S \rightarrow aA \quad 2. S \rightarrow bB \quad 3. A \rightarrow \varepsilon \quad 4. A \rightarrow cAd \quad 5. B \rightarrow \varepsilon$$

By applying Algorithm 5.4.1 on page 90 for constructing the *LR*(1) parsing tables, we get the finite automaton of Figure 5.5.3 on the next page and the *LR*(1) parsing table of Figure 5.5.4 on page 109. By applying Algorithm 5.5.1 on page 104 for constructing the *LALR*(1) parsing tables from the *LR*(1) parsing tables, we get the finite automaton of Figure 5.5.5 on page 109 and the *LALR*(1) parsing table of Figure 5.5.6 on page 110 (see also [23, pages 34,60]).

The states of the finite automaton of Figure 5.5.3 on the following page which have been fused are:

- (i) states q_6 and q_9 , fused into the state q_{69} ,
- (ii) states q_7 and q_{10} , fused into the state q_{710} , and
- (iii) states q_8 and q_{11} , fused into the state q_{811} .

□

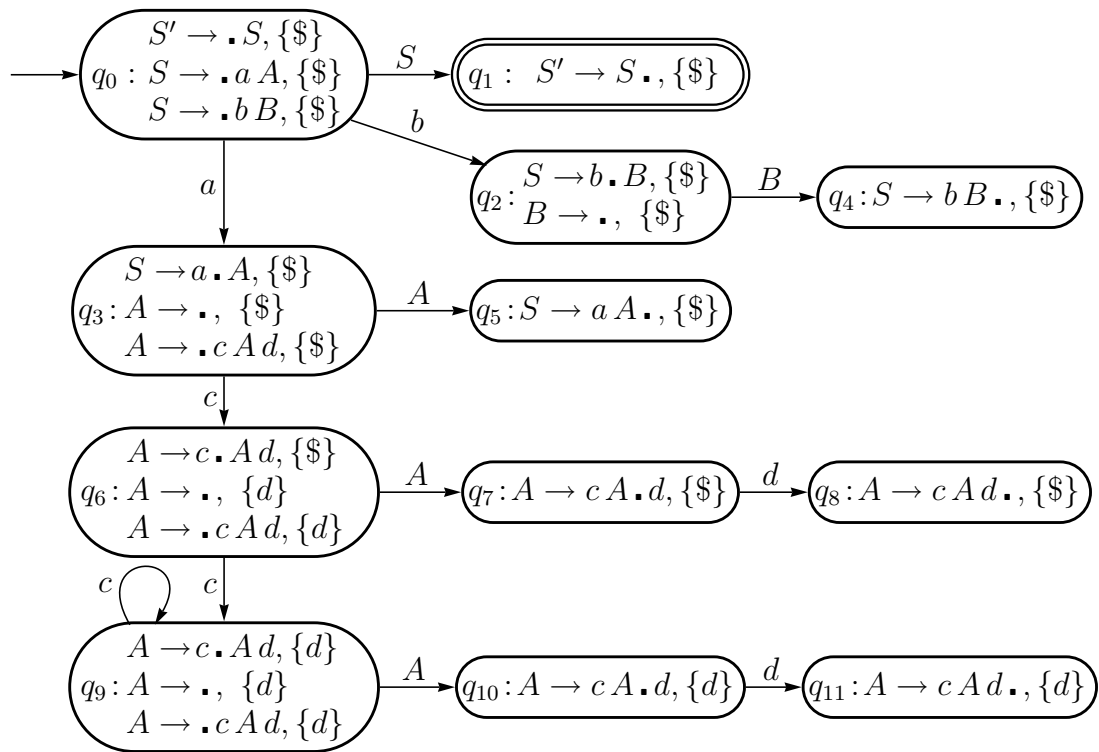


FIGURE 5.5.3. The finite automaton for the $LR(1)$ parsing of the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow a A$, 2. $S \rightarrow b B$, 3. $A \rightarrow \varepsilon$, 4. $A \rightarrow c A d$, 5. $B \rightarrow \varepsilon$.

	<u>action</u>					<u>goto</u>		
	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>\$</i>	<i>S</i>	<i>A</i>	<i>B</i>
<i>q</i> ₀	<i>sh q</i> ₃	<i>sh q</i> ₂				<i>q</i> ₁		
<i>q</i> ₁					<i>acc</i>			
<i>q</i> ₂					<i>red 5</i>			<i>q</i> ₄
<i>q</i> ₃			<i>sh q</i> ₆		<i>red 3</i>		<i>q</i> ₅	
<i>q</i> ₄					<i>red 2</i>			
<i>q</i> ₅					<i>red 1</i>			
<i>q</i> ₆			<i>sh q</i> ₉	<i>red 3</i>			<i>q</i> ₇	
<i>q</i> ₇				<i>sh q</i> ₈				
<i>q</i> ₈					<i>red 4</i>			
<i>q</i> ₉			<i>sh q</i> ₉	<i>red 3</i>			<i>q</i> ₁₀	
<i>q</i> ₁₀				<i>sh q</i> ₁₁				
<i>q</i> ₁₁				<i>red 4</i>				

- where:
0. $S' \rightarrow S$
 1. $S \rightarrow aA$
 2. $S \rightarrow bB$
 3. $A \rightarrow \varepsilon$
 4. $A \rightarrow cAd$
 5. $B \rightarrow \varepsilon$

FIGURE 5.5.4. LR(1) parsing table for the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow aA$, 2. $S \rightarrow bB$, 3. $A \rightarrow \varepsilon$, 4. $A \rightarrow cAd$, 5. $B \rightarrow \varepsilon$. It has been derived from the automaton of Figure 5.5.3 on the preceding page.

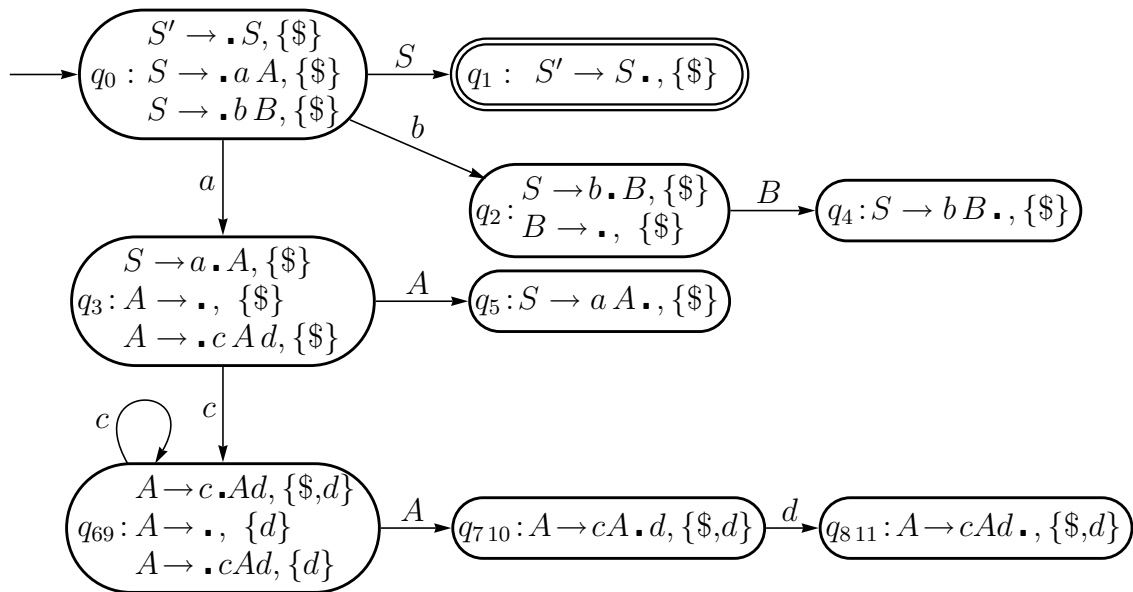


FIGURE 5.5.5. The finite automaton for the LALR(1) parsing of the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow aA$, 2. $S \rightarrow bB$, 3. $A \rightarrow \varepsilon$, 4. $A \rightarrow cAd$, 5. $B \rightarrow \varepsilon$.

	<u>action</u>					<u>goto</u>		
	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>\$</i>	<i>S</i>	<i>A</i>	<i>B</i>
<i>q</i> ₀	<i>sh q</i> ₃	<i>sh q</i> ₂				<i>q</i> ₁		
<i>q</i> ₁					<i>acc</i>			
<i>q</i> ₂					<i>red 5</i>			<i>q</i> ₄
<i>q</i> ₃			<i>sh q</i> ₆₉		<i>red 3</i>		<i>q</i> ₅	
<i>q</i> ₄					<i>red 2</i>			
<i>q</i> ₅					<i>red 1</i>			
<i>q</i> ₆₉			<i>sh q</i> ₆₉	<i>red 3</i>			<i>q</i> _{7 10}	
<i>q</i> _{7 10}				<i>sh q</i> _{8 11}				
<i>q</i> _{8 11}				<i>red 4</i>	<i>red 4</i>			

where:

0. $S' \rightarrow S$
1. $S \rightarrow a A$
2. $S \rightarrow b B$
3. $A \rightarrow \varepsilon$
4. $A \rightarrow c A d$
5. $B \rightarrow \varepsilon$

FIGURE 5.5.6. *LALR*(1) parsing table for the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow a A$, 2. $S \rightarrow b B$, 3. $A \rightarrow \varepsilon$, 4. $A \rightarrow c A d$, 5. $B \rightarrow \varepsilon$. It has been derived from the automaton of Figure 5.5.5 on the preceding page.

EXAMPLE 5.5.6. [*LALR(1) Parsing: Example 3*] Let us consider the augmented grammar G' :

0. $S' \rightarrow S$ 1. $S \rightarrow L = R$ 2. $S \rightarrow R$ 3. $L \rightarrow *R$ 4. $L \rightarrow a$ 5. $R \rightarrow L$

By applying Algorithm 5.4.1 on page 90 we get the finite automaton of Figure 5.5.7 and the $LR(1)$ parsing table of Figure 5.5.8 on the next page.

By applying Algorithm 5.5.1 on page 104 for constructing the $LALR(1)$ parsing table from the $LR(1)$ parsing table, we have to make no changes to that parsing table and thus, the table of Figure 5.5.8 is also the $LALR(1)$ parsing table of the grammar G' . \square

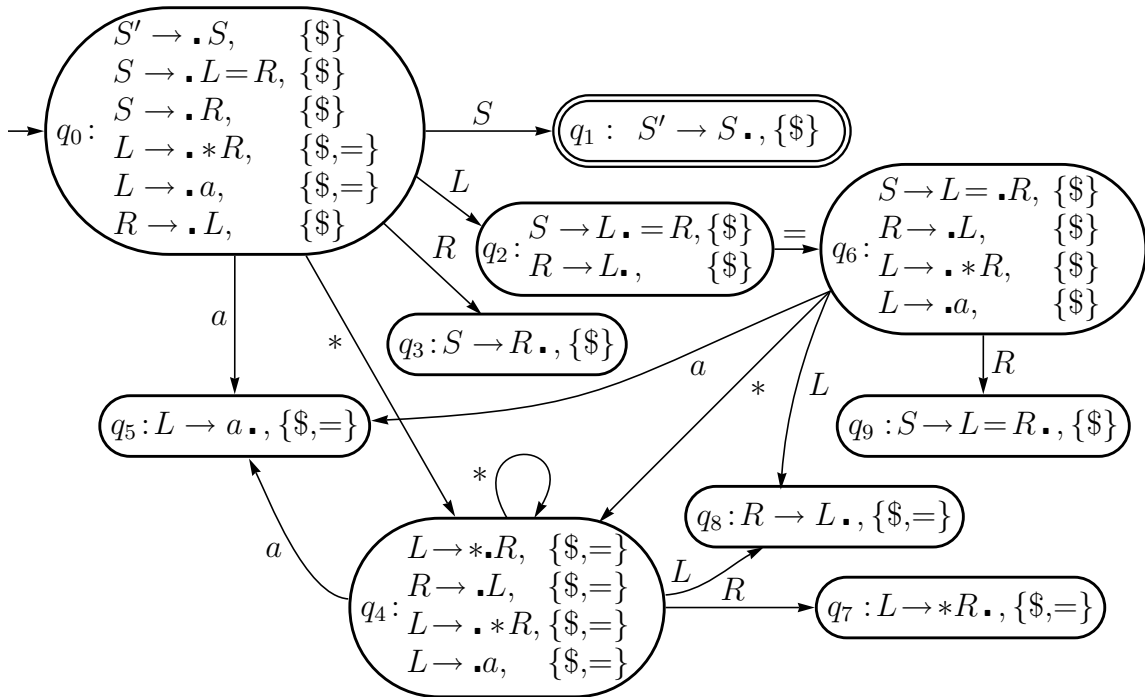


FIGURE 5.5.7. The finite automaton for the $LR(1)$ parsing of the grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow L = R$, 2. $S \rightarrow R$, 3. $L \rightarrow *R$, 4. $L \rightarrow a$, 5. $R \rightarrow L$. This automaton is also the $LALR(1)$ parsing automaton for this grammar G' .

	<u>action</u>				<u>goto</u>		
	<i>a</i>	=	*	\$	<i>S</i>	<i>L</i>	<i>R</i>
q_0	<i>sh</i> q_5		<i>sh</i> q_4		q_1	q_2	q_3
q_1				<i>acc</i>			
q_2		<i>sh</i> q_6		<i>red</i> 5			
q_3				<i>red</i> 2			
q_4	<i>sh</i> q_5	<i>sh</i> q_4				q_8	q_7
q_5		<i>red</i> 4		<i>red</i> 4			
q_6	<i>sh</i> q_5		<i>sh</i> q_4			q_8	q_9
q_7		<i>red</i> 3		<i>red</i> 3			
q_8		<i>red</i> 5		<i>red</i> 5			
q_9				<i>red</i> 1			

where:

0. $S' \rightarrow S$
1. $S \rightarrow L = R$
2. $S \rightarrow R$
3. $L \rightarrow *R$
4. $L \rightarrow a$
5. $R \rightarrow L$

FIGURE 5.5.8. $LR(1)$ parsing table for the augmented grammar G' with axiom S' and productions: 0. $S' \rightarrow S$, 1. $S \rightarrow L = R$, 2. $S \rightarrow R$, 3. $S \rightarrow *R$, 4. $L \rightarrow a$, 5. $R \rightarrow L$. This table is also the $LALR(1)$ parsing table for this grammar G' . It has been derived from the automaton of Figure 5.5.7 on the previous page.

EXAMPLE 5.5.7. [**$LALR(1)$ Parsing: Example 4**] Let us consider the grammar G with axiom S and the following productions:

1. $S \rightarrow B B a$
2. $B \rightarrow B b$
3. $B \rightarrow b$

Since the axiom S has one production only and S does not occur on the right hand side of any production, we can take the given grammar G to be the augmented grammar of G itself.

By applying Algorithm 5.4.1 on page 90 for constructing the $LR(1)$ parsing tables and recalling that $First_1(B) = \{c\}$, we get the finite automaton of Figure 5.5.9 on the next page and the $LR(1)$ parsing table of Figure 5.5.10 on the facing page. By applying Algorithm 5.5.1 on page 104 for constructing the $LALR(1)$ parsing tables from the $LR(1)$ parsing tables, we get the finite automaton of Figure 5.5.11 on page 114 and the $LALR(1)$ parsing table of Figure 5.5.12 on page 114.

The states of the finite automaton of Figure 5.5.9 on the facing page which have been fused are:

- (i) states q_4 and q_5 , fused into the state q_{45} , and
- (ii) states q_6 and q_7 , fused into the state q_{67} .

□

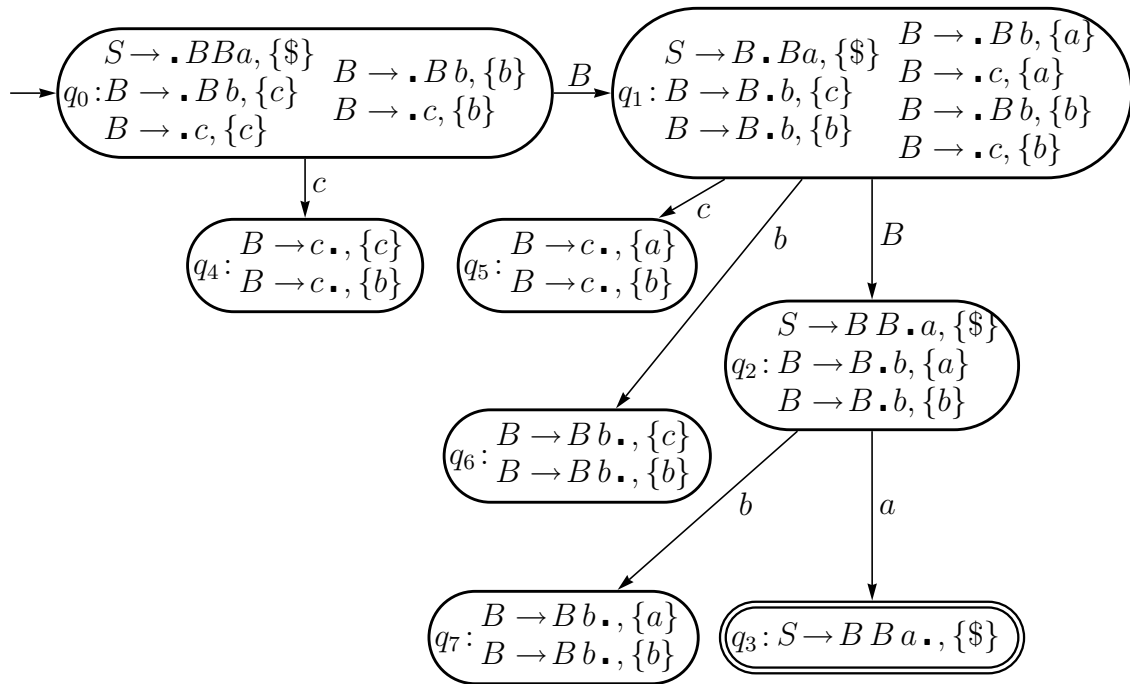


FIGURE 5.5.9. The finite automaton for the $LR(1)$ parsing of the grammar G with axiom S and productions: 1. $S \rightarrow BBa$, 2. $B \rightarrow Bb$, 3. $B \rightarrow b$.

	<u>action</u>			<u>goto</u>	
	a	b	c	$\$$	
q_0			$sh\ q_4$		q_1
q_1		$sh\ q_6$	$sh\ q_5$		q_2
q_2	$sh\ q_3$	$sh\ q_7$			
q_3				acc	
q_4		$red\ 3$	$red\ 3$		
q_5	$red\ 3$	$red\ 3$			
q_6		$red\ 2$	$red\ 2$		
q_7	$red\ 2$	$red\ 2$			

where: 1. $S \rightarrow BBa$
 2. $B \rightarrow Bb$
 3. $B \rightarrow c$

FIGURE 5.5.10. $LR(1)$ parsing table for the grammar G with axiom S and productions: 1. $S \rightarrow BBa$, 2. $B \rightarrow Bb$, 3. $B \rightarrow b$. It has been derived from the finite automaton of Figure 5.5.9.

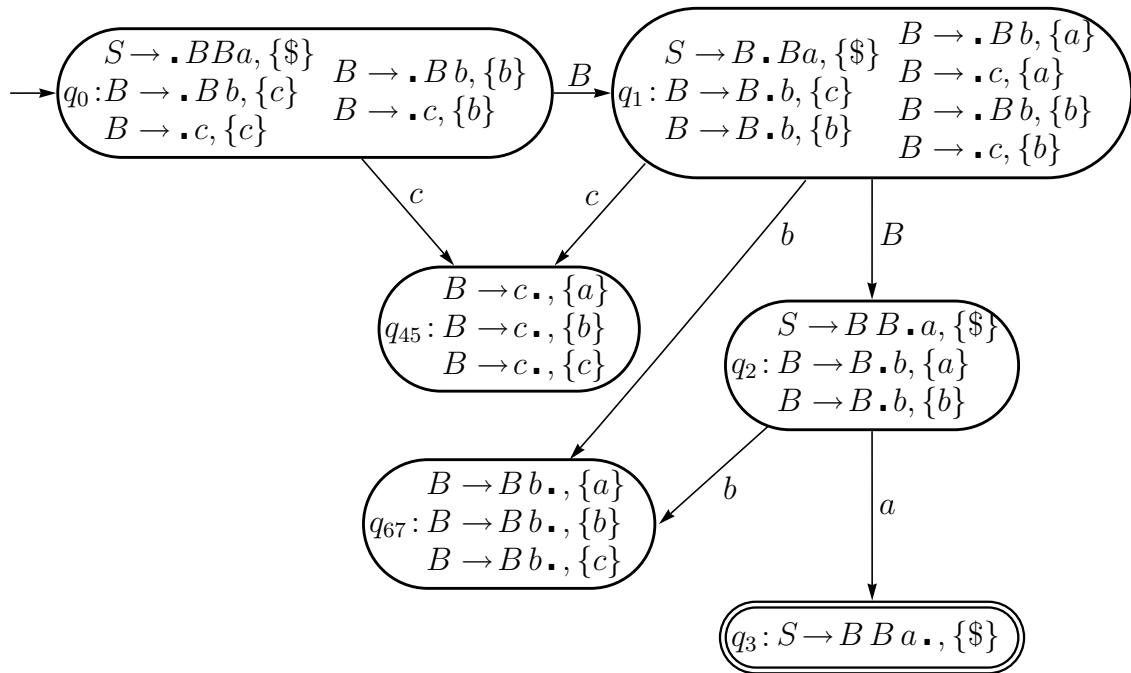


FIGURE 5.5.11. The finite automaton for the $LR(1)$ parsing of the grammar G with axiom S and productions: 1. $S \rightarrow BBa$, 2. $B \rightarrow Bb$, 3. $B \rightarrow b$.

	<u>action</u>			<u>goto</u>	
	a	b	c	$\$$	B
q_0			$sh\ q_4$		q_1
q_1		$sh\ q_6$	$sh\ q_5$		q_2
q_2	$sh\ q_3$	$sh\ q_7$			
q_3				acc	
q_{45}	$red\ 3$	$red\ 3$	$red\ 3$		
q_{67}	$red\ 2$	$red\ 2$	$red\ 2$		

where: 1. $S \rightarrow BBa$
 2. $B \rightarrow Bb$
 3. $B \rightarrow c$

FIGURE 5.5.12. $LALR(1)$ parsing table for the grammar G with axiom S and productions: 1. $S \rightarrow BBa$, 2. $B \rightarrow Bb$, 3. $B \rightarrow b$. It has been derived from the finite automaton of Figure 5.5.11.

5.6. Time and Space Complexity Results on Context-Free Parsing

In Table 1 we give the measures of the time-complexity and the space-complexity for testing a grammar and constructing a parser for the following subclasses of context-free grammars: $LL(1)$ grammars, $SLR(1)$ grammars, $LR(1)$ grammars, and $LALR(1)$ grammars [23, page 400]. In that table the size n of the given input grammar is assumed to be the total number of occurrences of symbols in its productions.

	deterministic time for testing a grammar	deterministic time for constructing the parser	size of the parser
$LL(1)$	$O(n^2)$	$O(2^{n^2+4 \log n})$	$O(2^{n^2+2 \log n})$
$SLR(1)$	$O(n^2)$	$O(2^{n+3 \log n})$	$O(2^{n+2 \log n})$
$LR(1)$	$O(n^3)$	$O(2^{n^2+4 \log n})$	$O(2^{n^2+2 \log n})$
$LALR(1)$	$O(2^{n+3 \log n})$	$O(2^{n+3 \log n})$	$O(2^{n+2 \log n})$

TABLE 1. Deterministic time and space complexities (that is, time and space taken by a deterministic Turing Machine) for testing a grammar of size n and for constructing a parser of various classes of grammars: $LL(1)$, $SRL(1)$, $LR(1)$, and $LALR(1)$.

We have that to test an arbitrary context-free grammar G for the $SLR(k)$ property, that is, to test whether or not, given an arbitrary context-free grammar G , there exists $k \geq 0$ such that G is $SLR(k)$ it takes deterministic polynomial time in the size of G . We also have that the problem of testing whether or not given an arbitrary context-free grammar G , there exists $k \geq 0$ such that G is an $LALR(k)$ grammar (or G is *not* an $LALR(k)$ grammar), is PSPACE-complete in the size of G [23, page 398].

5.7. Conventions for LL(k) and LR(k) parsing

In Table 2 on the following page below we recall some hypotheses we made concerning the parsing of various kinds of $LL(k)$ languages and $LR(k)$ languages and, in particular:

- (i) the use of a rightmost, new symbol \$ in the input string,
- (ii) the use of augmented grammars with the production of the axiom,
- (iii) the initial stack configuration, and
- (iv) the lookahead sets.

Obviously, when we consider augmented grammars, we have that the new axiom S' does not occur on the right hand side of any production.

5.8. Subclasses of Context-free Languages

This section is devoted to the illustration of Figure 5.8.1 on page 119 in which we consider various subclasses of the context-free languages and their relationships. We will also discuss some other relationships among subclasses of context-free languages.

	input string	augmented grammar	production of the axiom	initial stack	lookahead set
$LL(k)$	ended by $\$$	no	axiom S	$S\$$ ▲	none
$LR(0)$ and $SLR(1)$	ended by $\$$	yes	axiom S' add: $S' \rightarrow S\$$	q_0 ▲	none
$LR(1)$ and $LALR(1)$	ended by $\$$	yes	axiom S' add: $S' \rightarrow S$	q_0 ▲	$\{\$\}$

TABLE 2. Our conventions on the input string, the augmented grammar with the production of the axiom, the initial stack configuration (q_0 is the initial state), and the lookahead set for various classes of context-free grammars.

We have already seen on page 85 that the grammar G with axiom S and the following productions:

$$S \rightarrow A \mid B \quad A \rightarrow aAb \mid 0 \quad B \rightarrow aBbb \mid 1$$

is $LR(0)$ and it is *not* $LL(k)$, for any $k \geq 0$. This fact shows the proper containment between the deterministic context-free languages and the union of the $LL(k)$ languages, for all $k \geq 0$, as indicated in Figure 5.8.1.

The following notion is necessary for presenting Fact 5.8.2 below. It relates $SLR(k)$ grammars and $LR(k)$ grammars.

DEFINITION 5.8.1. [Structurally Equivalent Grammars] Two unambiguous context-free grammars G_1 and G_2 are said to be *structurally equivalent* iff $L(G_1) = L(G_2)$ and for every word $w \in L(G_1)$ (which is equal to $L(G_2)$) the derivation trees of w in G_1 and G_2 are the same except, possibly, for the labeling of the nodes with nonterminal symbols.

For instance, the following two grammars G_1 and G_2 are structurally equivalent: (i) the grammar G_1 has axiom S and productions

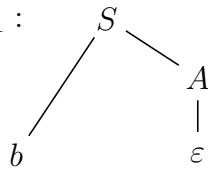
$$S \rightarrow Ac \mid bA \\ A \rightarrow \varepsilon$$

(ii) the grammar G_2 has axiom S and productions

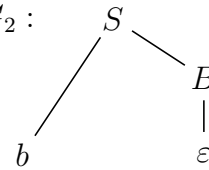
$$S \rightarrow Ac \mid bB \\ A \rightarrow \varepsilon \\ B \rightarrow \varepsilon$$

Indeed, in particular, for the word b we have the following two derivation trees which differ only for the labeling of a node with a nonterminal symbol.

In the grammar G_1 :



In the grammar G_2 :



FACT 5.8.2. For any $k \geq 0$, any context-free grammar G can be transformed into a structurally equivalent grammar which is $SLR(k)$ iff G is $LR(k)$ [23, page 84].

Note that, an ambiguous grammar may generate a deterministic context-free language. Consider, for instance, the grammar G with axiom S and productions:

$$S \rightarrow A \mid a$$

$$A \rightarrow a$$

It is an *ambiguous* grammar, but its language $L(G) = \{a\}$ is a deterministic context-free language (because it is a regular language) and it has an $LR(0)$ parser [23, page 45].

Now we will present some hierarchies of grammars (and languages) which are all subclasses of the context-free grammars (and languages). Unless otherwise specified, we will assume that the grammars are all reduced (see Definition 1.2.15 on page 15).

Let us first recall the different ways in which one can present a class of grammars, and indeed in this book various classes of grammars has been introduced in these different ways.

Possible presentations of classes of grammars.

A class of grammars can be introduced by either:

- (i) providing the rules for constructing their productions (for instance, the class of context-free grammars has been defined by stipulating that their productions are of the form: $A \rightarrow \sigma$, where $A \in V_N$ and $\sigma \in (V_T \cup V_N)^*$), or
- (ii) giving some conditions which should be satisfied by their derivations (for instance, the class of the $LR(k)$ grammars has been introduced in Definition 5.1.2 on page 75 in that way), or
- (iii) presenting an algorithm for parsing the languages generated by the grammars of the class, and giving some conditions on that algorithm (for instance, the class of $LR(0)$ grammars has been introduced in Definition 5.2.8 on page 84 in that way).

Every class of grammars defines a class of languages in the sense that, given a class \mathcal{A} of grammars, the corresponding class of languages is made out of all languages L such that there exists a grammar in the class \mathcal{A} which generates L . For instance, the class of the $LR(1)$ languages is made out of all languages L such that there exists an $LR(1)$ grammar which generates L .

We say that a class \mathcal{A} of grammars is contained in a class \mathcal{B} of grammars, and we write $\mathcal{A} \subseteq \mathcal{B}$, if every grammar of the class \mathcal{A} is also a grammar of the class \mathcal{B} . We say that a class \mathcal{A} of languages is contained in the class \mathcal{B} of languages, and we

write $\mathcal{A} \subseteq \mathcal{B}$, if for every language L in \mathcal{A} there is a grammar G of the class \mathcal{B} which generates L .

We have the following hierarchy of grammars and languages. As usual, (i) $\mathcal{A} = \mathcal{B}$ means $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$, and (ii) $\mathcal{A} \subset \mathcal{B}$ means $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$.

Hierarchy of grammars. For any $k > 1$:

$$LL(0) \subset LL(1) \subset \dots \subset LL(k) \subset LL(k+1) \subset \dots \subset LR(1) \subset \dots \subset LR(k)$$

Hierarchy of grammars:

$$LR(0) = SLR(0) \subset SLR(1) \subset LALR(1) \subset LR(1)$$

Hierarchy of languages. For any $k \geq 1$:

$$LL(0) \subset LL(1) \subset \dots \subset LL(k) \subset LL(k+1) \subset \dots \subset LR(1) = \dots = LR(k)$$

Some of the above results have been presented in Theorem 4.3.17 on page 73. We also have that:

- (i) $L \in LL(0)$ iff $L = \emptyset$ or there exists a word w such that $L = \{w\}$ (that is, L is a singleton), and
- (ii) $L \in LR(0)$ implies that $L = \{\varepsilon\}$ or $\varepsilon \notin L$ [9, page 352].

Figure 5.8.1 on the facing page shows the relationships among various subclasses of context-free languages. Recall that: (i) for every deterministic context-free language there exists a context-free grammar which is unambiguous, (ii) for all $k \geq 0$, any $LR(k)$ grammar is unambiguous, and (iii) if a word w has two distinct parse trees, then it has two distinct parse trees which are generated by leftmost derivations (or, equivalently, rightmost derivations).

FACT 5.8.3. [Relationships Between $SLR(k)$ Grammars and $LALR(k)$ Grammars] The class of $SLR(0)$ grammars coincides with the class of $LALR(0)$ grammars. For $k \geq 1$ the class of $SLR(k)$ grammars is properly contained in the class of $LALR(k)$ grammars.

EXAMPLE 5.8.4. The grammar with axiom S and productions

$$S \rightarrow Ac \mid bA \mid bc$$

$$A \rightarrow \varepsilon$$

is an $LALR(1)$ grammar which is *not* $SLR(k)$ for any $k \geq 0$ [23, page 72].

FACT 5.8.5. [Relationships Between $LALR(k)$ Grammars and $LR(k)$ Grammars] The class of $LALR(0)$ grammars coincides with the class of $LR(0)$ grammars. For $k \geq 1$ the class of $LALR(k)$ grammars is properly contained in the class of $LR(k)$ grammars.

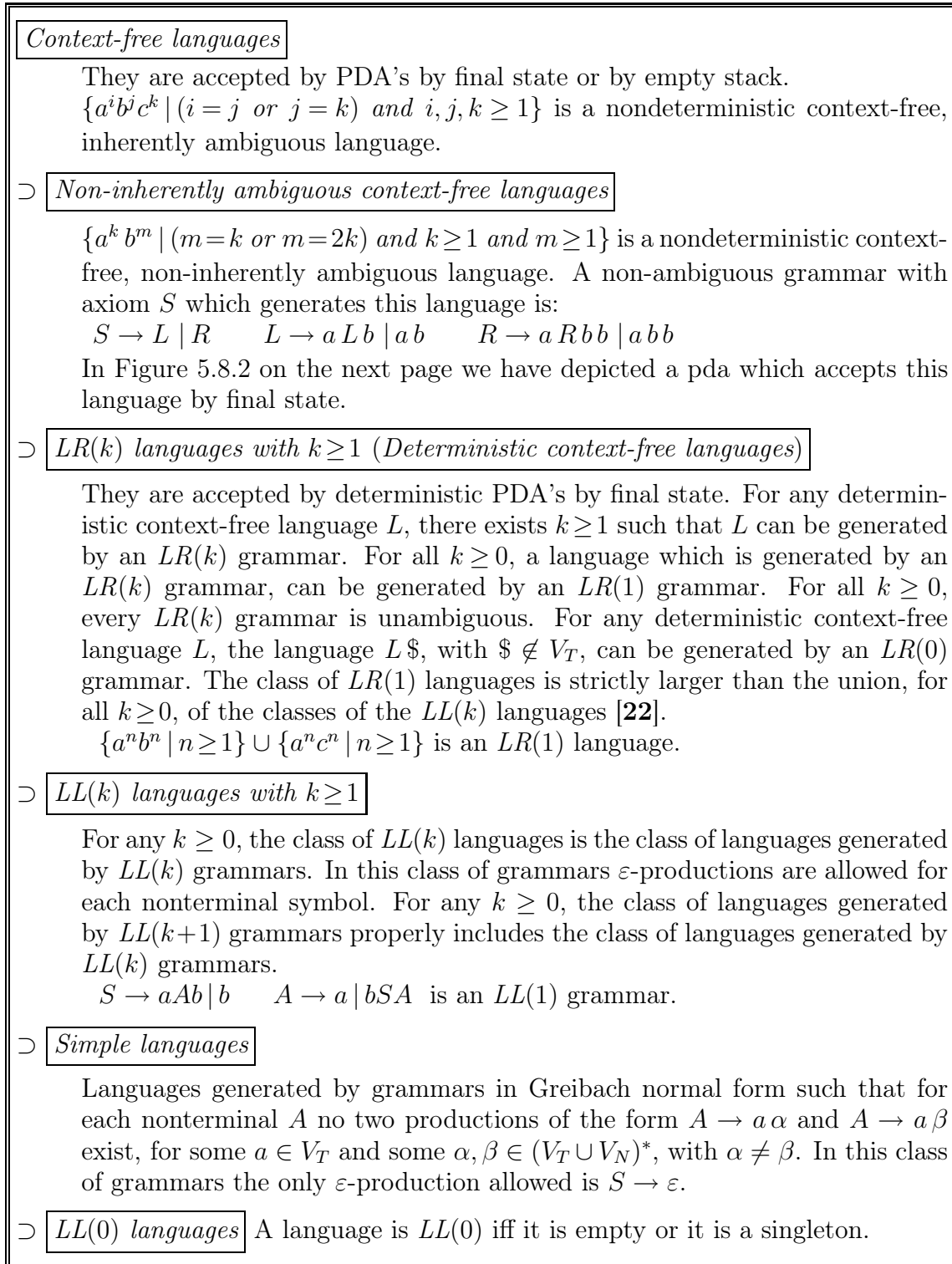


FIGURE 5.8.1. Hierarchy of subclasses of context-free languages. All containments \supset are proper. We allow ε -productions of the form $A \rightarrow \varepsilon$ with $A \in V_N$. We assume without loss of generality, that the start symbol S does *not* occur on the right hand side of any production.

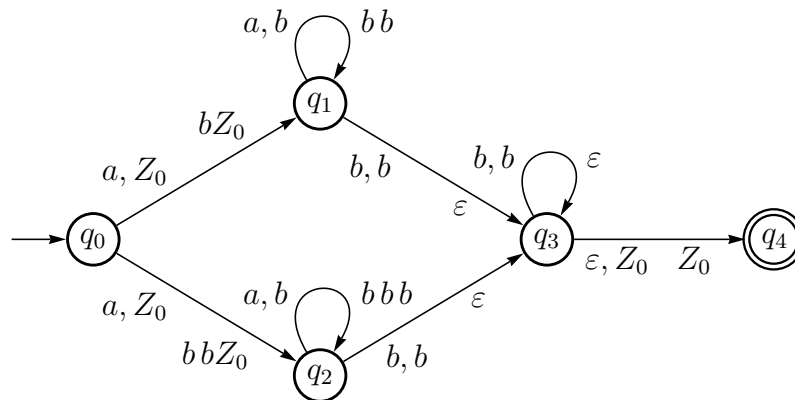


FIGURE 5.8.2. A nondeterministic pda which accepts by final state the language L generated by the grammar with axiom S and productions: $S \rightarrow L \mid R$, $L \rightarrow aLb \mid ab$, $R \rightarrow aRbb \mid abb$. We have that: $L = \{a^k b^m \mid (m = k \text{ or } m = 2k) \text{ and } k \geq 1 \text{ and } m \geq 1\}$. An arc from state q_i to state q_j with label ' $x, Z \ w$ ' means that the pda moves from state q_i to state q_j if: (i) the symbol x is read from the input, (ii) the top of the stack is Z , and (iii) the string w is pushed onto the stack so that the leftmost symbol of w will be the new top of the stack. In this move if $x \neq \epsilon$ then the input head moves one character to the right, otherwise it does not move.

EXAMPLE 5.8.6. The grammar with axiom S and productions

$$S \rightarrow aAa \mid bAb \mid aBb \mid bBa$$

$$A \rightarrow c$$

$$B \rightarrow c$$

is an $LR(1)$ grammar which is *not* $LALR(k)$ for any $k \geq 0$.

FACT 5.8.7. [**Hierarchy of the $LR(k)$ Grammars**] For all $k \geq 0$, the grammar with axiom S and the productions

$$S \rightarrow ab^k c \mid Ab^k d$$

$$A \rightarrow a$$

is an $LR(k+1)$ grammar which is *not* an $LR(k)$ grammar [23, page 53].

FACT 5.8.8. [**Relationships Between $LR(k)$ Grammars and $SLR(k)$ Grammars**] For any $k \geq 0$, any language generated by an $LR(k)$ grammar can be generated by an $SLR(k)$ grammar [23, page 75].

FACT 5.8.9. [**Relationships Between the Classes of $SLR(k)$ Languages, $LR(k)$ Languages, and $LALR(k)$ Languages**] For every $k \geq 0$, we have that the classes of $SLR(k)$ languages, $LR(k)$ languages, and $LALR(k)$ languages are all equal. [23, page 84].

FACT 5.8.10. [Computable Collapse of $LR(k)$ Languages for $k \geq 1$ to $LR(1)$ Languages and $SLR(1)$ Languages] (i) For any $k > 1$, given any $LR(k)$ grammar G there exists an equivalent $LR(1)$ grammar G_1 which is Turing computable from G (see also Theorem 5.4.11 on page 102). (ii) Given any $LR(1)$ grammar G_1 it can be transformed in a Turing computable way into an equivalent $SLR(1)$ grammar [23, page 84].

As a consequence of the above facts we have also the following fact.

FACT 5.8.11. Any deterministic context-free language can be generated by an $SLR(1)$ grammar.

We state without proof also the following results.

FACT 5.8.12. (i) A language L is an $LR(0)$ language iff (i.1) it is an $LR(1)$ language, and (i.2) for all u, v, x if $u \in L$ and $ux \in L$ and $v \in L$ and then $vx \in L$ [23]. (ii) Given an $LR(0)$ language L if $u \in L$ and $ux \in L$ then $ux^* \in L$.

FACT 5.8.13. [Regular Expressions in Right Hand Sides of Context-Free Grammars] Every context-free grammar whose productions have their right hand sides written as regular expressions over the alphabet $V_T \cup V_N$, can be transformed into an equivalent context-free grammar whose productions have their right hand sides written, as usual, as strings in $(V_T \cup V_N)^*$. This transformation can be done in linear time with respect to the size of the given grammar [23, page 116].

For instance, the grammar whose axiom is E and whose productions are:

$$\begin{aligned} E &\rightarrow T (+T)^* \\ T &\rightarrow F (\times F)^* \\ F &\rightarrow a + (E) \end{aligned} \tag{G_1}$$

can be transformed into the equivalent grammar whose axiom is E and whose productions are:

$$\begin{aligned} E &\rightarrow T \mid E + T \\ T &\rightarrow F \mid T \times F \\ F &\rightarrow a \mid (E) \end{aligned} \tag{G_2}$$

Obviously, during this transformation one should be careful about the use of overloaded symbols because, for instance, the symbols which are used for writing regular expressions, such as $+$, $($, and $)$, may also be used as terminal symbols of the given grammar. For instance, in the case of our grammar G_1 above, the round parentheses $($ and $)$ are symbols for writing regular expressions in the productions $E \rightarrow T (+T)^*$ and $T \rightarrow F (\times F)^*$, while they are terminal symbols in the production $F \rightarrow a + (E)$.

Parsers for Operator Grammars and Parser Generators

In this chapter we will present the parsers for operator grammars and we will show how to make use of the parser generators for generating parsers.

6.1. Operator-Precedence Parsers

Let us introduce the following concept of an operator grammar.

DEFINITION 6.1.1. [Operator Grammar] An *operator grammar* is a context-free grammar without ε -productions such that there are no two consecutive nonterminals on the right hand side of the productions.

We have the following fact whose proof is based on the existence of a procedure for generating an equivalent operator grammar from any given unambiguous context-free grammar [3].

FACT 6.1.2. Every context-free language L is such that $L - \{\varepsilon\}$ can be generated by an operator grammar.

Instead of presenting the general theory of parsing languages which are generated by operator grammars, we now present an example of parsing a particular expression language, which is generated by an operator grammar where each operator has an associated *precedence*. This notion of precedence is the usual notion of precedence used in Mathematics so that, for instance, the expression $5 + 2 \times 4$ is evaluated to 13 and not to 28, because \times has a higher precedence than $+$. In the literature the operator grammars whose operators have an associated precedence are called *operator-precedence grammars* and the techniques for parsing those grammars are referred to as *operator-precedence parsing techniques*.

Let us consider the operator-precedence grammar with axiom E and the following four productions:

$$E \rightarrow E + E \mid E \times E \mid E \uparrow E \mid a$$

where, as usual, $+$, \times , and \uparrow denote sum, product, and exponentiation, respectively.

We first introduce the following augmented grammar G' with axiom E' so to place every generated word between *two* occurrences of the symbol $\$$ which is assumed not to be in $V_T \cup V_N$. Thus, in G' we introduce the extra production $E' \rightarrow \$E\$$. Then we also introduce the following parsing table which encodes the precedence among the operators $+$, \times , and \uparrow , the symbol $\$$, and the identifier a of the given grammar:

	\ right	a	$+$	\times	\uparrow	$\$$
left \		a	$+$	\times	\uparrow	$\$$
T :	a		$>$	$>$	$>$	$>$
	$+$	$<$	$>$	$<$	$<$	$>$
	\times	$<$	$>$	$>$	$<$	$>$
	\uparrow	$<$	$>$	$>$	$<$	$>$
	$\$$	$<$	$<$	$<$	$<$	

This table T can be generated from the given grammar G' as we will indicate below. We have that: (i) the rows and the columns of that table are labeled by the terminal symbols of the grammar G' , (ii) the symbols $<$ and $>$ denote the precedence relation between operators, and (iii) the fact that, for instance, \times as a higher precedence with respect to $+$, is denoted in the table T by the two entries: $+\ltimes$ and $\times>+$, as indicated by the subtable:

	\dots	$+$	\times	\dots
\dots				
$+$			$<$	
\times		$>$		
\dots				

Now we will see how the table T is used for parsing a given word w of the language generated by G' .

Let us consider, for instance, the word $w = a + a \times a$. We add two extra $\$$ symbols as a first and last character of w and we get $\$a + a \times a\$$. Then we add the symbol $<$ or the symbol $>$ between any two symbols of $\$w\$$ according to the table T .

We get: $\$ \lt a \gt + \lt a \gt \times \lt a \gt \$$ (for instance, we have: $\$ \lt a$ because in the table T at row $\$$ and column a we have the symbol $<$, and we have: $+\lt a$ because in the table T at row $+$ and column a we have the symbol $<$).

Then the parsing process can be described as a left-to-right move from the left-most $\$$ until a symbol $>$ is found, followed by a right-to-left move until a symbol $<$ is found. Then the string between the symbols $<$ and $>$ (and including them) is reduced according to the grammar at hand and we erase also the symbols $<$ and $>$.

Then we proceed by a new left-to-right move until a symbol $>$ is found, followed by a new right-to-left move until a symbol $<$ is found. Again the string between the symbols $<$ and $>$ is reduced. We continue with these left-to-right and right-to-left moves and reductions, until all $<$'s and $>$'s have been erased.

At that point the first phase of the parsing process terminates. In the case of our grammar G' , we get from $\$w\$ = \$a + a \times a\$$ the following sequence of sentential forms:

1. $\$ \langle a \rangle + \langle a \rangle \times \langle a \rangle \$$
2. reducing $\langle a \rangle$: $\$E + \langle a \rangle \times \langle a \rangle \$$
3. reducing $\langle a \rangle$: $\$E + E \times \langle a \rangle \$$
4. reducing $\langle a \rangle$: $\$E + E \times E \$$

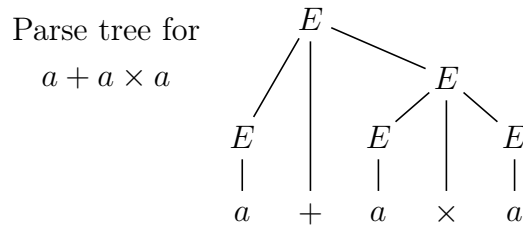
Then a second phase of the parsing process begins. We forget about the derived nonterminal symbols, E in our case, and we use again the table T to insert \langle 's and \rangle 's in the sentential form at hand. In our case by forgetting E , we get: $\$ + \times \$$, and after the insertion of the \langle 's and \rangle 's we get: $\$ \langle + \langle \times \rangle \$$.

Now we proceed as in the first phase of the parsing process and we perform a left-to-right move from the leftmost $\$$ until a symbol \rangle is found, followed by a right-to-left move until a symbol \langle is found. Then we reduce that part of the sentential form which is identified by the substring between the two symbols \langle and \rangle which have been found.

We repeat this process again by forgetting nonterminals, adding \langle 's and \rangle 's, performing moves, and reducing substrings until we are left only with the string $\$ \$$. In our case we get:

5. $\$ \langle + \langle \times \rangle \$$
6. reducing $\langle \times \rangle$: $\$ \langle + E \$$
7. forgetting E : $\$ \langle + \$$
8. adding \rangle : $\$ \langle + \rangle \$$
9. reducing $\langle + \rangle$: $\$ E \$$
10. forgetting E : $\$ \$$

By recalling the sequence of reductions we can reconstruct the parse tree of the given word w . In our case we get the following parse tree:



Now we list some heuristics for filling the table T . More can be found in [2, Chapter 4]. Empty entries in the table T denote errors and, during the parsing process, on those entries we have to invoke suitable *error recovery* routines.

(1) If an operator \times has higher priority than an operator $+$, we have the following subtable:

	+	×
+		<
×	>	

(2) If an operator $+$ has the same priority as an operator $-$, we have the following subtable:

	+	-
+		>
-	>	

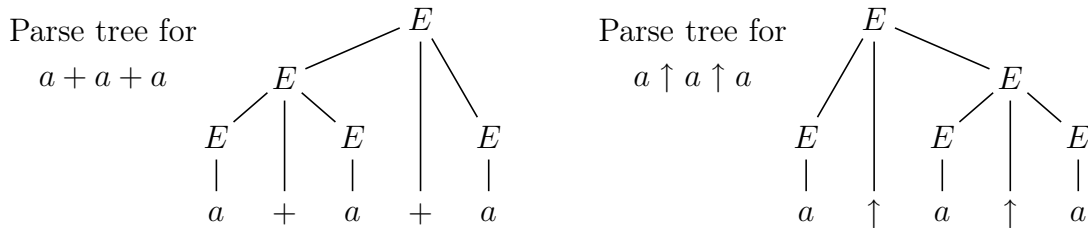
This table enforces a *left associative parsing*, that is, for instance, $a - a + a$ is parsed as $(a - a) + a$.

(3) If we want an operator to be right associative (such as, for instance, the exponentiation operator, denoted \uparrow), we have the following subtable:

	\uparrow
\uparrow	<

The reader may easily verify that, indeed, this subtable enforces a *right associative parsing*, that is, for instance, $a \uparrow a \uparrow a$ is parsed as $a \uparrow (a \uparrow a)$. Analogously, we use $>$ if we want an operator to be *left associative*.

By using the table T we get the following two parse trees for the words $a + a + a$ and $a \uparrow a \uparrow a$, respectively.



Instead of using the parsing table T which takes quadratic space with respect to the number of operators in the grammar, we may sometimes use two functions which we call the *left function*, denoted ℓ , and the *right function*, denoted r . To store those functions it takes linear space only.

In the rest of this section, by abuse of language when we say ‘operator’ we mean any element of the set $Op \cup \{\$, a\}$ which labels a row or a column of the table T .

The left function and the right function are used, instead of the parsing table T , as follows. For instance, given two operators α and β , in order to know whether $\alpha < \beta$ or $\alpha > \beta$, we compare the values of $\ell(\alpha)$ and $r(\beta)$, and

- (i) if $\ell(\alpha) < r(\beta)$ then we have $\alpha < \beta$, and
- (ii) if $\ell(\alpha) > r(\beta)$ then we have $\alpha > \beta$.

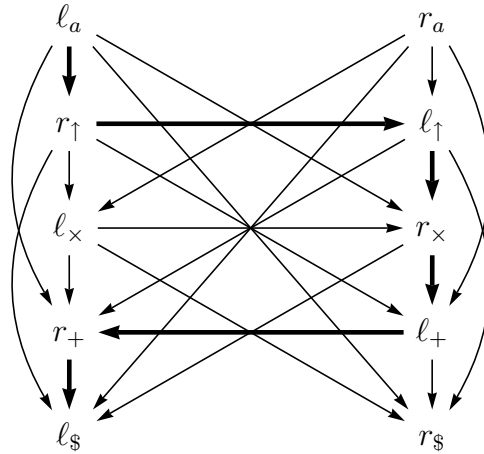
Now we illustrate how to derive the left function and the right function, if they exist, from a parsing table T [2, page 209] in the case when between two operators there may be only either the precedence relation $<$ or the precedence relation $>$. The interested reader may look at [2, Chapter 4] to see how to deal with the case in which extra relations, besides $<$ and $>$, exist between two operators of operator-precedence grammars.

The derivation of the left function and the right function is based on a graph which is constructed as follows.

For each operator α of the grammar we introduce two nodes: l_α and r_α . For each pair of operators α and β , if $\alpha > \beta$ then we draw an edge from l_α to r_β , and if $\alpha < \beta$ then we draw an edge from r_β to l_α .

If the graph we have constructed has a cycle than the *left function* and the *right function* do not exist. If the graph has no cycle, then for each operator α , $l(\alpha)$ is the length of the longest path in the graph starting from l_α , and $r(\alpha)$ is the length of the longest path in the graph starting from r_α .

For instance, in the case of the given expression grammar with operators $+$, \times , \uparrow , $\$$, and a , we get the following graph:



In this drawing of the graph for each operator α , the position of the node l_α and the position of the node r_α has been chosen for minimizing the number of the edge crossings. As the reader may verify, in our case we get the following values for the left function and the right function:

	a	$+$	\times	\uparrow	$\$$
left function l :	6	2	4	4	0
right function r :	5	1	3	5	0

For instance, the entry 6 for $l(a)$ is due to the path (in bold in the graph above):

$$l_a \rightarrow r_\uparrow \rightarrow l_\uparrow \rightarrow r_\times \rightarrow l_+ \rightarrow r_+ \rightarrow l_\$$$

which is the longest path in the graph starting from l_a and it has, indeed, 6 edges. In case when we have also the binary operators $-$ (minus) and $/$ (divide), and the round brackets, as usual for arithmetic expressions, we get the following values for the left function and the right function [2, page 209]:

	a	$+$	$-$	\times	$/$	\uparrow	$($	$)$	$\$$
left function l :	6	2	2	4	4	4	0	6	0
right function r :	5	1	1	3	3	5	5	0	0

As already mentioned, more information on the operator-precedence grammars and their parsing procedures can be found in [2, Chapter 4].

6.2. Use of Parser Generators

In this section we illustrate the use of the parser generator Bison [7] and also the use of the lexical analyzer generator Flex [18]. More information on these tools can be found in <http://dinosaur.compilertools.net/>. We will end this section by providing some suggestions for constructing parsers in practical cases.

All programs of this section have been executed using: (i) the Bison version 1.28 (GNU), (ii) the Flex version 2.5, and (iii) the C++ compiler i686-apple-darwin8-gcc-4.0.1, under the operating system Mac OS X 10.4.11.

6.2.1. Generation of Parsers Using Bison.

Bison is a *parser generator*, that is, a program which given a description of a grammar, produces a C or a C++ program that parses words of the language generated by that grammar. In particular, Bison is a parser generator for *LALR(1)* context-free grammars and it is compatible with Yacc (acronym for ‘Yet Another Compiler Compiler’) [11], a well-known parser generator for *LALR(1)* grammars, to which people usually refer when mentioning work in the area of *compiler-compilers*. Compiler-compilers are programs which given a description of the syntax of a programming language, automatically generate compilers for that language.

Bison is compatible with Yacc in the sense that almost all grammar descriptions which can be given in input to Yacc for producing a correct parser, can also be given in input to Bison which will also produce a correct parser.

For reasons of simplicity, we will not discuss here the behaviour of Bison when: (i) the given grammar is *not* an *LALR(1)* grammar, or (ii) the given grammar is an *ambiguous* grammar, or (iii) the word to be parsed *does not* belong to the language generated by the given grammar. For these issues the interested reader may refer to [7] for the case of Bison, and to [2, page 261–266] for the case of Yacc.

Let us see how to use the parser generator Bison by presenting a few examples.

EXAMPLE 6.2.1.

Generating a parser for a simple LALR(1) language.

Let us consider the following grammar G whose axiom is S and whose productions are:

$$1. S \rightarrow CC \quad 2. C \rightarrow cC \quad 3. C \rightarrow d$$

In order to get a parser for the language $L(G)$ generated for this grammar by Bison, we have first to prepare a file whose extension is ‘.y’. Let that file be called `grammar.y`. It should be structured in three parts separated by `%%` as follows:

```

declarations
%%
productions and semantic actions
%%
lexical analyzer and auxiliary C or C++ functions

```

For instance, in the case of our grammar G , we prepare the following file.

```

/**
 * =====
 *                               Use of Bison: an LALR(1) grammar
 * Filename: "grammar.y"
 *
 *                               S -> C C '\n'
 *                               C -> 'c' C | 'd'
 *
 * S is the axiom. 'c' and 'd' are the characters c and d, respectively.
 * '\n' is the carriage-return character. '\t' is the tab character.
 * =====
 */
%{
#include <stdio.h>    /* needed for using printf */
}%
%% /* Grammar and Semantic Actions.
 * The semantic value is the number of characters. */
S:      C C '\n' { printf ("--> result: %d\n", $1 + $2); }
;
C:      'c' C      { $$ = 1 + $2; }
      | 'd'      { $$ = 1; }
;
%%
/* --- Lexical Analyzer ---- */
/* The lexical analyzer returns the numeric code of the character read.
 * It skips all blanks and tabs. */
int yylex (void) {
int k;
/* Skip blanks and tabs. */
while ((k = getchar ()) == ' ' || k == '\t') ;
/* Return the numeric code k of a char read. */
return k;
}
/* -----
 * input:      output:
 * -----
 * $ bison grammar.y
 * $ cc grammar.tab.c -ly -o grammar
 * $ ./grammar
 * ccd cccd
 * --> result: 7
 * $ ./grammar
 * cdcdd
 * parse error
 * ----- */

```

In the first part of the file `grammar.y`, that is, the declarations part, besides some comments, we have only the command `#include <stdio.h>`. That command is required for using the printing function `printf` (see the second part of the file `grammar.y`).

In the second part we write the productions of the grammar. The nonterminal of the left hand side of the first production is assumed to be the axiom of the grammar. Each production is associated with a C++ command (or, in general, a sequence of C++ commands) which specifies the so-called *semantic actions*. In our case, we have chosen that the semantic actions compute the number of `c`'s and `d`'s which occur in the given string to parse.

Due to the presence of semantic actions, Bison produces, from a given grammar, ‘more’ than a simple parser. Indeed, it produces a *translator* which, when given in input a word of the language generated by the given grammar, constructs an output value (as it is done by the so-called *attribute grammars* [13]). In the specifications of the semantic actions: (i) the identifier `$$` denotes the value associated with the nonterminal in the left hand side of a production, and (ii) `$n` denotes the value associated with the nonterminal occurring on the n -th position on the right hand side of the production.

For instance, for the production $C \rightarrow cC$ (which is written as `C: 'c' C`), the associated semantic action `$$ = 1 + $2` computes for the nonterminal `C` on the left hand side a value which is one unit more than the value computed for the nonterminal `C` on the right hand side (which occurs in the second position, being the first position that of the character `'c'`). Indeed, since we want to count the number of the `c`'s and `d`'s in the string to parse, when we encounter a `'c'`, we have to add 1 to that number.

In the third part of the file `grammar.y` we write the *lexical analyzer*, called `yylex()`, that is, a piece of code for passing to the parser a terminal at a time. In our case a terminal is any character, excluding blank and tab (see the code of `yylex()`).

The functions `printf` and `getchar` are used in the second part and the third part of the file `grammar.y`.

The function `int printf(const char * format, x)` prints its second argument `x` as indicated by the string `format` specified in its first argument. In particular,

- (i.1) if `format` is `"%d\n"` then `x` is printed as a decimal integer,
- (i.2) if `format` is `"%g\n"` then `x` is printed as the shorter between a decimal floating point value or as a mantissa-exponent value, and
- (i.3) if `format` is `"%s\n"` then `x` is printed as a string of characters.

The newline character `\n` has the effect that, after printing `x` the next printing action will take place on the next line.

The function `int getchar(void)` returns the next character from the standard input as an integer value of type `int`.

Having prepared the file `grammar.y`, if we issue the following three commands (where `$` denotes the system prompt):

```
$ bison grammar.y
$ cc grammar.tab.c -ly -o grammar
$ ./grammar
```

and we give in input a word $w \in L(G)$, then we will get in output the value specified by the semantic actions. The option `-ly` in the second command invokes suitable library routines for the C++ compiler. In our case, if we input the word w is `'ccd cccd'`, then we get the number of characters different from blank and tab which are present in `'ccd cccd'`, that is, 7. In particular, as indicated in the comment lines at the end of the file `grammar.y`, we get:

```
ccd cccd          (typed in input)
--> result: 7     (obtained in output)
```

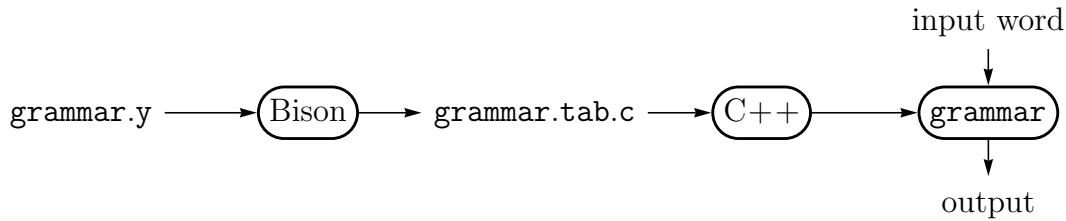



FIGURE 6.2.1. Generation of a parser using Bison for the grammar description in the file `grammar.y`. The suffixes `.y` and `.tab.c` cannot be modified. If the input word is: `ccd cccd` then the output is: `--> result: 7.`

In Figure 6.2.1 we have depicted the process of generating a parser using Bison as we have indicated above.

REMARK 6.2.2. When invoking Bison, if we add the option `-v`, that is, we type the command `bison -v grammar.y`, instead of `bison grammar.y`, we will also get in output a file named `grammar.output` with a description of the finite automaton which directs the *LALR*(1) parsing. Indeed, for our grammar G above, we get the following file which: (i) associates the rule numbers 1, 2, and 3 to the three productions $S \rightarrow CC$, $C \rightarrow cC$, and $C \rightarrow d$, respectively, and (ii) describes the finite automaton represented in Figure 5.5.1 on page 106:

```

Grammar
rule 1    S -> C C '\n'
rule 2    C -> 'c' C
rule 3    C -> 'd'
          ... omissis ...

state 0
  'c' shift, and go to state 1
  'd' shift, and go to state 2
  S   go to state 7
  C   go to state 3
state 1
  C -> 'c' . C (rule 2)
  'c' shift, and go to state 1
  'd' shift, and go to state 2
  C   go to state 4
state 2
  C -> 'd' . (rule 3)
  $default reduce using rule 3 (C)
state 3
  S -> C . C '\n' (rule 1)
  'c' shift, and go to state 1
  'd' shift, and go to state 2
  C   go to state 5
state 4
  C -> 'c' C . (rule 2)
  $default reduce using rule 2 (C)
state 5
  S -> C C . '\n' (rule 1)
  '\n' shift, and go to state 6
  
```

```

state 6
  S -> C C '\n' . (rule 1)
  $default reduce using rule 1 (S)
state 7
  $ go to state 8
state 8
  $ go to state 9
state 9
  $default accept

```

As the reader may verify, we have that the initial state 0 corresponds to state q_0 of Figure 5.5.1 on page 106 and, similarly, state 1 corresponds to state q_{36} , state 2 corresponds to state q_{47} , state 3 corresponds to state q_2 , state 4 corresponds to state q_{89} , state 5 corresponds to state q_5 , and state 7 corresponds to state q_1 .

In the above file `grammar.output` we have a description of the finite automaton of Figure 5.5.1 on page 106, but there are also the extra states 6, 8, and 9 which do not occur in that figure. The reason for these extra states is as follows:

- (i) state 6 is due to the fact that in the actual representation of the grammar G the production $S \rightarrow CC$ is terminated by the newline character '\n' and so a transition from state 5 to a new state 6 is required, and
- (ii) states 8 and 9 are due to the fact that in $LALR(1)$ parsers we actually use augmented grammars with the extra production $S' \rightarrow S$, where S' is the new axiom symbol. In our case this extra production requires a transition from state 7 to state 8 and from state 8 to state 9 (for more details on this point the reader may refer to the Bison manual [7]).

REMARK 6.2.3. When invoking Bison, if we add the option '-y', that is, we type the command '`bison -y grammar.y`', instead of '`bison grammar.y`', then Bison behaves like Yacc in the sense that we will get in output a file named `y.tab.c`, instead of `grammar.tab.c`. Then, in our case, after the command '`cc y.tab.c -ly`', we will get in output the file named `a.out`, instead of the file named `grammar`.

EXAMPLE 6.2.4.

Generating a parser for an expression calculator.

The following example illustrates the use of Bison in the presence of: (i) tokens, (ii) priorities of operators, and (iii) the `yyerror` routine which deals with the errors which may occur during parsing.

Let us consider the following grammar whose axiom is *line* and whose productions are:

$$\left[\begin{array}{l}
 line \rightarrow \varepsilon \quad | \quad line \backslash \mathbf{n} \quad | \quad line \ expr \ \backslash \mathbf{n} \\
 expr \rightarrow \mathbf{NUMBER} \quad | \quad expr + expr \quad | \quad expr - expr \quad | \quad expr * expr \quad | \quad expr / expr \\
 \quad \quad | \quad - expr \quad | \quad expr \wedge expr \quad | \quad (expr)
 \end{array} \right.$$

where: (i) ε is the empty word, (ii) $\backslash \mathbf{n}$ is the newline character, (iii) $+$, $-$, \times , $/$, $*$, and \wedge denote addition, subtraction, multiplication, division, and exponentiation, respectively, and (iv) `NUMBER` is a token string and it will be introduced below. We

have to construct the following file, named `calculator.y`, to be given in input to Bison.

```

/**
 * =====
 *                               Use of Bison: an expression calculator
 * Filename: "calculator.y"
 *
 * =====
 */

%{
#define YYSTYPE double /* YYSTYPE is the type of yylval          */
#include <stdio.h>      /* needed for using printf  */
#include <ctype.h>      /* needed for NUMBER        */
#include <math.h>       /* needed for using pow (exponential) */
%}

/* Precedence of operators is determined by the sequence of lines.
 * For instance, * ties more than +, and ^ ties more than UMINUS. */

%token NUMBER

/*                               PRECEDENCE          */
%left '+' '-' /* left associative: 2-1-1 = (2-1)-1   weakest */
%left '*' '/' /* left associative                       |          */
%left UMINUS /* left associative                       |          */
%right '^' /* right associative: 2^3^2 = 2^(3^2)     strongest */

%%
/* Grammar for input to calculator and Semantic Actions. */

line: // empty production //
    | line '\n'
    | line expr '\n' { printf ("--> result: %.10g\n", $2); }
    | error '\n' { yyerror(": bad line!\n"); } // <-- ERR
/* uncomment this line ERR if parsing should continue upon error */
;
expr: NUMBER { $$ = $1; } // default action //
    | expr '+' expr { $$ = $1 + $3; }
    | expr '-' expr { $$ = $1 - $3; }
    | expr '*' expr { $$ = $1 * $3; }
    | expr '/' expr { $$ = $1 / $3; }
    | '-' expr %prec UMINUS { $$ = - $2; } // unary minus //
    | expr '^' expr { $$ = pow($1,$3); } // exponentiation //
    | '(' expr ')' { $$ = $2; }
;
%%

/* ----- Lexical Analyzer ----- */
/* The lexical analyzer returns a double floating point number
 * on the stack and the token NUMBER, or the numeric code of
 * the character read, if not a number. It skips all blanks and tabs. */

int yylex (void) {
    int k;
    /* Skip blanks and tabs. */
    while ((k = getchar ()) == ' ' || k == '\t') ;

```

```

/* Process numbers.          */
if (k == '.' || isdigit (k))
  { ungetc (k, stdin);
    scanf ("%lf", &yylval); /* value of token */
    return NUMBER;         /* type of token */
  }
/* Return the numeric code k of a char read.  */
return k;
}

/* Function needed if line ERR above is uncommented */
/* Called by yyparse on error.                       */
int yyerror (char const *s) {
  printf ("%s", s);
}

/* -----
* input:      output:
* -----
* $ bison calculator.y
* $ cc calculator.tab.c -ly -o calculator
* $ ./calculator
* 3.2^2.3
*          --> result: 14.51593284
* b
*          parse error: bad line!    // if line ERR is uncommented
* 3+2
*          --> result: 5
* ----- */

```

In the first part of the file `calculator.y` we have, together with the command `#include <stdio>.h` which is needed for using the function `printf`, also the commands:

- (i) `#include <ctype>.h` which is needed for using token strings which are of the form `NUMBER`, and
- (ii) `#include <math>.h` which is needed for using the function `pow` which performs the exponentiation.

All these commands are included between the delimiters `%{` and `%}`. Between these delimiters we also have the command `#define YYSTYPE double` which specifies the type of the variable `yylval` which is used by Bison for returning the value of a token (see below). Outside the delimiters `%{` and `%}` in the first part of the file `calculator.y` there are also:

- (i) the token declarations, and
- (ii) the declarations of precedence between operators.

In our case we have the declaration of the token `NUMBER`. As specified by the third part of the file `calculator.y`, a token `NUMBER` is generated by the lexical analyzer `yylex()` from a sequence of non-blank and non-tab characters read from the standard input, by using the functions `ungetc` and `scanf` which we will present below.

A token `NUMBER` is a string generated by the following grammar (which is equivalent to a regular grammar):

$$\left[\begin{array}{l} \text{NUMBER} \rightarrow \textit{digit}^+ \mid \textit{digit}^* \cdot \textit{digit}^+ \mid \textit{digit}^+ \cdot \textit{digit}^* \\ \textit{digit} \rightarrow 0 \mid \dots \mid 9 \end{array} \right.$$

(Note that in view of Fact 5.8.13 on page 121 we have allowed ourselves to use regular expressions in the right hand sides of the productions.)

In general, the lexical analyzer returns: (i) a *value* which is stored in the variable `yylval`, and (ii) a *type* of that value which is the name of the token. In our case the type is `NUMBER` and it corresponds to a `double` number in C++, that is, a double-precision floating point number as specified by the command `#define YYSTYPE double` in the first part of the file `calculator.y`.

Tokens are generated by the lexical analyzer `yylex()` by calling the function `scanf`. However, before calling `scanf` the lexical analyzer calls the function `ungetc` for taking into account the last character read by the function `getchar` which was required for skipping blanks and tabs.

The function `int ungetc(int character, FILE * stream)` puts back the first argument `character` into the second argument `stream`, and it puts it back into the same position where the last character was read from the `stream`. When the next reading operation from the `stream` is performed, one gets that `character`. Note, however, that the physical content of the `stream` is *not* modified by the execution of the function `ungetc`: only the next reading operation is affected.

The function `int scanf(const char * format, &x)` reads a sequence of characters from the standard input. That sequence has to be understood as a value according to the first argument `format`. Then that value is stored in the location `x`. For instance, given in input `28.7` and having defined `yylval` as a double-precision floating point number by the command `#define YYSTYPE double`, the function call `scanf("%lf", &yylval)` assigns to the variable `yylval` a number which will be printed as `28.700000` when using the command `printf("%lf",yylval)` (`lf` stands for *long float*).

Analogously, given in input the string `abca`, `scanf("%s", str)` generates a string `str` whose value is `abca`.

Ambiguity of the grammar is resolved by adding priorities to the operators. These are specified in the first part of the file `calculator.y` after the part between the delimiters `%{` and `%}`. In particular, we write the command `%left op1` for specifying that the operator `op1` is left associative, and we write the command `%right op2` for specifying that the operator `op2` is right associative.

The precedence between operators is established by their relative order in the list of commands which specifies the associativity of the operators: the operator with the weakest precedence is listed first, and the operator with strongest precedence is listed last. The unary minus `'-'` is denoted by the identifier `UMINUS` and its precedence is indicated by the position of `UMINUS` in the list of precedences. In our case unary minus ties more than the binary minus, also denoted as `'-'`. For instance, the value of `-6-4` is `-10` (not `-2`), because it is equal to `(-6)-4`.

In the third part of the file `calculator.y` we write the function `yyerror`. This function is invoked when a parsing error occurs. In our case, when a parsing error

occurs, the parser skips symbols until the next ‘\n’ (the newline character) is encountered and then, after printing an error message as specified by the semantic action, if any, the parser continues as no error had occurred. In our case, we can type a new input line with a new expression to be evaluated.

At the end of the file `calculator.y` we have listed, as lines of comments, some examples of execution of the executable file named `calculator`. For the input `3.2^2.3` we get in output: `--> result: 14.51593284`, and for the input `b` we get in output: `parse error: bad line!`

EXAMPLE 6.2.5.

Generating a parser for a simple assignment language.

The following example illustrates: (i) the use of Flex [18] as a generator of lexical analyzers which can be used in conjunction with Bison, and (ii) the use of a symbol table during parsing.

The lexical analyzer, also called a *scanner*, or a *tokenizer*, produced by Flex passes to Bison a token at a time, so that Bison can perform the successive parsing phase according to a given *LALR*(1) grammar. Flex is a lexical analyzer generator similar to Lex [15].

In order to produce a lexical analyzer we have first to prepare for Flex a file whose extension is ‘.lex’. As in the case of Bison, that file is structured in three parts separated by `%%`. In the first part there are declarations, in the second part there are productions and semantic actions, and in the third part there are auxiliary functions. Each production takes the form of a regular expression.

Flex generates as output a C++ program named `lex.yy.c`, which defines a function `yylex()`. Once the program `lex.yy.c` is compiled, if from the standard input (or a given input file) `yylex()` recognizes a word which belongs to the language denoted by a regular expression of the list of productions, then `yylex()` executes the corresponding semantic actions.

If we use Flex in conjunction with Bison, the corresponding semantic action returns a *value* for the variable `yylval` and a *type* identifying the token which has been recognized (in our example either the token `VARIABLE` or the token `NUMBER`).

The grammar we will consider in this example is a grammar which defines a simple programming language of assignments.

$$\begin{aligned} \textit{line} &\rightarrow \varepsilon \mid \textit{line expr} \backslash \mathbf{n} \mid \textit{line VARIABLE} := \textit{expr} \backslash \mathbf{n} \\ \textit{expr} &\rightarrow \text{NUMBER} \mid \text{VARIABLE} \mid \textit{expr} + \textit{expr} \mid \textit{expr} - \textit{expr} \mid \textit{expr} * \textit{expr} \mid \textit{expr} / \textit{expr} \\ &\quad \mid - \textit{expr} \mid \textit{expr} \wedge \textit{expr} \mid (\textit{expr}) \end{aligned}$$

where: (i) ε is the empty word, (ii) $\backslash \mathbf{n}$ is the newline character, (iii) $+$, $-$, \times , $/$, $*$, and \wedge denote addition, subtraction, multiplication, division, and exponentiation, respectively, (iv) `NUMBER` is a double-precision floating point number, and (v) `VARIABLE` is a single character variable identifier in the set $\{\mathbf{a}, \dots, \mathbf{z}\}$.

There are three files to be prepared:

(i) one for Bison, which we name `fSAL.y` (the extension `.y` is mandatory),

(ii) one for Flex, which we name `fSAL.lex` (the extension `.lex` is mandatory), and
 (iii) one for operating on the symbol table storing the values of the variables. We name this file `fSAL.h` (the extension `.h` is mandatory).

In Figure 6.2.2 on page 139 we have depicted the process of generating a parser using Bison in conjunction with the lexical analyzer generator Flex.

Let us first consider the file `fSAL.y` which we now list.

```

/**
 * =====
 * Use of Bison: a simple assignment language SAL (1)
 * Filename: "fSAL.y"
 *
 * Parser with Lexical Analyzer. Running Flex and then Bison.
 * This file uses the file: "fSAL.lex"
 *
 * Note: variables have names made out of one character only.
 * =====
 */
%{
#include <stdio.h>
#include <math.h> /* needed for using pow (exponential) */
#include "fSAL.h" /* needed for using putVar, getVar, and printTable */

/* see file "fSAL.lex":
 * structure for information on VARIABLE's and NUMBER's */

struct tokenInfo
{ char *name; /* the name of a NUMBER is assumed to be NULL */
  double value; /* the initial value of a VARIABLE is assumed to be 0 */
};
%}

%union { struct tokenInfo ti; } // type of yylval
%token <ti> VARIABLE // VARIABLE is a token of type ti
%token <ti> NUMBER // NUMBER is a token of type ti
%type <ti> expr // expr has type ti

%token ASSIGN_SYMB PLUS MINUS MULTIPLY DIVIDE POWER OPEN CLOSED NEWLINE

%left PLUS MINUS /* left associative */
%left MULTIPLY DIVIDE /* left associative */
%left UMINUS /* left associative */
%right POWER /* right associative */

%%
line: // empty production //
    | line expr NEWLINE
      { printf("--> result: %.10g\n", $2.value); }
    | line VARIABLE ASSIGN_SYMB expr NEWLINE
      { putVar($2.name, $4.value); }
//    | error NEWLINE {yyerror(": bad input line!\n");} // <-- ERR
/* uncomment this line ERR if parsing should continue upon error. */
;
expr: NUMBER { $$ = $1; }
    | VARIABLE { if (getVar($1.name) != NULL) {
                  $$ .value = (getVar($1.name))->value; }
    }

```

```

| expr PLUS expr          { $$ .value = $1 .value + $3 .value; }
| expr MINUS expr         { $$ .value = $1 .value - $3 .value; }
| expr MULTIPLY expr      { $$ .value = $1 .value * $3 .value; }
| expr DIVIDE expr        { $$ .value = $1 .value / $3 .value; }
| MINUS expr %prec UMINUS { $$ .value = - $2 .value; }
| expr POWER expr         { $$ .value = pow($1 .value,$3 .value); }
| OPEN expr CLOSED        { $$ .value = $2 .value; }
;
%%
#include "lex.yy.c"
/* Function needed if the above line ERR is uncommented */
int yyerror (char const *s) {
    printf ("%s", s);
}
/* -----
* Execution tests with the line "ERR" (in this file "fSAL.y") and the two
* lines "SYMB_TABLE" (in the file "fSAL.h") uncommented: we notify errors
* and we print the Symbol Table after inserting new values for the
* variables.
* input:      output:
* -----
* $ flex fSAL.lex
* $ bison fSAL.y (or: bison -v fSAL.y)
* $ cc fSAL.tab.c -ly -o fSAL
* $ ./fSAL
* b := 3
*           Symbol Table after putVar: (b,3) nil
* a := 2
*           Symbol Table after putVar: (b,3) (a,2) nil
* c:=2+a
*           Symbol Table after putVar: (b,3) (a,2) (c,4) nil
* a:=1
*           Symbol Table after putVar: (b,3) (a,1) (c,4) nil
* a+b^(-2+c)
*           --> result: 10                               // 1+3^2 = 10
* b3
*           parse error: bad input line!
* 2+b
*           --> result: 5
* ----- */

```

The value of the variable `yyval` is *not* a double-precision floating point number as in the previous Example 6.2.1 on page 128, but it is a pointer to a structure named `tokenInfo`. That structure has two fields: (i) a *name* and (ii) a *value*.

(i) For a token `VARIABLE` the structure `tokenInfo` stores the name of the variable and its value (which is initialized to 0), and

(ii) for a token `NUMBER` the structure `tokenInfo` stores the fictitious name `NULL` and the numeric value of the token `NUMBER`.

The `%union` declaration specifies the list of the possible types of the variable `yyval`. In our case we have one type only: `ti` which is the type of the structure `tokenInfo`.

In the first part of the file `fSAL.y` we also have the list of the tokens which are returned by Flex (see the file `fSAL.lex`). For instance, `OPEN` denotes the open parenthesis `'(`.

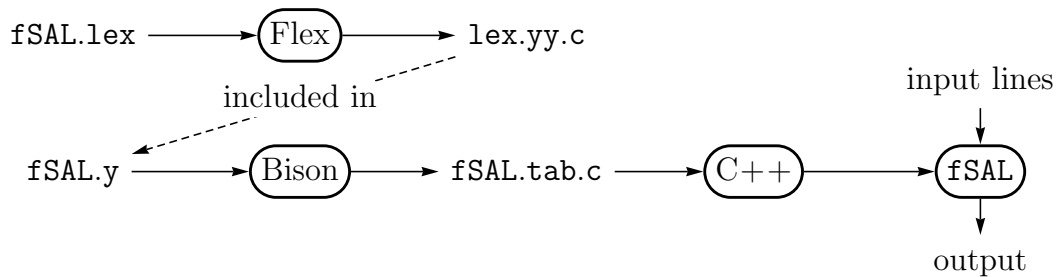


FIGURE 6.2.2. Generation of a parser using Bison, in conjunction with the lexical analyzer generator Flex, for the grammar description in the file `fSAL.y`. The name `lex.yy.c` and the suffixes `‘.lex’`, `‘.y’`, and `‘.tab.c’` cannot be modified. If the input lines are: `a:=1` and `2+a` then the output is: `--> result: 3`.

In the second part of the file `fSAL.y` we have the productions and the associated semantic actions. Note that, since the values of the variable `yylval` are instances of the structure `tokenInfo`, we need to refer to the fields `.name` and `.value`. If a line is an expression `expr`, the semantic action prints the value of that expression. If the line is an assignment of the form `a:=expr`, the semantic action initializes a record (or updates the record) for the variable `a` with the value of the expression `expr` using the command `‘putVar($2.name, $4.value)’`. If a record for the variable `a` is already present in the symbol table, then the value of the record for `a` is updated with the value of the expression `expr`.

If the expression is a `VARIABLE` token, say `a`, the semantic action looks in the symbol table for a structure `tokenInfo` whose name field is `a`, if any, and returns the value field of that structure.

In the third part of the file `fSAL.y` we have specified the lexical analyzer. The specification is *not* done via the definition of the function `yylex()`. It is done, instead, via the command `‘#include "lex.yy.c"’` which has the effect of including the lexical analyzer `lex.yy.c` generated by Flex.

Here is the file `fSAL.lex`. When it is given in input to Flex, we get in output the file `lex.yy.c` to be included in the file `fSAL.y`.

```

/**
 * =====
 *           Use of Bison: a simple assignment language SAL      (2)
 * file: "fSAL.lex" (flex_SimpleAssignmentLanguage)
 *
 * This file has to be given in input to Flex which will produce in output
 * the file "lex.yy.c" which contains the code of the lexical analyzer
 * (also called the scanner).
 *
 * The file "lex.yy.c" has to be included in the file "fSAL.y"
 * which is processed by Bison.
 * -----
 */

```

```

* This version of the lexical analyzer is NOT for STAND ALONE use.
* =====
*/
%{
#include <string.h> /* needed for using char *strdup( const char *str ) */
}%

%%
[a-z]  { yylval.ti.value = 0;
        /* yytext contains the matched character in [a-z]      */
        yyval.ti.name = (char *) strdup(yytext);
        return VARIABLE;
      }
[0-9]+|[0-9]*\.[0-9]+|[0-9]+\.[0-9]*  {
        /* yytext contains the matched number                  */
        yyval.ti.value = atof(yytext);      /* from string to double */
        yyval.ti.name = NULL;
        return NUMBER;
      }
:=      return ASSIGN_SYMB;
\+      return PLUS;
\-      return MINUS;
\*      return MULTIPLY;
\/      return DIVIDE;
\^      return POWER;
\<      return OPEN;
\>      return CLOSED;
\n      return NEWLINE;
[ \t]  ;      /* nothing to return in case of blank or tab      */
.      /* . (dot) matches any input character except \n      */
      printf("Invalid character.\n");

%%
int yywrap() {      // Flex terminates after processing the input file
  return 1;
}
/* -----
* input:      output:
* -----
* $ flex fSAL.lex
* $ bison fSAL.y (or: bison -v fSAL.y)
* $ cc fSAL.tab.c -ly -o fSAL
* $ ./fSAL
* 3+2
*      --> result: 5
* ----- */

```

As in the case of the file with extension ‘.y’ to be given in input to Bison, the file `fSAL.lex` is structured in three parts separated by `%%`. As already mentioned,

- (i) in the first part there are declarations,
- (ii) in second part there is a list of regular expressions and semantic actions which are self-explanatory, and
- (iii) in the third part of the file `fSAL.lex` there is the function `yywrap` which is required for making Flex to terminate when the input file has been processed.

Let us note only the following few points concerning the file `fSAL.lex`.

(i) When a character of the set $\{a, \dots, z\}$ is found, it is recognized as the name of a variable. A `tokenInfo` structure is constructed with name field which is the character found (stored in the variable `yytext`) and a value field initialized to 0. Then a token `VARIABLE` is returned.

(ii) When a string in the language denoted by the regular expression

$$\text{digit}^+ + \text{digit}^* \cdot \text{digit}^+ + \text{digit}^+ \cdot \text{digit}^*$$

is recognized as double-precision floating point number, a `tokenInfo` structure is constructed with the fictitious name field `NULL` and a value field initialized to the value produced by `atof(yytext)`.

The function `double atof(const char * str)`, given a string of characters, produces the corresponding floating point value. For instance, `double w = atof("43.3")` produces a double-precision floating point number `w` whose value will be printed, by using `printf("%f", w)`, as `43.3`.

Then a token `NUMBER` is returned.

(iii) `\+` denotes the symbol `'+'`.

(iv) `[\t]` denotes a regular expression which matches either the blank symbol `' '` (note the blank symbol between `'[` and `'\t'`) or the tab symbol `'\t'`.

(v) The dot `'.'` denotes a regular expression which matches any symbol except the newline symbol `'\n'`.

Now we list the file `fSAL.h` which is required for manipulating the symbol table which contains a record for each variable.

```
/**
 * =====
 * Use of Bison: a simple assignment language SAL (3)
 * Filename: "fSAL.h"
 * This file is required by "fSAL.y" (which is input to Bison)
 * It contains the code for the Symbol Table for variables
 * with the auxiliary functions: printTable, getVar, and putVar.
 * =====
 */
#include <stdlib.h>
#include <string.h> /* for using strlen, strcpy, and strcmp */

/* -----
 * varRec: a record for a variable in the Symbol Table */

struct varRec /* this structure is a record for a variable */
{ char *name; /* the name of the variable */
  double value; /* the value of the variable is a double */
  struct varRec *next;
  /* a pointer to the next record of the Symbol Table */
};
typedef struct varRec varRecType;

/* -----
 * varTable and lastRec are global variables initialized to NULL.
 * varTable points to the first varRec of the Symbol Table.
 * lastRec points to the last varRec of the Symbol Table. */
varRecType *varTable = NULL;
varRecType *lastRec = NULL;
```

```

/*
 * varTable                                lastRec
 * |                                         |
 * +-----+-----+-----+             +--->+-----+-----+-----+
 * +--->| name | value | next--|---->...---->| name | value | NULL |
 * +-----+-----+-----+             +-----+-----+-----+
 *
 * ----- */

/* printTable prints the Symbol Table. */
void printTable(varRecType *T) {
    if (T==NULL) {printf(" nil");}
    else {printf(" (%s",T->name); printf(",%g",T->value); printf(")");
          printTable(T->next);}
};
/* ----- */
/* getVar returns a pointer to the record of the argument variable name. */

varRecType *getVar(char *varName) {
    varRecType *ptr=NULL;

    for (ptr = varTable; ptr != NULL; ptr = (varRecType *) ptr->next) {
        // strcmp(s1,s2) returns 0 iff the string s1 is equal to the string s2
        if (strcmp(ptr->name, varName) == 0)
            return ptr;
    }
    return NULL;
};
/* ----- */
/* Given a variable name and a value, putVar updates the value in the
 * record of that variable in the Symbol Table. If there is no record
 * for that variable in the Symbol Table, putVar initializes a new
 * record for that variable using the given value.
 * ----- */
varRecType *putVar( char *varName, double varValue ) {
    varRecType *ptr = getVar(varName);

    if ( ptr == NULL ) {
        ptr = (varRecType *) malloc (sizeof(varRecType));

        if (varTable == NULL) { //If varTable == NULL the Symbol Table is empty
            varTable = ptr;
            lastRec = ptr;
        } else lastRec->next = ptr; // new record at the end of the list.

        ptr->name = (char *) malloc(strlen(varName) + 1);
        strcpy(ptr->name, varName);
        ptr->value = varValue;
        ptr->next = (varRecType *) NULL;
        lastRec = ptr;
    } else { ptr->value = varValue;
    }
    // printf("Symbol Table after putVar:"); // <-- SYMB_TABLE
    // printTable(varTable); printf("\n"); // <-- SYMB_TABLE
    return ptr;
};
/* ----- */

```

The symbol table is a list of structures each of which, called `varRec`, contains the information relative to a variable.

A structure `varRec` has three fields:

- (i) a field `name` for the name of the variable,
- (ii) a field `value` for the numeric value of the variable, and
- (iii) a field `next` which has a pointer to the next structure in the list.

The pointer to the first structure of the list is `varTable` and the pointer to the last structure in the list is `lastRec`. Both are initialized to `NULL`. The field `next` of the record pointed to by `lastRec` is always `NULL`.

6.2.2. Generation of Lexical Analyzers Using Flex.

In this section we see how Flex can be used for generating a lexical analyzer in a stand-alone mode, that is, without necessarily providing an input to Bison for generating a parser. We will see this use of Flex through an example. Let us assume that we want to construct a lexical analyzer for a simple assignment language similar to that of Example 6.2.5 on page 136.

The tokens to be recognized are:

- (i) single character variable names in the set $\{a, \dots, z\}$
- (ii) integer numbers as a sequence of 1 or more digits in the set $\{0, \dots, 9\}$
- (iii) `:=`
- (iv) `+`
- (v) `*`
- (vi) `\n` (newline)
- (vii) `' '` (blank)
- (viii) `\t` (tab).

When tokens are recognized they will not be returned to Bison as in Example 6.2.5 on page 136, but they will be printed out by a `printf(...)` command.

Below we list a file, called `faSAL.lex`, which can be given in input to Flex for producing of the desired lexical analyzer. This file is structured in three parts separated by `%%` as follows:

```

declarations
%%
productions and semantic actions
%%
auxiliary C or C++ functions and main program

/**
 * =====
 *                               SIMPLE ASSIGNMENT LANGUAGE
 * Filename: "faSAL.lex" (faSAL = flex_alone_SimpleAssignmentLanguage)
 *
 * Lexical Analyzer for the input to "fSAL.y" in STAND ALONE mode.
 *
 * -----
 * This version of the lexical analyzer is for STAND ALONE use.
 * =====
 */

```

```

%{
#include <stdlib.h>
#include <stdio.h>
#include <string.h>    // needed for using char *strdup( const char *str )

struct tokenInfo
{ char *name;
  int value;
};

struct tokenInfo  yylval;
}%

%%
[a-z]  {  yylval.value = 0;
        /* yytext contains the matched character in [a-z]      */
        yylval.name = (char *) strdup(yytext);
        printf("VARIABLE: %s\n", yylval.name);
      }
[0-9]+ {          /* yytext contains the matched number in [0-9]+      */
        yylval.value = atoi(yytext);          /* from string to int */
        yylval.name = NULL;
        printf("INTEGER: %d\n", yylval.value);
      }
:=     printf("ASSIGN_SYMB\n");
\+     printf("PLUS\n");
\*     printf("MULTIPLY\n");
\n     printf("NEWLINE\n");
[ \t]  ;          /* nothing to return in case of blank and tab      */
.      /* . (dot) matches any input character except \n          */
        printf("Invalid character.\n");
}%
int yywrap() {          // Flex terminates after processing the input file
  return 1;
}
int main() {
  yylex();
}
/* -----
 * Compiling and executing in STAND ALONE mode.
 * input:      output:
 * -----
 * $ flex faSAL.lex
 * $ cc lex.yy.c -o faSAL
 * $ ./faSAL
 * 3:=8a
 *          // This is not a valid expression, but the scanner works
 *          // correctly and the tokens are recognized in a correct way.
 *          INTEGER: 3
 *          ASSIGN_SYMB
 *          INTEGER: 8
 *          VARIABLE: a
 *          NEWLINE
 * ----- */

```

In Figure 6.2.3 on the next page we have represented the process of generating a lexical analyzer using Flex as we have illustrated above.

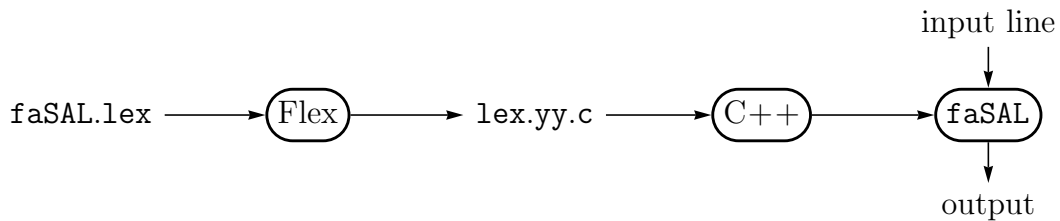


FIGURE 6.2.3. Generation of a lexical analyzer using Flex for the grammar description in the file `faSAL.y`. The suffix `.lex` and the name `lex.yy.c` cannot be modified. If the input line is: `3:=` then the output are the three lines: `INTEGER: 3`, `ASSIGN_SYMB`, and `NEWLINE`.

With respect to the file `fSAL.lex` that we have presented in Example 6.2.5 on page 139, the reader will note in the above file `faSAL.lex`:

- (i) the introduction of the definition of the structure `tokenInfo` and
- (ii) the declaration of the type `'struct tokenInfo'` of the variable `yytext`. This variable stores the string which matches the pattern of a production during the lexical analysis.

Variables are treated as in Example 6.2.5 on page 136, while numbers are now integer numbers, rather than double-precision floating point numbers. Given a string of characters, the function `int atoi(const char * str)` produces the corresponding integer value. For instance, `int n = atoi("52")` produces the integer number `n` whose value is 52. Note that the lexical analyzer works correctly also for inputs which are not legal expressions of Example 6.2.5. For instance, if we issue the following three commands (where `$` denotes the system prompt):

```

$ flex faSAL.lex
$ cc lex.yy.c -o faSAL
$ ./faSAL
  
```

and then we type in input the string `8c:=\n`, (that is, `8c:=` followed by the newline character), we will get the following output:

```

INTEGER: 8
VARIABLE: c
ASSIGN_SYMB
NEWLINE
  
```

Finally, in the file `fSAL.lex` we have the main function `main()` which consists only of the function call `yylex()` which performs the lexical analysis.

6.2.3. Suggestions for Constructing Parsers.

We close this section by providing suggestions for constructing parsers in some practical cases.

The first suggestion for constructing parsers is to use a *parser generator*, such as Yacc or Bison [7, 11]. If we decide not to use a parser generator, we have the choice between constructing either (1) a *top-down parser*, or (2) a *bottom-up parser*. This

choice depends on the manipulations we have to make after the parse tree has been generated by the parser (see [4, Chapter 9]).

Case (1). If we choose to construct a *top-down* parser then an $LL(1)$ parser is recommended. In this case we transform the grammar, if possible, into an $LL(1)$ grammar and then we produce the parsing table for the $LL(1)$ grammar we have generated. We may also try to reduce the size of the $LL(1)$ parsing table by applying the techniques indicated in [4, page 662].

Case (2). If we choose to construct a *bottom-up* parser then an $LALR(1)$ parser is recommended (or an $SLR(1)$ parser, if possible).

Visits of Trees and Graphs and Evaluation of Expressions

In this chapter we will consider a few programs for parsing expressions and visiting trees and graphs. In particular, we will consider programs for: (i) parsing strings written according to a given context-free grammar, and then visiting the trees they denote, (ii) parsing and evaluating boolean expressions, and (iii) testing whether or not propositional formulas are tautologies.

We will also describe an algorithm for encoding n -ary trees using binary trees, and two algorithms proposed by E. W. Dijkstra: (i) an algorithm which given an *undirected*, connected, weighted graph, constructs one of its minimal spanning trees, and (ii) an algorithm which given a *directed*, connected, weighted graph, and one of its nodes p , constructs all paths from the node p to every other node of the graph.

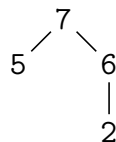
7.1. Depth First Visit of Trees

The program presented in this section: (i) first, parses a given input string denoting a tree, (ii) then, generates an internal form of that tree, and (iii) finally, visits that tree in a depth first manner according to the so called *postorder traversal* (not the preorder traversal we have described on page 23). In particular, for a binary tree we first visit the left son node, then the right son node, and finally, the root node. The productions of the context-free grammar which generates the input string are the following ones:

$$\left[\begin{array}{l} \text{tree} ::= \text{char} \mid - \text{char tree} \mid (\text{tree char tree}) \\ \text{char} ::= '0' \mid '1' \mid \dots \mid '9' \end{array} \right.$$

The axiom is `tree`. No blank characters are allowed before, in between, and after the input string.

A string which is a single character c , denotes a leaf with the character c . A string of the form `- c s` denotes a node with the character c and a single son node denoted by the string s (short for *son* node). A string of the form `(l c r)` denotes a node with the character c , a left son node denoted by the string l (short for *left* son node), and a right son node denoted by the string r (short for *right* son node). For instance, `(57-62)` denotes the tree



The internal tree which corresponds to a given input string, is printed as a string which is generated by the following productions with axiom `tree`:

$$\left[\begin{array}{l} \text{tree} ::= n \mid (n.\text{tree}) \mid (\text{tree}.n.\text{tree}) \\ n ::= 0 \mid 1 \mid \dots \mid 9 \end{array} \right.$$

(i) The first production `tree ::= n` is for the case of a leaf, (ii) the second production `tree ::= (n.tree)` is for the case of a unary tree, and (iii) the third production `tree ::= (tree.n.tree)` is for the case of a binary tree. For instance, the tree denoted by the string (57-62), is printed as: (5.7.(6.2))

If we issue the following two commands:

```
javac DepthFirstVisit.java
java  DepthFirstVisit
```

we get the following sentence in output:

Input a tree according to the grammar given in the program, please.

Then, if we type the following input string (note the absence of blank characters):

```
(-231(-564-78))
```

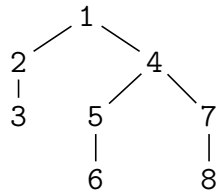
we get:

```
Tree in input:
((2.3).1.((5.6).4.(7.8)))
```

```
Depth first visit of the tree:
```

```
3 2 6 5 8 7 4 1
```

Indeed, the string (-231(-564-78)) denotes the tree:



whose depth first visit is given by the sequence: 3 2 6 5 8 7 4 1.

REMARK 7.1.1. [Terminology for Nodes and Trees] In our context here, we will consider the words ‘node’ and ‘tree’ as interchangeable, because every node in a tree can be viewed as the root of the subtree rooted in that node (see also Remark 7.4.1 on page 181).

In the class `DepthFirstVisit` of the program below, we have that the expression `Character.getNumericValue(c)` is the same as `c-48`, and returns the integer value corresponding to the character `c` representing that value. For instance, the integer value corresponding to the character ‘8’ is 8 (= 56-48).

Here is the program for parsing a string denoting a tree and computing the first depth visit of that tree.

```
/**
 * =====
 *                               DEPTH FIRST VISIT OF A TREE
 * * Filename: "DepthFirstVisit.java"
 * *
 * * Grammar G for the input tree:
 * *           tree ::= char | - char tree | (tree char tree)
 * *           char ::= '0' | '1' | ... | '9'
 * * Grammar for printing the input tree:
 * *           tree ::= n | (n.tree) | (tree.n.tree)
 * *           n    ::= 0 | 1 | ... | 9
 */
```

```

* No blanks are allowed before, in between, and after the input string.
* A tree is input as a string generated by the grammar G and then
* it is visited in the depth first manner.
* =====
*/
import java.io.*;
/**
* =====
*           Class which constructs a Node
*
* This class is used for constructing a Node of a binary tree.
* By default, a Node is a binary node.
*
* A Node can be extended for constructing a Node which is
* either a binary node, or a unary node, or a leaf node.
* =====
*/

class Node {
    public Object  value;      // value field
    public Node   left;       // reference to the left  child node
    public Node   right;      // reference to the right child node

    /**
    * @param v : value at the node
    * @param l : reference to the left  child node
    * @param r : reference to the right child node
    */
    // constructor of a node
    public Node( Object v, Node l, Node r) {
        if ( v == null ) throw new IllegalArgumentException( );
        value  = v;
        left   = l;
        right  = r;
    }
}
/**
* =====
*           Class which constructs a binary node with two child nodes
* =====
*/

class BinaryNode extends Node {
    // constructor of a binary node

    /**
    * @param v : value at the node
    * @param l : reference to the left  child node
    * @param r : reference to the right child node
    */
    public BinaryNode(Object v, Node l, Node r) {
        super(v, l, r);
    }
    /**
    * @return a string representing the tree whose root is the given node
    */
    public String toString(){
        return "(" + left.toString() + "." + value + "." + right.toString() + ")";
    }
}

/**
* =====

```

```

*           Class which constructs a unary node with one child node
* =====
*/

class UnaryNode extends Node {
                                // constructor of a unary node
/**
 * @param v      : value at the node
 * @param child  : reference to the child node
 */

    public UnaryNode(Object v, Node child) {
        super(v, child, null);
    }
/**
 * @return a string representing the tree whose root is the given node
 */
    public String toString() {
        return "(" + value + "." + left.toString() + ")";
    }
}
/**
 * =====
 *           Class which constructs a leaf node
 * =====
 */

class LeafNode extends Node {
                                // constructor of a leaf
/**
 * @param v : value at the node
 */
    public LeafNode(Object v) {
        super(v, null, null);
    }

/**
 * @return a string representing the leaf node
 */
    public String toString() {
        return "" + value;
    }
}

/**
 * =====
 *           Class ParseException
 * =====
 */

class ParseException extends Exception {
                                // constructor of a new instance of ParseException
    public ParseException() {
        super("Parse exception!");
    }
}
// -----
public class DepthFirstVisit {
// -----
    /**

```

```

* This class DepthFirstVisit transforms the input string into a tree
* of type Node.
* The string encodes the tree as indicated by the grammar G.
* @param string: the string which encodes the given tree
* @return the root of the tree generated by the parsing
* @throws IOException or ParseException
*/
    // stringReader is a sequence of characters, which are read one
    // at a time, by the method read() of the class StringReader
public static Node parse(BufferedReader stringReader)
    throws IOException, ParseException {
    char c = (char)stringReader.read();
    if ( ('0' <= c ) && ( c <= '9' ) ) {
        return new LeafNode(new Integer(Character.getNumericValue(c)));
    } else
    if ( c == '-' ) {
        char c2 = (char)stringReader.read();
        Integer value = new Integer(Character.getNumericValue(c2));
        Node childTree = parse(stringReader);
        UnaryNode node = new UnaryNode(value, childTree);
        return node;
    } else
    if ( c == '(' ) {
        Node leftTree = parse(stringReader);
        char c2 = (char)stringReader.read();
        Integer value = new Integer(Character.getNumericValue(c2));
        Node rightTree = parse(stringReader);
        BinaryNode node = new BinaryNode(value, leftTree, rightTree);
        c2 = (char)stringReader.read();
        return node;
    } else {
        throw new ParseException();
    }
}
/** Depth first visit of the given tree
* @param tree : tree to be visited in a depth first manner
* @return the sequence of nodes which is the depth first visit of
* the given tree
*/
public static String depthFirstVisit(Node tree) {
    StringBuffer buffer = new StringBuffer();
// ----- RECURSIVE VERSION -----
    if ( tree instanceof LeafNode ) { //
        buffer.append(((LeafNode)tree).value + " "); //
    } else //
    if ( tree instanceof UnaryNode ) { //
        buffer.append(depthFirstVisit(((UnaryNode)tree).left) //
            + tree.value + " "); //
    } else //
    if ( tree instanceof BinaryNode ) { //
        buffer.append(depthFirstVisit(((BinaryNode)tree).left) //
            + depthFirstVisit(((BinaryNode)tree).right) //
            + tree.value + " "); //
    }; //
// ----- END OF RECURSIVE VERSION -----
    return buffer.toString();
}
// -----
public static void main(String[] args) throws IOException {

```

```

        System.out.println("Input a tree according to the grammar given "
            + "in the program, please.");
        Node tree = null;
        try {
            tree = parse(new BufferedReader(new InputStreamReader(System.in)));
        } catch ( IOException e ) {
            e.printStackTrace();
        } catch ( ParseException pex ) {
            System.out.println(pex.getMessage());
        }
        if (tree != null) {
            System.out.println("\nTree in input:\n" + tree);
            System.out.println("\nDepth first visit of the tree:");
            System.out.println(depthFirstVisit(tree));
        }
    }
}

/**
 * input:  output:
 * -----
 * javac DepthFirstVisit.java
 * java  DepthFirstVisit
 *
 *      Input a tree according to the grammar given in the program, please.
 * (-231(-564-78))
 *
 *      Tree in input:
 *      ((2.3).1.((5.6).4.(7.8)))
 *
 *      Depth first visit of the tree:
 *      3 2 6 5 8 7 4 1
 * -----
 */

```

The following program is like the above one: it parses a string denoting a tree and computes the first depth visit of that tree, but it uses a graphical interface. It can be run by typing the following three commands:

```

javac DepthFirstVisit.java
javac DepthFirstVisitGUI.java
java  DepthFirstVisitGUI

/**
 * =====
 *
 *                      DEPTH FIRST VISIT OF A TREE
 *                      with a Graphical User Interface
 * Filename: "DepthFirstVisitGUI.java"
 *
 * This program should be compiled after the compilation of the program
 * DepthFirstVisit.java.
 * See also the initial comments in the file DepthFirstVisit.java
 *
 * This program uses the class DepthFirstVisit in the file named
 * DepthFirstVisit.java. The file DepthFirstVisit.java should be stored
 * in the same folder where this file DepthFirstVisitGUI.java is stored.
 *
 * To get a dialog window with font size 14 (instead of 28): set the
 * boolean variable bw (short for big_window) to false (see line ***)
 * =====
 */

```

```

*/
import java.io.*;
import javax.swing.JFrame;
import java.awt.Font;                               // needed for changing fonts

// -----
public class DepthFirstVisitGUI extends JFrame {
    /** Creates new form DepthFirstVisitGUI */
    public DepthFirstVisitGUI() {
        initComponents();
    }
    /** This method initComponents() is called from within the constructor
    * to initialize the form.
    */
    private void initComponents() { //GEN-BEGIN:initComponents
        jLabel1 = new javax.swing.JLabel();
        jLabel2 = new javax.swing.JLabel();
        jLabel3 = new javax.swing.JLabel();
        jLabel4 = new javax.swing.JLabel();
        txInput = new javax.swing.JTextField();
        txOutTree = new javax.swing.JTextField();
        txOutDPF = new javax.swing.JTextField();
        btParse = new javax.swing.JButton();
        menuBar = new javax.swing.JMenuBar();
        fileMenu = new javax.swing.JMenu();
        exitMenuItem = new javax.swing.JMenuItem();
// -----

        boolean bw = true; // true (false) for big (small) window. ***
        Font sansSerifFont =
            bw ? (new Font("SansSerif",Font.PLAIN,28)):
              (new Font("SansSerif",Font.PLAIN,14));
// -----
        txInput.setFont(sansSerifFont);
        txOutTree.setFont(sansSerifFont);
        txOutDPF.setFont(sansSerifFont);

        getContentPane().setLayout(null);
        setTitle("DEPTH FIRST VISIT OF A TREE");
        setLocationRelativeTo(null);
        addWindowListener(new java.awt.event.WindowAdapter() {
            public void windowClosing(java.awt.event.WindowEvent evt) {
                exitForm(evt);
            }
        });
// -----
        jLabel1.setBackground(new java.awt.Color(255, 255, 150));
        jLabel1.setFont(sansSerifFont);
        jLabel1.setText("<HTML>\n<PRE>\n"
            +" Grammar for the input tree:\n"
            +"         tree ::= char | - char tree |"
            +" (tree char tree)\n"
            +"         char ::= '0' | '1' | ... | '9'\n"
            +" Grammar for the internal tree:\n"
            +"         tree ::= n | (n.tree) | (tree.n.tree)\n"
            +"         n     ::= 0 | 1 | ... | 9\n"
            +"<PRE>\n</HTML>");

        jLabel1.setVerticalAlignment(javax.swing.SwingConstants.TOP);
        jLabel1.setOpaque(true);

```

```

        getContentPane().add(jLabel1);
        if (bw) {jLabel1.setBounds(10, 10, 1080, 190);}
        else {jLabel1.setBounds(5, 5, 525, 95);}
// -----
        jLabel2.setFont(sansSerifFont);
        jLabel2.setText("Input Tree:");
        getContentPane().add(jLabel2);
        if (bw) {jLabel2.setBounds(20, 208, 180, 32);}
        else {jLabel2.setBounds(10, 104, 90, 16);}
        getContentPane().add(txInput);
        if (bw) {txInput.setBounds(270, 208, 810, 40);}
        else {txInput.setBounds(140, 104, 380, 20);}
// -----
        jLabel3.setFont(sansSerifFont);
        jLabel3.setText("Internal Tree:");
        getContentPane().add(jLabel3);
        if (bw) {jLabel3.setBounds(20, 288, 200, 32);}
        else {jLabel3.setBounds(10, 144, 110, 16);}
        txOutTree.setBackground(new java.awt.Color(25, 255, 25));
        getContentPane().add(txOutTree);
        if (bw) {txOutTree.setBounds(270, 288, 810, 40);}
        else {txOutTree.setBounds(140, 144, 380, 20);}
// -----
        jLabel4.setFont(sansSerifFont);
        jLabel4.setText("Depth First Visit:");
        getContentPane().add(jLabel4);
        if (bw) {jLabel4.setBounds(20, 368, 280, 32);}
        else {jLabel4.setBounds(10, 184, 140, 16);}
        txOutDPF.setBackground(new java.awt.Color(25, 255, 25));
        getContentPane().add(txOutDPF);
        if (bw) {txOutDPF.setBounds(270, 368, 810, 50);}
        else {txOutDPF.setBounds(140, 184, 380, 20);}
// -----
        btParse.setFont(sansSerifFont);
        btParse.setText("Parse and Visit");
        btParse.addActionListener(new java.awt.event.ActionListener() {
            public void actionPerformed(java.awt.event.ActionEvent evt) {
                btParseActionPerformed(evt);
            }
        });
        getContentPane().add(btParse);
        if (bw) {btParse.setBounds(340, 440, 576, 52);}
        else {btParse.setBounds(240, 210, 138, 25);}
// -----
        fileMenu.setText("FILE_MENU");
        exitMenuItem.setText("Exit");
        exitMenuItem.addActionListener(
            new java.awt.event.ActionListener() {
                public void actionPerformed(java.awt.event.ActionEvent evt) {
                    exitMenuItemActionPerformed(evt);
                }
            });

        fileMenu.add(exitMenuItem); // adding a File Menu with an Exit button
        menuBar.add(fileMenu); //
        setJMenuBar(menuBar); //

        java.awt.Dimension screenSize =
            java.awt.Toolkit.getDefaultToolkit().getScreenSize();
        if (bw) {setBounds((screenSize.width-944)/2,

```



```

        (screenSize.height-700)/2, 1095, 600);}
    else {setBounds((screenSize.width-944)/2,
        (screenSize.height-700)/2, 544, 300);}
} //GEN-END:initComponents

private void btParseActionPerformed(java.awt.event.ActionEvent evt) {
    //GEN-FIRST:event_btParseActionPerformed
    // Add your handling code here:
    Node tree = null;
    try { // from String to StringReader and to BufferedReader
        BufferedReader inputline
            = new BufferedReader(
                new StringReader(txInput.getText()));
        tree = DepthFirstVisit.parse(inputline);
    } catch ( IOException e ) {
        e.printStackTrace();
        txOutTree.setText("IOException!");
    } catch ( ParseException pex ){
        txOutTree.setText(pex.getMessage());
    }
    if (tree != null) {
        txOutTree.setText(tree.toString());
        txOutDPF.setText(DepthFirstVisit.depthFirstVisit(tree));
    }
} //GEN-LAST:event_btParseActionPerformed

private void exitMenuItemActionPerformed(
    java.awt.event.ActionEvent evt) {
    //GEN-FIRST:event_exitMenuItemActionPerformed
    System.exit(0);
} //GEN-LAST:event_exitMenuItemActionPerformed

/** Exit the Application */
private void exitForm(java.awt.event.WindowEvent evt) {
    //GEN-FIRST:event_exitForm
    System.exit(0);
} //GEN-LAST:event_exitForm
/**
 * @param args the command line arguments
 */
public static void main(String args[]) {
    new DepthFirstVisitGUI().show();
}
// Variables declaration - do not modify //GEN-BEGIN:variables
private javax.swing.JMenuItem exitMenuItem;
private javax.swing.JMenu fileMenu;
private javax.swing.JLabel jLabel1;
private javax.swing.JLabel jLabel2;
private javax.swing.JLabel jLabel3;
private javax.swing.JLabel jLabel4;
private javax.swing.JMenuBar menuBar;
private javax.swing.JTextField txInput;
private javax.swing.JTextField txOutTree;
private javax.swing.JTextField txOutDPF;
private javax.swing.JButton btParse;
// End of variables declaration //GEN-END:variables
}
/**
 * input:          output:
 * -----

```

```
* javac DepthFirstVisit.java           // This compilation is needed
* javac DepthFirstVisitGUI.java
* java  DepthFirstVisitGUI
*
*           (see figures below)
* -----
*/
```

After typing the string $(-231(-564-78))$ we have:

```
/Users/adp/Desktop/ADP/TEACHING/LECTURE_NOTES_adp2004/AutomataLanguageTranslators/
```

Then by clicking the button 'Parse and Visit' we get:

```
/Users/adp/Desktop/ADP/TEACHING/LECTURE_NOTES_adp2004/AutomataLanguageTranslators/
```

7.2. Evaluator of Boolean Expressions

The program presented in this section: (i) first, parses a given input string denoting a boolean expression, (ii) then, generates the internal tree which represents that expression, and (iii) finally, returns the value of that boolean expression by evaluating every node of that internal tree.

In this section, as concrete instances of boolean expressions, we will take the formulas of the propositional calculus built out of the atoms `true` and `false` and the operators \neg , \vee , \wedge , and \rightarrow , which, respectively, denote negation, disjunction, conjunction, and implication.

The productions of the context-free grammar which generates the input string are the following ones:

<code>b</code>	<code>::= c c>b</code>	(<code>b</code> is a boolean expression)
<code>c</code>	<code>::= l l+c lxc</code>	(<code>c</code> is a condition)
<code>l</code>	<code>::= a -l</code>	(<code>l</code> is a literal)
<code>a</code>	<code>::= 't' 'f' (b)</code>	(<code>a</code> is an atom)

The axiom is `b`. `'t'` stands for `true`, `'f'` stands for `false`, `-` denotes negation, `+` denotes disjunction, `x` denotes conjunction, and `>` denotes implication. We assume that: (i) `+` and `x` bind more than `>`, and (ii) `-` binds more than `+` and `x`. We also assume right associativity for subexpressions involving `+` and `x`. Parentheses override priorities. No blanks are allowed before, in between, and after the input string.

For instance, the string `-t>f+t` denotes the boolean expression

$\neg \text{true} \rightarrow (\text{false} \vee \text{true})$

The internal tree which corresponds to the input string is then printed as a string which is generated by the following productions from the axiom `b`:

$\boxed{b} ::= \text{true} \mid \text{false} \mid (-b) \mid (b + b) \mid (b \times b) \mid (b > b)$

For instance, the boolean expression $\neg \text{true} \rightarrow (\text{false} \vee \text{true})$ is printed as

`((-true) > (false + true))`

If we execute the following two commands:

```
javac BooleanExpression.java
java BooleanExpression
```

we get the following sentence in output:

Input a boolean expression according to the given grammar, please.

Then, if we type the following input string (note the absence of blank characters):

`t>(f+f)+-tx-f`

we get:

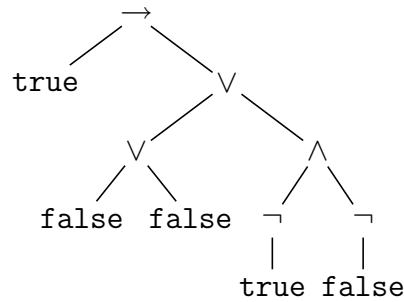
```
Parse tree of the boolean expression:
(true > ((false + false) + ((-true) x (-false))))
```

```
Value of the boolean expression:
```

```
false
```

Indeed, we have that the string `t>(f+f)+-tx-f` represents according to our conventions, the boolean expression $\text{true} \rightarrow ((\text{false} \vee \text{false}) \vee (\neg \text{true} \wedge \neg \text{false}))$.

That boolean expression can be depicted as a tree as follows:



Its value is: false.

The following program parses and evaluates boolean expressions.

```

/**
 * =====
 *                               EVALUATOR OF BOOLEAN EXPRESSIONS
 * Filename: "BooleanExpressionEvaluator.java"
 *
 * Grammar G for the input boolean expression b:
 *
 *   boolean expressions  b ::= c | c>b           <-- bexprParse
 *   conditions           c ::= l | l+c | lxc     <-- condParse
 *   literals             l ::= a | -l           <-- litParse
 *   atoms                a ::= 't' | 'f' | (b)   <-- atomParse
 *
 * 't' stands for true, 'f' stands for false, - denotes negation,
 * + denotes disjunction, x denotes conjunction, and > denotes implication.
 * The grammar should not be left recursive.
 * Internal tree:
 * -----
 *   b ::= true | false | (-b) | (b x b) | (b + b) | (b > b)
 *         (Leaf)  (Leaf)  (Unary) (Binary)  (Binary)  (Binary)
 * Output value:
 * -----
 *   value ::= true | false
 * -----
 * Priorities: + and x bind more than >
 *             - binds more than + or x
 *             Right associativity for + and x
 *             Parentheses override priorities.
 * No blanks are allowed before, in between, and after the input string.
 *
 * A boolean expression is given in input as a string generated by
 * the grammar G and then it is parsed and evaluated.
 * The word 'node' is assumed to be equivalent to the word 'tree'.
 *
 * -----
 * If "C2 extends C1" then the class C2 has the methods of the class C1.
 * This may not be desirable, but it allows the use of objects of the class
 * C1 and then the use of "instanceof" to test whether or not an object of
 * the class C1 is indeed of the class C2.
 *
 * -----
 * Line (1) below checks whether or not the input boolean expression is
 * terminated by a carriage-return '\n'.
 * =====
 */
import java.io.*;
/**

```

```

* =====
*                               Class which constructs a Node
* This class is used for constructing a Node of a binary tree.
* By default, a Node is a binary node.
* A Node can be extended for constructing a Node which is
* either a binary node, or a unary node, or a leaf node.
* =====
*/
class Node {
    public Object  value;      // value field
    public Node   left;       // reference to the left  child node
    public Node   right;      // reference to the right child node
                                // constructor
/**
 * @param v : value at the node
 * @param l : reference to the left  child node
 * @param r : reference to the right child node
 */
    public Node( Object v, Node l, Node r) {
        if ( v == null )  throw new IllegalArgumentException( );
        value  = v;
        left   = l;
        right  = r;
    }
}
/**
* =====
*                               Class which constructs a binary node with two child nodes
* =====
*/
class BinaryNode extends Node {
    /**                               // constructor of binary node
    * @param v : value at the node
    * @param l : reference to the left  child node
    * @param r : reference to the right child node
    */
    public BinaryNode(Object v, Node l, Node r) {
        super(v, l, r);
    }
    /**
    * @return a string representing the tree whose root is the given node
    */
    public String toString(){
        return "("+ left.toString() +" "+ value +" "+ right.toString() +)";
    }
}
/**
* =====
*                               Class which constructs a unary node with one child node
* =====
*/
class UnaryNode extends Node {
    /**                               // constructor of unary node
    * @param v      : value at the node
    * @param child  : reference to the child node
    */
    public UnaryNode(Object v, Node child) {
        super(v, child, null);
    }
}
/**

```

```

    * @return a string representing the tree whose root is the given node
    */
    public String toString() {
        return "(" + value + left.toString() + ")";
    }
}

/**
 * =====
 *                               Class which constructs a leaf node
 * =====
 */
class LeafNode extends Node {
    /**
     *                               // constructor of a leaf
     * @param v : value at the node
     */
    public LeafNode(Object v) {
        super(v, null, null);
    }
    /**
     * @return a string representing the leaf node
     */
    public String toString() {
        return "" + value;
    }
}

/**
 * =====
 *                               Class ParseException
 * =====
 */
class ParseException extends Exception {
    // constructor of a new instance of ParseException
    public ParseException() {
        super("parse exception!");
    }
}

/**
 * =====
 *                               Class EvalException
 * =====
 */
class EvalException extends Exception {
    // constructor of a new instance of EvalException
    public EvalException() {
        super("eval exception!");
    }
}

// -----
public class BooleanExpressionEvaluator {
// =====
    /**
     * This class BooleanExpressionEvaluator transforms the input string
     * into a tree of type Node.
     * The string encodes the tree as indicated by the grammar G.
     * @param string: the string which encodes the given tree
     * @return the root of the tree generated by the parsing
     * @throws IOException or ParseException or EvalException.
     */
    public static char c; // static field.
}

```

```

        // inputLine is a sequence of characters,
        // which are read one at a time, by
        // the method read() of the class StringReader
public static Node bexprParse(BufferedReader inputLine)
    throws IOException, ParseException {
    Node node1 = condParse(inputLine);
    if ( c == '>' ) {
        c = (char)inputLine.read();
        Node node2 = bexprParse(inputLine);
        return new BinaryNode(new Character('>'), node1, node2);
    } else { return node1; }
}

public static Node condParse(BufferedReader inputLine)
    throws IOException, ParseException {
    Node node1 = litParse(inputLine);
    if ( c == 'x' ) {
        c = (char)inputLine.read();
        Node node2 = condParse(inputLine);
        return new BinaryNode(new Character('x'), node1, node2);
    } else
    if ( c == '+' ) {
        c = (char)inputLine.read();
        Node node2 = condParse(inputLine);
        return new BinaryNode(new Character('+'), node1, node2);
    } else { return node1; }
}

public static Node litParse(BufferedReader inputLine)
    throws IOException, ParseException {
    Node node = null;
    if ( c == '-' ) {
        c = (char)inputLine.read();
        Node node1 = litParse(inputLine);
        node = new UnaryNode(new Character('-'), node1);
    } else { node = atomParse(inputLine); };
    return node;
}

public static Node atomParse(BufferedReader inputLine)
    throws IOException, ParseException {
    Node node1 = null;
    if ((c == 't') || (c == 'f')) {
        node1 = new LeafNode(new Boolean((c=='t')?true:false));
        c = (char)inputLine.read();
        return node1;
    } else {
    if ( c=='(' ) {
        c = (char)inputLine.read();
        node1 = bexprParse(inputLine);
        if ( c==')' ) { c = (char)inputLine.read(); } // reads past ')'
        else { throw new ParseException(); };
        return node1;
        };
    };
    throw new ParseException();
}
}
// -----

```

```

/** Evaluation of the tree which represents the input boolean expression
 * @param tree : tree to be evaluated
 * @return the boolean value of the tree
 */
public static boolean evalTree(Node tree)
                                throws EvalException {

// ----- RECURSIVE VERSION -----

if ( tree instanceof LeafNode ) {                                     //
    return ((Boolean)((LeafNode)tree).value).booleanValue();       //
} else                                                                 //
if ( tree instanceof UnaryNode ) {                                  //
    return !(evalTree(((UnaryNode)tree).left));                     //
} else                                                                 //
if ( tree instanceof BinaryNode ) {                                  //
    boolean left = evalTree(((BinaryNode)tree).left);              //
    boolean right = evalTree(((BinaryNode)tree).right);            //
    if (((Character)((BinaryNode)tree).value).charValue() == '+') //
        { return left || right; } else                             //
    if (((Character)((BinaryNode)tree).value).charValue() == 'x') //
        { return left && right; } else                             //
    if (((Character)((BinaryNode)tree).value).charValue() == '>') //
        { return (!left) || right; } else {                         //
        throw new EvalException();                                  //
    }                                                                 //
} else {                                                             //
    throw new EvalException();                                       //
}                                                                 //
// ----- END OF RECURSIVE VERSION -----
}

// -----
public static void main(String[] args)
                                throws IOException, ParseException {

    System.out.println("Input a boolean expression according to the "
        + "given grammar, please.");
    Node tree = null;
    try { // from String to StringReader and to BufferedReader
        BufferedReader inputLine =
            new BufferedReader(new InputStreamReader(System.in));
        c = (char)inputLine.read();
        tree = bexprParse(inputLine);
        if (c != '\n') { throw new ParseException(); }; // <--- (1)
    } catch ( IOException e ) {
        e.printStackTrace();
        System.out.println("IOException!");
    } catch ( ParseException pex ) {
        System.out.println(pex.getMessage());
        return;
    };
    // -----

    if (tree != null) {
        System.out.println("\nParse tree of the boolean expression:\n"
            + tree);
        System.out.println("\nValue of the boolean expression:");
    }
    try {

```



```

        System.out.println(evalTree(tree));
    } catch ( EvalException evalex ) {
        System.out.println(evalex.getMessage());
    }
}
}

/**
 * input: output:
 * -----
 * javac BooleanExpressionEvaluator.java
 * java BooleanExpressionEvaluator
 *
 *      Input a boolean expression according to the given grammar, please.
 * t>(f+f)+-tx-f
 *
 *      Parse tree of the boolean expression:
 *      (true > ((false + false) + ((-true) x (-false))))
 *
 *      Value of the boolean expression:
 *      false
 * -----
 */

```

The following program is like the above one: it parses and evaluates boolean expressions, but it uses a graphical interface. It can be run by typing the following three commands:

```

javac BooleanExpressionEvaluator.java
javac BooleanExpressionEvaluatorGUI.java
java BooleanExpressionEvaluatorGUI

```

```

/**
 * =====
 *                      EVALUATOR OF BOOLEAN EXPRESSIONS
 *                      with a Graphical User Interface
 * Filename: "BooleanExpressionEvaluatorGUI.java"
 *
 * This program should be compiled after the compilation of the program
 * BooleanExpressionEvaluator.java.
 * See also the initial comments of the file BooleanExpressionEvaluator.java
 *
 * This program uses the class BooleanExpressionEvaluator in the file named
 * BooleanExpressionEvaluator.java. The file BooleanExpressionEvaluator.java
 * should be stored in the same folder where this file
 * BooleanExpressionEvaluatorGUI.java is stored.
 *
 * To get a dialog window with font size 14 (instead of 28): set the
 * boolean variable bw (short for big_window) to false (see line ***)
 * =====
 */

import java.io.*;
import javax.swing.JFrame;
import java.awt.Font;                                // needed for changing fonts

// -----
public class BooleanExpressionEvaluatorGUI extends JFrame {

```

```

/** Creates new form BooleanExpressionEvaluatorGUI */
public BooleanExpressionEvaluatorGUI() {
    initComponents();
}
/** This method initComponents() is called from within the constructor
 * to initialize the form. */
private void initComponents() { //GEN-BEGIN:initComponents
    jLabel1 = new javax.swing.JLabel();
    jLabel2 = new javax.swing.JLabel();
    jLabel3 = new javax.swing.JLabel();
    jLabel4 = new javax.swing.JLabel();
    txInExpr = new javax.swing.JTextField();
    txOutExpr = new javax.swing.JTextField();
    txOutVal = new javax.swing.JTextField();
    btParse = new javax.swing.JButton();
    menuBar = new javax.swing.JMenuBar();
    fileMenu = new javax.swing.JMenu();
    exitMenuItem = new javax.swing.JMenuItem();
// -----
    boolean bw = true; // true (false) for big (small) window. ***
    Font sansSerifFont =
        bw ? (new Font("SansSerif",Font.PLAIN,28)):
            (new Font("SansSerif",Font.PLAIN,14));
// -----
    txInExpr.setFont(sansSerifFont);
    txOutExpr.setFont(sansSerifFont);
    txOutVal.setFont(sansSerifFont);

    getContentPane().setLayout(null);
    setTitle("EVALUATION OF A BOOLEAN EXPRESSION");
    setLocationRelativeTo(null);
    addWindowListener(new java.awt.event.WindowAdapter() {
        public void windowClosing(java.awt.event.WindowEvent evt) {
            exitForm(evt);
        }
    });
// -----
    jLabel1.setBackground(new java.awt.Color(255, 255, 150));
    jLabel1.setFont(sansSerifFont);
    jLabel1.setText("<HTML>\n<PRE>\n"
+" Grammar for the input boolean expression b:\n"
+"  b ::= c | c>b      c ::= l | l+c | lxc      l ::= a | -l\n"
+"  a ::= 't' | 'f' | (b)\n"
+"  't' stands for true. 'f' stands for false.\n"
+" Internal tree: \n"
+"  b ::= true | false | (-b) | (b + b)\n"
+"      | (b x b) | (b > b)\n"
+" Evaluation of the boolean expression:\n"
+"  true | false\n"
+"<PRE>\n</HTML>");
    jLabel1.setVerticalAlignment(javax.swing.SwingConstants.TOP);
    jLabel1.setOpaque(true);
    getContentPane().add(jLabel1);
    if (bw) {jLabel1.setBounds(10, 10, 1164, 288);}
    else {jLabel1.setBounds(5, 5, 582, 144);}
// -----
    jLabel2.setFont(sansSerifFont);
    jLabel2.setText("Input Expression:");
    getContentPane().add(jLabel2);
    if (bw) {jLabel2.setBounds(20, 336, 312, 40);}

```

```

        else {jLabel2.setBounds(10, 168, 156, 20);}
        getContentPane().add(txInExpr);
        if (bw) {txInExpr.setBounds(280, 336, 800, 50);}
        else {txInExpr.setBounds(10, 168, 156, 20);}
// -----
        jLabel3.setFont(sansSerifFont);
        jLabel3.setText("Internal Tree:");
        getContentPane().add(jLabel3);
        if (bw) {jLabel3.setBounds(20, 416, 312, 40);}
        else {jLabel3.setBounds(10, 208, 156, 20);}
        txOutExpr.setBackground(new java.awt.Color(25, 255, 25));
        getContentPane().add(txOutExpr);
        if (bw) {txOutExpr.setBounds(280, 416, 800, 50);}
        else {txOutExpr.setBounds(140, 208, 400, 25);}
// -----
        jLabel4.setFont(sansSerifFont);
        jLabel4.setText("Evaluation:");
        getContentPane().add(jLabel4);
        if (bw) {jLabel4.setBounds(20, 496, 280, 32);}
        else {jLabel4.setBounds(10, 253, 140, 16);}
        txOutVal.setBackground(new java.awt.Color(25, 255, 25));
        getContentPane().add(txOutVal);
        if (bw) {txOutVal.setBounds(280, 496, 800, 50);}
        else {txOutVal.setBounds(140, 253, 400, 25);}
// -----
        btParse.setFont(sansSerifFont);
        btParse.setText("Parse and Evaluate");
        btParse.addActionListener(new java.awt.event.ActionListener() {
            public void actionPerformed(java.awt.event.ActionEvent evt) {
                btParseActionPerformed(evt);
            }
        });
        getContentPane().add(btParse);
        if (bw) {btParse.setBounds(460, 596, 388, 50);}
        else {btParse.setBounds(230, 288, 194, 26);}
// -----
        fileMenu.setText("FILE_MENU");
        exitMenuItem.setText("Exit");
        exitMenuItem.addActionListener(new java.awt.event.ActionListener() {
            public void actionPerformed(java.awt.event.ActionEvent evt) {
                exitMenuItemActionPerformed(evt);
            }
        });
        fileMenu.add(exitMenuItem); // adding a File Menu with an Exit button
        menuBar.add(fileMenu); //
        setJMenuBar(menuBar); //
        java.awt.Dimension screenSize =
            java.awt.Toolkit.getDefaultToolkit().getScreenSize();
        if (bw) {setBounds((screenSize.width-944)/2,
            (screenSize.height-700)/2, 1112, 748);}
        else {setBounds((screenSize.width-944)/2,
            (screenSize.height-700)/2, 556, 374);}
    } //GEN-END:initComponents
    private void btParseActionPerformed(java.awt.event.ActionEvent evt) {
        //GEN-FIRST:event_btParseActionPerformed
        // Add your handling code here:
        // This part is like the main of the class BooleanExpressionEvaluator
        // in the file BooleanExpressionEvaluator.java.
        Node tree = null;
        try { // from String to StringReader and to BufferedReader. Added

```

```

        // an extra '\n' at the end of the input for terminating the
        // input as in the BooleanExpressionEvaluator class.
        BufferedReader inputLine
        = new BufferedReader(
            new StringReader((txInExpr.getText()) + '\n'));
        BooleanExpressionEvaluator.c = (char)inputLine.read();
        tree = BooleanExpressionEvaluator.bexprParse(inputLine);
        if (BooleanExpressionEvaluator.c != '\n')
            { throw new ParseException(); };
    } catch ( IOException e ) {
        e.printStackTrace();
        txOutExpr.setText("IOException!");
    } catch ( ParseException pex ){
        txOutExpr.setText(pex.getMessage());
        return;
    } // -----
    if (tree != null) {
        txOutExpr.setText(tree.toString());
    try { txOutVal.setText(
        (BooleanExpressionEvaluator.evalTree(tree))?"true":"false");
    } catch ( EvalException evalex ) {
        System.out.println(evalex.getMessage());
    }
}
} //GEN-LAST:event_btParseActionPerformed
private void
    exitMenuItemActionPerformed(java.awt.event.ActionEvent evt) {
    //GEN-FIRST:event_exitMenuItemActionPerformed
    System.exit(0);
} //GEN-LAST:event_exitMenuItemActionPerformed
/** Exit the Application */
private void exitForm(java.awt.event.WindowEvent evt) {
    //GEN-FIRST:event_exitForm
    System.exit(0);
} //GEN-LAST:event_exitForm
/**
 * @param args the command line arguments */
public static void main(String args[]) {
    new BooleanExpressionEvaluatorGUI().show();
}
// Variables declaration - do not modify //GEN-BEGIN:variables
private javax.swing.JMenuItem exitMenuItem;
private javax.swing.JMenu fileMenu;
private javax.swing.JLabel jLabel1;
private javax.swing.JLabel jLabel2;
private javax.swing.JLabel jLabel3;
private javax.swing.JLabel jLabel4;
private javax.swing.JMenuBar menuBar;
private javax.swing.JTextField txInExpr;
private javax.swing.JTextField txOutExpr;
private javax.swing.JTextField txOutVal;
private javax.swing.JButton btParse;
// End of variables declaration //GEN-END:variables
}
/**
 * input:                output:
 * -----
 * javac BooleanExpressionEvaluator.java // This compilation is needed
 * javac BooleanExpressionEvaluatorGUI.java
 * java BooleanExpressionEvaluatorGUI

```

```
*  
* (see figures below)  
*-----  
*/
```

After typing the string $t > (f+f) + -tx-f$ we have:

```
/Users/adp/Desktop/ADP/TEACHING/LECTURE_NOTES_adp2004/AutomataLanguageTranslators/
```

Then by clicking the button 'Parse and Evaluate' we get:

```
/Users/adp/Desktop/ADP/TEACHING/LECTURE_NOTES_adp2004/AutomataLanguageTranslators/
```

7.3. A Theorem Prover for the Propositional Calculus

The program presented in this section: (i) first, parses a given input string denoting a propositional formula, (ii) then, generates the internal tree which represents that formula, and (iii) finally, tells us whether or not that formula is a tautology. In case it is not a tautology the program returns the values of the propositional variables for which the given formula evaluates to **false**.

The productions of the context-free grammar which generates the input string are the following ones:

$$\left[\begin{array}{ll} \overline{p} ::= c \mid c>p & (\overline{p} \text{ is a propositional formula}) \\ c ::= 1 \mid 1+c \mid 1xc & (c \text{ is a condition}) \\ l ::= a \mid -l & (l \text{ is a literal}) \\ \overline{a} ::= '0' \mid \dots \mid '9' \mid (p) & (a \text{ is an atom}) \end{array} \right.$$

The axiom is \overline{p} . The propositional variables are denoted by the characters '0', ..., '9', and thus, we assume that there are at most ten of them.

As in Section 7.2 on page 157, - denotes negation, + denotes disjunction, x denotes conjunction, and > denotes implication. For reasons of brevity, we will also write **true** as **t** and **false** as **f**. We assume that: (i) + and x bind more than >, and (ii) - binds more than + and x. We also assume right associativity for subexpressions involving + and x. Parentheses override priorities. No blank characters are allowed before, in between, and after the input string.

For instance, the string $1x3+1$ denotes the propositional formula $P_1 \wedge (P_3 \vee P_1)$ where we have used the propositional variables P_1 and P_3 , instead of '1' and '3', respectively.

The internal tree which corresponds to the input string is then printed as a string which is generated by the following productions from the axiom \overline{p} :

$$\left[\begin{array}{l} \overline{p} ::= d \mid (-p) \mid (p + p) \mid (p \times p) \mid (p > p) \\ \overline{d} ::= 0 \mid 1 \mid \dots \mid 9 \end{array} \right.$$

Note that the propositional variables '0', ..., '9' are printed as the digits 0, ..., 9, respectively. For instance, the propositional formula $P_1 \wedge (P_3 \vee P_1)$ is printed as $(1 \times (3 + 1))$. If we execute the following two commands:

```
javac PropositionalTheoremProver.java
java PropositionalTheoremProver
```

we get the following sentence in output:

```
Input a propositional formula according to the given grammar, please.
The variables should be between 0 and 3.
```

The value 3 is due to the fact that in our implementation we have set the value of the variable DIM (which tells us the maximum number of the propositional variables which may occur in the input formula) to 4. Thus, our propositional variables are: '0', '1', '2', and '3'. The value of DIM can be changed by the programmer and in our implementation it is a positive integer which it can be at most 10.

Then, if we type the following input string (note the absence of blank characters):

```
0>(1+1)+-0x-2
```

we get:

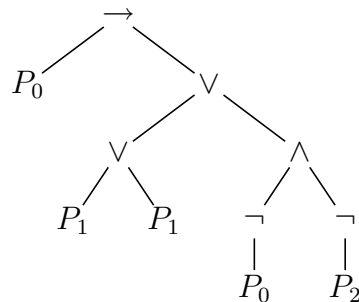
Parse tree of the propositional formula:
 $(0 > ((1 + 1) + ((-0) \times (-2))))$

Is it a tautology? NO.

The falsity assignment is:

0=true; 1=false; 2=false; 3=false;

Indeed, the string $(0 > ((1 + 1) + ((-0) \times (-2))))$ represents, according to our conventions, the propositional formula $P_0 \rightarrow ((P_1 \vee P_1) \vee (\neg P_0 \wedge \neg P_2))$, where we have used the propositional variables P_0 , P_1 , and P_2 , instead of '0', '1', and '2', respectively. That propositional formula can be depicted as a tree as follows:



The value of that formula is false if $P_0 = \text{true}$, and $P_1 = P_2 = \text{false}$.

Here is the program for parsing a given propositional formula and checking whether or not it is a tautology.

```
/**
 * =====
 *                      PROPOSITIONAL THEOREM PROVER
 * * Filename: "PropositionalTheoremProver.java"
 * *
 * * Grammar G for the input propositional formula p:
 * *
 * *   propositional formulas  p ::= c | c>p                <-- propParse
 * *   conditions              c ::= l | l+c | lxc         <-- condParse
 * *   literals                l ::= a | -l                <-- litParse
 * *   atoms                   a ::= '0' | ... | '9' | (p) <-- atomParse
 * *
 * * Propositional variables are named: '0', ..., '9'.
 * * The value of DIM is set to the maximum number of propositional variables
 * * which are actually present in any given formula.
 * * If DIM = 4, then the variables are: '0', '1', '2', and '3'.
 * * (see line (1) below). DIM should be at most 10.
 * * - denotes negation, + denotes disjunction, x denotes conjunction, and
 * * > denotes implication.
 * * Internal tree:
 * * -----
 * *   p ::= 0 | ... | 9 | (-p) | (p x p) | (p + p) | (p > p)
 * *         (Leaf)      (Leaf) (Unary)  (Binary)  (Binary)  (Binary)
 * *
 * * Output value of a propositional formula:
 * * -----
 * *   value ::= true | false
 * * -----
 * * Priorities: + and x bind more than >
 * *             - binds more than + or x
```

```

*           Right associativity for + and x
*           Parentheses override priorities.
* No blanks are allowed before, in between, and after the input string.
* The propositional formula in input should be a string generated by
* the grammar G and then it is parsed and evaluated for all assignments
* of the propositional variables.
*
* The word 'node' is assumed to be equivalent to the word 'tree'.
* -----
* If "C2 extends C1" then the class C2 has the methods of the class C1.
* This may not be desirable, but it allows the use of objects of the class
* C1 and then the use of "instanceof" to test whether or not an object of
* the class C1 is indeed of the class C2.
* -----
* Line (1) below depends on the number of propositional variables.
* Line (2) below checks whether or not the input propositional formula is
* terminated by a carriage-return '\n'.
* =====
*/
import java.io.*;
/**
 * =====
 *           Class which constructs a Node
 *
 * This class is used for constructing a Node of a binary tree.
 * By default, a Node is a binary node.
 * A Node can be extended for constructing a Node which is
 * either a binary node, or a unary node, or a leaf node.
 * =====
 */
class Node {
    public Object  value;      // value field
    public Node   left;       // reference to the left  child node
    public Node   right;      // reference to the right child node
                                // constructor
/**
 * @param v : value at the node
 * @param l : reference to the left  child node
 * @param r : reference to the right child node
 */
    public Node( Object v, Node l, Node r) {
        if ( v == null ) throw new IllegalArgumentException( );
        value  = v;
        left   = l;
        right  = r;
    }
}
/**
 * =====
 *           Class which constructs a binary node with two child nodes
 * =====
 */
class BinaryNode extends Node {
/**
 * @param v : value at the node
 * @param l : reference to the left  child node
 * @param r : reference to the right child node
 */
    public BinaryNode(Object v, Node l, Node r) {
        super(v, l, r);
    }
}

```



```

/**
 * @return a string representing the tree whose root is the given node
 */
public String toString(){
    return "("+ left.toString() +" "+ value +" "+ right.toString() +)";
}
}

/**
 * =====
 *                               Class which constructs a unary node with one child node
 * =====
 */
class UnaryNode extends Node {
    /**                               // constructor of unary node
     * @param v      : value at the node
     * @param child  : reference to the child node
     */

    public UnaryNode(Object v, Node child) {
        super(v, child, null);
    }
    /**
     * @return a string representing the tree whose root is the given node
     */
    public String toString() {
        return "(" + value + left.toString() + ")";
    }
}

/**
 * =====
 *                               Class which constructs a leaf node
 * =====
 */
class LeafNode extends Node {
    /**                               // constructor of a leaf
     * @param v : value at the node
     */
    public LeafNode(Object v) {
        super(v, null, null);
    }
    /**
     * @return a string representing the leaf node
     */
    public String toString() {
        return "" + value;
    }
}

/**
 * =====
 *                               Class ParseException
 * =====
 */
class ParseException extends Exception {
    /**                               // constructor of a new instance of ParseException
     * @return a string representing the leaf node
     */
    public ParseException() {
        super("parse exception!");
    }
}
}

```

```

/**
 * =====
 *                               Class EvalException
 * =====
 */
class EvalException extends Exception {
    // constructor of a new instance of EvalException
    public EvalException() {
        super("eval exception!");
    }
}
// -----
public class PropositionalTheoremProver {
// =====
    /**
     * This class PropositionalTheoremProver transforms the input string
     * into a tree of type Node.
     * The string encodes the tree as indicated by the grammar G.
     * @param string: the string which encodes the given tree
     * @return the root of the tree generated by the parsing
     * @throws IOException or ParseException or EvalException.
     */
    // Three static fields of the class PropositionalTheoremProver
        // DIM: maximum number of the distinct variables
        // in the propositional formula
    public static final int DIM = 4; // <--- (1)
        // At most DIM propositional variables in the
        // propositional formula. By default, every
        // element of tValues is initialized to "false"
    public static boolean [] tValues = new boolean [DIM];

    public static char c;
        // inputLine is a sequence of characters,
        // which are read one at a time, by
        // the method read() of the class StringReader

    public static Node propParse(BufferedReader inputLine)
        throws IOException, ParseException {
        Node node1 = condParse(inputLine);
        if ( c == '>' ) {
            c = (char)inputLine.read();
            Node node2 = propParse(inputLine);
            return new BinaryNode(new Character('>'), node1, node2);
        } else { return node1; }
    }

    public static Node condParse(BufferedReader inputLine)
        throws IOException, ParseException {
        Node node1 = litParse(inputLine);
        if ( c == 'x' ) {
            c = (char)inputLine.read();
            Node node2 = condParse(inputLine);
            return new BinaryNode(new Character('x'), node1, node2);
        } else
            if ( c == '+' ) {
                c = (char)inputLine.read();
                Node node2 = condParse(inputLine);
                return new BinaryNode(new Character('+'), node1, node2);
            } else { return node1; }
    }
}

```

```

public static Node litParse(BufferedReader inputLine)
                                throws IOException, ParseException {
    Node node = null;
    if ( c == '-' ) {
        c = (char)inputLine.read();
        Node node1 = litParse(inputLine);
        node = new UnaryNode(new Character('-'), node1);
    } else { node = atomParse(inputLine); };
    return node;
}

public static Node atomParse(BufferedReader inputLine)
                                throws IOException, ParseException {
    Node node1 = null;
    if (('0' <= c) && (c <= '9')) {
        // from the character to the integer: -48 //
        node1 = new LeafNode(new Integer(Character.getNumericValue(c)));
        c = (char)inputLine.read();
        return node1;
    } else {
        if ( c=='(' ) {
            c = (char)inputLine.read();
            node1 = propParse(inputLine);
            if ( c==')' ) { c = (char)inputLine.read(); } // reads past ')'
            else { throw new ParseException(); };
            return node1;
        };
        throw new ParseException();
    }
}

// -----
/** Evaluation of the tree which represents the input boolean expression
 * @param tree : tree to be evaluated
 * @return the boolean value of the tree
 */
public static boolean evalTree(Node tree)
                                throws EvalException {
// ----- RECURSIVE VERSION -----
if ( tree instanceof LeafNode ) { //
    return tValues[(Integer)((LeafNode)tree).value]; //
} else //
    if ( tree instanceof UnaryNode ) { //
        return !(evalTree(((UnaryNode)tree).left)); //
    } else //
        if ( tree instanceof BinaryNode ) { //
            boolean left = evalTree(((BinaryNode)tree).left); //
            boolean right = evalTree(((BinaryNode)tree).right); //
            if (((Character)((BinaryNode)tree).value).charValue() == '+') //
                { return left || right; } else //
            if (((Character)((BinaryNode)tree).value).charValue() == 'x') //
                { return left && right; } else //
            if (((Character)((BinaryNode)tree).value).charValue() == '>') //
                { return (!left) || right; } //
        }; //
        throw new EvalException(); //
// ----- END OF RECURSIVE VERSION -----
}
// -----

```

```

public static void main(String[] args)
    throws IOException, ParseException {
    boolean answer = true;
    boolean bval = true;
    System.out.println("Input a propositional formula according to the "
        + "given grammar, please.");
    System.out.println("The variables should be between 0 and "
        + (DIM-1) + ".");
    Node tree = null;
    try { // from String to StringReader and to BufferedReader
        BufferedReader inputLine =
            new BufferedReader(new InputStreamReader(System.in));
        c = (char)inputLine.read();
        tree = propParse(inputLine);
        if (c != '\n') { throw new ParseException(); }; // <--- (2)
    } catch ( IOException e ) {
        e.printStackTrace();
        System.out.println("IOException!");
    } catch ( ParseException pex ) {
        System.out.println(pex.getMessage());
        return;
    };
    // ----- (begin 0) -----
    if (tree != null) {
        System.out.println("\nParse tree of the propositional formula:\n"
            + tree);
        System.out.print("\nIs it a tautology? ");
    }
    /** -----
    * The following fragment computes the dispositions for tValues[0],...,
    * tValues[DIM-1] from false,...,false to true,...,true until we finds
    * a disposition which makes bval == false, if any. In that case the given
    * formula is NOT a tautology. If, otherwise, all dispositions make
    * bval == true, then the given formula is a tautology.
    * ----- */
    for (int i = 0; i < (int)Math.pow(2,DIM); i++) {
        try {
            bval = evalTree(tree);
        } catch ( EvalException evalex ) {
            System.out.println(evalex.getMessage());
        };
        if (!bval) { answer = false; break; }
        else {
            for (int k = 0; k < DIM; k++) {
                if (tValues[k]) { tValues[k] = false; }
                else { tValues[k] = true; break; }
            }
        }
    };
    // -----
    if (answer) { System.out.println("YES."); } else {
        System.out.println("NO.\nThe falsity assignment is:\n");
        for (int k = 0; k < DIM; k++) {
            System.out.print(k + "=" + tValues[k] + "; ");
        }
        System.out.println();
    }
    // -----
    }
    // ----- (end 0) -----
}

```

```

/**
 * input:  output:
 * -----
 * javac PropositionalTheoremProver.java
 * java  PropositionalTheoremProver
 *
 *      Input a propositional formula according to the given grammar, please.
 *      The variables should be between 0 and 3.
 * 0>1>0
 *
 *      Parse of the propositional formula:
 *      (0 > (1 > 0))
 *
 *      Is it a tautology? YES.
 * -----
 *      Input a propositional formula according to the given grammar, please.
 *      The variables should be between 0 and 3.
 * 0>-1
 *
 *      Parse of the propositional formula:
 *      (0 > (-1))
 *
 *      Is it a tautology? NO.
 *      The falsity assignment is:
 *
 *      0=true; 1=true; 2=false; 3=false;
 * -----
 *      Input a propositional formula according to the given grammar, please.
 *      The variables should be between 0 and 3.
 * 0>(1+1)+-0x-2
 *
 *      Parse tree of the propositional formula:
 *      (0 > ((1 + 1) + ((-0) x (-2))))
 *
 *      Is it a tautology? NO.
 *      The falsity assignment is:
 *
 *      0=true; 1=false; 2=false; 3=false;
 * -----
 */

```

The following program is like the above one: it parses a given propositional formula and checks whether or not it is a tautology, but it uses a graphical interface. It can be run by typing the following three commands:

```

javac PropositionalTheoremProver.java
javac PropositionalTheoremProverGUI.java
java  PropositionalTheoremProverGUI

```

```

/**
 * =====
 *
 *                      PROPOSITIONAL THEOREM PROVER
 *                      with a Graphical User Interface
 * Filename: "PropositionalTheoremProverGUI.java"
 *
 * This program should be compiled after the compilation of the program
 * PropositionalTheoremProver.java.
 *
 * See also the initial comments of the file PropositionalTheoremProver.java

```

```

*
* This program uses the class PropositionalTheoremProver in the file named
* PropositionalTheoremProver.java. The file PropositionalTheoremProver.java
* should be stored in the same folder where this file
* PropositionalTheoremProverGUI.java is stored.
* -----
* Lines (1) below depend on the number of propositional variables.
* Line (2) below checks whether or not the input propositional formula
* is terminated by a carriage-return '\n'.
* -----
* To get a dialog window with font size 14 (instead of 28): set the
* boolean variable bw (short for big_window) to false (see line ***)
* =====
*/

import java.io.*;
import javax.swing.JFrame;
import java.awt.Font;                               // needed for changing fonts

// -----
public class PropositionalTheoremProverGUI extends JFrame {
    /** Creates new form PropositionalTheoremProverGUI */
    public PropositionalTheoremProverGUI() {
        initComponents();
    }
    /** This method initComponents() is called from within the constructor
     * to initialize the form.
     */
    private void initComponents() { //GEN-BEGIN:initComponents
        jLabel1      = new javax.swing.JLabel();
        jLabel2      = new javax.swing.JLabel();
        jLabel3      = new javax.swing.JLabel();
        jLabel4      = new javax.swing.JLabel();
        txInForm     = new javax.swing.JTextField();
        txParsedForm = new javax.swing.JTextField();
        txAnsWTaut   = new javax.swing.JTextField();
        btParse      = new javax.swing.JButton();
        menuBar      = new javax.swing.JMenuBar();
        fileMenu     = new javax.swing.JMenu();
        exitMenuItem = new javax.swing.JMenuItem();

        getContentPane().setLayout(null);
        setTitle("PROPOSITIONAL THEOREM PROVER");
        setLocationRelativeTo(null);
        addWindowListener(new java.awt.event.WindowAdapter() {
            public void windowClosing(java.awt.event.WindowEvent evt) {
                exitForm(evt);
            }
        });
    }
}

// -----
boolean bw = true; // true (false) for big (small) window. ***
Font sansSerifFont =
    bw ? (new Font("SansSerif",Font.PLAIN,28)):
        (new Font("SansSerif",Font.PLAIN,14));

// -----
txInForm.setFont(sansSerifFont);
txParsedForm.setFont(sansSerifFont);
txAnsWTaut.setFont(sansSerifFont);

// -----

```

```

jLabel1.setBackground(new java.awt.Color(255, 255, 150));
jLabel1.setFont(sansSerifFont);
jLabel1.setText("<HTML>\n<PRE>\n"
+" Grammar for the input propositional formula p:\n"
+"   prop:  p ::= c | c>p      cond:  c ::= l | l+c | lxc\n"
+"   lit:   l ::= a | -l"
+"   atom:  a ::= '0' | ... | '3' | (p)\n"           // <--- (1)
+" Internal tree: p ::= 0 | ... | 3 | (-p) | (p x p)\n"// <--- (1)
+"           | (p + p) | (p > p)\n"
+" Abbreviations: f/false t/true -/not x/and +/or >/implies"
+"<PRE>\n</HTML>");
jLabel1.setVerticalAlignment(javax.swing.SwingConstants.TOP);
jLabel1.setOpaque(true);
getContentPane().add(jLabel1);

if (bw) {jLabel1.setBounds(10, 10, 1200, 190);}
else {jLabel1.setBounds(5, 5, 565, 97);}

// -----
jLabel2.setFont(sansSerifFont);
jLabel2.setText("Input Formula:");
getContentPane().add(jLabel2);
if (bw) {jLabel2.setBounds(20, 208, 200, 32);}
else {jLabel2.setBounds(10, 104, 130, 16);}
getContentPane().add(txInForm);
if (bw) {txInForm.setBounds(280, 208, 860, 40);}
else {txInForm.setBounds(130, 104, 430, 20);}

// -----
jLabel3.setFont(sansSerifFont);
jLabel3.setText("Parsed Formula:");
getContentPane().add(jLabel3);
if (bw) {jLabel3.setBounds(20, 288, 220, 32);}
else {jLabel3.setBounds(10, 144, 130, 16);}
txParsedForm.setBackground(new java.awt.Color(25, 255, 25));
getContentPane().add(txParsedForm);
if (bw) {txParsedForm.setBounds(280, 288, 860, 40);}
else {txParsedForm.setBounds(130, 144, 430, 20);}

// -----
jLabel4.setFont(sansSerifFont);
jLabel4.setText("Is it a tautology?");
getContentPane().add(jLabel4);
if (bw) {jLabel4.setBounds(20, 368, 280, 32);}
else {jLabel4.setBounds(10, 184, 140, 16);}
txAnswTaut.setBackground(new java.awt.Color(25, 255, 25));
getContentPane().add(txAnswTaut);
if (bw) {txAnswTaut.setBounds(280, 368, 860, 50);}
else {txAnswTaut.setBounds(130, 184, 430, 20);}

// -----
btParse.setFont(sansSerifFont);
btParse.setText("Parse and Evaluate");
btParse.addActionListener(new java.awt.event.ActionListener() {
    public void actionPerformed(java.awt.event.ActionEvent evt) {
        btParseActionPerformed(evt);
    }
});
getContentPane().add(btParse);
if (bw) {btParse.setBounds(430, 440, 576, 52);}
else {btParse.setBounds(240, 215, 158, 26);}

// -----
fileMenu.setText("FILE_MENU");

```

```

exitMenuItem.setText("Exit");
exitMenuItem.addActionListener(new java.awt.event.ActionListener() {
    public void actionPerformed(java.awt.event.ActionEvent evt) {
        exitMenuItemActionPerformed(evt);
    }
});
fileMenu.add(exitMenuItem); // adding a File Menu with an Exit button
menuBar.add(fileMenu); //
setJMenuBar(menuBar); //
java.awt.Dimension screenSize =
    java.awt.Toolkit.getDefaultToolkit().getScreenSize();
if (bw) {setBounds((screenSize.width-1300)/2,
    (screenSize.height-700)/2, 1160, 590);}
else {setBounds((screenSize.width-444)/2,
    (screenSize.height-300)/2, 580, 300);}
} //GEN-END:initComponents

private void btParseActionPerformed(java.awt.event.ActionEvent evt) {

    // GEN-FIRST:event_btParseActionPerformed
    // Add your handling code here:
    // This part is like the main of the class PropositionalTheoremProver
    // in the file PropositionalTheoremProver.java.

    boolean answer = true;
    boolean bval = true;
    int DIM = PropositionalTheoremProver.DIM; // <--- (1)
    Node tree = null;
    try { // from String to StringReader and to BufferedReader.
        // Added an extra '\n' at the end of the input for
        // terminating the input as in the
        // PropositionalTheoremProver class.
        BufferedReader inputLine
            = new BufferedReader(
                new StringReader((txInForm.getText()) + '\n'));
        PropositionalTheoremProver.c = (char)inputLine.read();
        tree=PropositionalTheoremProver.propParse(inputLine); // <--- (2)
        if (PropositionalTheoremProver.c != '\n')
            { throw new ParseException(); };
    } catch ( IOException e ) {
        e.printStackTrace();
        txParsedForm.setText("IOException!");
    } catch ( ParseException pex ){
        txParsedForm.setText(pex.getMessage());
        return;
    };

    // ----- (begin Parse and Evaluate)
    if (tree != null) {
        txParsedForm.setText(tree.toString());
    }

    // -----
    for (int i = 0; i < (int)Math.pow(2,DIM);
        i++) {
        try {
            bval = PropositionalTheoremProver.evalTree(tree);
        } catch ( EvalException evalex ) {
            txAnswTaut.setText(evalex.getMessage());
        };
        if (!bval) { answer = false; break; }
    }
}

```



```

        else {
            for (int k = 0; k < DIM; k++) {
                if (PropositionalTheoremProver.tValues[k])
                    { PropositionalTheoremProver.tValues[k] = false; }
                else { PropositionalTheoremProver.tValues[k] = true;
                    break; }
            }
        }
    };
// -----

    if (answer) { txAnswTaut.setText("YES."); } else {
        String outstring = "NO. False if: ";
        for (int k = 0; k < DIM; k++) {
            outstring = outstring + k + "="
                + ((PropositionalTheoremProver.tValues[k])?"t; ":"f; ");}
        txAnswTaut.setText(outstring);
    }

// -----
    }
    // ----- (end Parse and Evaluate)
} //GEN-LAST:event_btParseActionPerformed

private void
    exitMenuItemActionPerformed(java.awt.event.ActionEvent evt) {
    //GEN-FIRST:event_exitMenuItemActionPerformed
        System.exit(0);
    } //GEN-LAST:event_exitMenuItemActionPerformed

/** Exit the Application */

private void exitForm(java.awt.event.WindowEvent evt) {
    //GEN-FIRST:event_exitForm
        System.exit(0);
    } //GEN-LAST:event_exitForm

/**
 * @param args the command line arguments
 */

public static void main(String args[]) {
    new PropositionalTheoremProverGUI().show();
}

// Variables declaration - do not modify //GEN-BEGIN:variables
private javax.swing.JMenuItem exitMenuItem;
private javax.swing.JMenu fileMenu;
private javax.swing.JLabel jLabel1;
private javax.swing.JLabel jLabel2;
private javax.swing.JLabel jLabel3;
private javax.swing.JLabel jLabel4;
private javax.swing.JMenuBar menuBar;
private javax.swing.JTextField txInForm;
private javax.swing.JTextField txParsedForm;
private javax.swing.JTextField txAnswTaut;
private javax.swing.JButton btParse;
// End of variables declaration //GEN-END:variables
}

```

```
/**
 * input:                output:
 * -----
 * javac PropositionalTheoremProver.java      // This compilation is needed
 * javac PropositionalTheoremProverGUI.java
 * java  PropositionalTheoremProverGUI
 *
 *                (see figures below)
 * -----
 */
```

After typing the string $0 > (1+1) + -0x-2$ we have:

Then by clicking the button 'Parse and Evaluate' we get:

7.4. Encoding of n -ary Trees Using Binary Trees

Before formalizing the problem of encoding n -ary trees using binary trees, let us first present the following three data structures which are used in that formalization:

- (i) binary trees with objects of type T in the nodes,
- (ii) n -ary trees with objects of type T in the nodes, and
- (iii) lists of n -ary trees with objects of type T in the nodes.

For reasons of brevity, instead of saying ‘binary trees with objects of type T in the nodes’, we will also say: ‘binary trees with nodes of type T ’. Analogous terminology will also be used for n -ary trees and lists of n -ary trees.

REMARK 7.4.1. [Terminology for Nodes and Trees] When referring to trees, we will equivalently use the term ‘node’ and ‘tree’, because every node in a tree denotes the subtree rooted in that node (see also Remark 7.1.1 on page 148).

BINARY TREES WITH NODES OF TYPE T

Let V denote the set of objects of type T . We define the set B of the binary trees with objects of type T in the nodes, to be the solution of the following domain equation (see [20, pages 107–124]):

$$B = 1 + B \times V \times B \quad (1)$$

The empty binary tree in the domain 1 is the element $e \in B$, and a non-empty binary tree in the domain $B \times V \times B$ is a triple $(\mathbf{lb}, v, \mathbf{rb})$, where: $v \in V$ is an object of type T , and \mathbf{lb} and \mathbf{rb} are two binary trees in B with objects of type T . These two trees are said to be the *left* and *right* subtrees, respectively, of the given binary tree $(\mathbf{lb}, v, \mathbf{rb})$.

In the Java 1.5 implementation which we will present below, we have that:

- (•) the empty binary tree e with nodes of type T is constructed by the nullary constructor `BinNode<T>()` by using the command

```
new BinNode<T>(), and
```

- (•) the non-empty binary tree $(\mathbf{lb}, v, \mathbf{rb})$ with nodes of type T is constructed by the ternary constructor `BinNode<T>(lb,v,rb)` by using the command

```
new BinNode<T>(lb,v,rb)
```

where v is an object of type T , and \mathbf{lb} and \mathbf{rb} denote two already constructed binary trees with nodes of type T . The selector which tests whether or not a binary tree with nodes of type T is empty, is:

```
isEmpty()
```

Thus, for instance, in the case of binary trees with nodes of type `Integer` we have that

```
new BinNode<Integer>().isEmpty() returns true.
```

Here is a program which constructs binary trees with nodes of type T .

```
/**
 * =====
 *           Class which constructs a binary node with two child nodes
 * Filename: "BinNode.java"
 *
 * Binary trees with nodes of type T are internally represented as follows:
 *
 *   bt<T> ::= e | (bt<T>, val, bt<T>)   where val is of type T
 *
 */
```

```

* e is the empty binary node
* constructors: new BinNode<T>() | new BinNode<T>(-,-,-)
* selector:     new BinNode<T>().isEmpty() is true (the BinNode<T> is e)
*
* -----
* constructing e: new BinNode()
*
* constructing an non-empty binary node of type Integer, which is
* 7 with empty child nodes:
*
*     new BinNode<Integer>(new BinNode<Integer>(),7, new BinNode<Integer>())
*
* Notice that Java 1.5 accepts: 7, instead of: new Integer(7).
* =====
*/
public class BinNode<T> {

    private T value;           // value field of type T
    private BinNode left;     // reference to the left child node
    private BinNode right;    // reference to the right child node

    public BinNode ( ) {      // constructor 1: empty BinNode e
        this.value = null;    // for constructor 1: this.left == null is true
    }                          // for constructor 1: this.right == null is true

    /*
    * Constructor of a Node with:
    * - value v,
    * - parameters l and r, which are references to the left and right
    *   child nodes, respectively.
    */
    public BinNode ( BinNode l, T v, BinNode r ) { // constructor 2
        this.left = l;
        this.value = v;
        this.right = r;
    }

    /**
    * @return the value.
    */
    public T getValue() {
        return value;
    }

    /**
    * @return the left child node.
    */
    public BinNode getLeftChild(){
        return left;
    }

    /**
    * @return the right child node.
    */
    public BinNode getRightChild(){
        return right;
    }

    /**
    * @return true iff the value of the BinNode is the null reference.
    *
    * Each of the following prints: true

```

```

*      System.out.print(new BinNode().isEmpty());
*      System.out.print(new BinNode<Integer>().isEmpty());
*      System.out.print(new BinNode(null,null,null).isEmpty());
*      System.out.print(new BinNode<Integer>(null,null,null).isEmpty());
*
* The following two instructions:
*   BinNode<Integer> b = null;
*   System.out.println(b.isEmpty());
* give a NullPointerException
*/
public boolean isEmpty() {
    return value == null;
}
/**
* This method prints the tree whose root is the given node.
* We assume that 'value' can be printed as a string.
*/
public void binNodePrint(){
    if (value == null) {System.out.print("e");}
    else {
        System.out.print("("); left.binNodePrint();
        System.out.print(" "+value+" ");
        right.binNodePrint(); System.out.print(")");
    }
}
}
// -----

```

N-ARY TREES WITH NODES OF TYPE T

Let V denote the set of objects of type T . We define the set T of the n -ary trees with objects of type T in the nodes, to be the solution of the following domain equations:

$$\begin{aligned}
 T &= 1 + V \times Tlist \\
 Tlist &= 1 + T \times Tlist
 \end{aligned}
 \tag{2}$$

The empty n -ary tree with nodes of type T in the domain 1 is the element $E \in T$, and a non-empty n -ary tree with nodes of type T in the domain $V \times Tlist$ is a pair (v, L) , where: $v \in V$ is an object of type T , and $L \in Tlist$ is a list of n -ary trees with nodes of type T , which is constructed as indicated on page 186.

In the Java 1.5 implementation which we will present below, we have that:

(●) the empty n -ary tree E with nodes of type T is constructed by the nullary constructor `NaryNode<T>()` by using the command

```
new NaryNode<T>(), and
```

(●) the non-empty n -ary tree (v, L) with nodes of type T is constructed by the binary constructor `NaryNode<T>(v,L)` by using the command

```
new NaryNode<T>(v,L)
```

where v is an object of type T and L is a list of n -ary trees with nodes of type T . The selector which tests whether or not an n -ary tree with nodes of type T is empty, is:

```
isEmpty()
```

Thus, for instance, in the case of n -ary trees with nodes of type `Integer` we have that

```
new NaryNode<Integer>().isEmpty() returns true.
```

Here is a program which constructs n -ary trees with nodes of type T.

```

/**
 * =====
 *      Class which constructs an n-ary node with a list of child nodes
 *      Filename: "NaryNode.java"
 *
 *      N-ary trees with nodes of type T are internally represented as follows:
 *
 *      nt<T>      ::= E | (val, list_nt<T>)      where val is of type T
 *
 *      E is the empty n-ary tree.
 *      list_nt<T> is a list of n-ary trees with nodes of type T.
 *      constructors: new NaryNode<T>() | new NaryNode<T>(-,-)
 *      selector:     new NaryNode<Integer>().isEmpty() is true
 *                   (indeed, the NaryNode<Integer> is E)
 * -----
 * Lists of n-ary trees are internally represented as follows:
 *
 *      list_nt<T> ::= [] | nt<T> : list_nt<T>
 *
 *      cons is denoted by the infix ':'. In the implementation using ArrayList
 *      the last element which is cons-ed is to the right (not to the left).
 *
 *      constructors: new List<T>() | l.cons(nt) where l is a list of n-ary trees
 *                   of type <T> and nt is an n-ary tree of type <T>
 *      selector:     new List<Integer>().isEmpty() is true
 *                   (indeed, the List<Integer> is [])
 * -----
 * constructing E: new NaryNode<Integer>()
 *
 * constructing a non-empty n-ary node of type Integer, which is
 *      8 with an empty list of child nodes:
 *
 *      new NaryNode<Integer>(8, new List<NaryNode<Integer>>())
 *
 * Notice that Java 1.5 accepts: 8, instead of: new Integer(8).
 * =====
 */

import java.util.*;          // needed for using ArrayList<T> and Iterator

public class NaryNode<T> {

    private T value;          // value of the n-ary node of type T
    private List <NaryNode<T>> list; // list of child nodes

    /*
     * Constructor of a Node with:
     * - value v,
     * - parameter l, which is a reference to the list of child nodes.
     */

    public NaryNode ( ) { // constructor 1: empty NaryNode E
        this.value = null; // for constructor 1: this.list == null is true
    }

    public NaryNode ( T value, List<NaryNode<T>> list ) { // constructor 2
        this.value = value; // for: NaryNode <Integer> n1
        this.list = list; // = new NaryNode(1,new List())
    } // n1.list.isEmpty() is true
}

```

```

/**
 * @return the list of child nodes.
 */
public List <NaryNode<T>> getListOfChildren(){
    return list;
}

/**
 * @return the value at the node.
 */
public T getValue(){
    return value;
}

/**
 * @return true iff the value is the null reference.
 *
 * The following two instructions:
 *     System.out.println(new NaryNode().isEmpty());
 *     System.out.println(new NaryNode(null,null).isEmpty());
 * both return true.
 */
public boolean isEmpty() {
    return value == null;
}

/**
 * This method prints the tree whose root is the given node.
 * We assume that 'value' can be printed as a string.
 */
public void naryNodePrint(){
    if (value == null) { System.out.print("E"); } // empty n-ary tree
    else { System.out.print(value+" [");
        for (int i=list.size()-1; i>=0; i--) {
            ((NaryNode)(list.get(i))).naryNodePrint();
            if (i > 0) {System.out.print(", ");}
        };
        System.out.print("]");
    }
}
}

/**
 * -----
 * We have that:
 *
 * NaryNode <Integer> n0 = new NaryNode();
 * System.out.println(n0.list == null); // prints: true
 * System.out.println((n0.list).isEmpty()); // raises a NullPointerException!
 *
 * System.out.println(new NaryNode().isEmpty()); // prints: true
 * System.out.println(new NaryNode<Integer>().isEmpty()); // prints: true
 *
 * NaryNode <Integer> n1 = new NaryNode(1,new List());
 * System.out.println(n1.list == null); // prints: false
 * System.out.println(n1.list.isEmpty()) // prints: true
 * -----
 */

```

LISTS OF n -ARY TREES WITH NODES OF TYPE T

The lists of n -ary trees with objects of type T are elements of the domain $Tlist$ which has been defined by the domain equations (2) on page 183. For any type T , the empty list of n -ary trees with nodes of type T in the domain 1 is the element $[]$, and a non-empty list of n -ary trees with nodes of type T is a pair $h : H$, where:

- (i) $h \in T$ is an n -ary tree with nodes of type T ,
- (ii) $'.'$ denotes the infix *cons* operation on lists, and
- (iii) $H \in Tlist$ is a list of n -ary trees with nodes of type T .

In the Java 1.5 implementation of the lists with nodes of trees of type T which we will present below, we have that:

(•) the empty list $[]$ of n -ary trees with nodes of type T is constructed by the nullary constructor `List<NaryNode<T>>()` by using the command

```
new List<NaryNode<T>>(), and
```

(•) the non-empty list of n -ary trees with nodes of type T is constructed from an already constructed, shorter list H of n -ary trees with nodes of type T , by *cons*-ing onto it an n -ary tree nt with nodes of type T by using the following command:

```
H.cons(nt)
```

The selector which tests whether or not a list of n -ary trees with nodes of type T is empty, is:

```
isEmpty()
```

Thus, for instance, in the case of lists of n -ary trees with nodes of type `Integer` we have that

`new List<NaryNode<Integer>>().isEmpty()` returns `true`, because `new List<NaryNode<Integer>>()` is the empty list $[]$ of n -ary trees with nodes of type `Integer`.

Here is the program `List.java` which is the Java 1.5 implementation of the *generic class* of the lists of elements of type T . It is similar to the program, also called `List.java`, presented in Section 3.2 on page 39 where the reader may find some examples of use of the methods of the generic class `List`.

```
/** =====
 *          Generic Class List <T> (class used for encoding trees)
 * Filename: "List.java"
 *
 * Every object of the class List<T> is a list of elements of type T.
 * The following methods are available:
 *   cons(T d), head(), tail(), size(), get(int i), isSingleton(),
 *   isEmpty(), makeEmpty(), copy(), and listPrint().
 * Also cloneCopy() is available if we uncomment line (***) below and
 * comment the previous line.
 *
 * Notice that the tail() method is destructive.
 * After a tail operation, if we need the original value of the list, we
 * should reconstruct it by 'consing' the head of the list.
 * -----
 * Lists of elements nt of type T are internally represented as follows:
 *
 *   list_nt<T> ::= [] | nt<T> : list_nt<T>
 *
 * cons is denoted by the infix '.'. In the implementation using ArrayList
```



```

* the last element which is cons-ed is to the right (not to the left).
*
* constructors: new List<T>() | l.cons(t) where l is a list of objects
*               type <T> and t is an object of type <T>
* selector:     new List<Integer>().isEmpty() is true
*               (indeed, the List<Integer> is [])
*               new List().isEmpty() is true
* -----
* List la = new List();   la.cons(3);   la.listPrint(); // prints: [ 3 ]
* List lb = new List<Integer>();
* lb.cons(6);   lb.listPrint();           // prints: [ 6 ]
* =====
*/

import java.util.*;           // needed for using ArrayList<T> and Iterator

public class List<T> {       // it is ok if we do not use cloneCopy()

// public class List<T> implements Cloneable { // (***) In order to use
// cloneCopy(), uncomment this line and comment the previous one.

    private ArrayList<T> list;

    public List() {
        list = new ArrayList<T>(); // constructor
    }

    public void cons(T datum) { // the head of the list is to the right,
        list.add(datum);       // not to the left.
    }

    public T head() {          // the head of the list is to the right.
        if (list.isEmpty())    // It is the last element: >-----+
            {System.out.println("Error: head of empty list!");}; // |
            T obj = list.get(list.size() - 1); // <-----+
        return obj;           // get(int i) is a method of ArrayList<T>
    }

    public void tail() {       // destructive tail() method
        if (list.isEmpty())
            {System.out.println("Error: tail of empty list!");};
        list.remove(list.size() - 1);
    }

    public int size() {
        return list.size();   // size() is a method of ArrayList<T>
    }

    public T get(int i) {      // 0 <= i <= size() - 1
        return list.get(i);   // get(int i) is a method of ArrayList<T>
    }

    public boolean isSingleton() {
        return list.size() == 1;
    }

    public boolean isEmpty() {
        return list.isEmpty(); // isEmpty() is a method of ArrayList<T>
    }
}

```

```

public void makeEmpty() {
    list.clear();          // clear() is a method of ArrayList<T>
}

// ----- Printing a list -----
// We assume that an element of the list has a toString() method.

public void listPrint() {
    if ( !isEmpty() ){
        System.out.print("[ ");
        for (Iterator iter = list.iterator(); iter.hasNext(); ) {
            System.out.print((iter.next()).toString() + " ");
        };
        System.out.print("]");
    }
    else {System.out.print(" []");}
}

// ----- Making a copy of a list without clone() ----
// ----- The given list is NOT destroyed. -----
public List<T> copy() {
    List<T> copyList = new List<T>();
    for (Iterator<T> iter = list.iterator(); iter.hasNext(); ) {
        copyList.list.add(iter.next());
    };
    return copyList;
}

// ----- Making a copy of a list with clone() -----
// ----- The given list is NOT destroyed. -----
public List<T> cloneCopy() { // see (***) above
    try { List<T> copyList = (List<T>)super.clone();
        copyList.list = (ArrayList<T>)list.clone();
        return copyList;
    } catch (CloneNotSupportedException e) {
        throw new InternalError();
    }
}
}
}
/* ----- */

```

SOLVING THE ENCODING PROBLEM

Let us consider the set N of the natural numbers. In what follows by B and T we denote, respectively, the set of the binary trees and n -ary trees with natural numbers in the nodes which we have defined on pages 181 and 183.

Every n -ary tree in T can be encoded into a binary tree in B via the encoding function $\text{Kin} : T \rightarrow B$, and can be decoded from a binary tree via the decoding function $\text{Hin} : B \rightarrow T$. These two functions, which are defined in the Java program `EncodingTrees.java` listed on page 189, are such that for every n -ary tree nt , we have that: $\text{Hin}(\text{Kin}(\text{nt})) = \text{nt}$. In order to execute the program `EncodingTrees.java` we need the following Java programs:

- (i) `BinNode.java`,
- (ii) `NaryNode.java`, and
- (iii) `List.java`,

which implement, respectively, (i) the binary trees in B with nodes of type `T` (see page 181), (ii) the n -ary trees in T with nodes of type `T` (see page 183), and (iii) the

lists in *Tlist* of elements of type *T*, for some type parameter *T* (see page 186). The actual type of those trees and lists is given by instantiating the type parameter *T*, as allowed in Java 1.5. For instance, in the case of lists, if we instantiate the type *T* to `<NaryNode <Integer>>`, we get the lists of *n*-ary trees with nodes of type `Integer`.

In the program `EncodingTrees.java` an *n*-ary tree with nodes of type `Integer` to be encoded into a binary tree with nodes of type `Integer`, is given as a string `nt` generated by the following productions:

$$\begin{array}{l|l} \text{nt} & ::= E \quad | \quad \text{ne_nt} \\ \text{ne_nt} & ::= m \quad | \quad m(\text{ne_list}) \\ \text{ne_list} & ::= \text{ne_nt} \quad | \quad \text{ne-nt}, \text{ne_list} \end{array}$$

where: (i) *E* is the empty *n*-ary tree, (ii) comma and parentheses are terminal symbols, and (iii) *m* is any integer number in the set $\{\dots, -2, -1, 0, 1, 2, \dots\}$. In this representation we have used the notions of an `ne_nt` (that is, a non-empty *n*-ary tree) and `ne_list` (that is, a non-empty list of non-empty *n*-ary trees). This choice of using non-empty *n*-ary trees has the advantage that a node with value *m* and no son nodes, can be given in input as the string `m`, rather than the longer string `m([])`.

The internal representation of an *n*-ary tree `nt` is done according to domain equations (2) (see page 183), that is, it is generated by the following productions:

$$\begin{array}{l|l} \text{nt} & ::= E \quad | \quad m(\text{list}) \\ \text{list} & ::= [] \quad | \quad \text{nt}:\text{list} \end{array}$$

where we have used the empty list `[]` and the infix *cons* operation denoted ‘:’.

Here is a program for encoding *n*-ary trees into binary trees and decoding binary trees back to *n*-ary trees by using the functions `Kin` and `Hin`.

```
/**
 * =====
 *          ENCODING n-ARY TREES INTO BINARY TREES
 * Filename: "EncodingTrees.java"
 *
 * This program defines the encoding method Kin from n-ary trees to binary
 * trees. It also defines an inverse method Hin such that
 * for every n-ary tree nt, we have that Hin(Kin(nt)) = nt.
 * Grammar of the input n-ary tree nt:
 * n-ary tree:          nt      ::= E | ne_nt
 *
 * non-empty n-ary tree: ne_nt   ::= m | m(ne_list)
 *
 * non-empty list of non-empty n-ary trees:
 *                  ne_list ::= ne_nt | ne_nt , list_list
 *
 * where: E is the empty n-ary tree,
 *        m can be any integer in {...,-2,-1,0,1,2,...}
 * Comma and parentheses are terminal symbols.
 * =====
 * The internal representations of binary trees, n-trees, and lists of n-ary
 * trees are provided by objects of the classes: BinNode, NaryNode, and List
 * =====
 */

import java.io.*;
import java.util.*;          // needed for using ArrayList<T> and Iterator
/**
```

```

* =====
*                               Class ParseException
* =====
*/
class ParseException extends Exception {
    // constructor of a new instance of ParseException
    public ParseException() {
        super("parse exception!");
    }
};
public class EncodingTrees {
/**
* -----
* In order to define the encoding and decoding functions Kin, K, Hin, and H
* which are given below, we use the following representation of binary
* trees and n-ary trees. These representations make use of the ternary
* constructor b(-,-,-) and the binary constructor T(-,-). These
* constructors are used only for this purpose and in the comments below.
*
* binary tree:    bt = e | b(bt1,n,bt2)
* n-ary tree:    nt = E | T(n,list_nt) (list_nt is the list of sons of n)
*                list_nt = [] | nt : list_nt
*
* e is the empty binary tree
* E is the empty n-ary tree
* n is a natural number
* -----
* +++ Kin : nt -> bt                // Kin encodes an nt into a bt
* --- Kin(E)                        = e
* --- Kin(T(n,l))                   = K(T(n,l),[])
*
* +++ K : nt x list_nt -> bt        // nt is not equal to E
* --- K(T(n,[]),[])                 = b(e,n,e)           // ex: node 9
* --- K(T(n,[],h:1),h:1)            = b(e,n,K(h,1))      // ex: node 2
* --- K(T(n,h:1),[])                = b(K(h,1),n,e)      // ex: node 4
* --- K(T(n,h:1),h1:l1)             = b(K(h,1),n,K(h1,l1))// ex: node 3
* -----
*/

    public static BinNode<Integer> Kin (NaryNode<Integer> nt) {
        if (nt.isEmpty()) { return new BinNode(); }
        else { return K(nt, new List <NaryNode<Integer>> ());}
    }

    public static BinNode<Integer> K (NaryNode <Integer> nt,
        List <NaryNode <Integer>> lnt ) {

        if ( nt.getListOfChildren().isEmpty() && lnt.isEmpty() )
            { return new BinNode(new BinNode(), nt.getValue(), new BinNode());}
        else if ( nt.getListOfChildren().isEmpty() && !lnt.isEmpty() )
            { NaryNode<Integer> hd1 = lnt.head();
              lnt.tail();
              BinNode<Integer>
                res = new BinNode(new BinNode(), nt.getValue(), K(hd1, lnt));
              lnt.cons(hd1); // restoring the list
              return res; }
        else if ( !nt.getListOfChildren().isEmpty() && lnt.isEmpty() )
            { NaryNode<Integer> hd2 = (nt.getListOfChildren()).head();
              (nt.getListOfChildren()).tail();
              BinNode<Integer>

```

```

        res = new BinNode(K(hd2, nt.getListOfChildren()),
                          nt.getValue(),
                          new BinNode());
        (nt.getListOfChildren()).cons(hd2); // restoring the list
    return res; }
else { NaryNode<Integer> hd3 = (nt.getListOfChildren()).head();
      (nt.getListOfChildren()).tail();
      NaryNode<Integer> hd4 = lnt.head();
      lnt.tail();
      BinNode<Integer>
        res = new BinNode(K(hd3, nt.getListOfChildren()),
                          nt.getValue(),
                          K(hd4, lnt));
      (nt.getListOfChildren()).cons(hd3); // restoring the list
      lnt.cons(hd4); // restoring the list
    return res;
  }
}
}
/**
 * -----
 * +++ Hin : bt -> nt
 * --- Hin(e) = E
 * --- Hin(b(bt1,n,e)) = T(n,H(bt1)) //bt1 may be equal to e
 *
 * +++ H : bt -> list_nt
 * --- H(e) = []
 * --- H(b(bt1,n,bt2)) = T(n,H(bt1)) : H(bt2)//bt1 and bt2 may be equal to e
 * -----
 */
public static NaryNode <Integer> Hin (BinNode <Integer> bt) {
    if (bt.isEmpty()) { return new NaryNode(); }
    else { return new NaryNode(bt.getValue(), H(bt.getLeftChild()));}
}

public static List <NaryNode <Integer>> H (BinNode <Integer> bt) {
    if ( bt.isEmpty())
        { return new List(); }
    else { NaryNode<Integer>
          hd = new NaryNode(bt.getValue(), H(bt.getLeftChild()));
          List <NaryNode<Integer>> lnt = new List();
          lnt = H(bt.getRightChild());
          lnt.cons(hd);
          return lnt; }
}
// ----- parsing the input n-ary node -----
public static char c;
// inputLine is a sequence of characters,
// which are read one at a time, by
// the method read() of the class StringReader

public static NaryNode <Integer> nodeParse(BufferedReader inputLine)
    throws IOException, ParseException {
    NaryNode <Integer> node1 = new NaryNode();
    if ( c == 'E' ) { c = (char)inputLine.read(); return node1; } else
    if (('0' <= c) && (c <= '9')) {
        int v = c-48;
        node1 = new NaryNode(v, new List());
        c = (char)inputLine.read();
        if ( c == '(' ) {

```

```

        c = (char)inputLine.read();
        List <NaryNode <Integer>> listNodes = listNodeParse(inputLine);
        node1 = new NaryNode(v, listNodes);
        if ( c==' ' ) { c = (char)inputLine.read(); } // reads past ' '
        else { throw new ParseException(); };
        return node1;
    }
    else { return node1; }
}; throw new ParseException();
}

public static List <NaryNode <Integer>> listNodeParse(
    BufferedReader inputLine)
    throws IOException, ParseException {
    List <NaryNode <Integer>> listNodes = new List();
    NaryNode <Integer> node1 = nodeParse(inputLine);
    if ( c == ' ' ) {
        c = (char)inputLine.read();
        listNodes = listNodeParse(inputLine);
    };
    listNodes.cons(node1);
    return listNodes;
}
// ----- end of parsing -----
// ----- main -----
public static void main(String [] args ) {

    System.out.println("Input an n-ary tree nt according to the given "
        + "grammar, please.");
    NaryNode <Integer> nNode = new NaryNode();
    BinNode <Integer> bNode = new BinNode();
    try { // from String to StringReader and to BufferedReader
        BufferedReader inputLine =
            new BufferedReader(new InputStreamReader(System.in));
        c = (char)inputLine.read();
        nNode = nodeParse(inputLine);
        if (c != '\n') { throw new ParseException(); };
    } catch ( IOException e ) {
        e.printStackTrace();
        System.out.println("IOException!");
    } catch ( ParseException pex ) {
        System.out.println(pex.getMessage());
        return;
    };
    // -----
    System.out.print("\nParsing of the n-ary tree:\n  nt = ");
    nNode.naryNodePrint();
    System.out.print("\nBy applying Kin "
        + "we get the encoding binary tree:\n  bt = ");
    bNode = Kin(nNode);
    bNode.binNodePrint();
    NaryNode <Integer> nNode1 = new NaryNode();
    nNode1 = Hin(bNode);
    System.out.print("\nBack to the n-ary tree nt using Hin:\n"
        + "  nt = ");
    nNode1.naryNodePrint();
    System.out.println();
}
}
}

```

```

/**
 * input:    output
 * -----
 * javac EncodingTrees.java
 * java  EncodingTrees
 *      Input an n-ary tree nt according to the given grammar, please.
 * E
 *
 *      Parsing of the n-ary tree:
 *      nt = E
 *      By applying Kin we get the encoding binary tree:
 *      bt = e
 *      Back to the n-ary tree nt using Hin:
 *      nt = E
 * -----
 * 1(2)
 *
 *      Parsing of the n-ary tree:
 *      nt = 1 [2 []]
 *      By applying Kin we get the encoding binary tree:
 *      bt = ((e 2 e) 1 e)
 *      Back to the n-ary tree nt using Hin:
 *      nt = 1 [2 []]
 * -----
 * 1(2,3(5,6),4(7(9),8))
 *
 *      Parsing of the n-ary tree:
 *      nt = 1 [2 [], 3 [5 [], 6 []], 4 [7 [9 []], 8 []]]
 *      By applying Kin we get the encoding binary tree:
 *      bt = ((e 2 ((e 5 (e 6 e)) 3 (((e 9 e) 7 (e 8 e)) 4 e))) 1 e)
 *      Back to the n-ary tree nt using Hin:
 *      nt = 1 [2 [], 3 [5 [], 6 []], 4 [7 [9 []], 8 []]]
 * -----
 * that is:  nt
 *
 *          1
 *        / | \
 *       2 3 4
 *      /|  | \
 *     5 6 7 8
 *          |
 *          9
 *
 *          Kin
 *          ----->
 *          <-----
 *          Hin
 *
 *          bt
 *
 *          1
 *        / \
 *       2  e
 *      / \
 *     e  3
 *        / \
 *       5  4
 *      /|  | \
 *     e 6  7 e
 *      /|  | \
 *     e e  9 8
 *          /|  | \
 *         e e e e
 * -----
 */

```

We have the following properties.

(1) In $T(n, \text{list-nt})$, that is, in $\text{NaryNode}(n, \text{list-nt})$ we have that list-nt is the list of the son nodes, if any, of the node n in the left-to-right order.

(2) In $K(t, lt)$ the list lt is the list of the brother nodes, if any, of the node t in the left-to-right order.

(3) The first argument of K is never equal to the empty n -ary tree E . The proof of this property is based on the fact that for each call of $K(T(n, l1), l2)$, the tree E is a member of neither the list $l1$ nor the list $l2$. The easy proof by computational induction is left to the reader.

EXERCISE 7.4.2. Define the domain B' of the binary trees and the domain T' of the n -ary trees to be the solutions of the following domain equations, where N denotes the set of natural numbers and $Tlist'$ denotes the domain of the list of the n -ary trees in T' :

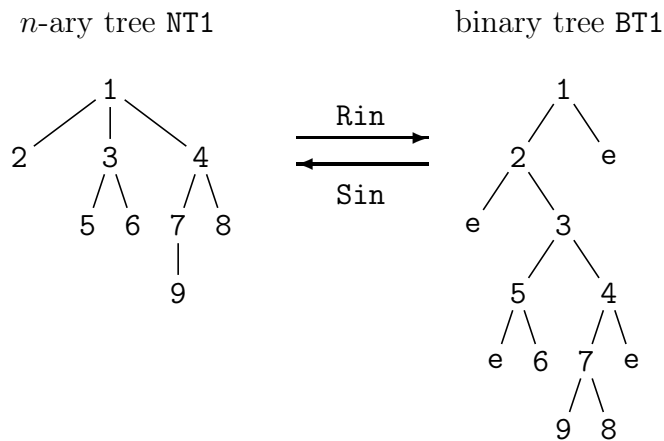
$$\begin{aligned} B' &= 1 + N + B' \times N \times B' \\ T' &= 1 + N + N \times Tlist' \\ Tlist' &= 1 + T' \times Tlist' \end{aligned}$$

Define the functions $Rin : T' \rightarrow B'$ and $Sin : B' \rightarrow T'$ such that for every n -ary tree nt we have that: $Sin(Rin(nt)) = nt$.

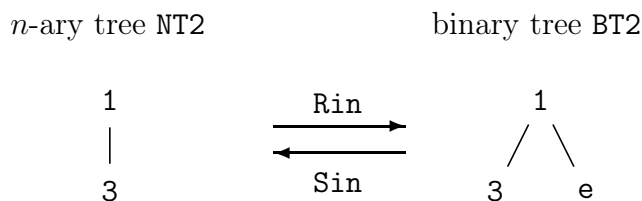
Hint: A possible implementation of these domains is as follows:

```
/**
 * -----
 *   binary tree:      bt = e | l(n) | b(bt1,n,bt2)
 *   n-ary tree:      nt = E | L(n) | T(n,list_nt)
 *                   list_nt = [] | nt : list_nt
 * -----
 */
```

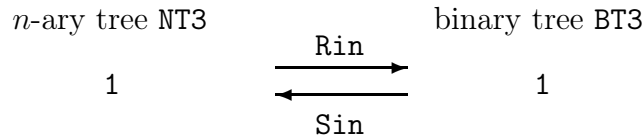
Here are some examples of how the functions Rin and Sin should behave:



The following picture shows that: $Rin(T(L(1), [L(3)])) = b(1(3), 1, e)$.



The following picture shows that: $\text{Rin}(L(1)) = l(1)$.



Here are the definitions of the functions `Rin` and `Sin`. They use the auxiliary functions `R` and `S`, respectively.

The example nodes 2, ..., 9 which we will consider in the comments of these auxiliary definitions, refer to the n -ary tree `NT1` and the binary tree `BT1` we have depicted above.

```
-----
+++ Rin : nt -> bt           // Rin encodes a nt into a bt
--- Rin(E)                   = e
--- Rin(L(n))                 = l(n)
--- Rin(T(n,l)) = R(T(n,l), []) // empty list of brothers of t

+++ R : nt x list_nt -> bt   // list_nt is the list of brothers of nt
--- R(L(n), [])              = l(n) // example: node 9
--- R(L(n), a:l)             = b(e,n,R(a,l)) // example: node 5
--- R(T(n,a:l), [])         = b(R(a,l),n,e) // example: node 4
--- R(T(n,a:l), a1:l1)     = b(R(a,l),n,R(a1,l1)) // example: node 3

-----

+++ Sin : bt -> nt
--- Sin(e)                   = E
--- Sin(l(n))                 = L(n)
--- Sin(b(bt1,n,e)) = T(n,S(bt1)) // bt1 is not equal to e

+++ S : bt -> list_nt
--- S(l(n))                   = [L(n)] // nodes 6, 9, and 8:
// leaves with no younger brothers.
--- S(b(e,n,bt1)) = L(n) : S(bt1) // nodes 2 and 5:
// leaves with younger brothers.
--- S(b(bt1,n,e)) = [T(n,S(bt1))] // node 4.
--- S(b(bt1,n,bt2)) = T(n,S(bt1)) : S(bt2) // nodes 3 and 7.

-----
```

We have the following properties.

(1) The first argument of `R` is never equal to the empty n -ary tree `E`. The proof of this property is based on the fact that for each call of `R(T(n,l1),l2)` and `R(L(n),l2)`, the tree `E` is a member of neither the list `l1` nor the list `l2`. The easy proof by computational induction is left to the reader.

(2) Let us assume that `Sin` takes as input a binary tree which can be obtained as the result of applying `Rin` to an n -ary tree. Then, in the above equations defining the function `S` (the ones preceded by `---`'s), the trees `bt1` and `bt2` are not equal to the empty binary tree `e`. \square

We will end this section by providing the definitions of the binary trees and the n -ary trees in Pascal and C++. In Pascal using variant records, binary trees are realized by the following type definition:

```

-----
type B = ^cell;
cell = record case tag : (no_son, two_sons) of
    no_son: ();
    two_sons: (value: integer; lson, rson: B)
end;
-----

```

In C++ binary trees are realized by the following class definition:

```

-----
class cell {
public :
    int value; cell* lson; cell* rson;
    cell(int v, cell* l, cell* r) : value(v), lson(l), rson(r) { };
};
typedef cell* B;
-----

```

EXERCISE 7.4.3. Define in Pascal and in C++ the domain B of the binary trees which are solutions of the following domain equation, where N denotes the set of natural numbers:

$$B = 1 + N + B \times N \times B \quad \square$$

In Pascal using variant records n -ary trees are realized by the following type definition:

```

-----
type    T = ^Tcell;
        Tlist = ^Tlistcell;
        Tcell = record case tag : (empty, non_empty) of
            empty: ();
            non_empty: (atom: integer; next: Tlist)
        end;
Tlistcell = record case tag : (emptyl, non_emptyl) of
            emptyl: ();
            non_emptyl: (value: T; sons: Tlist)
        end;
-----

```

We could have also defined the record Tlistcell as follows:

```
Tlistcell = record value: T; sons: Tlist end;
```

In this case: (i) a pointer to Tlistcell implements an element of the domain $T \times Tlist$ (see page 183), and (ii) the NULL pointer implements the element of the domain 1 in the equation $Tlist = 1 + T \times Tlist$ (see again page 183).

In C++ n -ary trees are realized by the following class definition:

```

-----
class Tlistcell;
typedef Tlistcell* Tlist;
class Tcell {
public : int atom; Tlist next;
        Tcell(int at, Tlist nx) : atom(at), next(nx) { };
};
typedef Tcell* T;
class Tlistcell {
public : T value; Tlist sons;
        Tlistcell(T val, Tlist ss) : value(val), sons(ss) { };
};
-----

```

EXERCISE 7.4.4. Define in Pascal and in C++ the domain T of the n -ary trees which are solutions of the following domain equations, where N denotes the set of natural numbers and $Tlist$ denotes the domain of the list of the n -ary trees in T :

$$T = 1 + N + N \times Tlist$$

$$Tlist = 1 + T \times Tlist$$

□

7.5. Minimal Spanning Tree of an Undirected Graph

In this section we describe an algorithm for constructing a minimal spanning tree of an undirected, connected, weighted graph with N nodes. The *weight* (also called *length*) of any arc is an integer which can be either positive or null or negative. The lengths can be *non-Euclidean*, in the sense that if the arc $i-j$ has length d_{ij} and the arc $j-k$ has length d_{jk} then the length d_{ik} of the arc $i-k$, if it exists, need not be related to the values of d_{ij} and d_{jk} . The length of an arc $i-j$ is also called the *distance* from node i to node j (which is equal to the distance from node j to node i because the graph is undirected). There could be cycles with total negative length. In the adjacency matrix $N \times N$ which represents the lengths of the arcs of the graph, the entry ‘ ∞ ’ at row i and column j , for $i, j = 0, \dots, N-1$, denotes the absence of the arc from node i to node j . We assume that there is no arc from node i to node i itself, for $i = 0, \dots, N-1$, but in the adjacency matrix at row i and column i , we have put the value 0 (because 0 is the neutral element for addition).

In the program below, for simplicity reasons, we have assumed that the lengths are all non-negative integers, and we have encoded the absence of an arc by the value ‘ -1 ’.

An brute force algorithm for constructing a minimal spanning tree of an undirected, connected, weighted graph consists in: (i) constructing all possible spanning trees, and then (ii) taking a minimal one. This obvious algorithm is very inefficient because, by Caley’s formula, there are N^{N-2} distinct spanning trees of a given undirected, connected graph with N nodes. In Figure 7.5.1 on the following page we have depicted the 16 spanning trees of a graph with four nodes placed at the vertexes of a square and the six arcs connecting every pair of nodes. Such graphs with all possible arcs are called *clicks*.

Now we will present an algorithm which is much more efficient than the brute force one. This efficient algorithm was first proposed by R. C. Prim and independently by E. W. Dijkstra. We will call this algorithm the *Minimal Spanning Tree algorithm*, or *MST algorithm*, for short.

The basic idea of the algorithm which is as follows. We start from a node, say p , of the given graph whose nodes can be colored either red or blue. We assume that initially all nodes are blue, except for the chosen node p which is red. Then we choose a blue node, say q , which is connected to node p by an arc which has a minimal length among all the arcs which have node p as one of their two vertexes. We color red this second node q . By doing so we have a first red arc between p and q which will be part of the minimal spanning tree to be constructed. We proceed by acquiring in the red part of the graph a new blue node which will then be colored red together with the arc which connects this new red node to the red part, until all nodes are colored red.

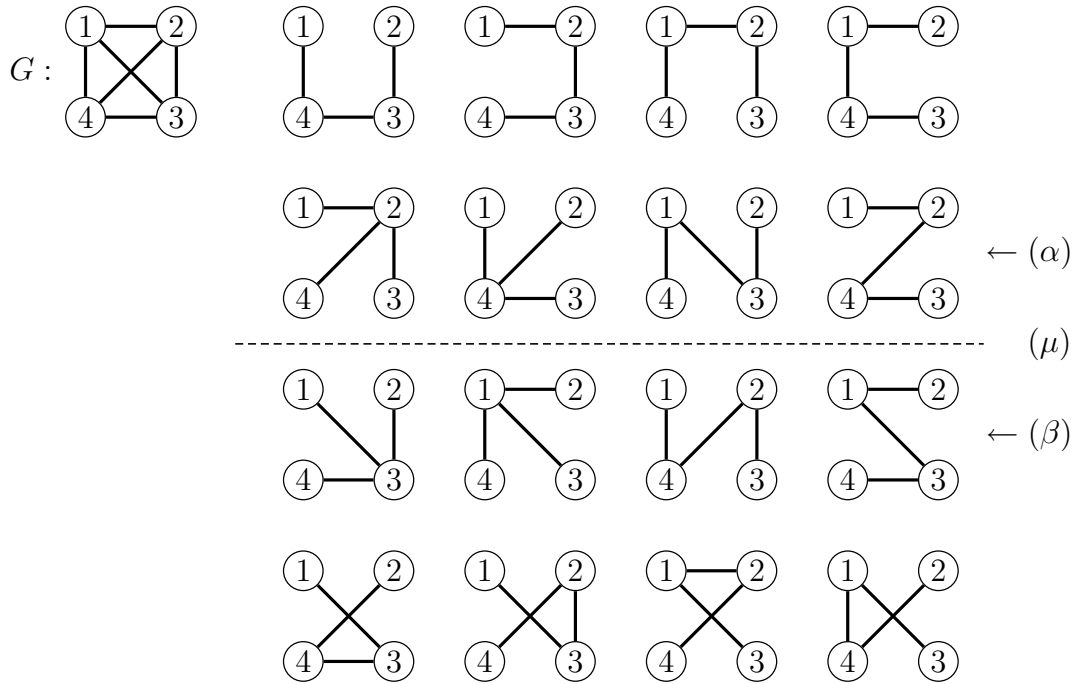


FIGURE 7.5.1. The 16 ($= 4^{4-2}$) spanning trees of a undirected, connected graph G with four nodes. The spanning trees of rows (α) and (β) are mirror images (modulo the names of the nodes) w.r.t. the line (μ) .

Thus, during the execution of the algorithm the given graph of which we must compute a minimal spanning tree, is divided into two parts: (i) a red part, called *Red*, for which we have already computed a minimal spanning tree, and (ii) a blue part, called *Blue*, from which we have to choose a node which will become red and by doing so, a new arc is added to the minimal spanning tree to be computed.

Ties of the length of the arcs can be resolved in any way we like, and this is why through this algorithm we actually compute *a* minimal spanning tree, rather than *the* minimal spanning tree. (Obviously, for all minimal spanning trees the total length of the arcs is the same.) In what follows, for reasons of simplicity, we will free to say ‘the minimal spanning tree’, instead of ‘a minimal spanning tree’. Similarly, we will feel free to say ‘the shortest arc’, instead of ‘a shortest arc’. The correctness of the MST algorithm can be derived, as illustrated in [6], from the following three points.

Point (i). The *local optimum* is a *global optimum*.

Point (ii). The use of the *function inversion technique*.

Point (iii). The use of an $(N-1) \times 2$ array.

Point (i). The *local optimum* is a *global optimum*.

This fact allows us to derive a *greedy algorithm* for computing the spanning tree which we compute ‘by addition of nodes’, as described above.

Suppose that V is the shortest arc connecting a red node to a blue node (see Figure 7.5.2 on the next page).

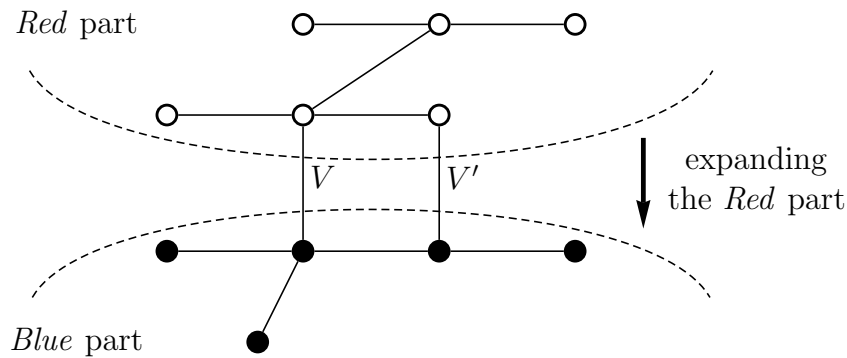


FIGURE 7.5.2. Construction of a minimal spanning tree. The *Red* part shows the portion of the final minimal spanning tree for the points which now are red. The *Blue* part shows the portion of the final minimal spanning tree for the points which now are blue.

This arc V must be an arc of the spanning tree. We prove this fact by contradiction. Let us assume that: (i) a different arc, call it V' , instead of V , is in the minimal spanning tree connecting a red node to a blue node, and (ii) the weight of V' is greater than the weight of V . Call T' this pretending minimal spanning tree which includes the arc V' , instead of the arc V . Now from the tree T' we construct a different spanning tree T which is shorter than T' by replacing the arc V' by the arc V . Indeed, (i) T connects all nodes of the graph because the blue node of V is connected to the blue node of V' , and (ii) T is a tree because no loops are generated by the substitution of V' by V , because if a loop is generated, it should go through another arc from the red nodes to the blue nodes and there is no such arc, simply because we took the arc V' away.

At this point the structure of the algorithm is as follows, where we called *violet* an arc which has a red node and a blue node. Recall that N is the number of nodes in the given undirected graph.

```

color red one node and color blue all the remaining nodes;
while the number of red nodes is less than  $N$  do
    | begin Step (1). Select the shortest violet arc.
    |     Step (2). Color it red and color red its blue node.
    | end

```

Point (ii). The use of the *function inversion technique*.

The choice of the shortest violet arc can be done in two ways: either (ii.1) for each red node we compute the shortest arc from it to a blue node, and then we take the shortest arc among these minimal arcs, one for each red node, or (ii.2) for each blue node we compute the shortest arc from it to a red node, and then we take the shortest arc among these minimal arcs, one for each blue node.

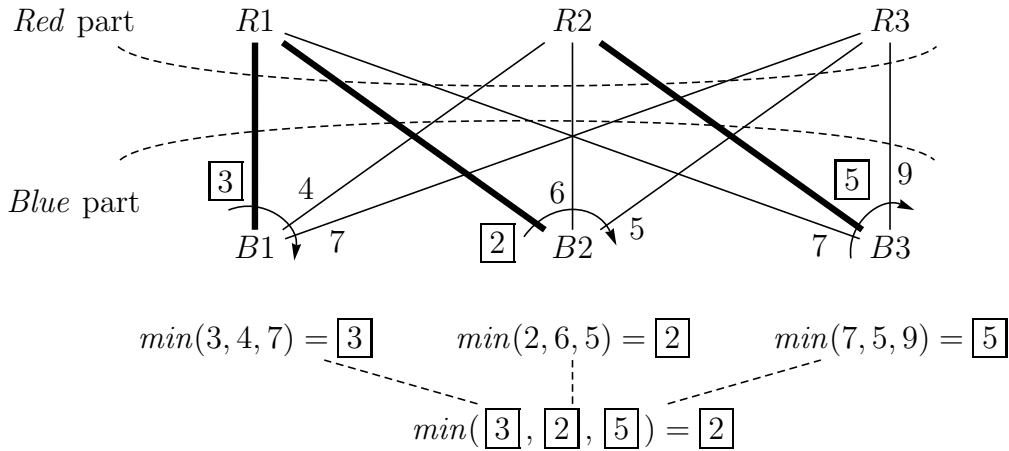


FIGURE 7.5.3. Computation of the minimal arc connecting the *Red* part to the *Blue* part. The arcs with thicker lines are the so called *ultraviolet arcs*. There is exactly one ultraviolet arc for each blue node. Given a blue node its ultraviolet arc is the shortest arc from that blue node to a red node. We did not draw every arc: more arcs may be present within the *Red* part or the *Blue* part.

Since the given graph is undirected, these two choices are both correct. However, from a complexity point of view they are *not* equivalent, because we are computing the spanning tree by making the *Red* part larger, while the *Blue* part is shrinking. Indeed, in the case of Choice (ii.2), when a blue node becomes red, the computation of the shortest arc for each blue node requires one comparison only, while in the case of Choice (ii.1), when a blue node becomes red, the computation of the shortest arc for each blue node requires a number of comparisons which is the product ‘current number of red nodes’ \times ‘current number of blue nodes’.

This technique which looks at the distances *for each blue node* towards to red nodes, instead of looking at the distances *from each red node* towards the blue nodes, is called *function inversion*. The word ‘inversion’ comes from the fact that we are extending the *Red* part, while looking at the distances from each *blue* node.

In Figure 7.5.3 we show how the computation of the shortest violet arc is done in the case of three red nodes: $R1$, $R2$, and $R3$, and three blue nodes: $B1$, $B2$, and $B3$. First, we compute the shortest violet arc for the node $B1$, the node $B2$, and the node $B3$. These three shortest arcs, which are called *ultraviolet arcs*, are drawn with thicker lines in that figure. The ultraviolet arc of a blue node is the shortest arc which connects that blue node to a red node. Then we compute the shortest arc among these three shortest ultraviolet arcs. In our case it is the arc from node $B2$ to node $R1$ whose length is 2. When the node $B2$ becomes red, we have to recompute: (•) the ultraviolet arc for the node $B1$, that is, the shortest violet arc from the blue node $B1$ to the *Red* part (this is done by comparing the length of the ultraviolet arc $B1-R1$ with the length of the arc $B1-B2$, if any), and

(•) the ultraviolet arc for the node $B3$, that is, the shortest violet arc from the blue node $B3$ to the *Red* part (this is done by comparing the length of the arc $B3-R2$ with the length of the arc $B3-B2$, if any).

Note that, instead of saying ‘*the* shortest violet (or ultraviolet) arc’ we should say ‘*a* shortest violet (or ultraviolet) arc’ because there may be ties. Indeed, as already mentioned, Algorithm 7.5.1 for computing minimal spanning trees is a nondeterministic algorithm and computes *a* minimal spanning tree.

Thus, while the computation of the minimal spanning tree progresses, in order to make Choice (ii.2), for each node which is presently blue, we have to store its ultraviolet arc and the length of this ultraviolet arc. This is done by keeping an array of ultraviolet arcs as described in Point (iii) below.

We have the following version of the algorithm.

ALGORITHM 7.5.1.

Computation of a Minimal Spanning Tree of an Undirected Graph. (Dijkstra’s MST algorithm)

color red one node and color blue all the remaining nodes;

while the number of red nodes is less than N **do**

begin Step (1). Select the shortest ultraviolet arc.

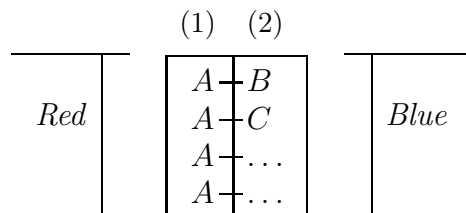
 Step (2). Color it red and color red its blue node, say P .

 Step (3). Adjust the set of ultraviolet arcs: for each blue node B , compare the length ℓ of its ultraviolet arc with the length m of the arc $B-P$ which, if $m < \ell$, becomes the new ultraviolet arc for the node B .

end

Point (iii). The use of an $(N-1) \times 2$ array.

This array, called `uvArcs` in the program `MST.java` on page 205, keeps the ultraviolet arcs, each arc being a pair of nodes which are stored in a row of the array. The array `uvArcs` is initialized by the arcs from the chosen initial red node, say A , to the other $N-1$ blue nodes. Thus, initially the array `uvArcs` looks like this:

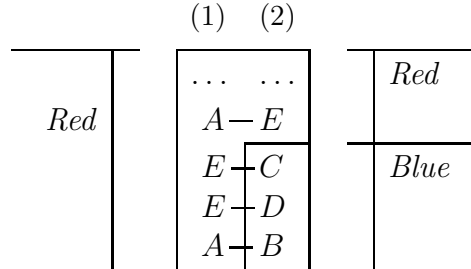


where the left part is of red nodes and the right part is of blue nodes. Initially, the only red node is A and all other nodes are blue nodes. The number of rows of the array `uvArcs` is the number of the nodes which are initially blue, that is, $N-1$.

Now let us see how the array `uvArcs` is modified during the computation of the minimal spanning tree (see also Example 7.5.2 on page 203).

Suppose that we have constructed the minimal spanning tree of the subgraph whose nodes are colored red, and supposed that the arcs of that minimal spanning

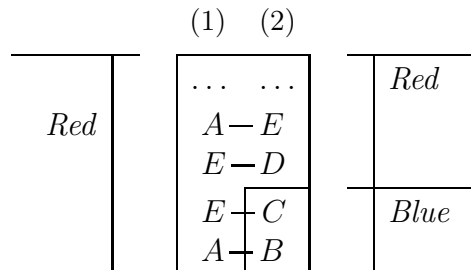
trees are stored in the upper part, named *Red*, of the array *uvArcs*. The lower part, named *Blue*, of that array stores the ultraviolet arcs, one for each blue node. In the following array *uvArcs* the blue nodes are *C*, *D*, and *B*.



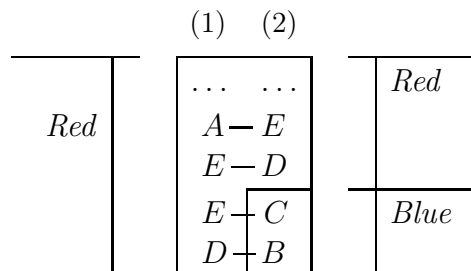
Every node which occurs in the left-upper part (that is, the nodes either in the column (1), or in the column (2) of the *Red* rows) of the array *uvArcs* is colored red, while every node which occur in the lower-right part (that is, the nodes in the column (2) of the *Blue* rows) of the array *uvArcs* is colored blue. In our case the only blue nodes are *C*, *D*, and *B*. Let us now consider the three steps of the body of the while-do loop of our Algorithm 7.5.1 on the previous page.

Step (1). *Select the shortest ultraviolet arc.* Among the arcs in the lower part of the array *uvArcs* (in our case, $E-C$, $E-D$, and $A-B$) we select the arc with the shortest length. Let us assume that it is the arc $E-D$.

Step (2). *Color it red and color red its blue node.* In our case we color red the arc $E-D$ and its blue node *D*. We do so by swapping the rows $E-D$ and $E-C$. We get the following array *uvArcs* where: (i) the blue part now has two arcs only, namely, $E-C$ and $A-B$, and (ii) the arc $E-D$ and the node *D* now are red and they belong to the *Red* part:



Step (3). *Adjust the set of ultraviolet arcs.* For each arc $X-Y$ in the set *Blue* of ultraviolet arcs (that is, in the *Blue* rows), we compare its length ℓ with the length m of the arc, if any, from the blue node *Y* to the node *D* which has been colored red at the preceding Step (2). If $m < \ell$ then we write *D*, instead of *X*. If we assume that $\text{length}(D-B) < \text{length}(A-B)$ and $\text{length}(D-C) \not< \text{length}(E-C)$, we get:



At the end of Step (3) we have extended the red part of the given graph by one more node (in our case the node D), having preserved the invariant that:

- (●) the *Red* part stores the arcs of the minimal spanning tree of the subgraph whose nodes are currently red, and
- (●) the *Blue* part stores the ultraviolet arcs of the nodes which are currently blue, that is, the shortest arc which connects the blue node in column (2) to a red node of the graph.

Note also that during the computation of the minimal spanning tree, we have the following invariant:

$$\text{number-of-red-arcs} + \text{number-of-blue-nodes} = N - 1$$

where: (i) *number-of-red-arcs* is the current number of the arcs of the minimal spanning tree of the subgraph whose nodes are colored red, and (ii) *number-of-blue-nodes* is the current number of the blue nodes (which is equal to the current number of the ultraviolet arcs).

The time complexity of our algorithm $O(N^2)$ and the space complexity is $O(N)$ where N is the number of nodes in the given graph.

Another possible algorithm for computing a minimal spanning tree of an undirected graph is Kruskal's algorithm (see, for instance, [20, page 101]) whose worst case time complexity is $O(N^2 \log N)$, where N is the number of nodes in the given graph and thus, the number of arcs is of order $O(N^2)$. In Kruskal's algorithm the extra $\log N$ factor is due to the need of sorting the arcs of the graph according to their length.

EXAMPLE 7.5.2. [Minimal Spanning Tree of an Undirected Graph] Let us consider the undirected, connected, weighted graph of Figure 7.5.4.

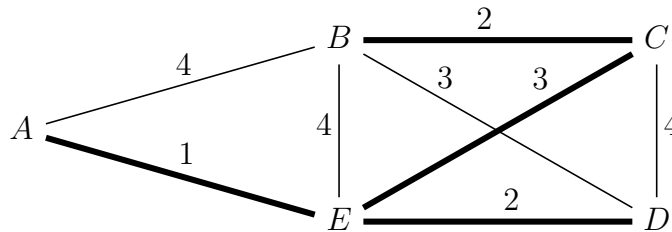
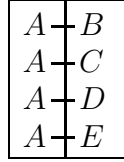


FIGURE 7.5.4. A five node graph of which we compute a minimal spanning tree starting from node A . The successive configurations of the array storing the ultraviolet arcs are shown on page 204. The arcs with thicker lines are those belonging to the resulting minimal spanning tree.

We have the following symmetric matrix of distances:

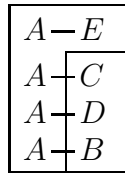
	A	B	C	D	E
A	0	4	∞	∞	1
B	4	0	2	3	4
C	∞	2	0	4	3
D	∞	3	4	0	2
E	1	4	3	2	0

Let us consider node A as first red node. All other nodes are blue. Thus, we have the following array $uvArcs$ of ultraviolet arcs (actually, some of them are not existing, that is, they have length ∞):

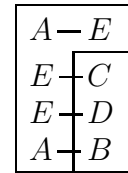


Then, we have the following sequence of pair of actions: (i) take a shortest ultraviolet arc, and (ii) adjust the ultraviolet arcs.

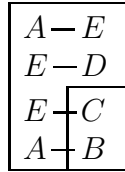
(1.i) Take $A-E$
which is a shortest
ultraviolet arc
and swap rows:



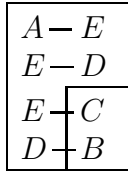
(1.ii) Adjust ultraviolet arcs
after node E became red:



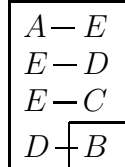
(2.i) Take $E-D$
which is a shortest
ultraviolet arc
and swap rows:



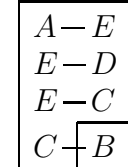
(2.ii) Adjust ultraviolet arcs
after node D became red:



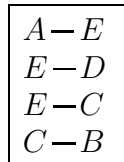
(3.i) Take $E-C$
which is a shortest
ultraviolet arc
and swap rows:



(3.ii) Adjust ultraviolet arcs
after node C became red:



Finally, we take the only available ultraviolet arc $C-B$ and we get the following arrays $uvArcs$ (no swap of rows is required and hence, the array is not changed):



Therefore, at the end of the MST algorithm (see Algorithm 7.5.1 on page 201) we get the minimal spanning tree which is indicated in Figure 7.5.4 on page 203 by ticker lines. That minimal spanning tree consists of the following four arcs:

$A-E$, $E-D$, $E-C$, and $C-B$. □

```

/**
 * =====
 *                MINIMAL SPANNING TREE OF A UNDIRECTED GRAPH
 *
 * Filename: "MST.java"
 *
 * N is the number of nodes of the graph. We assume that N > 1. N1 is N-1.
 * The element graph[i][j] holds the distance, which is an integer number,
 * from node i to node j of the graph.
 * The algorithm is correct also for distances which are negative, null, or
 * positive, and even for non Euclidean distances, but some changes should
 * be made to the code.
 * We assume that for any node i, distance[i][i]=0.
 * For distances which are real numbers, instead of integer numbers, we
 * should write 'double', instead of 'int'.
 * The algorithm is taken from E. W. Dijkstra:
 * 'A Short Introduction to the Art of Programming', EWD 316 August 1971.
 * Note. We stipulate that distance[i][i] == -1 when there is no arc from
 * node i to node j. Thus, when the algorithm is used for distances which
 * are negative, null, or positive, and even for non Euclidean distances,
 * suitable changes are to be made (see lines -1- and -2-) because -1 can no
 * longer be used for encoding the absence of an arc.
 * If all distances are >= 0 then
 * line -1- becomes: if (1 < uvMinLength) and
 * lines -2- should be suitably changed.
 * =====
 */
public class MST {                // MST : Minimal Spanning Tree

    public static void main(String[] args) {
        int N = 0;                // - to be read from the input: args[0]
        int initNode = 0;         // - to be read from the input: args[1]

// ----- N: number of Nodes of the graph.
//                               N > 1.
        try {N = Integer.parseInt(args[0]);
            if ( N < 2) { throw new RuntimeException (); };
        } catch (Exception e) {System.out.print("*** Wrong number of nodes!\n");}
// ----- initNode: initial node.
//                               0 <= initNode <= N-1.
        try {initNode = Integer.parseInt(args[1]);
            if ( ( 0 > initNode ) || ( initNode > N-1 ) ) {
                throw new RuntimeException ();
            };
        } catch (Exception e) {System.out.print("*** Wrong initial node!\n");}

// -----
// MAXLENGTH: maximum length of the arcs. 1 <= length <= MAXLENGTH.
        final int MAXLENGTH = 9;
// -----
// PerCentPROB: probability of connectivity. 0 < PerCentPROB <= 100.
// If PerCentPROB = k, every node is connected with probability k/100 to the
// other N-1 nodes.
        final int PerCentPROB = 60;
// -----
// Generation of a random graph with N nodes.
// The length of each arc belongs to the set {1,..., MAXLENGTH}.
// Every node has (N-1) x PerCentPROB / 100 successor nodes (in average).

        UGraph g =
            new UGraph(N, initNode, MAXLENGTH, PerCentPROB);
    }
}

```

```

g.graphPrint();
System.out.println("Every node is connected to the other nodes with " +
    "probability " + PerCentPROB + " percent.");
System.out.println("The length of each arc belongs to the set {1,..., "
    + MAXLENGTH + "}");
System.out.println("The initial node is: " + initNode);
//
int i, uvMin, p, N1 = N-1;
int l, uvMinLength; // 'double', not 'int', for real distances
//-----
// The g.uvArcs are the ultraviolet arcs of the graph g (see Dijkstra)
//-----
int k = 0;
while (k < N1) {g.uvArcs[k][0] = 0; g.uvArcs[k][1] = k+1; k++;};
k = 0;
while (k < N1) { // <<=====
    uvMin = k;
    uvMinLength = g.graph[g.uvArcs[k][0]] [g.uvArcs[k][1]]; //it may be -1
//-----

    i = k+1;
    while (i < N1) { // <<-----+
        l = g.graph[g.uvArcs[i][0]] [g.uvArcs[i][1]]; // |
        if ( (l < uvMinLength && l >= 0) || uvMinLength == -1 ) // | -1-
            { uvMin = i; uvMinLength = l; }; // |
        i++; // |
    }; // end of while <<-----+
    if (uvMinLength >= 0) { // // -2-
        p = g.uvArcs[uvMin][1]; // nearest blue node to red zone
//-----

        if (uvMin != k) {g.swap(k,uvMin);};
        i = k+1;
        while (i < N1) { // <<-----+
            if ( (g.graph[g.uvArcs[i][0]] [g.uvArcs[i][1]] // |
                > g.graph[p] [g.uvArcs[i][1]] // |
                && g.graph[p] [g.uvArcs[i][1]] >= 0 // | -2-
                ) // |
                || (g.graph[g.uvArcs[i][0]] [g.uvArcs[i][1]] == -1 // | -2-
                    && g.graph[p] [g.uvArcs[i][1]] >= 0) // | -2-
                ) { g.uvArcs[i][0] = p; }; // |
            i++; // |
        }; // end of while <<-----+
        k++;
    } else { break; } // // -2-
}; // end of while <<=====
g.uvPrint(k); }
}

/**
 * input:
 * -----
 * javac MST.java
 * java MST 9 0
 *
 * output:
 * -----
 * The random undirected graph has 9 node(s) (from node 0 to node 8).
 * The symmetric matrix of the arc lengths is:

```

```

* 0 : 0 4 7 - - - 2 4 9
* 1 : 4 0 - 1 7 - - 4 -
* 2 : 7 - 0 5 - 7 3 6 3
* 3 : - 1 5 0 - 5 8 3 4
* 4 : - 7 - - 0 8 5 4 6
* 5 : - - 7 5 8 0 - - 6
* 6 : 2 - 3 8 5 - 0 6 5
* 7 : 4 4 6 3 4 - 6 0 -
* 8 : 9 - 3 4 6 6 5 - 0
*
* Every node is connected to the other nodes with probability 60 percent.
* The length of each arc belongs to the set {1, ..., 9}.
* The initial node is: 0
*
* The arcs of the minimal spanning tree,
* starting from node 0 and reaching all reachable nodes, are:
* 0 -- 6
* 6 -- 2
* 2 -- 8
* 0 -- 1
* 1 -- 3
* 3 -- 7
* 7 -- 4
* 3 -- 5
* -----
*/

/**
 * =====
 *                               The UGraph class
 * Filename: "UGraph.java"
 *
 * This class UGraph defines a undirected, weighted graph.
 * The class UGraph is not a monitor, because not all fields are "private".
 * However, the non-private fields are read, but never assigned to.
 * All methods are "synchronized".
 * Thus, this class UGraph is "almost" a monitor.
 * For real distances, instead of integer distances, write 'double', instead
 * of 'int'.
 * =====
 */
import java.util.Random;

public class UGraph {
    final int N;           // N = number of nodes of the graph
    final int initNode;    // initial node of the minimal spanning tree
    static int [][] graph; // an N x N array
    static int [][] uvArcs; // an N-1 x 2 array: arcs of the minimal
                           // spanning tree

// ----- beginning of constructor -----

    public UGraph(int N, int initNode, int MAXLENGTH, int PerCentPROB) {

/** Generation of a random, undirected graph with N nodes.
 *
 * The random variable randomConnect controls the connectivity:
 * node i is connected with node j, for i != j, with percent
 * probability PerCentPROB: 0 <= PerCentPROB <= 100
 * The random variable randomLength controls the length of the arcs:
 * 1 <= length of an arc <= MAXLENGTH
 */

```

```

this.N                = N;
this.initNode        = initNode;
this.graph           = new int [N][N];
this.uvArcs          = new int [N-1][2];

Random randomConnect = new Random(); // random graph connectivity
Random randomLength  = new Random(); // random arc length

// initializing the graph: -1 (printed as "-") means "unconnected node"
// graph[i][j] = m with m >= 0
// iff m is the length of the arc from node i to node j
for(int i=0;i<N;i++) {
    for(int j=i;j<N;j++) {
        if ((i != j) && (randomConnect.nextInt(100) < PerCentPROB))
            { graph[i][j] = 1 + randomLength.nextInt(MAXLENGTH);
              graph[j][i] = graph[i][j]; // symmetric graph
            }
        else if (i == j) { graph[i][j] = 0; }
        else { graph[i][j] = -1; graph[j][i] = -1; }; // symmetric graph
    }
}
}
// ----- end of constructor -----

// ----- swapping two rows of the array uvArcs -----
public static void swap (int i, int j) {
    int temp;
    temp = uvArcs[i][0]; uvArcs[i][0] = uvArcs[j][0]; uvArcs[j][0] = temp;
    temp = uvArcs[i][1]; uvArcs[i][1] = uvArcs[j][1]; uvArcs[j][1] = temp;
}

// ----- printing the graph -----
public synchronized void graphPrint () {
    System.out.println("The random undirected graph has "+ N +" node(s) " +
        "(from node 0 to node " + (N-1) + ").\n" +
        "The symmetric matrix of the arc lengths is:");
    for(int i=0;i<N;i++) {
        System.out.print(i + " : ");
        for(int j=0;j<N;j++) {
            if (graph[i][j] < 0) { System.out.print("- "); }
            else System.out.print( graph[i][j] + " ");
        }; System.out.println();
    }; System.out.println();
}

// ----- printing the arcs of the minimal spanning tree -----
public synchronized void uvPrint (int N) {
    if (N==0) {System.out.println("\nThe minimal spanning tree is the "+
        "initial node "+ initNode +" only.");}
    else { System.out.print("\nThe arcs of the minimal spanning tree,\n" +
        "starting from node " + initNode + " and reaching " +
        "all reachable nodes, are: \n");
        for (int k=0; k<N; k++)
            {System.out.print(uvArcs[k][0] + " -- " +
                uvArcs[k][1] + "\n");};}
}
}
// -----

```

Path Problems in Directed Graphs

In this chapter we present some techniques for solving path problems in directed graphs. We will first recall some algorithms for matrix multiplication because, as we will see, matrix multiplication can be used for solving path problems, when directed graphs are represented by adjacency matrices (see, for instance, [20, pages 8–10]).

By definition, we have that given a graph G with n nodes, its adjacency matrix M is an $n \times n$ matrix made out of 0's and 1's such that for all i, j , with $1 \leq i, j \leq n$, $M[i, j] = 1$, that is, an element of M in position $\langle i, j \rangle$ is 1, iff in the graph G there is an arc from node i to node j . In particular, $M[i, i] = 1$ iff there exists an arc from node i to node i itself. We assume that the length of a path is given by the number of its arcs. Thus, in particular, in any graph G there is a path of length 0 from any node of G to that same node.

We have that there exists in G a path of length k (≥ 0) from node i to node j iff the in k -th power of M , denoted M^k , the element in position $\langle i, j \rangle$, denoted $M^k[i, j]$, is 1.

8.1. Matrix Multiplication Algorithms

The matrix multiplication problem can be formalized as follows.

PROBLEM 8.1.1. *Matrix Multiplication.*
Input: two $n \times n$ matrices of integers

$$A = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{vmatrix}$$

Output: the $n \times n$ matrix $C = A \times B$, that is,
 for every $i, j = 1, \dots, n$, $c_{ij} = \sum_{1 \leq k \leq n} a_{ik} b_{kj}$.

In the sequel we will feel free to write the multiplication of the matrices A and B as AB , instead of $A \times B$.

The elementary matrix multiplication algorithm is as follows.

ALGORITHM 8.1.2. *Elementary Algorithm for Matrix Multiplication.*

Given any two $n \times n$ matrices A and B , the matrix $C = AB$ is obtained by performing the n multiplications and $n - 1$ additions for each $c_{ij} = \sum_{1 \leq k \leq n} a_{ik} b_{kj}$, as indicated in the specification of the Matrix Multiplication Problem (see Problem 8.1.1).

This elementary algorithm takes $n^2(2n-1) = 2n^3 - n^2$ arithmetic operations for $n \geq 2$. Let us now consider the following algorithm for matrix multiplication.

ALGORITHM 8.1.3. *Matrix Multiplication by Partition.*

Consider the following $n \times n$ matrices A , B , and C , and the partitions of each of these matrices into four $(n/2) \times (n/2)$ blocks:

$$A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}, B = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix}, \text{ and } C = \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix}.$$

Perform the 8 multiplications of $(n/2) \times (n/2)$ matrices and the 4 additions of $(n/2) \times (n/2)$ matrices as indicated by the equalities:

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21}, & C_{12} &= A_{11}B_{12} + A_{12}B_{22}, \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21}, & C_{22} &= A_{21}B_{12} + A_{22}B_{22}. \end{aligned}$$

This algorithm takes $O(n^3)$ arithmetic operations, because their number satisfies the following recurrence relation for $n \geq 2$:

$$T(n) = 8T(n/2) + 4(n^2/4), \text{ with } A(1) = 1,$$

whose solution is: $T(n) = 2n^3 - n^2$.

We have also the following algorithm.

ALGORITHM 8.1.4. *Strassen Matrix Multiplication.*

Consider the following $n \times n$ matrices A , B , and C , and the partitions of each of these matrices into four $(n/2) \times (n/2)$ blocks:

$$A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}, B = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix}, \text{ and } C = \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix}.$$

Perform the 7 multiplications of $(n/2) \times (n/2)$ matrices and the 10 additions of $(n/2) \times (n/2)$ matrices (we count the subtractions as additions of negated values) indicated by the following equalities:

$$\begin{aligned} m_1 &= (A_{12} - A_{22})(B_{21} + B_{22}), & m_2 &= (A_{11} + A_{22})(B_{11} + B_{22}), \\ m_3 &= (A_{11} - A_{21})(B_{11} + B_{12}), & m_4 &= (A_{11} + A_{12})B_{22}, \\ m_5 &= A_{11}(B_{12} - B_{22}), & m_6 &= A_{22}(B_{21} - B_{11}), \\ m_7 &= (A_{21} + A_{22})B_{11}, \end{aligned}$$

Then, by performing the following 8 additions of $(n/2) \times (n/2)$ matrices, we get:

$$\begin{aligned} C_{11} &= m_1 + m_2 - m_4 + m_6, & C_{12} &= m_4 + m_5, \\ C_{21} &= m_6 + m_7, & C_{22} &= m_2 - m_3 + m_5 - m_7. \end{aligned}$$

This algorithm due to V. Strassen in 1969 takes $O(n^{\log_2 7})$, which is about $O(n^{2.81})$, because their number satisfies the following recurrence relation for $n \geq 2$:

$$A(n) = 7A(n/2) + 18(n^2/4), \text{ with } A(1) = 1.$$

Indeed, we have that:

$$\begin{aligned} A(n) &= (18/4)[n + (7/4)n^2 + (7/4)^2n^2 + \dots + (7/4)^{(\log_2 n)-1}n^2] + 7^{\log_2 n}A(1) = \\ &= (18n^2/4)[((7/4)^{\log_2 n} - 1)/((7/4) - 1)] + 7^{\log_2 n} = \\ &= 6n^{\log_2 7} - 6n^2 + n^{\log_2 7} < 7n^{\log_2 7}. \end{aligned}$$

(Recall that $x^{\log_y z} = z^{\log_y x}$ for all $x, y, z > 0$ and $y \neq 1$.)

If n is not an exact power of 2 we can pad the given matrix with extra rows (on the bottom) and columns (on the right) made out of all 0's, so that it becomes an $m \times m$ matrix, where $m = 2^{\lceil \log_2 n \rceil}$.

A better method is as follows: if n is even then make a recursive call to $n/2$, if n is odd then pad with one row and one column of 0's (getting an $(n+1) \times (n+1)$ matrix) and make a recursive call to $(n+1)/2$.

Following Strassen's idea, new algorithms have been discovered which multiply matrices even faster than $O(n^{\log_2 7})$. The current asymptotic upper bound is about $O(n^{2.55})$. However, we cannot hope to multiply two $n \times n$ matrices faster than $O(n^2)$, because there are n^2 elements to compute.

8.2. Comparing Matrix Multiplication Algorithms

Let us recall some definitions from Abstract Algebra for comparing the above algorithms for matrix multiplication.

DEFINITION 8.2.1. [Semiring] A *semiring* is an algebra $(S, +, \cdot)$ with carrier set S and the two operations $+$ and \cdot , such that:

- (i) $+$ is associative and commutative,
- (ii) \cdot is associative, and
- (iii) \cdot distributes over $+$ on both sides, that is, for all $a, b, c \in S$,
 $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$.

An element u of S is an *identity* for the operation op iff

$$\text{for all } x \in S, \quad x \text{ op } u = u \text{ op } x = x.$$

An element z of S is an *absorbent element* for the operation op iff

$$\text{for all } x \in S, \quad x \text{ op } z = z \text{ op } x = z.$$

It can be shown that in any semiring:

- if an identity for $+$ exists then it is unique,
- if an identity for \cdot exists then it is unique, and
- if an absorbent element for \cdot exists then it is unique.

DEFINITION 8.2.2. [Semiring with 0 and 1] A *semiring with 0 and 1* is a semiring $(S, +, \cdot)$ with the identity for \cdot , denoted 1, and the absorbent element for \cdot , denoted 0, which is also the identity for $+$. We will write $(S, +, \cdot, 0, 1)$ -semiring to denote such a semiring.

DEFINITION 8.2.3. [Ring] A *ring* is a semiring $(S, +, \cdot)$ where the algebra $(S, +)$ is a commutative group, that is, $+$ is associative and commutative, $+$ has an identity, denoted 0, and every element x has an additive inverse, denoted $-x$, that is, for all $x \in S$, $x + (-x) = (-x) + x = 0$.

As for any group, we have that in a ring the additive inverse of any element is unique. One can prove that in any ring the identity for $+$ is the absorbent element for \cdot , that is, for all $x \in S$, $x \cdot 0 = 0 \cdot x = 0$. As proved in [8], one can also show that:

$$\text{for all } x \in S, \quad x \cdot (-y) = (-x) \cdot y = -(x \cdot y) \text{ and } (-x) \cdot (-y) = x \cdot y .$$

DEFINITION 8.2.4. [Ring with 1] A ring $(S, +, \cdot)$ with the identity for \cdot , denoted 1, is said to be a *ring with 1*. Such a ring will be denoted by $(S, +, \cdot, 0, 1)$ -ring.

DEFINITION 8.2.5. [Field] An $(S, +, \cdot, 0, 1)$ -ring is said to be a *field* iff \cdot is commutative, and every element x different from 0, has a multiplicative inverse, denoted x^{-1} , that is, for all $x \in S$, $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

Given any $(S, +, \cdot, 0, 1)$ -semiring, we may consider the set, denoted M_n , of the $n \times n$ matrices whose elements are taken from the set S . M_n is said to be the set of the $n \times n$ matrices over the $(S, +, \cdot, 0, 1)$ -semiring (or over S , for short).

Let 0_n denote the $n \times n$ matrix with all 0's, and I_n denote the $n \times n$ matrix which is like 0_n , except that for all i , with $1 \leq i \leq n$, $(I_n)_{ii} = 1$, that is, each element of the main diagonal of I_n is 1.

One can show that $(M_n, +_n, \cdot_n, 0_n, I_n)$ is a semiring with 0 and 1, denoted $(M_n, +_n, \cdot_n, 0_n, I_n)$ -semiring, where:

(i) $+_n$ is defined elementwise, that is, for all $n \times n$ matrices A and B , we have that $A +_n B$ is the $n \times n$ matrix C such that for all i, j , with $1 \leq i, j \leq n$, $C_{ij} = A_{ij} + B_{ij}$, and

(ii) \cdot_n is defined as follows: for all $n \times n$ matrices A and B , we have that $A \cdot_n B$ is the $n \times n$ matrix C such that

$$\text{for all } i, j, \text{ with } 1 \leq i, j \leq n, \quad C_{ij} = (A_{i1} \cdot B_{1j}) + (A_{i2} \cdot B_{2j}) + \dots + (A_{in} \cdot B_{nj}).$$

Analogously, we have that, given any $(S, +, \cdot, 0, 1)$ -ring, instead of any $(S, +, \cdot, 0, 1)$ -semiring, we get the ring with 1 of the $n \times n$ matrices over S , instead of the semiring with 0 and 1 of the $n \times n$ matrices over S . The ring of the $n \times n$ matrices over a given $(S, +, \cdot, 0, 1)$ -ring will be denoted by $(M_n, +_n, \cdot_n, 0_n, I_n)$ -ring.

Notice that any $(M_n, +_n, \cdot_n, 0_n, I_n)$ -ring of matrices is *not* commutative even if the corresponding $(S, +, \cdot, 0, 1)$ -ring of their elements is commutative.

FACT 8.2.6. [Comparing the Matrix Multiplication Algorithms] The Elementary Algorithm and the Partition Algorithm for matrix multiplication work for all $n \times n$ matrices whose elements are taken from any $(S, +, \cdot, 0, 1)$ -semiring. Strassen algorithm for matrix multiplication works for all $n \times n$ matrices whose elements are taken from any $(S, +, \cdot, 0, 1)$ -ring.

8.3. Fast Boolean Matrix Multiplication

If we look for an asymptotically fast method for multiplying boolean matrices, we cannot use Strassen algorithm because the boolean values 1 and 0 with the additive operation \vee (that is, the least upper bound) and the multiplicative operation \wedge (that is, the greatest lower bound), do *not* form a ring with 1. Indeed, there is no inverse w.r.t. the operation \vee .

However, given two $n \times n$ boolean matrices A and B , we can multiply them together as they were matrices of elements in \mathbb{Z}_{n+1} , that is, the integers modulo $n+1$, for $n \geq 2$, where we consider \vee as $+$ (i.e., addition), and \wedge as \cdot (i.e., multiplication).

It can be shown that if we multiply the two $n \times n$ boolean matrices A and B as they were matrices of elements in \mathbb{Z}_{n+1} , thereby getting the matrix \tilde{C} , then the boolean matrix $C = A \wedge_n B$ can be computed as follows:

for all i, j , with $1 \leq i, j \leq n$, if $\tilde{C}_{ij} = 0$ then $C_{ij} = 0$ and if $1 \leq \tilde{C}_{ij} \leq n$ then $C_{ij} = 1$.

Thus, also boolean matrices can be multiplied within the same time complexity (counting the number of arithmetic operations) which is required for multiplying two integer matrices, that is, they can be multiplied with time complexity $O(n^{\log_2 7})$.

If, instead, we count the number of bit operations, as it is reasonable for boolean matrices, then the time complexity is $O(n^{\log_2 7} I_M(\log_2 n))$, where $I_M(n)$ is the time complexity for multiplying two integers, each of which being made out of n digits in base 2. (Recall that for all i, j , with $1 \leq i, j \leq n$, \tilde{C}_{ij} has at most $\lceil \log_2(n+1) \rceil$ bits, because \tilde{C}_{ij} is at most n .)

Recalling that by the Schönhage-Strassen method we have that

$$I_M(n) = O(n (\log_2 n) \log_2(\log_2 n)),$$

we get that the time complexity for multiplying two $n \times n$ boolean matrices is:

$$O(n^{\log_2 7} (\log_2 n) (\log_2 \log_2 n) (\log_2 \log_2 \log_2 n)).$$

We will see below that fast algorithms for boolean matrix multiplication allow us to construct fast algorithms for computing both the transitive closure of binary relations and the paths between nodes in graphs.

8.4. IC-Semirings and Path Problems in Directed Graphs

In order to study path problems in labeled directed graphs it is convenient to consider the $n \times n$ matrices whose elements are taken from suitable semirings, called *ic-semirings*, which we will introduce in this section. We will have that the algebra of the $n \times n$ matrices whose elements are taken from an ic-semiring, is an ic-semiring as well.

We assume that paths of labeled directed graphs are represented as elements of matrices associated with the graphs and, in particular, we assume that a path with label p from node h to node k of a graph G with n nodes, is represented by the element p taken from the semiring $(S, +, \cdot, 0, 1)$ and placed in row h and column k of an $n \times n$ matrix M associated with the graph G . Thus, for the matrix M we have that $M_{hk} = p$.

In order to motivate our definitions below, we now list a few properties that we want to hold for the paths of labeled directed graphs whose labels are taken from the semiring $(S, +, \cdot, 0, 1)$.

- (i) Two paths from node m to node n , the first one with label p and the second one with label q , are equal to a single path from m to n with label $p + q$.
- (ii) For any node n_1, n_2 , and n_3 , a path from n_1 to n_2 with label p followed by a path from n_2 to n_3 with label q , is equal to a path from n_1 to n_3 with label $p \cdot q$.
- (iii) The absence of a path from node m to node n is equal to a path from node m to node n with label 0.

- (iv) The path of length 0 (that is, a path of no arcs) from a node to itself is equal to a path from that node to itself with label 1.
- (v) For any node n_1 , n_2 , and n_3 , a path from n_1 to n_2 with label p , followed by a path from n_2 to n_3 with label $q_1 + q_2$, is equal to a path from n_1 to n_3 with label $(p \cdot q_1) + (p \cdot q_2)$ (this property is the distributivity of \cdot over $+$).
- (vi) Two paths both with label p , between the same two nodes, are equal to a single path with label p (this property is the idempotency of $+$).
- (vii) A path from a node to itself with label p is equal to a path from that node to itself with label p^* , that is, $1 + p + p \cdot p + p \cdot p \cdot p + \dots$ (this property motivates the need for countably infinite sums).

DEFINITION 8.4.1. [**ic-semiring**] An $(S, +, \cdot, 0, 1)$ -semiring, that is, a semiring with 0 and 1, is said to be *idempotent complete semiring* (or *ic-semiring*, for short) iff

- (i) $+$ is idempotent, that is, for all $x \in S$, $x + x = x$, and
- (ii) there are countably infinite sums, that is, there exists the value of $\sum_{i \geq 0} a_i$, that is, $a_0 + a_1 + \dots$ for any infinite sequence $\langle a_0, a_1, \dots \rangle$ of elements of S , and
- (iii) \cdot distributes on both sides over countably infinite sums, that is,

$$\begin{aligned}
 (a_0 + a_1 + a_2 + \dots) \cdot (b_0 + b_1 + b_2 + \dots) &= (a_0 \cdot b_0) + (a_0 \cdot b_1) + (a_0 \cdot b_2) + \dots \\
 &+ (a_1 \cdot b_0) + (a_1 \cdot b_1) + (a_1 \cdot b_2) + \dots \\
 &+ (a_2 \cdot b_0) + (a_2 \cdot b_1) + (a_2 \cdot b_2) + \dots \\
 &+ \dots
 \end{aligned}$$

These semirings are called *closed semirings* in [1]. We prefer the name *idempotent complete semirings* to recall that the operation the idempotency of $+$ and the existence of countably infinite sums.

For ic-semirings we will use the unary postfix operation $*$ to denote the infinite sum of all powers, that is, for all $s \in S$, $s^* =_{\text{def}} \sum_{i \geq 0} s^i$, that is, $s^* =_{\text{def}} 1 + s + s \cdot s + s \cdot s \cdot s + \dots$ (recall that there is no need for parentheses among the summands, because \cdot is associative). Thus, $0^* = 1$. (One can informally state this property by saying that a loop of no path from a node m to itself is the same as the path of length 0 from m to itself.)

Here are three examples of ic-semirings with 0 and 1.

- (i) $(\{0, 1\}, \wedge, \vee, 0, 1)$ -semiring, called the *boolean ic-semiring*, where:

$$\begin{aligned}
 x \wedge y &=_{\text{def}} \text{if } x = 1 \text{ and } y = 1 \text{ then } 1 \text{ else } 0, \text{ and} \\
 x \vee y &=_{\text{def}} \text{if } x = 1 \text{ or } y = 1 \text{ then } 1 \text{ else } 0.
 \end{aligned}$$

This ic-semiring is used for testing whether or not in a given directed graph there is a path from node i to node j .

- (ii) $(R^{\geq 0} \cup \{+\infty\}, \min, +, +\infty, 0)$ -semiring, called the *non-negative ic-semiring*, where $R^{\geq 0}$ is the set of the non-negative reals, and for all x in $R^{\geq 0} \cup \{+\infty\}$ we have that $\min(x, +\infty) = \min(+\infty, x) = x$.

This ic-semiring is used for computing, given any directed graph whose arcs are labeled by non-negative reals, the minimum distance from any node to any node of the given graph.

(iii) $(REG(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ -semiring, called the *Regular Expressions ic-semiring*, where Σ is any given finite alphabet, Σ^* is the set of all words over Σ , that is, the set of all sequences of symbols in Σ , and $REG(\Sigma^*)$ is the set of all regular languages over Σ .

In what follows we will denote a regular language also by its corresponding regular expression (hence the name *Regular Expressions ic-semiring* for this ic-semiring).

Recall that, given two languages L_1 and L_2 , their concatenation $L_1 \cdot L_2$ is defined in terms of concatenation of words, also denoted \cdot , as follows:

$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1 \text{ and } w_2 \in L_2\}.$$

This ic-semiring can be used for computing, given any directed graph whose arcs are labeled by regular expressions, the regular expression which denotes the set of all paths from any node to any node of the given graph.

From any given ic-semiring $(S, +, \cdot, 0, 1)$ we may get, as specified in Section 8.2 on page 212, the $(M_n, +_n, \cdot_n, 0_n, I_n)$ -semiring of the $n \times n$ matrices whose elements are taken from the given semiring. It can be shown that $(M_n, +_n, \cdot_n, 0_n, I_n)$ -semiring is an ic-semiring.

Let us now look at some problems for which the matrices over ic-semirings we have introduced, are particularly useful.

8.5. Transitive Closure in Directed Graphs: the Reachability Problem

Given a directed graph G with n nodes, the reachability problem in G consists in testing for each pair of nodes i and j whether or not there exists a path from i to j . We can solve this problem by considering the ic-semiring of the $n \times n$ matrices over the boolean ic-semiring as follows.

We associate with G the $n \times n$ boolean matrix $A(G)$, called *adjacency matrix*, (see, for instance, [20, Chapter 1]) such that for all i and j with $1 \leq i, j \leq n$,

$$A(G)_{ij} =_{\text{def}} \text{if there is an arc from node } i \text{ to node } j \text{ then } 1 \text{ else } 0.$$

In particular, $A(G)_{ii} = 0$ if there is no arc from node i to itself. From now on the matrix $A(G)$ will also be simply called A when the graph to which it refers, is understood from the context.

We assume that $A^0 = I_n$ (where I_n is the identity matrix, that is, the $n \times n$ matrix with all elements 0's except those in the main diagonal which are 1's), $A^1 = A$, and $A^{n+1} = A^n \wedge_n A$. We have that: $A^{n+1} = A \wedge_n A^n$.

The definition of the identity matrix I_n is based on the facts that:

- (•) for all i and j with $1 \leq i, j \leq n$, there is no path of one or more arcs from a node i to a node j , and
- (•) if $i = j$ then node j is reachable from node i (and this fact is encoded by the values 1's in the main diagonal of I_n) else node j is not reachable from node i (and this is encoded by the values 0's outside the main diagonal of I_n).

Below we will feel free to omit writing the operators \wedge_n and \wedge , and we will use juxtaposition, instead. Often, for reasons of simplicity, we will also write \wedge , instead of \wedge_n . Analogously, we will also write \vee , instead of \vee_n .

Recall that, as usual, for all i, j with $1 \leq i, j \leq n$, we have that:

$$(A \wedge_n B)_{ij} = (A_{i1} \wedge B_{1j}) \vee (A_{i2} \wedge B_{2j}) \vee \dots \vee (A_{in} \wedge B_{nj}).$$

One can easily show by induction on $r \geq 0$ that $A_{ij}^r = 1$ iff in the graph G there exists a path of length r from node i to node j . Thus, let us consider the matrix

$$A^* =_{def} I_n \vee A \vee A^2 \vee A^3 \vee \dots \vee A^n \vee \dots$$

We have that given any two nodes i and j , the node j is reachable from node i (that is, there is a path from node i to node j) iff $A_{ij}^* = 1$. One can show that:

$$A^* = I_n \vee A \vee A^2 \vee A^3 \vee \dots \vee A^{n-1}$$

This result follows from the fact that any path which, so to speak, touches more than n nodes of the graph G , should touch at least twice a node of G . We also have that:

$$A^* = (I_n \vee A)^{n-1}. \quad (\#1)$$

This result follows from the fact that $A^* = I_n \vee A \vee A^2 \vee A^3 \vee \dots \vee A^{n-1}$ and $I_n \vee A \vee A^2 \vee A^3 \vee \dots \vee A^{n-1} = (I_n \vee A)^{n-1}$ because any path p of length k , with $0 \leq k \leq n-1$, can be viewed as a path of length $n-1$ which is the concatenation of the given path p of length k and a path of length $(n-1)-k$ from the final node of p to that same final node.

As a consequence of Equation (#1), the matrix A^* can be computed by successive squaring of $(I_n \vee A)$ for $\lceil \log_2(n-1) \rceil$ times. (For any real number x , by $\lceil x \rceil$ we denote the least integer m such that $m \geq x$.) This takes $T(n) \leq M(n) \lceil \log_2 n \rceil$ steps, where by $M(n)$ we denoted the time complexity for boolean matrix multiplication. In the following section we will see that the $\log_2 n$ factor is actually redundant, that is, $T(n) \leq M(n)$.

8.6. Reducing Transitive Closure to Boolean Matrix Multiplication

We have the following fact.

FACT 8.6.1. [Reducing Boolean Transitive Closure to Boolean Matrix Multiplication] Given a directed graph G with n nodes, let us consider the corresponding $n \times n$ adjacency, boolean matrix A , and its partition into four $(n/2) \times (n/2)$

blocks (assume that n is even): $A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$. We have that:

$$A^* = \begin{vmatrix} A_{11}^* \vee (A_{11}^* \wedge A_{12} \wedge E^* \wedge A_{21} \wedge A_{11}^*) & A_{11}^* \wedge A_{12} \wedge E^* \\ E^* \wedge A_{21} \wedge A_{11}^* & E^* \end{vmatrix} \quad (\#2)$$

where $E = (A_{21} \wedge A_{11}^* \wedge A_{12}) \vee A_{22}$.

PROOF. It is based on the interpretation of the matrix E and the four submatrices in the block partition of A^* via the diagrams of Figure 8.6.1 on the facing page and Figure 8.6.2 on the next page.

Let A_{11} denote a subgraph G_1 of G with $n/2$ nodes, and A_{22} denote the subgraph G_2 of G with the remaining $n/2$ nodes. In Figure 8.6.1 on the facing page and Figure 8.6.2 on the next page we use the following conventions:

- (i) a *solid* arrow \longrightarrow denotes an arc of the graph G ,
- (ii) a *solid double* arrow \Longrightarrow in the graph G_1 or G_2 denotes a path of any length (≥ 0) within the subgraph G_1 or G_2 , respectively, and
- (iii) a *dashed double* arrow \Longrightarrow denotes a path of any length (≥ 0) going anywhere through the graph G .

Thus, for instance, the left upper corner of A^* , that is, $A_{11}^* \vee (A_{11}^* \wedge A_{12} \wedge E^* \wedge A_{21} \wedge A_{11}^*)$, can be interpreted as a path going from a node in G_1 to a node in G_1 (see

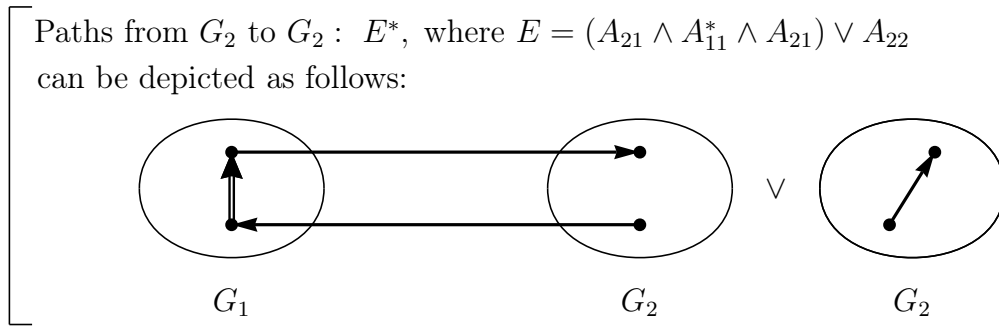


FIGURE 8.6.1. The set E of paths that go from the subgraph G_2 to the same subgraph G_2 . They provide an interpretation of the matrix $E = (A_{21} \wedge A_{11}^* \wedge A_{12}) \vee A_{22}$.

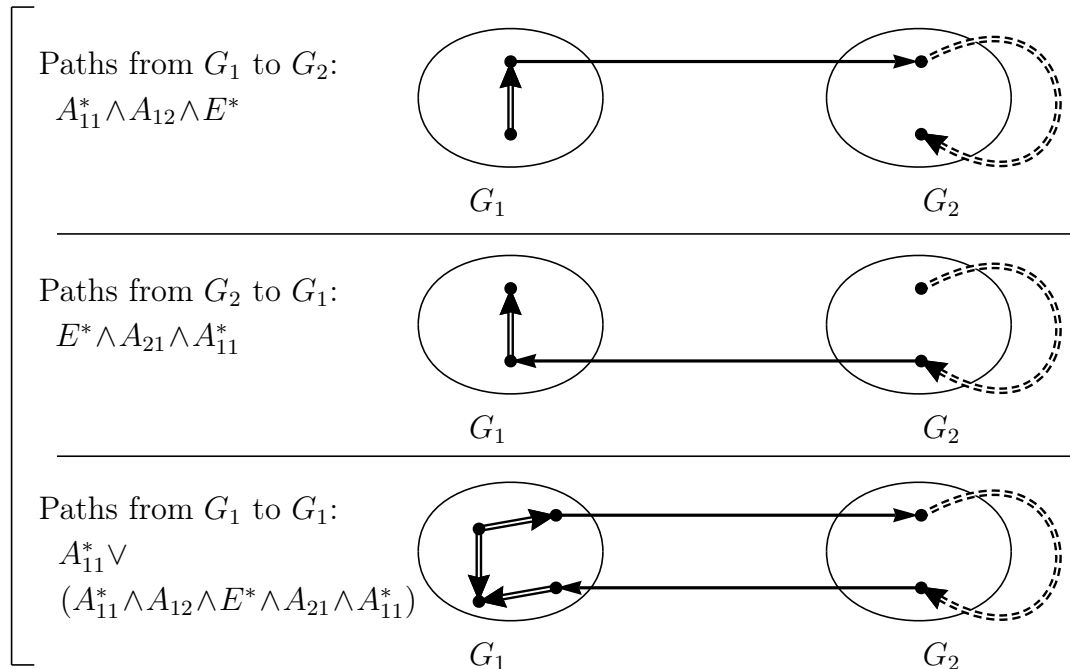


FIGURE 8.6.2. Three sets of paths that provide an interpretation of the three matrices $A_{11}^* \wedge A_{12} \wedge E^*$, $E^* \wedge A_{21} \wedge A_{11}^*$, and $A_{11}^* \vee (A_{11}^* \wedge A_{12} \wedge E^* \wedge A_{21} \wedge A_{11}^*)$, respectively.

also Figure 8.6.2). It is *either* entirely inside G_1 (see A_{11}^*) *or* it is inside G_1 (see A_{11}^*) and then it goes to G_2 (see A_{12}), it returns zero or more times to G_2 (see E^*), it returns to G_1 (see A_{21}), and finally, it continues entirely inside G_1 (see A_{11}^*). Analogous interpretation can be done for the other blocks of the matrix A^* (see again Figure 8.6.2). \square

FACT 8.6.2. Let $M(n)$ denote the time complexity for $n \times n$ boolean matrix multiplication. If $M(n) = O(n^\alpha)$ with $\alpha > 2$, then the time for computing the transitive closure of an $n \times n$ boolean matrix is $T(n) = O(n^\alpha)$.

PROOF. From Fact 8.6.1 on page 216 we get that:

$$T(n) \leq 2T(n/2) + 6M(n) + O(n^2).$$

Thus, $T(n) = O(n^\alpha)$ if $\alpha > 2$. The factor 6 comes from the following six matrix multiplications which are necessary for obtaining the elements of the matrix A^* :

$$\begin{array}{lll} M_1 = A_{11}^* \wedge A_{12} & M_2 = A_{21} \wedge A_{11}^* & M_3 = M_1 \wedge E^* \\ M_4 = M_3 \wedge M_2 & M_5 = E^* \wedge M_2 & M_6 = A_{21} \wedge M_1. \end{array}$$

If n is not a power of 2 then we can embed a given $n \times n$ boolean matrix, say A , into an $m \times m$ boolean matrix, say D , with m being a power of 2, with $m > n$, which has the form:

$$D = \left| \begin{array}{cc} A & 0 \\ 0 & I \end{array} \right|$$

where I is the $(m-n) \times (m-n)$ boolean identity matrix. Since m is not larger than $2n$ the time complexity will increase of at most 2^3 times, which is a constant factor. \square

From this Fact 8.6.2 on the previous page it follows that, in order to obtain a fast algorithm for computing the transitive closure of a graph, we simply need a fast algorithm for boolean matrix multiplication.

REMARK 8.6.3. The matrix A^* in Equation (#2) of the above Fact 8.6.1 on page 216 is *not* symmetric with respect to the partition of the given graph G into the two subgraphs G_1 and G_2 in the sense that the second row *cannot* be derived from the first row by interchanging the subscripts 1 and 2 and interchanging the two columns.

On the contrary, the following expression of the matrix A^* is symmetric with respect to that partition (and thus, in this expression of the matrix A^* the second row *can* be derived from the first row by interchanging the subscripts 1 and 2 and interchanging the two columns):

$$A^* = \left| \begin{array}{cc} D^* & D^* \wedge A_{12} \wedge A_{22}^* \\ E^* \wedge A_{21} \wedge A_{11}^* & E^* \end{array} \right| \quad (\#3)$$

where $D = (A_{12} \wedge A_{22}^* \wedge A_{21}) \vee A_{11}$ and $E = (A_{21} \wedge A_{11}^* \wedge A_{12}) \vee A_{22}$.

Note also that the value of E can be derived from the value of D by interchanging the subscripts 1 and 2. Using the symmetric expression (#3), the evaluation of A^* requires $T(n)$ time units where $T(n)$ satisfies the following inequation:

$$T(n) \leq 4T(n/2) + 6M(n) + O(n^2).$$

Thus, again, we have that $T(n) = O(n^\alpha)$ if $\alpha > 2$. The factor 4 comes from the four transitive closures we have to compute: E^* , D^* , A_{11}^* , and A_{22}^* . The factor 6 comes from the following six matrix multiplications we have to perform:

$$\begin{array}{lll} M_1 = A_{12} \wedge A_{22}^* & M_2 = M_1 \wedge A_{21} & M_3 = D^* \wedge M_1 \\ M_4 = A_{21} \wedge A_{11}^* & M_5 = M_4 \wedge A_{12} & M_6 = E^* \wedge M_4. \quad \square \end{array}$$

The proof of Fact 8.6.1 on page 216 is based only on the property that the boolean ic-semiring is indeed an ic-semiring. Thus, if we consider directed graphs with labels which are elements of a non-negative ic-semiring (see page 214) or a Regular Expressions ic-semiring (see page 215), we have that the statements analogous to Fact 8.6.1

on page 216 and Fact 8.6.2 on page 217 hold also for the matrices whose elements are non-negative reals or regular expressions.

The following fact shows that we can reduce the multiplication of two boolean matrices to the computation of the transitive closure of a boolean matrix.

FACT 8.6.4. [Reducing Boolean Matrix Multiplication to Boolean Transitive Closure] Let $T(n)$ denote the time complexity for computing the transitive closure of an $n \times n$ boolean matrix. If $T(n) = O(n^\alpha)$ with $\alpha > 2$, then the time complexity for the $n \times n$ boolean matrix multiplication is $M(n) = O(n^\alpha)$.

PROOF. It is easy to verify that

$$\begin{vmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{vmatrix}^* = \begin{vmatrix} I & 0 & A \wedge B \\ 0 & I & 0 \\ 0 & 0 & I \end{vmatrix}$$

Thus, in order to multiply two given $n \times n$ boolean matrices A and B , it is enough to compute the transitive closure of the $(3n) \times (3n)$ boolean matrix $T = \begin{vmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{vmatrix}$. It takes $O((3n)^\alpha)$ time. Now $O((3n)^\alpha)$ is equal to $O(n^\alpha)$, and this time complexity is the time complexity required for computing the transitive closure of an $n \times n$ boolean matrix. Having computed that transitive closure T^* of the matrix T , we then consider the top right corner of T^* (which is equal to $A \wedge B$). \square

8.7. Transitive Closure in IC-Semirings: the Shortest Path Problem

Given a labeled (or weighted) directed graph G with n nodes, the problem of finding the shortest paths in G consists in finding for each pair of nodes i and j , a path with minimum sum of the labels of its arcs. If the labels of G form a non-negative ic-semiring (see Definition 8.4.1 on page 214), we can solve this problem by considering the $n \times n$ matrices over that semiring as follows.

Given a labeled directed graph G with n nodes, we associate with G the $n \times n$ matrix $M(G)$ such that for all i and j , with $1 \leq i, j \leq n$, $M(G)_{ij} = \lambda(i, j)$, where $\lambda(i, j)$ is the label of the arc from node i to node j . We assume that for all i and j , with $1 \leq i, j \leq n$, $\lambda(i, j) = +\infty$ if there is no arc from node i to node j (even if $i = j$).

In what follows the matrix $M(G)$ will also be called M when the graph to which it refers, is understood from the context.

Then, for any two nodes i and j of G , we can find a path from i to j with the minimum sum of the labels of its arcs by applying the analogous of Fact 8.6.1 on page 216 for matrices over a non-negative ic-semiring.

Notice that in the identity matrix over the non-negative ic-semiring, each element of the main diagonal is 0 (which is the identity for $+$), while every other element is $+\infty$ (which is the identity for min). This definition of the identity matrix for the non-negative ic-semiring $(R^{\geq 0} \cup \{+\infty\}, min, +, +\infty, 0)$ is based on the fact that for all i and j , with $1 \leq i, j \leq n$,

(•) if $i = j$ then a shortest path from node i to node j is a path of length 0 whose sum of the labels is 0 and, otherwise,

(•) if $i \neq j$ and there is no path from node i to node j then a shortest path from i to j is assumed to have the sum of the labels equal to $+\infty$ (because from node i it is not possible to reach node j).

It can be shown that one can compute the shortest paths between every pair of nodes of a labeled directed graph G by computing the transitive closure of the matrix M over the non-negative ic-semiring of the labels of G .

We have the following fact which is analogous to Fact 8.6.2 on page 217.

FACT 8.7.1. Let $M(n)$ denote the time complexity for multiplying $n \times n$ matrices. If $M(n) = O(n^\alpha)$ with $\alpha > 2$, then the time for computing a shortest path between any two nodes in labeled direct graph whose labels are non-negative reals is $T(n) = O(n^\alpha)$.

EXAMPLE 8.7.2. Let us consider the labeled directed graph with three nodes depicted in Figure 8.7.1. We have that:

$$\min(I, M) = \begin{vmatrix} 0 & 8 & 5 \\ 3 & 0 & \infty \\ \infty & 2 & 0 \end{vmatrix} \quad \text{and} \quad (\min(I, M))^* = (\min(I, M))^{3-1} = \begin{vmatrix} 0 & 7 & 5 \\ 3 & 0 & 8 \\ 5 & 2 & 0 \end{vmatrix}. \quad \square$$

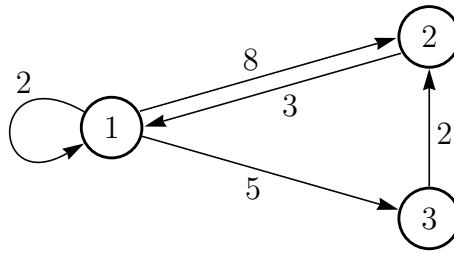


FIGURE 8.7.1. A directed graph with non-negative labels on the arcs.

We leave it to the reader to show that if the labels of the arcs of the given directed graph G are not taken from a non-negative ic-semiring, as for instance in the case of labels which may have negative values, the technique based on the computation of $(\min(I, M))^*$ for finding the shortest paths between every pair of nodes of G , is still correct, provided that there are no *negative cycles*, that is, cycles whose sum of the labels of the arcs is a negative value.

If there are negative cycles in the given graph, the technique based on the computation of $(\min(I, M))^*$ is *not* correct, and we have to apply other algorithms, like the Floyd-Warshall's algorithm or a variant of Algorithm 8.8.1 (Dijkstra's SSSP algorithm) which we will present on page 221. If there are negative cycles, the time complexity of those algorithms goes up to $O(n^3)$ [16, Chapter 6, page 133].

8.8. Single Source Shortest Paths in Directed Graphs

In this section we consider the problem of finding, given a labeled directed graph, the shortest paths starting from an initial node, also called the *source*, to all other nodes of the graph. This problem is called the *Single Source Shortest Path problem*. For solving this problem there exists an algorithm which is faster than the one based on the computation of the transitive closure of the adjacency matrix associated with the

graph, which we have considered in the previous Section 8.7. We will call this faster algorithm the *Single Source Shortest Path algorithm*, or *SSSP algorithm*, for short.

This algorithm is based on the Dynamic Programming idea. It is due to E. W. Dijkstra, as the algorithm we have presented in Section 7.5 for computing a minimal spanning tree of an undirected graph (see Algorithm 7.5.1 on page 201), and it takes $O(n^2)$ time for graphs of n nodes, if the labels on the arcs are taken from a non-negative ic-semiring. Dijkstra's SSSP algorithm is correct also for any labeled directed graph provided that there are no negative cycles, that is, there are no cycles with negative sum of the labels of their arcs.

Dijkstra's SSSP algorithm can be viewed as a particular instance of the *Primal-Dual algorithm* (see [16, Chapter 5]) and thus, its correctness can be derived from that of the Primal-Dual algorithm.

Note that between any two given nodes of the graph there could be more than one path with the minimal sum of the labels of its arcs. In this case Dijkstra's SSSP algorithm computes one of these minimal paths. However, by abuse of language, in what follows we will feel free to say 'the shortest path', instead of 'a shortest path'.

Now, for reasons of simplicity, we will present Dijkstra's SSSP algorithm in the case when the labels of the arcs are non-negative reals.

ALGORITHM 8.8.1.

Computation of the Single Source Shortest Paths in Directed Graphs. (Dijkstra's SSSP algorithm)

Input: We are given a labeled, directed, connected graph G with n nodes. We are also given a node n_0 of G , called the *source*. No assumptions are made on the arcs which depart from n_0 or arrive at n_0 .

For each arc $\langle p, q \rangle$, we assume that there exists a label $\lambda(p, q)$ which is a non-negative real number. That label is also called the *length* of the arc.

For each node p , we assume that there exists the arc $\langle p, p \rangle$ whose label $\lambda(p, p)$ is 0.

Output: for each node q in G , we compute a *shortest path* from node n_0 to node q , that is, a path from n_0 to q with minimal *length*. That path has minimal sum of the labels of its arcs, being the minimum taken over all paths from n_0 to q .

A. With each node $q \in G$ we associate a label $\Lambda(q)$ which is the length of a shortest path from n_0 to q .

1. $\Lambda(n_0) := 0$; $Reach := \{n_0\}$;
2. for each arc $\langle n_0, q \rangle$, $\Lambda(q) := \lambda(n_0, q)$;
3. Repeat

among all nodes not in $Reach$ with a label, take a node, say p ,
with <i>minimal</i> label $\Lambda(p)$;
3.1 $Reach := Reach \cup \{p\}$;
3.2 for each arc $\langle p, m \rangle$ with m not in $Reach$,
if q has already a label
then (3.2.1) $\Lambda(m) := \min\{\Lambda(p) + \lambda(p, m), \Lambda(m)\}$
else (3.2.2) $\Lambda(m) := \Lambda(p) + \lambda(p, m)$;

B. We find by backtracking a shortest path from node n_0 to a node $q \in G$.

4. Choose *any* node p , different from q , such that $\Lambda(p) = -\lambda(p, q) + \Lambda(q)$;
 5. Recursively, find a shortest path from n_0 to p .
-

The following points illustrate a few important facts about of the above SSSP algorithm.

- (i) The source node n_0 can be any node of the given graph G . In particular, there may be arcs going into the source node.
- (ii) At Point 4 if there exists more than one node p , different from q , such that $\Lambda(p) = -\lambda(p, q) + \Lambda(q)$, then we may choose *any* one of them. If this is the case, then there exists more than one path from n_0 to q with minimal length.
- (iii) If $\langle n_0, n_1, \dots, n_{k-1}, n_k \rangle$ is a shortest path from n_0 to n_k ,
then $\Lambda(n_k) = \sum_{i \geq 0}^{k-1} \lambda(n_i, n_{i+1})$.
- (iv) This SSSP algorithm is similar to Dijkstra's MST algorithm [6] for computing a minimal spanning tree of undirected graphs with labels on the arcs (see also Algorithm 7.5.1 on page 201). Analogously to Dijkstra's MST algorithm:
 - (•) the SSSP algorithm is a *greedy* algorithm, that is, the local optimum is a global optimum (recall what has been said for the MST algorithm on page 198),
 - (•) according to what we have said about the MST algorithm, the nodes in *Reach* may be viewed as *red* nodes and the nodes not in *Reach* may be viewed as *blue* nodes, and
 - (•) the SSSP algorithm, after transforming a blue node p into a red node (see Point 3.1), readjusts (see Point 3.2.1) or generates (see Point 3.2.2) the label of each blue node m such that there exists an arc $\langle p, m \rangle$.

Now we present the invariants of the loops of the above Dijkstra's SSSP Algorithm 8.8.1. Those invariants clarify the way in which that algorithm works and are the basis of the proof of its correctness which we leave to the reader.

The invariant of the *Repeat* statement at Point 3 is:

«the label of any node q in the set *Reach* is the length of a shortest path from the source node n_0 to q ».

The invariant at Point 3.2 is:

«each node m *not* in *Reach* such that there is an arc $\langle p, m \rangle$ with $p \in \text{Reach}$, has a label which is the length of a shortest path $\langle n_0, \dots, p, m \rangle$ from the source node n_0 to m such that the nodes n_0, \dots, p are all in *Reach*».

The invariant at Point 5 is:

«in a shortest path from the source node n_0 to the node m , the last arc is $\langle p, m \rangle$ ».

The correctness of Dijkstra's SSSP algorithm is based on the following property.

Let us assume that we have computed a shortest path from the source node n_0 to every node in a subset W of the nodes such that $n_0 \in W$. Let us consider a shortest path, say π , from the node n_0 to any node y not in W such that for every node x if

there is the arc $\langle x, y \rangle$ in G then $x \in W$. Then the path π has the property that it goes through nodes of W (except for y itself).

Note that if at Point 3 we take a node p whose label $\Lambda(p)$ is not *minimal*, the algorithm is *not* correct.

EXAMPLE 8.8.2. Given the graph of Figure 8.8.1 and the node n_0 , after Point 3 of Dijkstra's SSSP algorithm we get the 'boxed labels' at each node. Then, having labeled all nodes with the boxed labels, we may find *by backtracking* every arc whose label is exactly the difference of the boxed labels of its nodes. By doing so (see Points 4 and 5 of the SSSP algorithm), we get the following two sequences of values:

- (i) $\boxed{12} \xleftarrow{6} \boxed{6} \xleftarrow{1} \boxed{5} \xleftarrow{2} \boxed{3} \xleftarrow{3} \boxed{0}$ and
 (ii) $\boxed{12} \xleftarrow{3} \boxed{9} \xleftarrow{4} \boxed{5} \xleftarrow{2} \boxed{3} \xleftarrow{3} \boxed{0}$

which correspond to the following two shortest paths from node n_0 to node n_6 (see the thicker arcs of the graph of Figure 8.8.1):

- (i) $n_0 \longrightarrow n_2 \longrightarrow n_3 \longrightarrow n_5 \longrightarrow n_6$ and
 (ii) $n_0 \longrightarrow n_2 \longrightarrow n_3 \longrightarrow n_4 \longrightarrow n_6$, respectively.

The total length of each of those shortest paths is 12 which is, indeed, the boxed label of the final node n_6 . Note that by backtracking, from node n_6 we do *not* go back to node n_3 (because $12 - 9 \neq 5$) and, instead, we go back either to node n_4 (because $12 - 6 = 6$) or to node n_5 (because $12 - 3 = 9$). \square

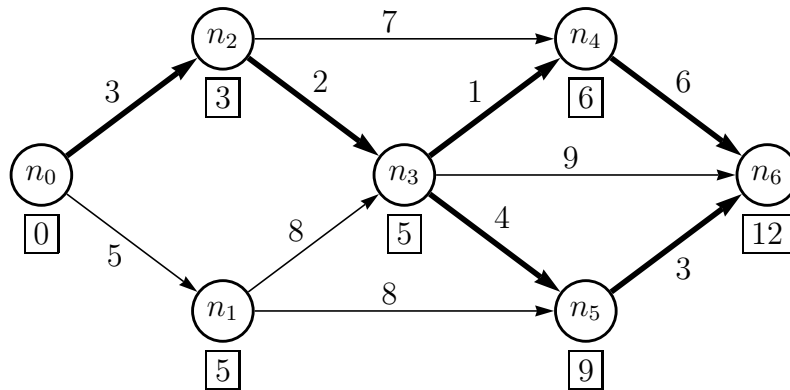


FIGURE 8.8.1. A directed graph with non-negative labels (also called lengths) on the arcs. The thicker arcs are those belonging to the two shortest paths of length 12 from node n_0 to node n_6 .

For a graph of n nodes the time complexity of Dijkstra's SSSP algorithm is $O(n^2)$ and this is due to the fact that: (i) the body of the *Repeat* statement (see Point 3) is performed at most n times because there are at most n nodes not in *Reach*, and

Worst Case Time Complexity	
Shortest paths from <i>any</i> node to <i>any</i> node: (the complexity is that of matrix multiplication)	$O(n^{2.81})$
Shortest paths from a <i>fixed</i> node to <i>any</i> node:	$O(n^2)$
Shortest path from a <i>fixed</i> node to a <i>fixed</i> node: (it is an open problem whether or not one can do better)	$O(n^2)$

FIGURE 8.8.2. Time complexity of the computation of the shortest paths in directed graphs with n nodes.

(ii) for each node p not in $Reach$ there exist at most n arcs of the form $\langle p, m \rangle$ with m not in $Reach$ (see Point 3.2).

Dijkstra's SSSP algorithm shows that if we fix the initial node and the labels of the given graph are taken from a non-negative ic-semiring, we may solve the Single Source Shortest Path problem in $O(n^2)$ time. This is an improvement over the time complexity for solving the shortest path problem between any two nodes. Indeed, as we have shown in Section 8.6 on page 216, this last problem requires the same amount of time which is needed for matrix multiplication, say $O(n^{2.81})$ time (see also Figure 8.8.2).

One may wonder whether or not it is possible to derive an even faster algorithm for the shortest path problem when we fix both the initial node and the target node. This is the so called *Single Source Single Target Shortest Path problem*. The answer, to our knowledge, is still open, in the sense that no algorithm has been found for this last problem which has a better time performance, in the worst case, than the best algorithm known for the Single Source Shortest Path problem which has an $O(n^2)$ time complexity for graphs of n nodes.

A final point is worth noticing. Let us consider an *undirected graph* with weights on its arcs as a particular case of directed graph, where for each arc from node x to node y with label w , there exists a symmetric arc from node y to node x with the same label w . In an undirected graph any pair of directed arcs from node x to node y and vice versa will be denoted by $x-y$. Now, given an undirected graph G with non-negative weights on its arcs, we have that a shortest path between two nodes may be made out of arcs none of which belongs to the minimal spanning tree of G . This fact is illustrated in the graph of Figure 8.8.3 on the facing page. In that figure we use the usual convention of drawing every pair of directed arcs of an undirected graph by a single arc without arrowheads. In Figure 8.8.3 a shortest path from node b to node c is the arc $b-c$ with weight 3, while a minimal spanning tree of the graph G has total weight 4 and it is made out of the arcs $a-b$ and $a-c$.

In what follows we present a Java program which implements Dijkstra's Single Source Shortest Path algorithm. It is made out of two files: `SSSP.java` and `Graph.java`, which are located in the same folder.

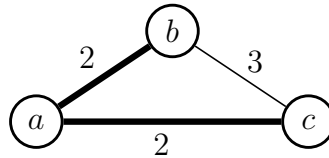


FIGURE 8.8.3. An undirected graph with non-negative labels on the arcs. The thicker arcs are those of the minimal spanning tree.

```

/**
 * =====
 *           Single Source Shortest Path: Main Program
 * Filename: "SSSP.java"
 *
 * This program computes a shortest path from a node to any other node
 * of a weighted, directed graph. The weight of an arc is the length of the
 * arc.
 *
 * In what follows we will feel free to say 'the shortest path' (or 'the
 * shortest distance'), instead of 'a shortest path' (or 'a shortest
 * distance'), even if the shortest path may not be unique.
 *
 * In the command:
 *   java SSSP nn in fn
 * the integers nn, in, and fn denote the number of nodes, the initial
 * node, and the final node, respectively.
 * For instance, we should issue the command: java SSSP 5 0 3
 * for a graph of 5 nodes in which we look for a shortest path from node 0
 * to node 3.
 * =====
 */

import java.util.ArrayList;
import java.util.Iterator;

public class SSSP {           // SSSP : Single Source Shortest Path

    public static int min (int a, int b) {
        if (a>b) {return b;} else {return a;}
    }

    public static void main(String[] args) {

        int N = 0;           // - to be read from the input: args[0]
        int initNode = 0;    // - to be read from the input: args[1]
        int finalNode = 0;   // - to be read from the input: args[2]

        // ----- N: number of Nodes of the graph.
        //                N >= 1.

        try {N = Integer.parseInt(args[0]);
            if ( N < 1) { throw new RuntimeException (); };
        } catch (Exception e)
            {System.out.print("*** Wrong number of nodes!\n");}

```

```

// ----- initNode: initial node.
//          0 <= initNode <= N-1.

    try {initNode = Integer.parseInt(args[1]);
        if ( ( 0 > initNode ) || ( initNode > N-1 ) ) {
            throw new RuntimeException ();
        };
    } catch (Exception e)
        {System.out.print("*** Wrong initial node!\n");}

// ----- finalNode: final node.
//          0 <= finalNode <= N-1.

    try {finalNode = Integer.parseInt(args[2]);
        if ( ( 0 > finalNode ) || ( finalNode > N-1 ) ) {
            throw new RuntimeException ();
        };
    } catch (Exception e) {System.out.print("*** Wrong final node!\n");}

// -----
// MAXLENGTH: maximum length of the arcs. 1 <= length <= MAXLENGTH.
//          final int MAXLENGTH = 9;

// -----
// PerCentPROB: probability of connectivity. 0 < PerCentPROB <= 100.
// If PerCentPROB = k, every node is connected with probability k/100 to
// the other N-1 nodes.
//          final int PerCentPROB = 40;

// -----
// Generation of a random graph with N nodes.
// The length of each arc belongs to the set {1,..., MAXLENGTH}.
// Every node has (N-1) x PerCentPROB / 100 successor nodes (in average).
    Graph g =
        new Graph(N, initNode, finalNode, MAXLENGTH, PerCentPROB);
    g.graphPrint();
    System.out.println("Every node is connected to the other nodes with " +
        "probability " + PerCentPROB + " percent.");
    System.out.println("The length of each arc belongs to the set {1,..., "
        + MAXLENGTH + "}");
    System.out.println("The initial node is: " + initNode);
    System.out.println("The final node is: " + finalNode);

// -----
    int p = g.nearestOutsideNode();

    while (p >= 0) { // <-----+
        g.reach[p] = true; // Reach = Reach U {p} // |
        for (int m=0; m<N; m++) { // |
            if (!g.reach[m] && g.graph[p][m] >= 0) { // |
                if (g.shortestDistances[m] >= 0) // |
                    {g.shortestDistances[m] = // |
                        min (g.shortestDistances[p] + g.graph[p][m], // |
                            g.shortestDistances[m]);} // |
                else {g.shortestDistances[m] = // |
                    g.shortestDistances[p] + g.graph[p][m];}; // |
            }; // |
        }; // |
        p = g.nearestOutsideNode(); // |
    } // end of while <-----+

```



```

// -----

// add(0,e) inserts the element e at position 0, and shifts
// the element at position 0 and all subsequent elements to the right
// (adds 1 to their indices).

g.itinerary.add(0, new Integer(finalNode));
int m = finalNode;
while (m != initNode) { // <<-----+
    int j=0; // |
    while ( j == m || g.shortestDistances[j] != // |
           -g.graph[j][m]+g.shortestDistances[m] ) { j++;}; // |
    g.itinerary.add(0, new Integer(j)); // |
    m=j; // |
}; // end of while <<-----+
g.shortestDistancesPrint();
g.itineraryPrint();
}
}

// -----

/**
 * input:
 * -----
 * javac SSSP.java
 * java SSSP 5 0 3
 *
 * output:
 * -----
 *
 * The random graph has 5 node(s) (from node 0 to node 4).
 * The matrix of the arc lengths is:
 * 0 : 0 - 9 - -
 * 1 : - 0 - - 7
 * 2 : - - 0 4 -
 * 3 : - - 6 0 -
 * 4 : - - 6 - 0
 *
 * Every node is connected to the other nodes with probability 40 percent.
 * The length of each arc belongs to the set {1, ..., 9}.
 * The initial node is: 0
 * The final node is: 3
 *
 * The shortest distances from the initial node 0 to every node of
 * the graph are ('-' means 'unreachable node'):
 * 0 - 9 13 -
 *
 * The nodes of the shortest path from node 0 to node 3 are:
 * 0 2 3
 *
 * The cumulative distances of the shortest path from node 0 to node 3 are:
 * 0 9 13
 *
 * -----
 */

```

```

/**
 * =====
 *                               The Graph class
 * Filename: "Graph.java"
 *
 * This class Graph is not a monitor, because not all fields are "private".
 * However, the non-private fields are read, but never assigned to.
 * All methods are "synchronized".
 * We can say that this class Graph is "almost" a monitor.
 *
 * In what follows we will feel free to say 'the shortest path' (or 'the
 * shortest distance'), instead of 'a shortest path' (or 'a shortest
 * distance'), even if the shortest path may not be unique.
 * =====
 */

import java.util.Random;
import java.util.ArrayList;
import java.util.Iterator;

public class Graph {

    final int N;           // N = number of nodes of the graph
    final int initNode;    // 0 <= initNode <= N-1
    final int finalNode;   // 0 <= finalNode <= N-1
    static int [][] graph; // an N x N array

    // shortestDistances is an array of N integers belonging to the object
    // of the class Graph

    static int [] shortestDistances;
    static boolean [] reach;
    static ArrayList itinerary = new ArrayList();

// ----- beginning of constructor -----

    public Graph(int N, int initNode, int finalNode, int MAXLENGTH,
                 int PerCentPROB) {

/** Generation of a random, directed graph with N nodes.
 * The random variable randomConnect controls the connectivity:
 *     node i is connected with node j, for i != j, with percent
 *     probability PerCentPROB: 0 <= PerCentPROB <= 100
 * The random variable randomLength controls the length of the arcs:
 *     1 <= length of an arc <= MAXLENGTH
 */

        this.N = N;
        this.initNode = initNode;
        this.finalNode = finalNode;
        this.graph = new int [N] [N];
        this.shortestDistances = new int [N]; // \Lambda(n) for each node n
        this.reach = new boolean [N]; // the Reach set
        Random randomConnect = new Random(); // random graph connectivity
        Random randomLength = new Random(); // random arc length

```

```

// initializing the graph: -1 (printed as "-") means "unconnected node"
// graph[i][j] = m with m >= 0
// iff m is the length of the arc from node i to node j
for(int i=0;i<N;i++) {
    for(int j=0;j<N;j++) {
        if ((i != j) && (randomConnect.nextInt(100) < PerCentPROB))
            { graph[i][j] = 1 + randomLength.nextInt(MAXLENGTH); }
        else if (i == j) { graph[i][j] = 0; }
        else { graph[i][j] = -1; };
    }
}

// initializing the array shortestDistances:
// -1 (printed as "-") means "unreached node".
// shortestDistances[i] = d with d >= 0
// iff d is the shortest distance from the initial node to node i
// computed so far.

for(int j=0;j<N;j++) shortestDistances[j] = graph[initNode][j];

// initializing the array reach
// -1 (printed as "-") means "node not in the array reach"
// initially

for(int j=0;j<N;j++) reach[j] = false; reach[initNode] = true;

// initializing the shortest path, called 'itinerary',
// from initNode to finalNode

    ArrayList itinerary = new ArrayList();
}
// ----- end of constructor -----

// ----- updating the array shortestDistances -----
public synchronized void shortestDistancesUpdate (int i, int distance) {
    shortestDistances[i] = distance;
}

// ----- accessing the array shortestDistances -----
public synchronized int shortestDistancesAt (int i) {
    return shortestDistances[i];
}

// ----- finding a p node -----
// if node == -1 there is no such nearest node outside the array reach.
public synchronized int nearestOutsideNode () {
    int min = -1; int node = -1;
    for(int j=0; j<N; j++)
        { if (reach[j] == false && shortestDistances[j] >= 0
            && min == -1)
            { node = j; min = shortestDistances[j]; }
          else
            if (reach[j] == false && shortestDistances[j] >= 0
                && shortestDistances[j] < min)
                { node = j; min = shortestDistances[j]; };
        };
    return node;
}

```

```
// ----- printing the graph -----

public synchronized void graphPrint () {
    System.out.println("The random graph has "+ N +" node(s) " +
        "(from node 0 to node " + (N-1) + ").\n" +
        "The matrix of the arc lengths is:");

    for(int i=0;i<N;i++) {
        System.out.print(i + " : ");
        for(int j=0;j<N;j++) {
            if (graph[i][j] < 0) { System.out.print("- "); }
            else System.out.print( graph[i][j] + " ");
        }; System.out.println();
    }; System.out.println();
}

// ----- printing the array shortestDistances -----

public synchronized void shortestDistancesPrint () {
    System.out.println("\nThe shortest distances from the initial node "
        + initNode + " to every node of\nthe graph " +
        "are ('-' means 'unreachable node'):");

    for(int i=0;i<N;i++) {
        if (shortestDistances[i] < 0) { System.out.print("- "); }
        else System.out.print( shortestDistances[i] + " ");
    }; System.out.println();
}

// ----- printing the array reach -----

// This method is NOT needed. This method is useful for debugging.

public synchronized void reachPrint () {
    System.out.print("\nThe reach array is: \n");
    for(int i=0;i<N;i++) {
        if (!reach[i]) { System.out.print("- "); }
        else System.out.print("1 ");
    };
    System.out.println();
}

// ----- printing the shortest path from initNode to finalNode ---

// the shortest path from initNode to finalNode is called 'itinerary'
//
public synchronized void itineraryPrint () {
    if ( shortestDistances[finalNode] >= 0 ) {
        System.out.println("\nThe nodes of the shortest path from node "
            + initNode + " to node "+finalNode + " are:");

        for (Iterator iter = itinerary.iterator(); iter.hasNext(); ) {
            System.out.print((iter.next()).toString() + " ");
        };
        System.out.println();
    }
}
```

```

// -----
System.out.println("\nThe cumulative distances of the shortest "
                  + "path from node "
                  + initNode + " to node "+finalNode + " are:");

int d = 0; int i = initNode; int j = 0;

for (Iterator iter1 = itinerary.iterator(); iter1.hasNext(); ) {
    j = ((Integer)iter1.next()).intValue();
    d = d + graph[i][j]; i = j;
    System.out.print(d + " ");
};
System.out.println();
// -----
}
else { System.out.println("There is no path from node " + initNode +
                          " to node " + finalNode + ".");
};
}
}
// -----

```

8.9. From Nondeterministic Finite Automata to Regular Expressions

Given a directed graph G of n nodes with every arc labeled by a regular expression over an alphabet Σ (in particular, a single symbol of Σ), let us consider the problem of finding for each pair of nodes i and j in G , the regular expression which denotes the language made out of all words which take from node i to node j as we now specify. A word w takes from node i to node j iff (i) w is $a_0 \dots a_{m+1}$, where for all $i = 0, \dots, m$, $a_i \in \Sigma$, and (ii) there is a path $\langle n_0, n_1, \dots, n_{m+1} \rangle$ such that: (ii.1) $n_0 = i$, (ii.2) $n_{m+1} = j$, and (ii.3) for all $i = 0, \dots, m$, a_i belongs to the language denoted by the regular expression which is the label of the arc $\langle n_i, n_{i+1} \rangle$.

This problem of finding regular expressions can be solved by applying the analogous of Fact 8.6.1 on page 216 for matrices over the Regular Expressions ic-semiring (see page 215).

Given a directed graph G with n nodes with regular expressions as labels of the arcs, let us associate with G the $n \times n$ matrix $R(G)$ such that for all i, j , with $1 \leq i, j \leq n$, $R(G)_{ij} = \text{expr}(i, j)$, where $\text{expr}(i, j)$ is the regular expression which labels the arc from node i to node j .

For all i, j (not necessarily distinct), with $1 \leq i, j \leq n$, we have that $\text{expr}(i, j) = \emptyset$ if there is no arc from node i to node j .

It can be shown that, given any pair $\langle i, j \rangle$ of nodes of the graph G , one may compute the language made out of the words which take from i to j by computing the transitive closure of the matrix $R(G)$ over the Regular Expressions ic-semiring of the labels of G .

Notice that in the identity matrix over the Regular Expressions ic-semiring each element of the main diagonal is $\{\varepsilon\}$ (which is the identity for \cdot), while every other element is \emptyset (which is the identity for \cup). This definition of the identity matrix is based on the facts that:

- (1) for all i and j with $1 \leq i, j \leq n$, there is no path of one or more arcs from a node i to a node j , and
- (2) *if* $i = j$ *then* node i is reachable from node i itself via a path of length 0 (and this explains why $\{\varepsilon\}$ is in the main diagonal of the identity matrix) *else* node j is not reachable from node i (and this is why \emptyset is outside the main diagonal of the identity matrix).

For the case of matrices over the Regular Expressions ic-semiring it is not natural to establish a fact analogous to Fact 8.6.2 on page 217, because the operations \cup and \cdot cannot reasonably be considered as atomic computation steps. However, by using a fact analogous to Fact 8.6.1 on page 216, we can prove Kleene Theorem which states that for every finite automaton there exists an equivalent regular expression, that is, a regular expression which denotes the language accepted by the finite automaton. Indeed, if we assume that each element of the matrices A_{11} , A_{12} , A_{21} , and A_{22} is a symbol in Σ , the graph is a (deterministic or nondeterministic) finite automaton.

Let that automaton be called A . The transitive closure of $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$ computes the language from a node to any other node by using the operations \cdot , $+$, and $*$, instead of the operations \wedge , \vee , and $*$ occurring in the statement of Fact 8.6.1. Thus, in particular, we get a regular expression from the initial state to any final state of the finite automaton A . The regular expression denoting the language accepted by A is the sum of all regular expressions from the initial state to a final state of A .

String Matching

In this chapter we present two matching algorithms for testing whether or not a pattern occurs in string. The first algorithm is a naive algorithm (see page 233) whose Prolog implementation will be given in Section 9.2 on page 243. The second algorithm is the Knuth-Morris-Pratt algorithm [14].

9.1. Knuth-Morris-Pratt Pattern Matching

In this section we present a pattern matching algorithm taken from [14]. This algorithm detects all occurrences, possibly overlapped, of a substring p , called the *pattern*, in a string s , called the *subject*.

The main idea of this algorithm is the construction of a finite automaton. We will clarify this point in what follows. The time complexity of this algorithm is $O(m+n)$, where m is the length of the string s and n is the length of the substring p .

First we present the naive pattern matching algorithm which has time complexity $O(nm)$. It can be expressed as follows.

ALGORITHM 9.1.1. Naive Pattern Matcher.

Input: the *subject* string $s[0], \dots, s[n-1]$ and the *pattern* string $p[0], \dots, p[m-1]$. We assume that $n \geq 1$ and $m \geq 1$.

Output: a list $\lambda = [i_1, \dots, i_k]$ of non-negative integers such that for all i in λ , we have that p occurs in position i in the subject s , that is, for all $j = 0, \dots, m-1$ we have that $s[i+j] = p[j]$.

$\lambda := [];$

for $i = 0, \dots, n-m$ **do**

$occurs := true;$	
for $j = 0, \dots, m-1$ do if $s[i+j] \neq p[j]$ then $occurs := false;$	
if $occurs$ then $\lambda := append(\lambda, [i])$	(†)

In the instruction (†) of the external for-loop we have used the *append* function, instead of the less expensive *cons* function, because we want that a pattern occurrence i_1 which is located in the subject string on the left of another pattern occurrence i_2 , that is, $i_1 < i_2$, should be listed in λ on the left of i_2 .

This naive pattern matcher works by repeating, for each position in the subject string, a test which checks whether or not the substring of the subject string starting at that position coincides with the given pattern.

We can improve over this naive pattern matcher algorithm by using the Knuth-Morris-Pratt pattern matcher which we now present. This algorithm improves the

naive pattern matcher because, if during the process of checking an occurrence of the pattern in the subject we find a mismatch between the character of the pattern and the character of the subject string, then we resume the matching process *not* by a right-shift of the pattern by one position only, but by performing a right-shift of r (≥ 1) positions as indicated by the so called *failure function* which we will introduce below.

This failure function determines the value of r by taking into account the structure of the pattern and by doing so, it allows us to reduce the total number of character comparisons during the matching process.

The Knuth-Morris-Pratt pattern matcher is made out of two parts which have a very similar structure:

- (i) computation of the failure function, and
- (ii) computation of the occurrences of the pattern.

ALGORITHM 9.1.2. *Knuth-Morris-Pratt Pattern Matcher.*

Input: the subject string $s[0], \dots, s[n-1]$ and the pattern string $p[0], \dots, p[m-1]$. We assume that $n \geq 1$ and $m \geq 1$.

Output: a list $\lambda = [i_1, \dots, i_k]$ of non-negative integers such that for all i in λ , we have that the pattern p occurs at position i in the subject s , that is, for all $j = 0, \dots, m-1$ we have that $s[i+j] = p[j]$.

Computation of the failure function: $\pi[0], \dots, \pi[m-1]$.

```

 $\pi[0] := 0;$ 
 $j := 0;$ 
for  $i = 1, \dots, m-1$  do
    while  $j > 0$  and  $p[j] \neq p[i]$  do  $j := \pi[j-1];$ 
    if  $p[j] = p[i]$  then  $j := j+1;$ 
     $\pi[i] := j;$ 

```

(1.α)
(1.β)
(1.γ)
(1.δ)

Computation of the occurrences of the pattern: $p[0], \dots, p[m-1]$.

```

 $\lambda := [];$ 
 $j := 0;$ 
for  $i = 0, \dots, n-1$  do
    while  $j > 0$  and  $p[j] \neq s[i]$  do  $j := \pi[j-1];$ 
    if  $p[j] = s[i]$  then  $j := j+1;$ 
    if  $j = m$  then begin  $\lambda := \text{append}(\lambda, [i-m+1]);$   $j := \pi[j-1]$  end

```

(2.α)
(2.β)
(2.γ)
(2.δ)

Now we introduce the failure function π . It is represented as an array $\pi[0, \dots, m-1]$ which has the same length m as the pattern p . We first need the following notions.

DEFINITION 9.1.3. [**Prefix and Suffix**] Let us consider a pattern $p[0, 1, \dots, m-1]$ of length m . The non-empty *prefixes* of that pattern are: $p[0]$, $p[0, 1]$, \dots , and $p[0, 1, \dots, m-1]$. For $j = 0, \dots, m-1$, the prefix $p[0, \dots, j]$ is also denoted by p_{0j} . A *proper prefix* of that pattern is a prefix which is different from the pattern itself.

The *suffixes* of the pattern $p[0, 1, \dots, m-1]$ are: $p[0, 1, \dots, m-1]$, $p[1, \dots, m-1]$, \dots , $p[m-2, m-1]$, $p[m-1]$. A *proper suffix* of that pattern is a suffix which is different from the pattern itself.

Given the pattern $p[0, 1, \dots, m-1]$, for $j = 0, \dots, m-1$, we have that the prefix p_{0j} is the sequence of the $j+1$ leftmost characters of the pattern. The following tables shows the prefixes of the pattern $p[0, 1, \dots, m-1]$. For $j = 0$, the prefix p_{00} is the leftmost character $p[0]$ of the pattern.

	prefixes p_{0j} of the pattern $p[0, 1, \dots, m-1]$	length of the prefix
$j = 0$	$p_{00} = p[0]$	1
$j = 1$	$p_{01} = p[0, 1]$	2
\dots	\dots	\dots
$j = m-1$	$p_{0m-1} = p[0, 1, \dots, m-1]$	m

DEFINITION 9.1.4. [**Failure Function**] Given a pattern p of length m , its *failure function* π is defined for each $j = 0, \dots, m-1$, and $\pi[j]$ is the length of the longest *proper prefix* of p_{0j} which is also a *proper suffix* of p_{0j} .

We have that:

$$\begin{aligned} \pi[0] &= 0 \\ \pi[1] &= \text{if } p[0] = p[1] \text{ then } 1 \text{ else } 0 \end{aligned}$$

If we consider the pattern $ababa$, whose length m is 5, we get the failure function which is represented in the following table where, for $j = 0, \dots, m-1$, the value of $p[j]$ is written in column j of the row named ‘*pattern p*’, and the value of $\pi[j]$ is written in column j of the row named ‘*failure function π* ’:

	$p[0]$	$p[1]$	$p[2]$	$p[3]$	$p[4]$
<i>pattern p:</i>	a	a	b	a	a
<i>failure function π:</i>	0	1	0	1	2
	$\pi[0]$	$\pi[1]$	$\pi[2]$	$\pi[3]$	$\pi[4]$

Let us explain the entries of the above table. We have that:

- (i) p_{00} is a and the longest proper prefix of a which is also a proper suffix of a , is ϵ and the length of ϵ is 0. Thus, $\pi[0] = 0$.
- (ii) p_{01} is aa and the longest proper prefix of aa which is also a proper suffix of aa , is a and the length of a is 1. Thus, $\pi[1] = 1$.
- (iii) p_{02} is aab and the longest proper prefix of aab which is also a proper suffix of aab , is ϵ and the length of ϵ is 0. Thus, $\pi[2] = 0$.
- (iv) p_{03} is $abaa$ and the longest proper prefix of $abaa$ which is also a proper suffix of $abaa$, is a and the length of a is 1. Thus, $\pi[3] = 1$.
- (v) p_{04} is $abaaa$ and the longest proper prefix of $abaaa$ which is also a proper suffix of $abaaa$, is aa and the length of aa is 2. Thus, $\pi[4] = 2$.

If all characters in the pattern p are equal, that is, $p[0] = p[1] = \dots = p[m-1]$, then the failure function π is such that: $\pi[0] = 0, \pi[1] = 1, \pi[2] = 2, \dots, \pi[m-1] = m-1$. Thus, if we assume that the pattern p is a sequence of a 's, we get the following table, where for $j = 0, \dots, m-1$, the j -th column indicates that $p[j] = a$ and $\pi[j] = j$:

	$p[0]$	$p[1]$	$p[2]$	\dots	$p[m-1]$
<i>pattern p:</i>	a	a	a	\dots	a
<i>failure function π:</i>	0	1	2	\dots	$m-1$
	$\pi[0]$	$\pi[1]$	$\pi[2]$	\dots	$\pi[m-1]$

The failure function derived at the end of the first part of the Knuth-Morris-Pratt algorithm, called *Computation of the failure function*, is used during the second phase, called *Computation of the occurrences of the pattern*, as we now indicate.

Let us consider the above pattern $abaa$ and suppose that we have found the matching substring $abaa$ and we get a mismatch on the last character which should be a . That is, $s[i] \neq a$. The value of j is 4. This mismatch tells us that we have to consider a new value for j which is $\pi[j-1]$ (see instruction (2. β)). In our case we get $j=1$. Then, we check whether or not the current character in the subject string is a (because $p[1]$ is a). We fail again and we get the new value of j which is 0. Indeed, according to the instruction (2. β), we have to take as new value of j the value of $\pi[j-1]$, that is, $\pi[0]$ which is 0. At this point it is not the case that $j > 0$, we exit the loop of instruction (2. β), and we perform instruction (2. γ).

Since $p[0] = a$ and $s[i] \neq a$ and $j \neq m$, we increase i by 1 (which corresponds to a shift of the pattern on the subject string by one character) and we execute again the instructions (2. β), (2. γ), and (2. δ) with this new value of i (actually, in our case, since $j=0$, the instruction (2. β) will not be executed).

The failure function for a pattern $p[0]p[1] \dots p[m-1]$ can be represented as a finite automaton whose states are $0, 1, \dots, m-1, m$ (see Figure 9.1.1 on the next page for the failure function for the pattern $abaa$). For each $k = 0, \dots, m-1$, there is a *forward arc* from state k to state $k+1$ labeled by the predicate ' $= p[k]$ '. For each $k = 1, \dots, m$, there is a *backward arc* from state k to state $\pi[k-1]$ labeled by the predicate ' $\neq p[k]$ '. The label of the backward arc from state m to state $\pi[m-1]$ is labeled by the predicate *true*. From state 0 there is an arc to state 0 itself labeled by ' $\neq p[0]$ '.

This automaton is used as an acceptor of the subject string in the second part of the Knuth-Morris-Pratt algorithm called *Computation of the occurrences of the pattern*, according to the following rule:

if for the input subject string s the automaton gets from the initial state 0 to the final state m , *then* there is an occurrence of the pattern p in the subject string s .

The predicate in each arc of the automaton allows the transition iff it is satisfied by the current character of the subject string.

For the pattern $abaa$ of length $m=5$ the automaton which represents the failure function is depicted in Figure 9.1.1 on the facing page. For the pattern $ababaaa$

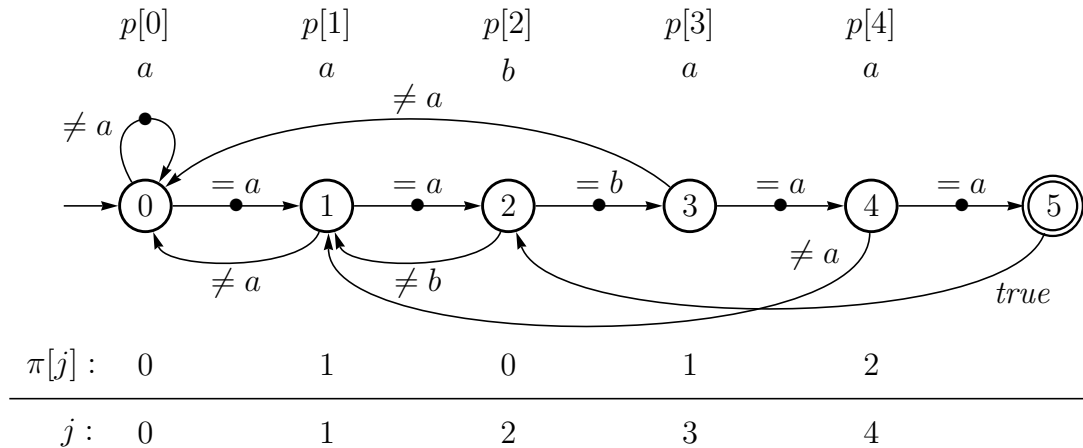


FIGURE 9.1.1. The finite automaton corresponding to the failure function for the pattern $abaa$. We have that $m=5$. When the characters of the subject string are fed into this automaton for finding the occurrences of the pattern, we consider the next character in input only when we make a transition with the dot \bullet .

of length $m = 7$ the automaton which represents the failure function is depicted in Figure 9.1.2 on the next page.

At each state the automaton which represents the failure function, reads the character at hand and that character is used for making the transition to the next state. As already stated, in the figures representing the failure functions as finite automata, the failure function is used for drawing the backward arcs according to the following rules:

- (i) for all i, j , with $i \geq j$, if $\pi[i-1]=j$ then there is a backward arc from state i to state j , and
- (ii) there is a backward arc from state 0 to itself labeled by the negation of the label of the arc from state 0 to state 1.

For instance, in Figure 9.1.1 we have drawn a backward arc from state 4 to state 1 because $\pi[4-1]=1$.

Now we would like to make the following two remarks.

REMARK 9.1.5. When an occurrence of the pattern has been found (see, for instance, the arc labeled by *true* from state 5 to state 2 in the automaton of Figure 9.1.1), the automaton representing the failure function makes a state transition as if a failure occurred. Indeed, the pattern occurrence we have found and the next pattern occurrence may overlap.

REMARK 9.1.6. Some of the tests which the failure function forces us to make, can be avoided because we know in advance the results of these tests. Let us consider, for instance, the case of the failure function represented by the automaton of Figure 9.1.1. If in state 4 we have a character of the subject which is different from a we go to state 1 where the test ‘ $= a$ ’ is bound fail and we have to go to state 0. By avoiding all

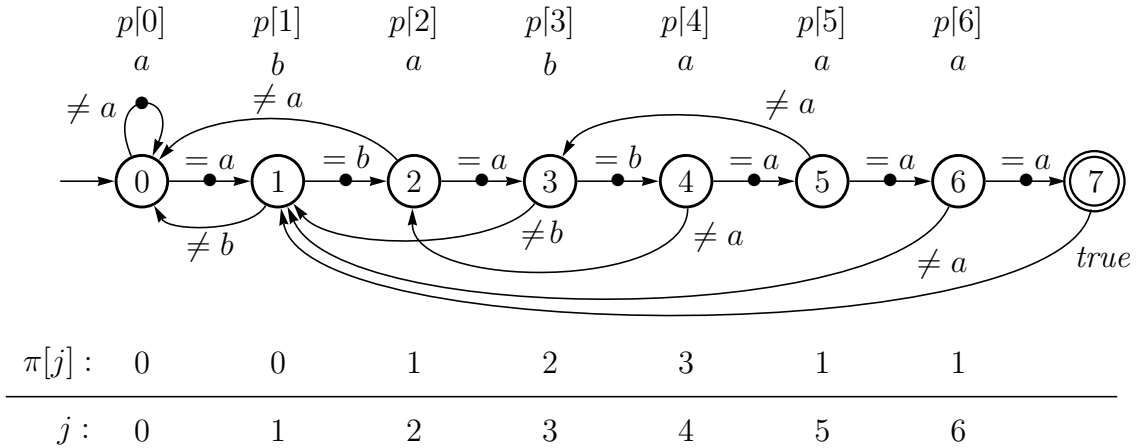


FIGURE 9.1.2. The finite automaton corresponding to the failure function for the pattern $ababaaa$. We have that the length m of the pattern is 7. When the characters of the subject string are fed into this automaton for finding the occurrences of the pattern, we consider the next character in input only when we make a transition with the dot \bullet .

redundant tests such as the test ‘ $= a$ ’, we get the automaton of Figure 9.1.3, instead of the automaton of Figure 9.1.1 on the preceding page.

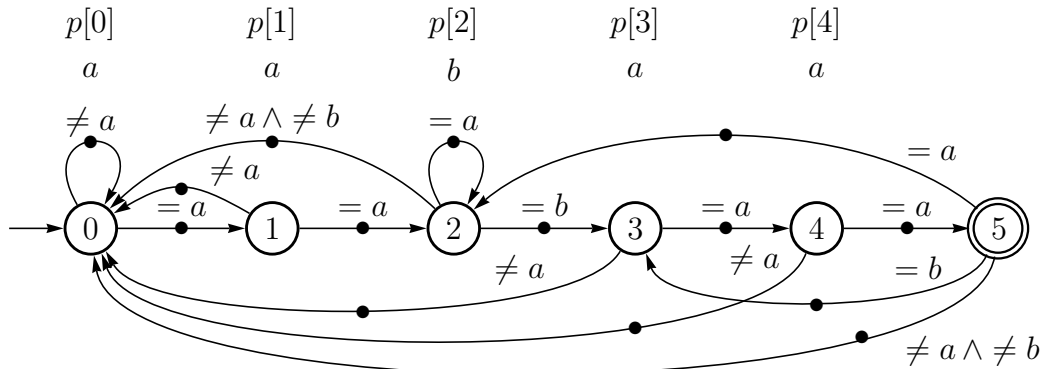


FIGURE 9.1.3. The finite automaton corresponding to the failure function for the pattern $aabaa$. When the characters of the subject string are fed into this automaton for finding the occurrences of the pattern, we always consider the next character in input (indeed, all transitions have the dot \bullet).

Note, however, that if we use the automaton of Figure 9.1.3, instead of the one of Figure 9.1.1, we do not improve the asymptotic time complexity of the Knuth-Morris-Pratt algorithm (see Section 9.1.1 on the next page).

It is interesting to notice that the first part of the Knuth-Morris-Pratt algorithm which computes the failure function, that is, the length of the longest proper prefix

of the pattern which is also a proper suffix of pattern, is very similar to the second part of the algorithm which computes the occurrences of the pattern in the subject string. The only differences between the two parts are the following ones.

- (i) The for-do loop in instruction (1.α) starts with $i = 1$ and not $i = 0$, because for every pattern the value of $\pi[0]$ is 0 and thus, we can skip the loop for $i = 0$.
- (ii) We record the fact that an occurrence of the pattern has been found, by inserting in the list λ the initial position of the occurrence of the pattern in the subject string (see instruction (2.δ)). Then we continue our matching process by looking for a new, possibly overlapping, occurrence of the pattern, and we do so as if a matching failure did occur. Indeed, as in the case of a failure, we update the value of j and the new value of j is set to $\pi[j - 1]$.

We have that the Knuth-Morris-Pratt algorithm for the input subject

$$s = a b a c a b a a b a a$$

with length $n = 13$ and the input pattern

$$p = a b a a$$

computes the list [5, 8] of occurrence positions (see Figure 9.1.4).

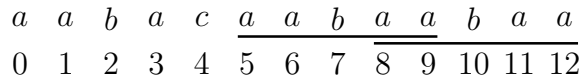


FIGURE 9.1.4. The two occurrences at positions 5 and 8 of the pattern $a b a a$ in the subject string $a b a c a b a a b a a$.

9.1.1. Time Complexity Analysis of the Knuth-Morris-Pratt Algorithm.

We start off by evaluating the time complexity for computing the failure function. We will measure this time complexity in terms of the number of the assignments to the variable j .

The for-loop of the computation of the failure function is performed from $i = 1$ to $i = m - 1$. The while-loop at instruction (1.β) in the body of the for-loop is controlled by the value of j which may be at most $m - 1$ (and this case occurs when the characters in the pattern are all equal) and at each execution of the body of the loop, as we will prove in Lemma 9.1.7 below, the value of j can only be decreased. Thus, we conclude that the time complexity of the computation of the failure function is $O(m^2)$.

The following analysis, based on an amortized analysis, shows that the cost of the computation of the failure function is, in fact, $O(m)$. First we prove the following lemma.

LEMMA 9.1.7. For all $i = 1, \dots, m - 1$, for all $h = 0, \dots, i$, $0 \leq \pi[h] \leq h$ and $0 \leq j \leq i$.

PROOF. By induction.

(Basis: $i = 1$). We want to show that after the execution of instruction (1.δ) in the first iteration of the for-loop, that is, for $i = 1$, we have that: for $h = 0, 1$, $0 \leq \pi[h] \leq h$ and $0 \leq j \leq 1$. Indeed, since initially $j = 0$, instruction (1.β) is not performed. Thus, (i) by (1.γ) we have that if $p[0] = p[1]$ then $j = 1$ else $j = 0$, and (ii) by (1.δ) we have

that $\pi[1] = j$. By recalling that: (i) $\pi[0] = 0$, and (ii) if $p[0] = p[1]$ then $\pi[1] = 1$, we get, as desired, that $0 \leq \pi[0] \leq 0$, $0 \leq \pi[1] \leq 1$, and $0 \leq j \leq 1$.

(*Step: $i > 1$*) We assume that for all $h = 0, \dots, i$, $0 \leq \pi[h] \leq h$ and $0 \leq j \leq i$ holds for $i = k$ and we want to show that for all $h = 0, \dots, i$, $0 \leq \pi[h] \leq h$ and $0 \leq j \leq i$ holds for $i = k+1$. By using Hoare's triples we have to show the validity of the following triple:

$$\begin{aligned} & \{ \text{for } h=0, \dots, k, 0 \leq \pi[h] \leq h \text{ and } 0 \leq j \leq k \} \\ & \quad \mathbf{while } j > 0 \text{ and } p[j] \neq p[k+1] \mathbf{ do } j := \pi[j-1]; & (1.\beta) \\ & \quad \mathbf{if } p[j] = p[k+1] \mathbf{ then } j := j+1; & (1.\gamma) \\ & \quad \pi[k+1] = j & (1.\delta) \\ & \{ \text{for } h=0, \dots, k+1, 0 \leq \pi[h] \leq h \text{ and } 0 \leq j \leq k+1 \} \end{aligned}$$

We have the following two points.

Point (1). At instruction (1. β) the value of the variable j never increases and may only decrease or remain unchanged. Indeed, by the precondition, we have that for $h = 0, \dots, k$, $0 \leq \pi[h] \leq h$ and thus, for $h' = 1, \dots, k+1$, $0 \leq \pi[h'-1] \leq h' - 1$. As a consequence, for $h' = 1, \dots, k+1$, $0 \leq \pi[h'-1] < h'$ and the assignment $j := \pi[j-1]$, which is performed only if j is such that $1 \leq j \leq k$ and $p[j] \neq p[k+1]$ (see the precondition $0 \leq j \leq k$ and the test of the while-loop), decreases the value of j .

Point (2). At instruction (1. γ) the variable j increases by at most one unit.

From Points (1) and (2) and the precondition $0 \leq j \leq k$, after instruction (1. δ) we get that: $0 \leq j \leq k+1$.

In order to get the postcondition it remains to show that for $h = 0, \dots, k+1$, we have that $0 \leq \pi[h] \leq h$. By the precondition and the fact that for $h = 0, \dots, k$, the values of $0 \leq \pi[h] \leq h$ are not changed, we get that for $h = 0, \dots, k$, $0 \leq \pi[h] \leq h$. Moreover, we have that $0 \leq \pi[k+1] \leq k+1$, because at instruction (1. δ) $\pi[k+1]$ is assigned the value of j and we have already proved that $0 \leq j \leq k+1$. This completes the proof of the lemma. \square

During the computation of the failure function, j is initially 0 and it may be increased by at most $m-1$ units during the $m-1$ iterations of the body of the for-loop. During the execution of the while-loop at instruction (1. β) we have that j is decreased (by Lemma 9.1.7 on the previous page). Thus, during the whole computation of the failure function the variable j is assigned at most $2(m-1)$ times, that is, the time complexity of the failure function is $O(m)$.

Since the first part of the Knuth-Morris-Pratt algorithm which compute the failure function, has the same structure of the second part which computes the occurrences of the pattern, in order to measure the time complexity of the Knuth-Morris-Pratt algorithm we can measure the computation of two failure functions, one for a pattern of length m and one for a pattern of length n . We conclude that the overall time complexity of the Knuth-Morris-Pratt algorithm is $O(m+n)$.

Note that the time complexity is still $O(m+n)$ if we measure the number of tests performed by the Knuth-Morris-Pratt algorithm, instead of the number of assignments to j .

Indeed, for all values of j , we make an assignment to j after at most two tests, which are: (i) $j > 0$, and (ii) $p[j] \neq p[i]$. We do not have to count the test $p[j] = p[i]$

because if we know the value of the test $p[j] \neq p[i]$ we also know the value of the test $p[j] = p[i]$.

It is easy to see that the same time complexity of $O(m+n)$ for the Knuth-Morris-Pratt algorithm holds if we measure the sum of the number of the assignments to j and the number of tests.

9.1.2. Java Implementation of the Knuth-Morris-Pratt Algorithm.

Below we present a Java program which implements to Knuth-Morris-Pratt algorithm for pattern matching. It runs under the Java compiler is 1.5.0-13 and Mac OS X. In this program, instead of constructing the list λ of positions, we have printed out the position of every occurrence of the pattern as soon as it has been computed.

The test case presented in the final comments of the program shows the values of the variables i and j during the execution of the Knuth-Morris-Pratt algorithm for the subject *aabacaabaabaa* and the pattern *aabaa* which occurs at positions 5 and 8.

```
/**
 * =====
 *                               KNUTH-MORRIS-PRATT PATTERN MATCHER
 * Filename: "KMP.java"
 *
 * =====
 */

public class KMP {

    private static void printar(int[] array, int size) {

        for (int k=0; k<size; k++) {System.out.print(array[k]+" ");}

        System.out.println();
    }

    /**
     * -----
     *                               main
     * -----
     */

    public static void main(String[] args) {

        boolean traceon = true;    // traceon: tracing variable

        String s, p;                // s: given subject
        int sl,pl;                  // p: pattern to search

        s = "aabacaabaabaa";
        p = "aabaa";

        sl = s.length();
        pl = p.length();

        System.out.println("\nThe given subject string s of " + sl +
            " elements is:\n " + s);
    }
}
```

```

System.out.println("\nThe given pattern p of "+pl+" elements is:\n "+ p);

/**
 * -----
 * Recall that indexing in Java begins with 0 (as in C++). Thus,
 *
 *      stp=abcab  length(stp)=5  stp.charAt(2) is c.
 * index:      01234
 *
 * lstp1=4
 * -----
 */

/**
 * -----
 *                               computing the failure function pi
 * -----
 */
int [] pi = new int [pl]; // the length of pi is the length of the pattern
int i, j;

pi[0]=0;
j=0;
for (i=1; i<pl; i=i+1) { // <<-----+
    while (j>0 && p.charAt(j)!=p.charAt(i)) {j=pi[j-1];}; // |
    if (p.charAt(j)==p.charAt(i)) {j=j+1}; // |
    pi[i]=j; // |
}; // end of for // <<-----+

//-----

System.out.print(
    "\nThe failure function is given by the following array pi" +
    "\nof " + pl + " elements:\n ");
printar(pi,pl); System.out.println();

//-----

/**
 * -----
 *                               finding the occurrences of the pattern
 * -----
 */

j=0;
for (i=0; i<sl; i=i+1) { // <<-----+
    while (j>0 && p.charAt(j)!=s.charAt(i)) {j=pi[j-1];}; // |
    if (p.charAt(j)==s.charAt(i)) {j=j+1}; // |
    if (tracoon) {System.out.println(" i="+i" j="+j);}; // |
    if (j==pl) { // |
        System.out.println("The pattern occurs at position "+(i-pl+1)+"\n"); // |
        j=pi[j-1]; // |
    }; // |
}; // end of for // <<-----+
} // end of main
}

```



```

/**
 * input:      output:  (with traceon == true)
 * -----
 * javac KMP.java
 * java  KMP
 *
 *           The given subject string s of 13 elements is:
 *           aabacaabaabaa
 *
 *           The given pattern p of 5 elements is:
 *           aabaa
 *
 *           The failure function is given by the following array pi
 *           of 5 elements:
 *           0 1 0 1 2
 *
 *           i=0 j=1
 *           i=1 j=2
 *           i=2 j=3
 *           i=3 j=4
 *           i=4 j=0
 *           i=5 j=1
 *           i=6 j=2
 *           i=7 j=3
 *           i=8 j=4
 *           i=9 j=5
 *           The pattern occurs at position 5
 *
 *           i=10 j=3
 *           i=11 j=4
 *           i=12 j=5
 *           The pattern occurs at position 8
 * -----
 */

```

9.2. String Matching in Prolog

The following Prolog program tests whether or not a pattern P (viewed as list of characters) occurs in a string S (also viewed as a list of characters). The pattern P occurs in S iff in S there exists a consecutive list of characters which is a substring of S and is equal to P .

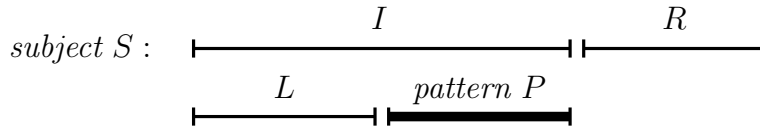
In order to understand the way in which this program works it is enough to recall the usual definition of the *append* predicate, that is, $append(Xs, Ys, Zs)$ holds iff the string Zs is the concatenation of the strings Xs and Ys . For instance, we have that:

$append([2, 3], [4, 2], Zs)$ holds iff $Zs = [2, 3, 4, 2]$
 $append(Xs, [4, 4], [4, 3, 4, 4])$ holds iff $Xs = [4, 3]$.

String Matching in Prolog: basic version.

1. $occur(P, S) \leftarrow append(I, R, S), append(L, P, I)$
2. $append([], Ys, Ys) \leftarrow$
3. $append([X|Xs], Ys, [X|Zs]) \leftarrow append(Xs, Ys, Zs)$

We have that $occur(P, S)$ holds iff the pattern P occurs in the string S . Indeed, clause 1 says that P occurs in S iff there exist three lists of characters I (short for *Intermediate*), R (short for *right*), and L (short for *left*) such that both $append(I, R, S)$ and $append(L, P, I)$ hold. We can depict this situation as in the following figure, where the list I is the concatenation of the lists L and P , and the list S is the concatenation of the lists I and R .



Note that, although the position of the atoms does not influence the logical meaning of a clause, it may influence its operational behaviour. In particular, due to the implementation of the *append* predicate in Prolog, the position of the atoms $append(I, R, S)$ and $append(L, P, I)$ in the body of clause 1 should not be interchanged because, otherwise, the computation may not terminate.

From the length of the list L we can get the position of the pattern P in the subject string S .

Then the various positions in which the pattern occurs in the subject, can be obtained by using the backtracking facility available in Prolog. This can be done by using the *findall* predicate as indicated in the following program.

String Matching in Prolog: finding all occurrence positions.

1. $occur(P, S, N) \leftarrow append(I, R, S), append(L, P, I), length(L, N)$
2. $append([], Ys, Ys) \leftarrow$
3. $append([X|Xs], Ys, [X|Zs]) \leftarrow append(Xs, Ys, Zs)$
4. $length([], 0).$
5. $length([H|T], N1) \leftarrow length(T, N), N1 \text{ is } N+1$

Recall that in Prolog $findall(N, Goal, Ns)$ returns a list Ns which represents the multiset of all instances of the variable N such that $Goal$ is satisfied. The variable N should appear within the clause only in the term $Goal$. By submitting the query:

$\leftarrow findall(N, occur([a b a a], [a b a c a a b a a b a a], N), Ns)$

we get the value $[5, 8]$ for the variable Ns (see also Figure 9.1.4 on page 239).

CHAPTER 10

Appendices

10.1. Simple Prolog Programs and Parsing Sentences in Prolog

In this section we present a few simple Prolog programs for: (i) checking natural numbers, (ii) sorting strings, and (iii) parsing strings.

Checking Natural Numbers.

The following program tests whether or not a given term X is a natural number in the set $\{0, s(0), s(s(0)), \dots\}$. X is a natural number iff $nat(X)$. The following two clauses define the unary predicate $nat(X)$.

- | |
|----------------------------------|
| Natural Numbers |
| 1. $nat(0) \leftarrow$ |
| 2. $nat(s(X)) \leftarrow nat(X)$ |

For instance, we have that the query $\leftarrow nat(s(s(s(0))))$ evaluates to true.

Sorting Strings.

The following program sorts a given list of natural numbers. Each natural number is an element of the set $\{0, 1, 2, \dots\}$. This program is a very slow program because it constructs all possible permutations of the elements of the given list and then it outputs the permutation which is ordered. Since there are $n!$ distinct permutations of n distinct elements, this program has worst case time complexity of order $O(n!)$.

Recall that: (i) $2^n = O(n!)$ and (ii) $n! = O(n^n)$, while (iii) $n! \neq O(2^n)$ and (iv) $n^n \neq O(n!)$.

- | |
|--|
| Permutation Sort |
| 1. $sort(L1, L2) \leftarrow permutation(L1, L2), sorted(L2)$ |
| 2. $permutation([], []) \leftarrow$ |
| 3. $permutation(L, [H T]) \leftarrow$
$append(U, [H V], L),$: place H in a position in W
$append(U, V, W),$: break W into U and V
$permutation(W, T)$: take a permutation W of T |
| 4. $sorted(L) \leftarrow sorted1(0, L)$ |
| 5. $sorted1(N, []) \leftarrow$ |
| 6. $sorted1(N, [H T]) \leftarrow N \leq H, sorted1(H, T)$ |

For instance, we have that the query $\leftarrow sort([3, 2, 2, 4], L2)$ evaluates to true with the variable $L2$ bound to $[2, 2, 3, 4]$.

Now we present a different Prolog program for sorting a given list of natural numbers. This program is based on the Quicksort algorithm, and it is much faster than the previous program, because its worst case time complexity is of order $O(n^2)$, where n is the length of the list.

<ol style="list-style-type: none"> 1. $partition(H, [A X], [A Y], Z) \leftarrow A \leq H, partition(H, X, Y, Z)$ 2. $partition(H, [A X], Y, [A Z]) \leftarrow A > H, partition(H, X, Y, Z)$ 3. $partition(H, [], [], []) \leftarrow$ 4. $quicksort([], []) \leftarrow$ 5. $quicksort([H T], S) \leftarrow partition(H, T, A, B),$ $quicksort(A, A1), quicksort(B, B1),$ $append(A1, [H B1], S)$ 	Quicksort
--	-----------

Parsing Sentences.

The following programs show how to parse sentences generated by context-free grammars. We follow the approach described in [5, Chapter 9]. Here we will not present the general parsing methodology and the reader may refer to [5, Chapter 9] for more information on this topic.

Let us consider the context-free grammar G with axiom *sentence* and the following productions:

<i>sentence</i>	\rightarrow	<i>noun-phrase verb-phrase</i>
<i>noun-phrase</i>	\rightarrow	<i>determiner noun</i>
<i>verb-phrase</i>	\rightarrow	<i>verb noun-phrase</i>
<i>verb-phrase</i>	\rightarrow	<i>verb</i>
<i>determiner</i>	\rightarrow	<i>the</i>
<i>noun</i>	\rightarrow	<i>apple man</i>
<i>verb</i>	\rightarrow	<i>eats sings</i>

Here is a simple Prolog program for parsing strings of the language generated by that grammar.

<ol style="list-style-type: none"> 1.1 $sentence(X) \leftarrow append(Y, Z, X), noun_phrase(Y), verb_phrase(Z)$ 1.2 $noun_phrase(X) \leftarrow append(Y, Z, X), determiner(Y), noun(Z)$ 1.3 $verb_phrase(X) \leftarrow append(Y, Z, X), verb(Y), noun_phrase(Z)$ 1.4 $verb_phrase(X) \leftarrow verb(X)$ 1.5 $determiner([the]) \leftarrow$ 1.6 $noun([apple]) \leftarrow$ 1.7 $noun([man]) \leftarrow$ 1.8 $verb([eats]) \leftarrow$ 1.9 $verb([sings]) \leftarrow$ 	Parsing Sentences. Program P1
--	-------------------------------

If we want to parse, for instance, the string ‘*the man eats the apple*’, we have to submit to the above Prolog program *P1* the following query:

```
← sentence([the, man, eats, the, apple])
```

By evaluating that query we get the empty clause, and the trace of that evaluation gives us the derivation tree of the given string ‘*the man eats the apple*’.

Here is an alternative Prolog program for parsing strings of the language generated by the grammar *G* given above. It runs faster than Program *P1*, because it avoids the use of *append* and uses, instead, the so called *difference lists* [24, page 283].

Parsing Sentences (Use of difference lists). Program *P2*

```
2.1 sentence(S0, S) ← noun_phrase(S0, S1), verb_phrase(S1, S)
2.2 noun_phrase(S0, S) ← determiner(S0, S1), noun(S1, S)
2.3 verb_phrase(S0, S) ← verb(S0, S1), noun_phrase(S1, S)
2.4 verb_phrase(S0, S) ← verb(S0, S)
2.5 determiner([the|S], S) ←
2.6 noun([apple|S], S) ←
2.7 noun([man|S], S) ←
2.8 verb([eats|S], S) ←
2.9 verb([sings|S], S) ←
```

For this program if we want to parse the string ‘*the man eats the apple*’, we have to submit the following query:

```
← sentence([the, man, eats, the, apple], [])
```

Now we present a third Prolog program for parsing strings of the language generated by the same context-free grammar *G* given above. This program uses the so called *definite clause grammars* (or *DCG*’s) which, indeed, were introduced for making it easier the specification and the solution of parsing problems [5, Section 9.1]. Rules 3.1–3.9 below encode the rules of the given context-free grammar. We write each rule using the arrow ‘*-->*’ to separate the left hand side from the right hand side, and we write a dot at the right end of each rule.

Parsing Sentences (Use of definite clause grammars). Program *P3*

```
3.1 sentence      --> noun-phrase, verb-phrase.
3.2 noun-phrase  --> determiner, noun.
3.3 verb-phrase  --> verb, noun-phrase.
3.4 verb-phrase  --> verb.
3.5 determiner   --> [the].
3.6 noun         --> [apple].
3.7 noun         --> [man].
3.8 verb         --> [eats].
3.9 verb         --> [sings].
```

If we want to parse the string ‘*the man eats the apple*’, we have to submit the following query:

```
← sentence([the, man, eats, the, apple], [])
```

If we also want to construct the parse tree of the sentence being parsed, we need the following program which uses a definite clause grammar with arguments:

Parsing Sentences with Parse Trees (Use of definite clause grammars). Program P4

```
4.1 sentence(st(NP, VP)) --> noun-phrase(NP), verb-phrase(VP).
4.2 noun-phrase(np(D, N)) --> determiner(D), noun(N).
4.3 verb-phrase(vp(V, NP)) --> verb(V), noun-phrase(NP).
4.4 verb-phrase(vp(V)) --> verb(V).
4.5 determiner(d(the)) --> [the].
4.6 noun(n(apple)) --> [apple].
4.7 noun(n(man)) --> [man].
4.8 verb(v(eats)) --> [eats].
4.9 verb(v(sings)) --> [sings].
```

If we want to parse the string ‘*the man eats the apple*’ and we want to construct the parse tree of that string, we have to submit the following query:

```
← sentence(X, [the, man, eats, the, apple], [])
```

and we get the following parse tree represented as a term bound to the variable X (see also Figure 10.1.1):

```
X = st(np(d(the), n(man)), vp(v(eats), np(d(the), n(apple))))
```

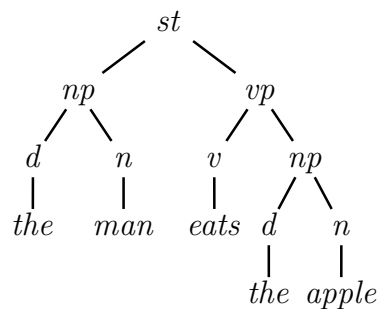


FIGURE 10.1.1. The parse tree of the string ‘*the man eats the apple*’. □

10.2. Decidability Results for $LL(k)$ and $LR(k)$ Grammars and Languages

In this section we present some decidability and undecidability results about the $LL(k)$ languages and the $LR(k)$ languages (see also [10, Section 8.5]). Some other decidability and undecidability results concerning the context-free languages have been reported in [21].

It is undecidable whether or not given a context-free grammar G there exist $k \geq 0$ and an $LR(k)$ grammar G_1 such that G and G_1 generate the same language (see [12] and [3, pages 397–399, Exercise 5.2.12]).

This undecidability result can be derived as follows. We have that: (i) the class of the $LR(0)$ languages is contained in the class of the $LR(1)$ languages (which is the class of deterministic context free languages), and (ii) for any $k > 0$ the class of the $LR(k)$ languages is equal to the class of the $LR(1)$ languages. Thus, undecidability holds because, otherwise, it would have been decidable the problem of knowing whether or not a context free grammar is equivalent to a deterministic context-free grammar.

It is undecidable whether or not given a context-free grammar G and a number $k \geq 0$, G equivalent to an $LR(k)$ grammar [12, pages 397–399 Exercise 5.2.12]. For $k > 0$ we can reason as follows. Since for any $k > 0$, the class of the $LR(k)$ languages is equal to the class of the $LR(1)$ languages (which is the class of deterministic context-free languages), this problem is undecidable for $k > 0$ because otherwise, it would have been decidable the problem of testing whether or not a context-free grammar generates a deterministic context-free language.

It is undecidable whether or not given a context-free grammar G there exists $k \geq 0$ such that G is an $LR(k)$ grammar [23, page 399].

We have, however, the following result.

THEOREM 10.2.1. (i) It is decidable whether or not given $k \geq 0$ and a context-free grammar G , G is an $LL(k)$ grammar. (ii) It is decidable whether or not given $k \geq 0$ and a context-free grammar G , G is an $LR(k)$ grammar.

The two decidable problems of this Theorem 10.2.1 are solvable in nondeterministic 1-exponential time [23, page 399] when k is given in binary, that is, they are solvable in $O(2^{p(n)})$ time by a nondeterministic Turing Machine, where $p(n)$ is a polynomial on the size of the input grammar G (that is, the sum of the number of the characters occurring in its productions) plus the size of k written in binary.

It is undecidable whether or not given a context-free grammar G , there exists $k \geq 0$ such that G is an $LL(k)$ grammar (see [22, Theorem 11] and [23, page 399]). However, it is decidable whether or not given an $LR(h)$ grammar G , for some $h \geq 0$, there exists $k \geq 0$ such that G is an $LL(k)$ grammar [22, Theorem 12] and [4, page 690]. (In [3, page 397] it is incorrectly stated that the latter problem is undecidable.)

It is undecidable whether or not given a context-free grammar G , there exist $k \geq 0$ and an $LL(k)$ grammar G_1 such that $L(G) = L(G_1)$ (see [22, Theorem 13] and [3, page 361]).

It is undecidable whether or not given a context-free grammar G and a number $k \geq 0$, there exists an $LL(k)$ grammar $G1$ such that $L(G) = L(G1)$ ([**22**, Theorem 13] and [**3**, page 361]).

Given two $LL(k)$ grammars $G1$ and $G2$, for some $k \geq 0$, it is decidable whether or not $L(G1) = L(G2)$ [**3**, page 362]. Note that if $G1$ is an $LL(k)$ grammar, for some $k \geq 0$, and $G2$ is an $LL(h)$ grammar, for some $h \geq 0$, we can consider both grammars $G1$ and $G2$ to be $LL(\max(k, h))$ grammars.

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