THE SCHRÖDINGER REPRESENTATION ON HILBERT BUNDLES

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ABSTRACT

We construct a representation of the path groupoid over the group of diffeomorphisms of the base space of a Hilbert bundle (equipped with a unitary parallel transport). The Schrödinger representation (in Weyl form) appears as a *flat* and *measure-invariant* particular case of this representation. We prove that our representation induces a representation of Mensky's path group and that generators of the representation of one-parameter groups of the path group are given by the covariant derivative plus a volume correction. We prove that, as in the case of locally compact groups, the imprimitivity system of this representation is irreducible if and only if the diffeomorphisms group acts transitively on the manifold and the inducing representation is irreducible. In our case the inducing subgroup is the loop group and the inducing representation is the holonomy group of the parallel transport.

1. Introduction

The main goal of the present investigation is to construct a generalization of the Schrödinger representation, in bounded form, for the class of manifolds which are the natural enviroment for gauge theories (Hilbert bundles). We shall also prove an irreducibility criterium for this representation, analogue to Mackey's criterium for imprimitivity systems based on homogeneous spaces.

The deep insight provided by Mackey's generalization of the Schrödinger representation as a system of imprimitivity[17,7] comes from the fact that it relates the observables of a quantum system to the kinematical symmetries of its base space (hereinafter also called its *position manifold*).

All the attempts to generalize Mackey's approach (or its more recent C^*-algebraic variants) to position manifolds of a more general type had to face the difficulty that the natural symmetry groups for these manifolds are either *too small* (e.g. the group of global isometries of a Riemannian manifold) or *too big* (e.g. the group of all diffeomorphisms of a manifold). For this reason the theory could not go much beyond the framework of homogeneous spaces, which is too narrow for the applications to quantum physics.

The difficulties of constructing concrete examples (not to speak of a general classification theory) of unitary representations of non locally compact groups such as the group of diffeomorphisms of a manifold are well known and they are related to the non-existence of a satisfactory analogue of the Haar measure on such groups as well as to the non-existence of a natural exponential map onto these groups from the formal analogue of their Lie algebras, i.e. the Lie algebra of all the smooth vector fields on the manifold.

On the other hand, as soon as one tries to restrict oneself to sub-classes of vector fields, better behaved with respect to the exponential map (complete vector fields), one runs into difficulties, provided by well known counterexamples, which show that these vector fields are not even a linear space [12].

For this reason, the considerable information presently available on infinite dimensional Lie algebras, is essentially of an algebraic type and does not allow to construct a class of unitary representations sufficiently wide for the basic applications to physics.

A new approach [1] to the problem was based on the idea of constructing a representation, not of the Lie algebra of vector fields, but of the family of *flows* associated to them. If v is a vector field (all objects mentioned are smooth, unless otherwise is specified) on a manifold M, the flow associated to it, in the time interval $[s,t]$, shall be denoted by $T_v(s,t)$. Thus, by definition, $T_v(s,t) \in Diff(M)$ and for any point $x \in M$, the curve

$$X(z) := T_v(s,z)X \ ; \ z \in [s,t] \tag{1.1}$$

is the trajectory of the vector field starting at x at time s.

Notice that, in this approach, a flow is determined by a diffeomorphism of M *and* a time interval $[s,t]$. The family of flows, associated to a given family of vector fields, has a natural structure of *groupoid* (heuristically a groupoid is a group in which the compostion law is not defined for all pairs of elements).

This structure is defined by the prescription that the composition of the flow $T_v(z,s)$ with the flow $T_w(z',s')$ is well defined if and only if $s = z'$ and in this case it is given by the piecewise smooth flow $T(z,s')$, defined by

$$T(z,r) = \begin{cases} T_v(z,r) & \text{if } r \in [z,s] \\ T_w(s,r) & \text{if } r \in [s,s'] \end{cases} \tag{1.2}$$

which, intuitively, means that a trial particle would follow the trajectory of the v-field, in the time interval $[z,s]$, and the trajectory of the w-field in the time-interval $[s,s']$ (recall that $z' = s$)).

The groupoid obtained by applying this construction to the family of all vector fields, was called [1] the *symmetry groupoid* of the manifold M and it was suggested that the kinematical quantum observables associated to the manifold M arise from unitary representations of this symmetry groupoid through considerations similar to those used to deduce the form of standard kinematical observables (position, momentum, spin, ...) from symmetry arguments based on unitary representations of a locally compact Lie group (the symmetry group of the theory).

The first step to realize this program was to construct a family of concrete non-trival unitary representations of this symmetry groupoid and to study their generators. It was shown [1] that, if M is an orientable manifold and $\pi : H \to M$ is a Hilbert bundle over M (i.e. a fiber bundle with base M and fiber a Hilbert space), then to every unitary parallel transport on the bundle H, one can associate, in a natural way, a unitary representation of the symmetry groupoid of M acting on the space of square integrable (for any fixed volume form on M) cross-sections of the bundle H.

In the present paper we start from a generalization of this construction based on the following remark: To the flow $T_v(s,t)$ on M, we can associate the continuous oriented curve (cf.section(5))

$$r \in [s,t] \mapsto T_v(s,r) \in Diff(M) \tag{1.3}$$

on $Diff(M)$. Now, the family of all continuous curves on $Diff(M)$ has a natural structure of groupoid: the curve γ can be composed with the curve δ if the end point of γ is the starting point of δ; in this case their composition gives simply the curve which, as a set, coincides with the union of γ and δ (i.e. of their points) with an orientation determined by the fact that the points of γ *preceed* those of δ. We call this *the path groupoid* of M (see section (3) for a precise definition).

From the above considerations it is clear that the path groupoid includes the symmetry groupoid of M. It is therefore natural to try to extend to the path groupoid the above mentioned construction of the unitary representation of the symmetry groupoid.

We achieve this extension by means of a generalization to groupoids of the theory of induced representations, as described, e.g., by Warner [16]. In fact our results can be considered as a confirmation of Warner's statement [16]: "*...it is desirable to have at hand a geometrical interpretation of the inducing mechanism;... What is the point of the procedure? It appears to be this: once the... definition of induced representation has been formulated in the language of fiber bundles, important generalization of the entire process immediately suggest themselves...*"

Having constructed a class of unitary representations of the path groupoid, acting on the space of square integrable sections of the bundle H, we prove that if V is any such representation and γ any continuous curve on $Diff(M)$, the unitary operator V_γ does not depend on γ itself, but only on its equivalence class (with respect to a relation described in section (3)).

The quotient groupoid of the path groupoid by this equivalence relation is shown to be a group, and in fact it is nothing but the *path group* introduced by M.Mensky,[9] starting from a completely different point of view. In fact Mensky had also considered, in some particular cases the construction of the unitary representation mentioned above.

REMARK. One should be aware that we are using the words "path group" and "loop group" with a meaning different from that of Pressley and Segal [13] where the product is defined pointwise.

2. Unitary induced representations and Hilbert G-bundles

In this section we shortly recall the geometric version [15,16] of the induced representation and outline its equivalence to the usual way of introducing it.

In what follows M shall denote an orientable manifold, ω a fixed volume form on M and μ the associated measure on M. Let G be a topological group acting transitively on M by diffeomorphisms. For $g \in G$ there is a unique function $c(g, \cdot)$ on M characterized by

$$(g^*\omega)_x = c(g,x)\omega_x \qquad g \in G\,, \quad x \in M$$

$$(g^*w)_x(v_1, \dots, v_n) := w_{g(x)}(g_{*x}(v_1), \dots, g_{*x}(v_n))$$

The function $c(g, \cdot)$ is related to the change of variable formula by:

$$\int_M f(g(x))|c(g,x)|d\mu(x) = \int_M f(x)\, d\mu(x)$$

In the following, unless otherwise is specified, all the bundles we are going to consider will be ***Hilbert bundles***, i.e. bundles whose fiber is a Hilbert space. To every Hilbert bundle with base M, one associate the Hilbert space of square-integrable cross-sections:

$$\mathcal{H} := L^2(M, \mu; H) =$$

$$= \{f : M \to H : \pi(f(x)) = x, \quad \forall x \in M, \quad \int_M \|f(x)\|^2{}_{H_x}\, d\mu(x) < +\infty\}$$

DEFINITION (2.1) Let $H \xrightarrow{\pi} M$ be a Hilbert bundle. An action of G on H is called ***admissible*** if:

i) H is a G–space and $gH_x = H_{gx} \quad \forall x \in M$;

ii) $g : H_x \to H_{gx}$ is a unitary operator. A fiber bundle, with an admissible action on it of a group G, is called a *G-bundle*.

The *induced representation* $V : g \in G \to V_g \in Un(\mathcal{H})$ is defined by

$$(V_g f)(x) = |c(g^{-1}, x)|^{1/2} g f(g^{-1}x) \qquad f \in \mathcal{H} \tag{2.2}$$

REMARK (2.1). Notice that, in the above definition, the group structure of G plays no role at all: up to now G can be *any set* with an admissible action on H. In particular the induced representation (2.2), can be defined also if G is a groupoid and in this case it will be a representation of the groupoid.

Starting from an induced representation defined as above the usual way to introduce the induced representation can be recovered as follows: fix an arbitrary $x_0 \in M$ and consider its isotropy group $K_{x_0} \subseteq G$. Since H is a G-bundle then K_{x_0} acts on the fiber H_{x_0}, in fact if $k \in K_{x_0}$ then $kH_{x_0} = H_{kx_0} = H_{x_0}$. The action of K_{x_0} on H_{x_0}, denoted by L, is called the ***inducing representation*** and the induced representation V is usually denoted by $V_g{}^L$. The *imprimitivity system* associated to V^L is by definition the pair (V^L, P^L) where P^L is the projection valued measure on $\mathcal{H}$ defined by

$$(P_B{}^L f)(x) = \chi_B(x) f(x), \qquad f \in \mathcal{H}, \quad B \in Borel(M) \tag{2.3}$$

where χ_B denotes the characteristic function of B, i.e. $\chi_B(x) = 0$ if $x \in B$ and $= 1$ if $x \notin B$. From (2.2) and (2.3) one easily deduces that the following commutation relation holds for any $g \in G$ and $B \in Borel(M)$

$$V_g{}^L P_B{}^L = P_{gB}{}^L V_g{}^L \tag{2.4}$$

Equivalently (2.4) can be written in the form

$$V_g{}^L P_B{}^L V_{g^{-1}}{}^L = P_{gB}{}^L \tag{2.5}$$

which evidentiates the *covariance* of the measure P^L under the action V^L of the group G. If $M = \mathbf{R}$ with the Lebesgue measure and $G = \mathbf{R}$, acting by traslations on itself, then the above construction can be applied and the covariance condition (2.4) is equivalent to Weyl's formulation of Heisenberg commutation relations.

Summing up: to every induced representation of a group G, as defined geometrically at the beginning of this Section, one can associate a closed subgroup $K(= K_{x_0})$ of G and a unitary representation L of K. Conversely, to any such a pair, one can associate an induced representation, as defined geometrically, with the following procedure. Given a closed subgroup K of a group G and a unitary representation L of K, acting on a Hilbert space H one defines a right action of K on $G \times H$ by:

$$(g, v)k := (gk, L_k{}^{-1} v) \qquad g \in G; k \in K; v \in H$$

Let $E := (G \times H)/K$ denote the orbit space of this action and let $[(g, v)]$ be the equivalence class of $(g, v) \in G \times H$. Defining the manifold M by

$$M := G/K$$

and the projection map $\pi : E \to M$ by

$$\pi([(g, v)]) = gK \in M = G/K$$

the bundle

$$E = (G \times H)/K \xrightarrow{\pi} M = G/K$$

becomes a Hibert G-bundle with the action of G given by:

$$g_0([(g, v)]) = [(g_0 g, v)] \qquad g, g_0 \in G; \quad v \in H$$

The above construction establishes a one-to-one correspondence between Hilbert G–bundles and unitary representations of closed subgroups K of G.

3. Groupoids of curves and their actions

Let X be a topological space. A ***parameterized oriented curve on*** X, shortly ***a curve on*** X, is a continuous function $\gamma : [a,b] \to X$ where $a, b \in R$. Let $\Gamma(X)$ denote the family of curves on X. If $\gamma : [a,b] \to X$ is such a curve, we denote

$$i_\gamma = \gamma(a) = \text{initial point of } \gamma; \qquad \varphi_\gamma = \gamma(b) = \text{final point of } \gamma \tag{3.1}$$

Composition between two curves $\gamma : [a,b] \to X$; $\delta : [b',c'] \to X$ is defined if and only if

$$b = b'; \qquad \gamma(b) = \delta(b')$$

In this case the composite curve is denoted by $\delta \circ \gamma$ and is defined by

$$(\delta \circ \gamma)t = \begin{cases} \gamma(t) & \text{if } t \in [a,b] \\ \delta(t) & \text{if } t \in [b,c] \end{cases} \tag{3.2}$$

Notice that the *orientation* should be read from *right to left*: first one runs along γ, then along δ. This composition law defines, on $\Gamma(X)$, a natural structure of groupoid. If $\Gamma \subseteq \Gamma(X)$ is a subgroupoid and $H \xrightarrow{\pi} M$ is a Hilbert bundle, a ***parallel transport on*** H ***along the curves of*** Γ is a map U such that

$$\gamma \in \Gamma \to U_\gamma \in Iso(H_{i_\gamma}; H_{\varphi_\gamma}) \tag{3.3}$$

(where $Iso(H_x; H_y)$ denotes the unitary isomorphisms from H_x to H_y) with the property that, if $\gamma, \delta \in \Gamma$ can be composed in the sense of (3.2), then

$$U_{\delta\circ\gamma} = U_\delta \circ U_\gamma \in Iso(H_{i_\gamma}; H_{\varphi_\gamma}) \tag{3.4}$$

We shall be interested in the case in which it is given a manifold M and one considers groupoids of curves on

$$X := G = \text{a subgroup of } Diff(M)$$

If $\gamma : [a,b] \to G$ is such a curve we associate to γ a family of curves on M defining

$$\gamma_p(t) := (\gamma(t))(p) \qquad t \in [a,b], p \in M$$

For each $p \in M$, $\gamma_p : [a,b] \to M$ is called the ***p-projection of*** γ. Notice that, in the notations (3.1):

$$i_{\gamma_p} = i_\gamma(p) = \gamma(a)(p) =: \gamma(a,p); \qquad \varphi_{\gamma_p} = \varphi_\gamma(p) = \gamma(b)(p) =: \gamma(b,p); \qquad \forall p \in M$$

Moreover, if $\gamma : [a,b] \to G$ and $\delta : [b,c] \to G$ can be composed in the sense of (3.2) then for any $p \in M$ one has

$$(\delta \circ \gamma)_p = \delta_{\gamma(b)p} \circ \gamma_p$$

This identity implies that, for any $p \in M$, and for δ and γ as above,

$$\varphi_{(\delta\circ\gamma)_p} = \delta_{\gamma(b)p}(c) = \delta(c)[\gamma(b)p] = \varphi_\delta\varphi_\gamma(p)$$

i.e. the map $\gamma \mapsto \varphi_\gamma$ defines an action on M, of the groupoid of all the curves on G.

Let $\mathcal{G}$ be a groupoid of curves on G and denote by $\mathcal{P}_M(\mathcal{G})$ the groupoid of all the p-projections of the curves of $\mathcal{G}$ on M, when p varies among all the points of M. If $H \xrightarrow{\pi} M$ is a Hilbert bundle and U is a parallel transport H along the curves of $\mathcal{P}_M(\mathcal{G})$, one can define an action of $\mathcal{G}$ on H by:

$$\gamma v := U_{\gamma_p} v \qquad \gamma \in \mathcal{G}, \qquad p \in M, \qquad v \in H_{i_\gamma p} = \pi^{-1}(i_\gamma p) \tag{3.5}$$

The action is well defined because any point $x \in M$ can be written in the form $x = i_\gamma p$ for some $p \in M$, since i_γ is a diffeomorphism of M. Moreover this action is admissible, in the sense of Definition (2.1), in fact:

$$\gamma H_{i_{\gamma_p}} = U_{\gamma_p} H_{i_{\gamma_p}} = H_{\varphi_{\gamma_p}}; \qquad \gamma \in \mathcal{G}; p \in M$$

THEOREM (3.1)

Let be given:

- a manifold M;
- a subgroup G of $Diff(M)$;
- a groupoid $\mathcal{G}$ of curves on G
- a Hilbert bundle $H \xrightarrow{\pi} M$.

Then any parallel transport U on H, along the curves of $\mathcal{P}_M(\mathcal{G})$, defines a structure of $\mathcal{G}$-bundle on H.

PROOF.

From the discussion above.

If M is orientable and ω is a fixed volume form on M, we can associate to the $\mathcal{G}$-bundle H, the induced representation (2.2). This representation will be called ***the representation induced by the parallel transport*** U. It is a natural generalization of the representation constructed by Accardi [1]. In the following Section we shall prove that, in the conditions of the Theorem (3.1) above, there exists a ***group*** $P(G)$ canonically associated to $\mathcal{G}$ and an homomorphism α, from $\mathcal{G}$ onto $P(G)$ with the property that for ***any*** parallel transport U the projection of $\mathcal{G}$ on M and the representation of $\mathcal{G}$, induced by U, factorizes through the homomorphism α and a representation $\tilde{V}$ of $P(G)$ so that $\tilde{V}_{\alpha(\gamma)} = V_\gamma$.

Moreover, we shall prove that $P(G)$ coincides with the ***path group*** introduced by M.Mensky [9].

The path group and its subgroups

We outline here the construction of the path group. We refer to the works of Mensky [9,10,11] for more details. Let X and $\Gamma(X)$ be defined as before. Two curves of $\Gamma(X)$ are called ***equivalent*** if one of them can be obtained from the other by the following operations:

(i) reparameterizations;
(ii) cancellation or insertion of appendices (by definition an *appendix* is a piece of curve travelled back and forth).

The quotient $\hat{P}(X) = \Gamma(X)/\sim$ has a groupoid structure induced by that of $\Gamma(X)$. If $X = M$ is a manifold we may consider groupoids of piecewise smooth curves. If $X = G$ is a topological group, then a stronger equivalence relation can be obtained by adding to the above operations (i),(ii) the following:

(iii) translations by elements of G.

If $\gamma : [a,b] \to G$ is a curve on G its *translate* by an element $g \in G$ is the curve γg defined by

$$(\gamma g)(t) = \gamma(t)g; \qquad t \in [a,b]$$

We denote by $\sim$ the equivalence relation defined by (i), (ii), (iii). The equivalence class of γ will be denoted by $[\gamma]$. The quotient $P(G) = \Gamma(G)/\sim$ has a natural group structure since any two equivalence classes can now be composed. This is the *path group* of M.B.Mensky.

REMARK (3.1).

We may identify $P(G)$ with the set of continuous curves starting from the origin of G. Composition of such curves is given by composition of curves after translation in G. Namely if

$$\gamma : [0,1] \to G, \qquad \gamma(0) = e = 1_G; \qquad \delta : [1,2] \to G, \qquad \delta(1) = e$$

then the product in the path group is defined by

$$\delta \circ \gamma : [0,2] \to G$$

$$(\delta \circ \gamma)t = \begin{cases} \gamma(t) & \text{if } t \in [0,1] \\ \delta(t)\gamma(1) & \text{if } t \in [1,2] \end{cases}$$

When no confusion is possible we will omit the notation $[\gamma]$, that is $\gamma \in P(G)$ will mean that γ is a curve starting from the origin. With this notation $P(G)$ is identified to a subset of $\Gamma(G)$ and therefore on it the map φ, associating to a curve γ its end point φ_γ is well defined.

REMARK (3.2)

Let G_e be the arcwise connected component of the identity $e \in G$. G_e is a normal subgroup of G. Moreover it is obvious that

$$P(G) = P(G_e)$$

In what follows we restrict ourselves to the case $G = G_e$.

REMARK (3.3) In the notations above $\varphi : P(G) \to G$ is a group homomorphism. The kernel of this homomorphism is the loop group $L(G) := \varphi^{-1}(e)$. Therefore

$$P(G)/L(G) = G$$

In what follows we restrict for technical reasons (i..e. to avoid pathologies) to piecewise injective paths.

DEFINITION (3.1) A topology on $P(G)$ is called *admissible* if $\varphi : P(G) \to G$ is continuous and $L(G)$ is closed.

THEOREM (3.2) If G is a metric group with distance d then, $P(G)$ has an admissible, metrizable topology induced by the metric

$$d''(p,q) := \inf_{\substack{\gamma \in A_p \\ \gamma \in A_q}} d'(\gamma, \delta)$$

where for any $p \in P(G)$ and $\gamma, \delta \in \Gamma(G)$

$$d'(\gamma,\delta) := \sup_{t \in [0,1]} d(\gamma(t), \delta(t))$$

$$A_p := \{\gamma : [0,1] \to G, \qquad \gamma(0) = 1_G, \quad [\gamma] = p\}$$

PROOF

Here we shall only proof the triangle inequality. Let $p, q \in P(G)$ and fix $\gamma_0 \in A_p$. Then for any pair $\gamma \in A_p$, $\delta \in A_q$ there exists a $\hat{\delta} \in A_q$ such that

$$d'(\gamma,\delta) = d'(\gamma_0, \hat{\delta})$$

To see this fix γ_0 in A_p. Since γ_0 and γ are both in A_p there exists $f \in$ Homeo $([0,1])$ such that $f(0) = 0$, $f(1) = 1$ and

$$\gamma(t) = \gamma_0(f(t)), \quad \forall t \in [0,1]$$

Define $\hat{\delta}(t) := (\delta_0 f^{-1})(t)$. By definition $\hat{\delta} \in A_q$. So we get

$$\begin{aligned} d'(\gamma,\delta) &= \sup_{t\in[0,1]} d(\gamma(t), \delta(t)) \sup_{t\in[0,1]} d(\gamma_0(f(t)), \delta(t)) \\ &= \sup_{s\in[0,1]} d(\gamma_0(s), (\delta_0(f^{-1}(s)) = d'(\gamma_0.\hat{\delta}) \end{aligned}$$

This implies that for any pair $\gamma_0 \in A_p$, $\delta_0 \in A_q$, one has

$$d''(p,q) = \inf_{\substack{\gamma\in A_p \\ \delta\in A_q}} d'(\gamma,\delta) = \inf_{\delta\in A_q} d'(\gamma_0,\delta) = \inf_{\gamma\in A_p} d'(\gamma,\delta_0)$$

Now we are ready for the triangle inequality. For any $\gamma_0 \in A_p$, $\delta \in A_q$, $r \in P(G)$, $\delta_0 \in A_q$ and $\eta \in A_r$ one has:

$$d''(p,q) = \inf_{\substack{\gamma\in A_p \\ \delta\in A_q}} d'(\gamma,\delta) \le d'(\gamma,\delta_0) \le d'(\gamma_0,\eta) + d'(\eta,\delta_0)$$

Therefore for any $\eta_0 \in A_r$ one has

$$d''(p,q) - d'(\gamma,\eta_0) \leq d'(\eta_0,\delta) \qquad \forall \gamma \in A_p\,, \quad \forall \delta \in A_q$$

so

$$d''(p,q) - d'(\gamma,\eta_0) \leq \inf_{\gamma \in A_q} d'(\eta_0,\delta) = d''(r,q) \quad \forall \gamma \in A_p$$

This implies

$$d''(p,q) - d''(r,q) \leq d'(\gamma,\eta_0) \qquad \forall \gamma \in A_p$$

and therefore

$$d''(p,q) - d''(r,q) \leq \inf_{\gamma \in A_p} d'(\gamma,\eta_0) = d''(p,r)$$

which is the triangle inequality.

The following result [5] will play a crucial role in the investigation of the irreducibility conditions for imprimitivity systems associated to induced representations of G-bundles.

THEOREM (3.3). If G is a polish group, i.e. a metrizable, separable, complete topological group, then also the path group $P(G)$ is polish.

The importance of polish groups in representation theory is due to the following

THEOREM (3.4). (Dixmier [4]) If G is a polish group and K is closed normal subgroup then there exists a Borel section for the canonical projection $\pi : G \to G/K$

For the use of Borel section see section (6) in the present work. More extensive treatment is given elsewhere [5].

4. Induced representation of path groups and parallel transports

Suppose now that $H \xrightarrow{\pi} M$ is a Hilbert fiber bundle on the orientable manifold M with the volume form ω and associated measure μ. Let G be a transitive group of diffeomorphisms of M and U a unitary parallel transport on a groupoid $\mathcal{A}$ of curves in G.

PROPOSITION (4.1). On $H \xrightarrow{\pi} M$ there is a natural structure of Hilbert $P(G)$–bundle.

PROOF.
We have to construct a compatible action of $P(G)$ on M and H. To do this one has just to mimic the preceding procedure for the path groupoid. If we define $c(\gamma,x) := c(\varphi_\gamma, x)$, where $c(\cdot,\cdot)$ is the function defined in section (2), then the induced representation has the canonical form

$$(V_\gamma f)(x) = |c(\gamma^{-1},x)|^{1/2}\gamma f(\gamma^{-1}(x)) \qquad \gamma \in P(G), \qquad f \in L^2(M,d\mu;H) \tag{4.1}$$

In the case $M = G$ the isotropy group of any point $p \in M$ is given by $L(G)$, the loop group. The image under the inducing representation is just $Hol(U)$ i.e. the holonomy group of the parallel transport.

5. One parameter subgroups and generators

There is a natural way to associate one parameter groups of $P(G)$ to one parameter groups of G. Let $g_{(\cdot)} : \mathbf{R} \to G$ be a one parameter group of G. Define

$$\gamma_t : [0,t] \to G \quad ; \qquad \gamma_t(s) = g_s \quad ; \qquad \forall s \in [0,t] \tag{5.1}$$

It is possible to prove that different s,t give rise to different elements of the path group, i.e.

$$s \neq t \Rightarrow [\gamma_t] \neq [\gamma_s] \tag{5.2}$$

This allows us to define

$$q_t := [\gamma_t] \tag{5.3}$$

Moreover one can prove that

$$q_t q_s = q_{t+s} \tag{5.4}$$

Thus $q_{(\cdot)} : \mathbf{R} \to P(G)$ is a one parameter subgroup of $P(G)$. Now let g_t be a one-parameter subgroup associated to a complete vector field v so that

$$v_x := \left.\frac{dg_t(x)}{dt}\right|_{t=0} \qquad ; \qquad \forall x \in M \tag{5.5}$$

and let γ_t, q_t be as in (5.3),(5.4). Then (q_t) is a one-parameter subgroup of $P(G)$ and it is not difficult to see that for any $f,h \in L^2(M,\mu;H)$ the map $t \in \mathbf{R} \mapsto (V_{q_t}f,h) = (V_{\gamma_t}f,h)$ is measurable (where V is the representation (4.1)). Von Neumann theorem implies that $V_{\gamma_t} = V_{q_t}$ is strongly continuous so we can apply Stone theorem to get the existence of a selfadjoint operator A such that

$$V_{\gamma_t} = V_{q_t} = e^{itA}$$

In order to compute the explicit form of A we need two preliminary results.

LEMMA (5.1)
In the above notations, if ∇ denotes the covariant derivative, then:

$$(\nabla_v f)(x) = \lim_{t\to 0}\frac{1}{t}(\gamma_t{}^{-1}f(\gamma_t x) - f(x))$$

PROOF. By definition of covariant derivative

$$\begin{aligned}(\nabla_v f)(x) &= \\ &= \lim_{t\to 0}\frac{1}{t}((U_x{}^g_t x((\gamma_t)_x))^{-1}f(g_t(x)) - f(x)) \\ &= \lim_{t\to 0}\frac{1}{t}((U_x{}^{(\varphi\gamma_t)(x)}((\gamma_t)_x))^{-1}f(\varphi\gamma_t(x)) - f(x)) \\ &= \lim_{t\to 0}\frac{1}{t}(\gamma_t{}^{-1}f(\gamma_t x) - f(x))\end{aligned}$$

DEFINITION (5.1) The *Lie derivative* of ω with respect to v is defined by

$$(L_v\omega)_x := \frac{d}{dt}\Big|_{t=0} (g_t^*\omega)_x \qquad x \in M$$

The *ω-divergence* of the vector field V is defined by the equation:

$$(L_v\omega)_x = (\text{div } v)_x\omega_x$$

In the following Lemma $c(\cdot,\cdot)$ is defined as in section (2) and v is the vector field on M defined by (5.5).

LEMMA (5.2).
1) $c(g_0, x) = c(1_G, x) = 1 \quad \forall x \in M;$
2) $c(g_t, x) > 0 \quad \forall t \in R, \quad \forall x \in M;$
3) $c(g_t^{-1}, x)^{1/2} \to 1 \quad$ as $\quad t \to 0;$
4) $\frac{d}{dt}\Big|_{t=0} c(g_t^{-1}, x)^{-1/2} = -\frac{1}{2}(\text{div } v)_x \qquad \forall x \in M$

PROOF.
Since $c(\cdot,\cdot)$ is defined by

$$c(g, x)\omega_x = (g^*\omega)_x$$

we get

$$\omega_x = (1_G{}^*)_x = (g_0{}^*)_x = c(g_0, x)\omega_x$$

and this proves i). From i) and from the continuity of the function $t \to c(g_t^{-1}, x)^{1/2}$ follows iii). To prove ii) observe that g_t is in the arcwise connected component of the identity and therefore g_t is orientation preserving for any t. This implies that $c(g_t, x) > 0 \; \forall t \in \mathbf{R}$, $\forall x \in M$.

To prove iv) notice that

$$(\lim_{t\to 0} \frac{1}{t}(c(g_t, x) - 1))\omega_x = \lim_{t\to 0} \frac{1}{t}(c(g_t, x)\omega_x - \omega_x) = \lim_{t\to 0} \frac{1}{t}((g_t^*\omega)_x - \omega_x)$$

$$= \frac{d}{dt}\Big|_{t=0} (g_t^*\omega)_x = (L_v\omega)_x = (\text{div } v)_x\omega_x$$

Now let $f(t) = c(g_t, x)$ and $u(t) = f(-t)^{-1/2}$. Then $f(0) = 1$ and $f'(0) = (\text{div } v)_x$. Moreover

$$u'(t) = -\frac{1}{2} f(-t)^{-3/2} f'(-t)(-1) = \frac{1}{2} f(-t)^{-3/2} f'(t)$$

Therefore

$$u'(0) = \frac{1}{2} f(0)^{-3/2} f'(0) = \frac{1}{2} f'(0) = \frac{1}{2} f'(0) = \frac{1}{2}(\text{div } v)_x$$

This implies

$$\lim_{t\to 0} \frac{1}{t}(1 - c(g_t^{-1}, x)^{-1/2}) = -\lim_{t\to 0} \frac{1}{t}(c(g_{-t}, x)^{-1/2} - 1) = -\lim_{t\to 0} \frac{1}{t}(u(t) - u(0))$$

$$= -u'(0) = -\frac{1}{2}(\text{div } v)_x$$

and this end the proof.

THEOREM (5.1). The infinitesimal generator of the 1-parameter unitary group (γ_t) is:

$$A = \frac{1}{i}\left(-\left(\nabla_v + \frac{1}{2}\text{div } v\right)\right) = i\left(\nabla_v + \frac{1}{2}\text{div } v\right)$$

PROOF.
Applying Lemma (5.1) and Lemma (5.2) to the identity:

$$\frac{1}{t}[(V_{\gamma_t} f)(x) - f(x)] = \frac{1}{t}[|c(\gamma_t^{-1}, x)|^{1/2}\gamma_t f(\gamma_t^{-1}x) - f(x)]$$

$$= \frac{1}{t}c(\gamma_t^{-1}, x)^{1/2}[\gamma_t f(\gamma_t^{-1}x) - f(x) + f(x) - c(\gamma_t^{-1}, x)^{-1/2} f(x)]$$

$$= \underbrace{c(g_t^{-1}, x)^{1/2}}\left[\underbrace{\frac{1}{t}(\gamma_t f(\gamma_t^{-1}x) - f(x))} + \underbrace{\frac{1}{t}(1 - c(g_t^{-1}, x)^{-1/2}) f(x)}\right]$$

One deduces that, as $t \to 0$, the scalar factor tends to 1, the first term in braces tends to $(\nabla_{-v} f)(x)$ and the second one to $(-\frac{1}{2}\text{div } v)_x f(x)$. Therefore

$$\lim_{t\to 0} \frac{1}{t}[(V_{\gamma_t} f)(x) - f(x)] = ([-(\nabla_v + \frac{1}{2}\text{div } v)f])(x)$$

and this proves the theorem.

6. Irreducibility of the Schrödinger representation

Let G be a locally compact, separable group and K a closed subgroup. The main irreducibility result in Mackey's imprimitivity theorem [7] is expressed as follows.

THEOREM (6.1). If L is a representation of K then the commutants of L and of (U^L, P^L) are isomorphic.

Theorem (6.1) implies that a transitive, imprimitive system (U^L, P^L) is irreducible if and only if the inducing representation L is irreducible. In our case the induced representation is defined for a non-locally compact group (the path group). Therefore, in order to find the irreducibility conditions for the Schrodinger representation on a general Hilbert bundle H, it is necessary to extend Theorem (6.1) to a class of non-locally compact groups including the class of path groups. Let us give a

DEFINITION (6.1). Let G be a topological group and K a closed subgroup of G. We say that the pair (G, K) has the selection property if there exists a Borel transversal for the quotient G/K.

We proved the following result [5].

THEOREM (6.2) Let G be a topological group, K a closed subgroup of G such that (G, K) has the selection property. Suppose that G/K carries a Borel, regular, σ–finite, G–quasi-invariant measure. If L, L' are unitary representations of K then the associated imprimitivity systems (U^L, P^L), $(U^{L'}, P^{L'})$ are equivalent if and only if L and L' are equivalent.

We proved that the above irreducibility result is true for the induced representation of the path group given by (4.1) provided the group is polish [5]. We want to underline the geometrical content of this result. As we said in section (4), if $G = M$ then the isotropy group in the action (3.5) is the loop group $L(G)$ and the image of $L(G)$ under the inducing representation turns out to be $Hol(U)$ the holonomy group of the parallel transport. So Theorem (6.2) implies that the imprimitivity system of the representation (4.1) is irreducible if and only if the holonomy group acts irreducibly on the fibers. But the Ambrose-Singer theorem says that the (restricted) holonomy group is directly related to the curvature (*field strenghts*) of the given parallel transport. This shows the deep relation between the geometry of fiber bundles and the algebraic properties of the induced representations of path groups.

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REFERENCES

[1] Accardi L., *The Weyl-Scrödinger representation on curved spaces and functional integrals on connections*, Unpublished notes, Princeton (1986)

[2] Barut A.O.-Racka R., *Theory of group representations and applications*, World scientific (1986)

[3] Bleuler K.-Werner M. (eds.), *Differential Geometrical methods in theoretical physics*, Kluwer Academic Publishers (1988)

[4] Dixmier J., *Dual et quasi-dual d'une algebre de Banach involutive*,Trans. Amer. Math. Soc. 104 (1962), 278-283

[5] Gibilisco P., *Induced representations of path groups and parallel transport*, Ph.d.Thesis (submitted), Università di Roma II (1991)

[6] Kobayashi S.-Nomizu K., *Foundations of differential geometry*, John Wiley & Sons, (1963)

[7] Mackey G.W., *Unitary representations in Physics, Probability and number theory*, The Bejamin-Cummings Publ. Comp. (1978)

[8] Mackey G.W., *Weyl program and modern physics*, in [3], 11-36

[9] Mensky M.B., *Group of parallel transports and description of particles in curved spacetime*, Lett.Math. Phys., 2 (1978), 175-180

[10] Mensky M.B., *The group of paths: Measurement, fields, particles*, Nauka, Moscow (1983) (in Russian)

[11] Mensky M.B., *Application of path group in gauge theory, Gravitation and String theory*, in *Gauge theories of fundamental interaction*, World Scientific (1990)

[12] Palais R., *A global fomulation of the Lie theory of trasformation groups*, Mem. Amer. Math. Soc. 22, (1957)

[13] Pressley A.-Segal G., *Loop Groups*, Clarendon Press, Oxford, (1986)

[14] Rieffel M.A., *Induced representations of C^*-algebras*, Adv.Math.13, 176-257 (1974)

[15] Wallach N.R., *Harmonic analysis on homogeneous space*, Marcel Dekker, New york (1973)

[16] Warner G., *Harmonic analysis on semisimple Lie groups*, vol.I, Springer (1972)

[17] Wightman A.S., *On the localizability of quantum mechanical Systems*, Rev. Modern Phys.34, 845-872 (1962)