Giuseppe Tomassetti

On configurational balance in slender bodies

Abstract We propose a new derivation of the evolution equation of a sharp, coherent interface in a two-phase body having elongated shape, a body which we regard as a one-dimensional micropolar continuum. To this aim, we introduce a system of forces acting at the interface, and we apply the method of virtual powers to derive a balance law involving these forces. By exploiting the dissipation inequality, we manage to write this balance law in terms of a scalar field whose form is reminiscent of a well-known expression for the configurational stress in three dimensional micropolar continua.

Keywords Beam theories · Configurational forces · Material forces · Eshelby stress · Virtual power

1 Introduction

In this paper, we provide a short, self-contained derivation of the evolution equation governing the motion of a sharp interface in a *two-phase slender body*, *i.e.*, a two-phase body having *elongated shape*. As customary in structural mechanics, we model this body as a one-dimensional continuum whose material particles, which we call *sections*, are endowed with translational and rotational degrees of freedom. For simplicity, we restrict attention to planar motions, and we leave thermal effects out of the picture.

Our viewpoint is the same as Gurtin's [7]: a sharp phase interface should be treated as a material structure ruled by a *configurational balance* law standing on the same footing as the standard balance laws of continuum physics; the actual law governing the evolution of the interface should result from the combination of the configurational balance law with constitutive prescriptions accounting for the diversity of materials. Our approach is different from Gurtin's: the treatment of configurational forces presented in [7] is based on a notion of working performed on control volumes migrating through a body's reference shape, combined with invariance of that working under changes in material observer; the approach we adopt in this paper, by developing a method proposed in [15], relies instead on a non-standard version of the Principle of Virtual Powers.

Basic to our derivation is the introduction of an *internal force* $\mathcal{G}$ and an *external force* $\mathcal{F}$ which enter in the expressions of the internal and external powers as work conjugates of the referential velocity of the interface. Thanks to these extra terms, the application of the method of virtual powers yields, besides the standard balance laws in the bulk (*i.e.* away from the interface), the *interface condition* (*cf.* Sect. 3.2):

$$\mathcal{G} + [\![u'\mathsf{n} + v'\mathsf{t} + \vartheta'\mathsf{m}]\!] = \mathcal{F}. \tag{1}$$

Here, u, v, and ϑ are scalar fields delivering the *axial displacement*, the *transverse displacement*, and the *rotation* of the typical section; a prime sign denotes partial differentiation with respect to arc length; n, t, and

G. Tomassetti (✉)
Dipartimento di Ingegneria Civile, Università degli Studi di Roma "Tor Vergata",
Via Politecnico 1, 00133 Rome, Italy
E-mail: tomassetti@ing.uniroma2.it

m are scalar fields accounting for, respectively, *axial force*, *shear force*, and *bending moment*. A pair of double brackets denotes the jump of the enclosed scalar field across the interface.

Following a procedure of Podio–Guidugli [13], we identify inertial interactions using the requirement that the power they expend on any part of the body be equal to minus the rate of change of kinetic energy of the same part. By doing so, we find that the inertial part of the external force acting on the interface is:

$$\mathcal{F}^{\mathrm{in}} = \frac{1}{2}\left[\!\left[\varrho u'^2 + \varrho v'^2 + \iota \vartheta'^2\right]\!\right] \mathcal{V}^2, \tag{2}$$

where ϱ and ι are positive constants accounting for linear and rotational inertia, and $\mathcal{V}$ is the (referential) *velocity of the interface*.

We decompose the internal force into its *equilibrium* and *non-equilibrium* parts: $\mathcal{G}^{\mathrm{eq}}$ and $\mathcal{G}^{\mathrm{ne}}$, respectively. Then, we exploit the dissipation inequality to show that the equilibrium part must be equal to minus the jump of the free-energy density across the interface, and that the non-equilibrium part expends non-negative power during every process [*cf.* (44)]:

$$\mathcal{G}^{\mathrm{eq}} = -[\![\psi]\!], \quad \mathcal{G}^{\mathrm{ne}}\mathcal{V} \geq 0.$$

On setting:

$$\mathsf{c} = \psi - u'\mathsf{n} - v'\mathsf{t} - \vartheta'\mathsf{m}, \tag{3}$$

and on dispensing of the non-inertial part of the external force by choosing:

$$\mathcal{F}^{\mathrm{ni}} = 0,$$

we can write the interface condition (1) as:

$$[\![\mathsf{c}]\!] - \mathcal{G}^{\mathrm{ne}} = -\frac{1}{2}\left[\!\left[\varrho u'^2 + \varrho v'^2 + \iota\theta'^2\right]\!\right] \mathcal{V}^2, \tag{4}$$

which is the sought-for *configurational balance*.

The treatment of configurational forces in strings, bars, and beams is not new, and its many applications span from fracture mechanics to structural optimization [3,8,9,12,14,16]. In particular, an equation ruling the motion of a sharp interface in a setting similar to ours has been derived by O'Reilly [12] by postulating a balance of configurational forces.

The scalar field c is the one-dimensional analogue of the *configurational stress* in a three-dimensional micropolar continuum [10,11]. In the more standard setting of Cauchy continua, the counterpart of (3) is the *Eshelby relation*:

$$\mathbf{C} = \varphi\mathbf{I} - \nabla\mathbf{u}^{\mathsf{T}}\mathbf{S},$$

which defines the *Eshelby stress* [5] in the so-called displacement-based formulation [7, Chap. 13]. Here, φ is the free energy per unit volume, $\mathbf{I}$ is the identity tensor, $\mathbf{u}$ is the displacement, and $\mathbf{S}$ is the stress. In the same setting, the motion of a sharp, coherent interface is ruled by the *normal configurational balance* (*cf.* [7, Eq. (14-4)] and [6, Eq. (6.2)]):

$$\mathbf{m}\cdot[\![\mathbf{C}]\!]\,\mathbf{m} + \gamma = -\frac{1}{2}\left[\!\left[\rho|(\nabla\mathbf{u})\mathbf{m}|^2\right]\!\right] V^2. \tag{5}$$

Here, $\mathbf{m}$ is the unit vector normal to the interface, ρ is the referential mass density, V is the normal velocity of the interface, and γ is the *internal force acting on the interface*. The latter must satisfy:

$$\gamma V \leq 0,$$

as a consequence of the dissipation inequality. When the balance of linear momentum is taken into account, (5) implies:

$$-[\![\varphi]\!] + \langle\mathbf{S}\rangle\mathbf{m}\cdot[\![\nabla\mathbf{u}]\!]\,\mathbf{m} = -\gamma, \tag{6}$$

whose left-hand side is the *driving traction* at the interface [1,18]. Inasmuch (5) implies (6), the configurational balance (4) implies (we give a proof in the Appendix):

$$-[\![\psi]\!] + \langle\mathsf{n}\rangle\left[\!\left[u'\right]\!\right] + \langle\mathsf{t}\rangle\left[\!\left[v'\right]\!\right] + \langle\mathsf{m}\rangle\left[\!\left[\vartheta'\right]\!\right] = -\mathcal{G}^{\mathrm{ne}}, \tag{7}$$

which can be regarded as a special case of the jump condition given in [12, Eq. (31)] when no point supplies of linear and angular momentum are present.

2 Evolution equations in the bulk

In this section, we assemble the balance equations in the bulk, i.e., away from the interface. We accomplish this task by restricting attention to parts of the body that *do not contain the interface*.

2.1 Bulk kinematics

We take as body manifold the interval $\mathcal{B} = (0, l)$ of the real line, and we call its elements *sections*. At time t, the typical section s occupies a *position* $P(s, t)$ in a two-dimensional Euclidean point space, and is endowed with an *orientation* $\mathbf{d}(s, t)$, a planar vector with unit norm. Next, as customary in solid mechanics, we select a *reference placement* for $\mathcal{B}$. This we do by choosing a position A along with two mutually orthogonal unit vectors $\mathbf{a}$ and $\mathbf{b}$, and by setting (Fig. 1):

$$P_0(s) = A + s\mathbf{a}, \quad \mathbf{d}_0(s) = \mathbf{b}.$$

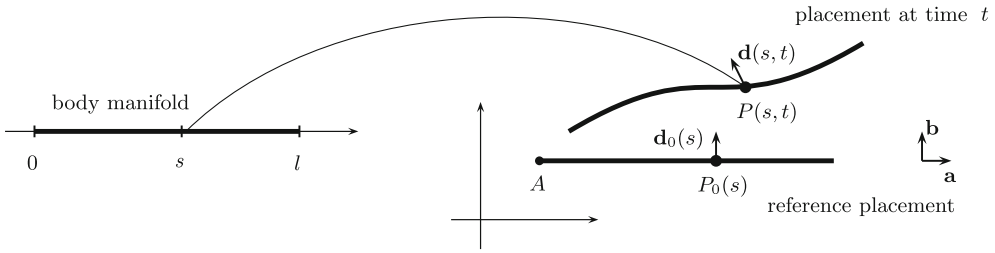

Fig. 1 Placement and orientation of the typical section

On writing:

$$P(s, t) = P_0(s) + \mathbf{u}(s, t), \quad \mathbf{d}(s, t) = \cos(\vartheta(s, t))\,\mathbf{a} + \sin(\vartheta(s, t))\,\mathbf{b},$$

we represent position and orientation of the typical section s at time t through its *displacement* $\mathbf{u}(s, t)$ and its *rotation* $\vartheta(s, t)$ with respect to the reference placement. Finally, we decompose the displacement into its *axial* and *transverse* components (Fig. 2):

$$u = \mathbf{u} \cdot \mathbf{a}, \quad v = \mathbf{u} \cdot \mathbf{b},$$

and, we introduce the *strain measures*:[1]

$$\varepsilon = u', \quad \gamma = v' + \vartheta, \quad \chi = \vartheta',$$

which account for *axial extension*, *shear*, and *bending*, respectively; implicit in this choice is the assumption that both displacements and rotations are *small*.

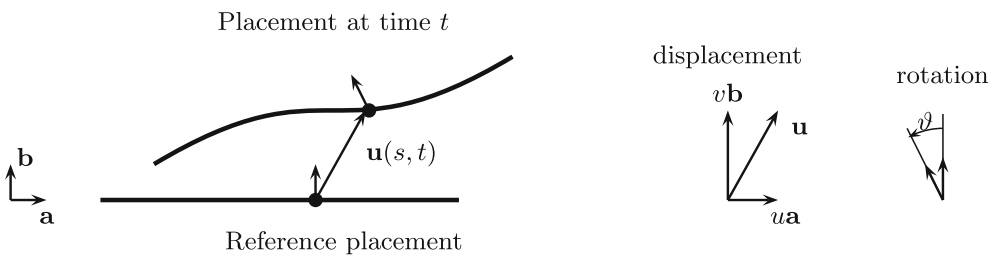

Fig. 2 Displacement and rotation of the typical section

[1] With a prime mark we indicate partial differentiation with respect to s.

2.2 Balance equations

By a *part* of the body, we mean an open interval $\mathcal{P} \subset (0, l)$. By a *virtual velocity*, we mean an ordered list $(\dot{u}_\mathrm{v}, \dot{v}_\mathrm{v}, \dot{\vartheta}_\mathrm{v})$ of smooth scalar fields on $(0, l)$. We stipulate that, given a part *not containing the interface* and a virtual velocity with *support compactly contained*[2] in the same part, the internal and the external powers have the form:

$$W^{\mathrm{int}}(\mathcal{P}) = \int_{\mathcal{P}} \mathsf{n}\dot{\varepsilon}_\mathrm{v} + \mathsf{t}\dot{\gamma}_\mathrm{v} + \mathsf{m}\dot{\chi}_\mathrm{v}\,\mathrm{d}s \quad \text{and} \quad W^{\mathrm{ext}}(\mathcal{P}) = \int_{\mathcal{P}} p\dot{u}_\mathrm{v} + q\dot{v}_\mathrm{v} + r\dot{\vartheta}_\mathrm{v}\,\mathrm{d}s,$$

respectively, where

$$\dot{\varepsilon}_\mathrm{v} = \dot{u}_\mathrm{v}', \quad \dot{\gamma}_\mathrm{v} = \dot{v}_\mathrm{v}' + \dot{\vartheta}_\mathrm{v}, \quad \dot{\chi}_\mathrm{v} = \dot{\vartheta}_\mathrm{v}',$$

are the *virtual strain rates* associated with the virtual velocity $(\dot{u}_\mathrm{v}, \dot{v}_\mathrm{v}, \dot{\vartheta}_\mathrm{v})$. Here, p, q, and r are, respectively, the *external axial force*, the *external transverse force*, and the *external couple* (per unit referential length). Moreover, we require that the internal and the external powers be equal:

$$W^{\mathrm{int}}(\mathcal{P}) = W^{\mathrm{ext}}(\mathcal{P}) \tag{8}$$

for every such pair of a part $\mathcal{P}$ and a virtual velocity $(\dot{u}_\mathrm{v}, \dot{v}_\mathrm{v}, \dot{\vartheta}_\mathrm{v})$. A standard argument based on by-parts integration and localization yields the *balance equations in the bulk*:[3]

$$\mathsf{n}' + p = 0, \quad \mathsf{t}' + q = 0, \quad \mathsf{m}' - \mathsf{t} + r = 0. \tag{9}$$

2.3 Inertia

We split the *external forces* p and q, and the *external couple* r into their ***in**ertial* and ***non-i**nertial* parts:

$$p = p^{\mathrm{in}} + p^{\mathrm{ni}}, \quad q = q^{\mathrm{in}} + q^{\mathrm{ni}}, \quad r = r^{\mathrm{in}} + r^{\mathrm{ni}}. \tag{10}$$

Then, we state that the power expended by inertial forces on any part during any realizable process be equal to minus the temporal change of kinetic energy of the same part:

$$\int_{\mathcal{P}} p^{\mathrm{in}}\dot{u} + q^{\mathrm{in}}\dot{v} + r^{\mathrm{in}}\dot{\vartheta}\,\mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} k\,\mathrm{d}s. \tag{11}$$

We accompany this statement with the usual choice for the *kinetic energy* per unit referential length:

$$k = \frac{1}{2}\varrho\left(\dot{u}^2 + \dot{v}^2\right) + \frac{1}{2}\iota\dot{\vartheta}^2, \tag{12}$$

where $\varrho > 0$ and $\iota > 0$ account for linear and rotational inertia. In view of (12), the statement (11) becomes:

$$\int_{\mathcal{P}} \left(p^{\mathrm{in}} + \varrho\ddot{u}\right)\dot{u} + \left(q^{\mathrm{in}} + \varrho\ddot{v}\right)\dot{v} + \left(r^{\mathrm{in}} + \iota\ddot{\vartheta}\right)\dot{\vartheta}\,\mathrm{d}s = 0. \tag{13}$$

A consequence of (13) and of the arbitrariness of $\mathcal{P}$ is that

$$\left(p^{\mathrm{in}} + \varrho\ddot{u}\right)\dot{u} + \left(q^{\mathrm{in}} + \varrho\ddot{v}\right)\dot{v} + \left(r^{\mathrm{in}} + \iota\ddot{\vartheta}\right)\dot{\vartheta} = 0 \tag{14}$$

must hold identically at all sections and at all times. In order to meet (14), we choose:

$$p^{\mathrm{in}} = -\varrho\ddot{u}, \quad q^{\mathrm{in}} = -\varrho\ddot{v}, \quad r^{\mathrm{in}} = -\iota\ddot{\vartheta}. \tag{15}$$

[2] Should we consider virtual velocities that do not vanish on $\partial\mathcal{P}$, then we must include additional terms in the expression of the external power to account for work expenditure by contact forces and contact couples acting on the boundary of $\mathcal{P}$.

[3] Equation (9) may also be derived by asking that the total force and the total moment on any part be null, as in standard Strength-of-Materials textbooks (*cf. e.g.* [17]).

2.4 Dissipation inequality

For notational convenience, we write:

$$\mathbf{s} = (\mathsf{n}, \mathsf{t}, \mathsf{m}), \quad \text{and} \quad \mathbf{e} = (\varepsilon, \gamma, \chi).$$

We consider constitutive equations of the form:

$$\mathbf{s} = \hat{\mathbf{s}}(\mathbf{e}, \dot{\mathbf{e}}), \tag{16}$$

with $\hat{\mathbf{s}}$ a smooth function. Likewise, we postulate a constitutive equation for the *free energy*:[4]

$$\psi = \hat{\psi}(\mathbf{e}), \tag{17}$$

with $\hat{\psi}$ smooth. Next, we assume that, during every process, the *dissipation inequality*:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}} \psi \, \mathrm{d}s \leq \int_{\mathcal{P}} \mathbf{s} \cdot \dot{\mathbf{e}} \, \mathrm{d}s \tag{18}$$

holds at all times for every part $\mathcal{P}$ not containing the interface. At those times, we have $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}} \psi \, \mathrm{d}s = \int_{\mathcal{P}} \dot{\psi} \, \mathrm{d}s$. Then, by taking into account of (16) and (17) and using the arbitrariness of $\mathcal{P}$, we obtain:

$$\left(\partial \hat{\psi}(\mathbf{e}) - \hat{\mathbf{s}}(\mathbf{e}, \dot{\mathbf{e}}) \right) \cdot \dot{\mathbf{e}} \leq 0, \tag{19}$$

an inequality to be satisfied at all points away from the interface during every evolution process.

Following Coleman & Noll [4], we argue that (19) must hold whatever the choice of $\mathbf{e} \in \mathbb{R}^3$ and $\dot{\mathbf{e}} \in \mathbb{R}^3$. By splitting $\hat{\mathbf{s}}$ into its ***e**quilibrium* and ***n**on-**e**quilibrium* parts, respectively:

$$\hat{\mathbf{s}}^{\mathrm{eq}}(\mathbf{e}) = \hat{\mathbf{s}}(\mathbf{e}, \mathbf{0}) \quad \text{and} \quad \hat{\mathbf{s}}^{\mathrm{ne}}(\mathbf{e}, \dot{\mathbf{e}}) = \hat{\mathbf{s}}(\mathbf{e}, \dot{\mathbf{e}}) - \hat{\mathbf{s}}^{\mathrm{eq}}(\mathbf{e}),$$

we can use the algebraic lemma in Appendix B of [2] to obtain the following representation for the equilibrium part of $\hat{\mathbf{s}}$:

$$\hat{\mathbf{s}}^{\mathrm{eq}}(\mathbf{e}) = \partial \hat{\psi}(\mathbf{e}). \tag{20}$$

An immediate consequence of (20) is that the local version (19) of the dissipation inequality turns into a restriction on the sole non-equilibrium part of $\hat{\mathbf{s}}$:

$$0 \leq \hat{\mathbf{s}}^{\mathrm{ne}}(\mathbf{e}, \dot{\mathbf{e}}) \cdot \dot{\mathbf{e}}.$$

3 Derivation of the interface condition

3.1 Interfacial kinematics

We denote by $\mathcal{I}(t) \in (0, l)$ the section where the interface is located at time t (Fig. 3). We allow the velocities $(\dot{u}, \dot{v}, \dot{\vartheta})$, the strains $(\varepsilon, \gamma, \chi)$, and the stresses $(\mathsf{n}, \mathsf{t}, \mathsf{m})$ to jump across the moving interface, but we require that $u(\cdot, t)$, $v(\cdot, t)$, and $\vartheta(\cdot, t)$ be continuous across the interface at each time t. This requirement characterizes the interface as *coherent* and has two well-known consequences. First, the velocities $\dot{u}$, $\dot{v}$, $\dot{\vartheta}$, and the (referential) *velocity of the interface* $\mathcal{V} = \dot{\mathcal{I}}$ must satisfy the *compatibility conditions* (*cf.* [7, Eq. (10-2a)])

$$[\![\dot{u}]\!] + [\![u']\!] \mathcal{V} = 0, \quad [\![\dot{v}]\!] + [\![v']\!] \mathcal{V} = 0, \quad [\![\dot{\vartheta}]\!] + [\![\vartheta']\!] \mathcal{V} = 0; \tag{21}$$

second, the *transported velocities*:

$$\overset{\square}{u}(t) := \frac{\mathrm{d}}{\mathrm{d}t} u(\mathcal{I}(t), t), \quad \overset{\square}{v}(t) := \frac{\mathrm{d}}{\mathrm{d}t} v(\mathcal{I}(t), t), \quad \overset{\square}{\vartheta}(t) := \frac{\mathrm{d}}{\mathrm{d}t} \vartheta(\mathcal{I}(t), t)$$

[4] A slight modification of the argument that follows would rule out any dependence of ψ on $\dot{\mathbf{e}}$.

are well defined, and satisfy:

$$\overset{\square}{u} = \langle \dot{u} \rangle + \langle u' \rangle \mathcal{V}, \quad \overset{\square}{v} = \langle \dot{v} \rangle + \langle v' \rangle \mathcal{V}, \quad \overset{\square}{\vartheta} = \langle \dot{\vartheta} \rangle + \langle \vartheta' \rangle \mathcal{V}, \tag{22}$$

where the brackets $\langle \cdot \rangle$ denote the average of the enclosed field at either side of the interface.

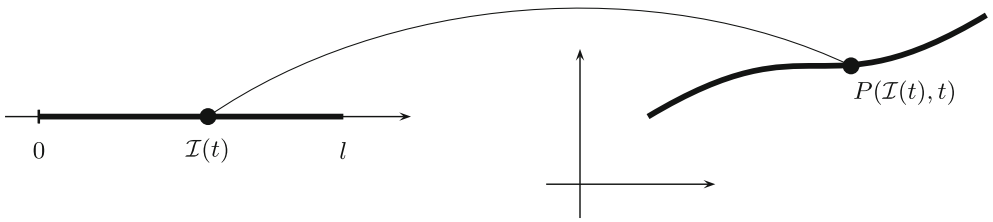

Fig. 3 The moving interface

Remark 1 After introducing the interfacial kinematics, we must update the constitutive equations introduced in Sect. 2.4 so as to allow for a different mechanical response in either phase. In particular, we must specify two functions $\hat{\mathbf{s}}^+$ and $\hat{\mathbf{s}}^-$ and replace the constitutive Eq. (16) with

$$\mathbf{s}(s,t) = \begin{cases} \hat{\mathbf{s}}^-(\mathbf{e}(s,t),\ \dot{\mathbf{e}}(s,t)) & \text{if } s < \mathcal{I}(t), \\ \hat{\mathbf{s}}^+(\mathbf{e}(s,t),\ \dot{\mathbf{e}}(s,t)) & \text{if } s > \mathcal{I}(t). \end{cases}$$

3.2 Balance equations at the interface

We generalize the notion of virtual velocity by augmenting $\left(\dot{u}_{\mathrm{v}}, \dot{v}_{\mathrm{v}}, \dot{\vartheta}_{\mathrm{v}}\right)$ with a quadruplet of real numbers $\left(\mathcal{V}_{\mathrm{v}}, \overset{\square}{u}_{\mathrm{v}}, \overset{\square}{v}_{\mathrm{v}}, \overset{\square}{\vartheta}_{\mathrm{v}}\right)$. We refer to $\mathcal{V}_{\mathrm{v}}$ as the *virtual velocity of the interface*, and to $\left(\overset{\square}{u}_{\mathrm{v}}, \overset{\square}{v}_{\mathrm{v}}, \overset{\square}{\vartheta}_{\mathrm{v}}\right)$ as the *virtual transported velocities* of (u, v, ϑ). On account of (21) and (22), we require that the elements of a generalized virtual velocity satisfy:

$$[\![\dot{u}_{\mathrm{v}}]\!] + [\![u']\!]\,\mathcal{V}_{\mathrm{v}} = 0, \quad [\![\dot{v}_{\mathrm{v}}]\!] + [\![v']\!]\,\mathcal{V}_{\mathrm{v}} = 0, \quad [\![\dot{\vartheta}_{\mathrm{v}}]\!] + [\![\vartheta']\!]\,\mathcal{V}_{\mathrm{v}} = 0 \tag{23}$$

and

$$\overset{\square}{u}_{\mathrm{v}} = \langle \dot{u}_{\mathrm{v}} \rangle + \langle u' \rangle \mathcal{V}_{\mathrm{v}}, \quad \overset{\square}{v}_{\mathrm{v}} = \langle \dot{v}_{\mathrm{v}} \rangle + \langle v' \rangle \mathcal{V}_{\mathrm{v}}, \quad \overset{\square}{\vartheta}_{\mathrm{v}} = \langle \dot{\vartheta}_{\mathrm{v}} \rangle + \langle \vartheta' \rangle \mathcal{V}_{\mathrm{v}}. \tag{24}$$

In order to capture the physics underlying the evolution of the interface, we stipulate that *for every part $\mathcal{P}$ containing the interface* the internal power be:

$$W_{\mathrm{int}}(\mathcal{P}) = \int_{\mathcal{P}} \mathsf{n}\dot{\varepsilon}_{\mathrm{v}} + \mathsf{t}\dot{\gamma}_{\mathrm{v}} + \mathsf{m}\dot{\chi}_{\mathrm{v}}\,\mathrm{d}s + \mathcal{G}\mathcal{V}_{\mathrm{v}}, \tag{25}$$

where we interpret $\mathcal{G}$ as the *internal force at the interface*. Then, we stipulate that the virtual external power expended on $\mathcal{P}$ be:[5]

$$W_{\mathrm{ext}}(\mathcal{P}) = \int_{\mathcal{P}} p\dot{u}_{\mathrm{v}} + q\dot{v}_{\mathrm{v}} + c\dot{\vartheta}_{\mathrm{v}}\,\mathrm{d}s + \mathcal{F}\mathcal{V}_{\mathrm{v}} + P\overset{\square}{u}_{\mathrm{v}} + Q\overset{\square}{v}_{\mathrm{v}} + R\overset{\square}{\vartheta}_{\mathrm{v}}, \tag{26}$$

[5] In Configurational Mechanics, the point of view that the external power should be a linear functional of both referential and transported velocities is not new (*cf.* [6, Sect. 4]).

whenever $(\dot{u}_{\mathrm{v}}, \dot{v}_{\mathrm{v}}, \dot{\vartheta}_{\mathrm{v}})$ has support contained in $\mathcal{P}$. We interpret $\mathcal{F}$, P, Q, and R as *external forces acting at the interface*. We show in the Appendix that:

$$W_{\mathrm{int}}(\mathcal{P}) = -\int_{\mathcal{P}} \mathsf{n}'\dot{u}_{\mathrm{v}} + \mathsf{t}'\dot{v}_{\mathrm{v}} + (\mathsf{m}' - \mathsf{t})\dot{\vartheta}_{\mathrm{v}}\,\mathrm{d}s + \left(\mathcal{G} + \left[\!\left[\mathsf{n}u' + \mathsf{t}v' + \mathsf{m}\vartheta'\right]\!\right]\right)\mathcal{V}_{\mathrm{v}} - [\![\mathsf{n}]\!]\overset{\square}{u}_{\mathrm{v}} - [\![\mathsf{t}]\!]\overset{\square}{v}_{\mathrm{v}} - [\![\mathsf{m}]\!]\overset{\square}{\vartheta}_{\mathrm{v}}. \tag{27}$$

By using (26) and (27), and by taking into account (9), we deduce the following consequence of the balance of powers (8):

$$\left(\mathcal{G} + \left[\!\left[\mathsf{n}u' + \mathsf{t}v' + \mathsf{m}\vartheta'\right]\!\right] - \mathcal{F}\right)\mathcal{V}_{\mathrm{v}} - ([\![\mathsf{n}]\!] + P)\overset{\square}{u}_{\mathrm{v}} - ([\![\mathsf{t}]\!] + Q)\overset{\square}{v}_{\mathrm{v}} - ([\![\mathsf{m}]\!] + R)\overset{\square}{\vartheta}_{\mathrm{v}} = 0. \tag{28}$$

By requiring that (28) holds for every generalized virtual velocity, we obtain the *interface condition*:

$$\mathcal{G} + \left[\!\left[u'\mathsf{n} + v'\mathsf{t} + \vartheta'\mathsf{m}\right]\!\right] = \mathcal{F}, \tag{29}$$

along with the *jump conditions*:

$$[\![\mathsf{n}]\!] + P = 0, \quad [\![\mathsf{t}]\!] + Q = 0, \quad [\![\mathsf{m}]\!] + R = 0. \tag{30}$$

3.3 Inertia

Following the line of reasoning adopted in Sect. 2.3, we split the work conjugates of the velocities appearing in (26) into their ***in****ertial* and ***non-i****nertial* parts by writing:

$$P = P^{\mathrm{in}} + P^{\mathrm{ni}}, \quad Q = Q^{\mathrm{in}} + Q^{\mathrm{ni}}, \quad R = R^{\mathrm{in}} + R^{\mathrm{ni}}, \tag{31}$$

and

$$\mathcal{F} = \mathcal{F}^{\mathrm{in}} + \mathcal{F}^{\mathrm{ni}}. \tag{32}$$

We characterize the inertial parts by requiring that the *inertial power*:

$$W^{\mathrm{in}}(\mathcal{P}) = \int_{\mathcal{P}} p^{\mathrm{in}}\dot{u} + q^{\mathrm{in}}\dot{v} + r^{\mathrm{in}}\dot{\vartheta}\,\mathrm{d}s + \mathcal{F}^{\mathrm{in}}\mathcal{V} + P^{\mathrm{in}}\overset{\square}{u} + Q^{\mathrm{in}}\overset{\square}{v} + R^{\mathrm{in}}\overset{\square}{\vartheta}$$

be equal to minus the time derivative of the kinetic energy of $\mathcal{P}$, that is to say:

$$\int_{\mathcal{P}} p^{\mathrm{in}}\dot{u} + q^{\mathrm{in}}\dot{v} + r^{\mathrm{in}}\dot{\vartheta}\,\mathrm{d}s + \mathcal{F}^{\mathrm{in}}\mathcal{V} + P^{\mathrm{in}}\overset{\square}{u} + Q^{\mathrm{in}}\overset{\square}{v} + R^{\mathrm{in}}\overset{\square}{\vartheta} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} k\,\mathrm{d}s. \tag{33}$$

By a standard transport theorem,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} k\,\mathrm{d}s = -\int_{\mathcal{P}} \dot{k}\,\mathrm{d}s + [\![k]\!]\mathcal{V}. \tag{34}$$

On combining (33) and (34) with the identity (for a derivation, see the Appendix):

$$[\![k]\!] = \frac{1}{2}\left[\!\left[\varrho u'^2 + \varrho v'^2 + \iota\vartheta'^2\right]\!\right]\mathcal{V}^2 + \left[\!\left[\varrho\dot{u}\right]\!\right]\overset{\square}{u} + \left[\!\left[\varrho\dot{v}\right]\!\right]\overset{\square}{v} + \left[\!\left[\varrho\dot{\vartheta}\right]\!\right]\overset{\square}{\vartheta}, \tag{35}$$

and on recalling (15), we conclude that:

$$\left(\mathcal{F}^{\mathrm{in}} - \frac{1}{2}\left[\!\left[\varrho u'^2 + \varrho v'^2 + \iota\vartheta'^2\right]\!\right]\mathcal{V}^2\right)\mathcal{V} + \left(P^{\mathrm{in}} - \left[\!\left[\varrho\dot{u}\right]\!\right]\mathcal{V}\right)\overset{\square}{u} + \left(Q^{\mathrm{in}} - \left[\!\left[\varrho\dot{v}\right]\!\right]\mathcal{V}\right)\overset{\square}{v} + \left(R^{\mathrm{in}} - \left[\!\left[\iota\dot{\vartheta}\right]\!\right]\mathcal{V}\right)\overset{\square}{\vartheta} = 0.$$

By arguing as in Sect. 2.3, we are led to

$$\mathcal{F}^{\text{in}} = \frac{1}{2}\left[\!\left[\varrho u'^2 + \varrho v'^2 + \iota \vartheta'^2\right]\!\right] \mathcal{V}^2, \tag{36}$$

[*cf.* (2)] alongside with:

$$P^{\text{in}} = \left[\!\left[\varrho \dot{u}\right]\!\right] \mathcal{V}, \quad Q^{\text{in}} = \left[\!\left[\varrho \dot{v}\right]\!\right] \mathcal{V}, \quad R^{\text{in}} = \left[\!\left[\iota \dot{\vartheta}\right]\!\right] \mathcal{V}, \tag{37}$$

as the appropriate constitutive equations for the inertial interactions at the interface.

Remark 2 It is worth noticing that $\mathcal{F}^{\text{in}}$ is not independent from P^{in}, Q^{in}, and R^{in}. In fact, by using the jump conditions (21) and the first of (37) one can verify that:

$$\frac{1}{2}\left[\!\left[\rho u'^2\right]\!\right] \mathcal{V}^2 = -\langle u' \rangle \left[\!\left[\rho \dot{u}\right]\!\right] \mathcal{V} = -\langle u' \rangle P^{\text{in}}.$$

Likewise, one can see that:

$$\mathcal{F}^{\text{in}} = -\langle u' \rangle P^{\text{in}} - \langle v' \rangle Q^{\text{in}} - \langle \vartheta' \rangle R^{\text{in}}. \tag{38}$$

When looking at (38) one may wonder whether a relation such as:

$$\mathcal{F}^{\text{ni}} = -\langle u' \rangle P^{\text{ni}} - \langle v' \rangle Q^{\text{ni}} - \langle \vartheta' \rangle R^{\text{ni}}$$

should involve the non-inertial forces that act at the interface. Our answer is in the negative. The identity (38) follows from the special structure of the inertial power, as encoded in the right-hand side of (34). Non-inertial terms, on the other hand, should be independent from each other. Indeed, the Coleman–Noll argument we used in Sect. 2.4— and we shall use again in Sect. 3.4— requires the realizability of whatever evolution process one may conceive. In order to support an arbitrary evolution process, we must have at our avail a collection of independent, constitutively–unspecified forcing terms, one for each balance equation appearing in (9) and (29–30).

3.4 Dissipation inequality

For the internal force, we restrict attention to constitutive equations of the form:

$$\mathcal{G} = \hat{\mathcal{G}}\left(\mathbf{e}^+, \mathbf{e}^-, \mathcal{V}\right), \tag{39}$$

where $\hat{\mathcal{G}}$ is a *smooth* function and where $\mathbf{e}^{\pm}$ denote the limiting values attained by $\mathbf{e} = (\varepsilon, \gamma, \chi)$ at either side of the interface. We now consider a part $\mathcal{P}$ that *contains the interface*. In order to be consistent with (25), we replace (18) with:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \psi \, \mathrm{d}s \leq \int_{\mathcal{P}} \mathbf{s} \cdot \dot{\mathbf{e}} \, \mathrm{d}s + \mathcal{G}\mathcal{V},$$

when writing the dissipation inequality for $\mathcal{P}$. By a standard transport theorem, we have:

$$\int_{\mathcal{P}} \dot{\psi} \, \mathrm{d}s - \left[\!\left[\psi\right]\!\right] \mathcal{V} \leq \int_{\mathcal{P}} \mathbf{s} \cdot \dot{\mathbf{e}} \, \mathrm{d}s + \mathcal{G}\mathcal{V} \tag{40}$$

By localizing (40) at the interface, we obtain:

$$(\mathcal{G} + \left[\!\left[\psi\right]\!\right])\mathcal{V} \geq 0. \tag{41}$$

By introducing the ***equilibrium*** and ***non-equilibrium*** parts of $\mathcal{G}$:

$$\hat{\mathcal{G}}^{\text{eq}}\left(\mathbf{e}^+, \mathbf{e}^-\right) = \hat{\mathcal{G}}\left(\mathbf{e}^+, \mathbf{e}^-, 0\right),$$
$$\hat{\mathcal{G}}^{\text{ne}}\left(\mathbf{e}^+, \mathbf{e}^-, \mathcal{V}\right) = \hat{\mathcal{G}}\left(\mathbf{e}^+, \mathbf{e}^-, \mathcal{V}\right) - \hat{\mathcal{G}}^{\text{eq}}\left(\mathbf{e}^+, \mathbf{e}^-\right),$$

and by using the constitutive equations (17) and (39), we write (41) as:

$$\left(\hat{\mathcal{G}}^{\mathrm{eq}}\left(\mathbf{e}^{+},\mathbf{e}^{-}\right)+\hat{\mathcal{G}}^{\mathrm{ne}}\left(\mathbf{e}^{+},\mathbf{e}^{-},\mathcal{V}\right)+\left[\!\left[\hat{\psi}\left(\mathbf{e}\right)\right]\!\right]\right)\mathcal{V}\geq 0.$$

By arguing as in Sect. 2.4, we conclude that the equilibrium part must have the form:

$$\hat{\mathcal{G}}^{\mathrm{eq}}\left(\mathbf{e}^{+},\mathbf{e}^{-}\right)=-\left[\!\left[\hat{\psi}\left(\mathbf{e}\right)\right]\!\right]\quad\left(=\hat{\psi}\left(\mathbf{e}^{-}\right)-\hat{\psi}\left(\mathbf{e}^{+}\right)\right),\tag{43}$$

and that the non-equilibrium part must satisfy:

$$\hat{\mathcal{G}}^{\mathrm{ne}}\left(\mathbf{e}^{+},\mathbf{e}^{-},\mathcal{V}\right)\mathcal{V}\geq 0\tag{44}$$

for every $\mathbf{e}^{+}$, $\mathbf{e}^{-}$, and $\mathcal{V}$. We can read into (43) and (44) what we anticipated in the Introduction, namely:

$$\mathcal{G}^{\mathrm{eq}}=-\left[\!\left[\psi\right]\!\right],\quad\mathcal{G}^{\mathrm{ne}}\mathcal{V}\geq 0.\tag{45}$$

On combining (29, 32, 36), and (45), and on setting $\mathcal{F}^{\mathrm{in}}=0$ and

$$\mathsf{c}=\psi-u'\mathsf{n}-v'\mathsf{t}-\vartheta'\mathsf{m},$$

we eventually arrive at the *configurational balance*:

$$\left[\!\left[\mathsf{c}\right]\!\right]-\mathcal{G}^{\mathrm{ne}}=-\frac{1}{2}\left[\!\left[\varrho u'^{2}+\varrho v'^{2}+\iota\vartheta'^{2}\right]\!\right]\mathcal{V}^{2},$$

which is the principal result of this paper.

4 Appendix

In this section, we prove (27) and (35), and we show that (7) follows from (4) when $P^{\mathrm{ni}}=0$, $Q^{\mathrm{ni}}=0$, and $R^{\mathrm{ni}}=0$.

Since $\dot{u}_{\mathrm{v}}$ has support compactly contained in $\mathcal{P}$, a by-parts integration yields:

$$\int_{\mathcal{P}}\mathsf{n}\dot{\varepsilon}_{\mathrm{v}}\,\mathrm{d}s=-\int_{\mathcal{P}}\mathsf{n}'\dot{u}_{\mathrm{v}}\,\mathrm{d}s-\left[\!\left[\mathsf{n}\dot{u}_{\mathrm{v}}\right]\!\right].$$

On recalling the identity:

$$\left[\!\left[\alpha\beta\right]\!\right]=\left[\!\left[\alpha\right]\!\right]\left\langle\beta\right\rangle+\left\langle\alpha\right\rangle\left[\!\left[\beta\right]\!\right],\tag{46}$$

and using (23) and (24) we obtain:

$$\begin{aligned}-\left[\!\left[\mathsf{n}\dot{u}_{\mathrm{v}}\right]\!\right]&=-\left[\!\left[\mathsf{n}\right]\!\right]\left\langle\dot{u}_{\mathrm{v}}\right\rangle-\left\langle\mathsf{n}\right\rangle\left[\!\left[\dot{u}_{\mathrm{v}}\right]\!\right]=-\left[\!\left[\mathsf{n}\right]\!\right]\overset{\square}{u}_{\mathrm{v}}+\left[\!\left[\mathsf{n}\right]\!\right]\left\langle u'\right\rangle\mathcal{V}_{\mathrm{v}}+\left\langle\mathsf{n}\right\rangle\left[\!\left[u'\right]\!\right]\mathcal{V}_{\mathrm{v}}\\&=-\left[\!\left[\mathsf{n}\right]\!\right]\overset{\square}{u}_{\mathrm{v}}+\left[\!\left[\mathsf{n}u'\right]\!\right]\mathcal{V}_{\mathrm{v}},\end{aligned}$$

whence:

$$\int_{\mathcal{P}}\mathsf{n}\dot{\varepsilon}_{\mathrm{v}}\,\mathrm{d}s=-\int_{\mathcal{P}}\mathsf{n}'\dot{u}_{\mathrm{v}}\,\mathrm{d}s+\left[\!\left[\mathsf{n}u'\right]\!\right]\mathcal{V}_{\mathrm{v}}-\left[\!\left[\mathsf{n}\right]\!\right]\overset{\square}{u}_{\mathrm{v}}.$$

By handling in the same fashion the other terms in the integral sign on the right-hand side of (25), we obtain (27).

As to (35), the chain of equations:[6]

$$\begin{aligned}\frac{1}{2}\left[\!\left[\varrho\dot{u}^{2}\right]\!\right]&=\left[\!\left[\varrho\dot{u}\right]\!\right]\left\langle\dot{u}\right\rangle=\left[\!\left[\varrho\dot{u}\right]\!\right]\overset{\square}{u}-\left[\!\left[\varrho\dot{u}\right]\!\right]\left\langle u'\right\rangle\mathcal{V}=\left[\!\left[\varrho\dot{u}\right]\!\right]\overset{\square}{u}+\left[\!\left[\varrho u'\right]\!\right]\left\langle u'\right\rangle\mathcal{V}^{2}\\&=\frac{1}{2}\left[\!\left[\varrho u'^{2}\right]\!\right]\mathcal{V}^{2}+\left[\!\left[\varrho\dot{u}\right]\!\right]\overset{\square}{u}\end{aligned}$$

should satisfy the Reader.

[6] Here we are using (21, 22), and (46), and we are bearing in mind that ϱ is a constant.

We close with the derivation of (7) from (4). Let $P^{\rm ni}=0$, $Q^{\rm ni}=0$, and $R^{\rm ni}=0$. Using the identity (46), the jump conditions (30), the constitutive equations (37), and the compatibility conditions (21) we get:

$$\begin{aligned}-\left[\!\left[u'\mathsf{n}\right]\!\right] &= -\langle u'\rangle\left[\!\left[\mathsf{n}\right]\!\right]-\left[\!\left[u'\right]\!\right]\langle\mathsf{n}\rangle=\langle u'\rangle P^{\rm in}-\left[\!\left[u'\right]\!\right]\langle\mathsf{n}\rangle=\langle u'\rangle\left[\!\left[\varrho\dot{u}\right]\!\right]\mathcal{V}-\left[\!\left[u'\right]\!\right]\langle\mathsf{n}\rangle\\ &= -\frac{1}{2}\left[\!\left[\varrho u'^2\right]\!\right]\mathcal{V}^2-\left[\!\left[u'\right]\!\right]\langle\mathsf{n}\rangle.\end{aligned}$$

It is now apparent that:

$$-\left[\!\left[u'\mathsf{n}+v'\mathsf{t}+\vartheta'\mathsf{m}\right]\!\right]=-\frac{1}{2}\left[\!\left[\varrho u'^2+\varrho v'^2+\iota\vartheta'^2\right]\!\right]\mathcal{V}^2-\left[\!\left[u'\right]\!\right]\langle\mathsf{n}\rangle-\left[\!\left[v'\right]\!\right]\langle\mathsf{t}\rangle-\left[\!\left[\vartheta'\right]\!\right]\langle\mathsf{m}\rangle,$$

an identity which we combine with (3) and (4) to arrive at the desired result.

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