Math. Ann. 291, 275–280 (1991)

Mathematische Annalen

Computing the equivariant cohomology of group compactifications

Elisabetta Strickland

Dipartimento di Matematica, II. Università di Roma "Tor Vergata", I-00133 Rome, Italy

Received May 21, 1990; in revised form March 15, 1991

Introduction

In [S] we have given a method to compute the equivariant Betti numbers of any wonderful symmetric variety in terms of some simple contributions which can be computed effectively using the classification of symmetric pairs. In this paper we shall use the full strength of the result in [BDP] to give an algorithmical way to determine the $G\times G$-equivariant cohomology ring of the regular $G\times G$-equivariant embeddings of a semisimple adjoint group G.

This will be done on one hand in terms of some combinatorial objects associated to the fan of the embedding which we shall call the Stanley-Reisnèr constant system of the fan, and on the other hand, of a sheaf on the fundamental Weyl chamber for G, which is independent from the choice of the embedding. One can more generally treat in a similar way also the case of any regular symmetric variety. We plan to do this in a subsequent work, in order to keep here the notational as well as the technical apparatus to a minimum (in the general case, at the moment, a lengthy case by case examination is needed, using the classification), and also to stress the computational aspects of our results.

1 Regular embeddings of G

Let G be a semisimple algebraic group of adjoint type over the complex numbers $\mathbb{C}$. We fix once and for all a maximal torus $T\subset G$, and a Borel subgroup $T\subset B\subset G$. We denote by $R\subset X^*(T)\otimes\mathbb{Q}=V$ the corresponding root system, by $\Delta\subset R$ the corresponding set of simple roots and by $W=N(T)/T$ the Weyl group. W acts on the space $V^*\cong X_*(T)\otimes\mathbb{Q}$ by its reflection representation and our choice gives us a canonically chosen fundamental domain C for the W action which we shall call the fundamental Weyl chamber. Now $G\times G$ acts on G by left and right multiplication, and by an embedding of G we shall mean a point $G\times G$ variety (X,p) such that the $G\times G$ orbit X_0 of p is dense and open in X and the stabilizer of p in $G\times G$ is the subgroup G diagonally embedded, so that

$$X_0 \cong X \times G/G_\Delta \cong G,$$

where G_Δ is the diagonal in $G \times G$. Under this identification the $G \times G$ action on X_0 becomes the $G \times G$ action on G by left and right multiplication. We shall say that (X, p) or briefly X is regular if the closure of every $G \times G$-orbit in X is smooth, if $X - X_0$ is a divisor with normal crossings each of whose components is an orbit closure which will be called a boundary divisor and given an orbit $\theta \subset X$, $\theta \neq X_0$, then $\bar{\theta}$ is the intersection of the boundary divisors containing θ.

One has a classification for such embeddings which goes as follows [DP].

Let F be a rational regular fan in V^* with support in C (we shall refer to F as to a fan in C). Using the W action, we can clearly construct a W-stable fan in V^*, call it $\tilde{F}$. Now to $\tilde{F}$ corresponds (see [O]), a smooth T embedding $Z_{\tilde{F}}$ and one has that there is one and only one regular G embedding $(X_{F,p})$ such that $\bar{T}_p = Z_{\tilde{F}}$. So the map associating to the fan F the variety X_F is a bijection between regular fans in C and regular G embeddings.

Indeed one can describe a good deal of the geometry of X_F just using F. We give here just the results we shall need below. First of all there is a bijective correspondence between faces of F and $G \times G$ orbits in X_F which reserves incidence relations between classes so that to the face $\{0\}$ corresponds the open orbit X_0. Using this, one can explicitly describe the orbit θ_f corresponding to a face f of F as a $G \times G$-homogeneous space $G \times G/H_f$.

Indeed fix a face f and let γ be the smallest face of C containing f. To γ we can associate a parabolic group $T \subset P \subset G$ (here we index parabolics in such a way that to $\{0\}$ there corresponds G and to C the Borel subgroup B). Let $U \subset P$ be the unipotent radical of P, $L = P/U$. Let Z be the center of L and consider in $Z \times Z$ the subgroup Γ_f generated by the one parameter subgroup (λ, λ^{-1}), $\lambda \in f \cap X_*(T)$ [recall that $X_*(T)$ is the space of one parameter subgroups in T and that under the projection $P \to P/U = L$, T maps isomorphically onto a maximal torus of L which hence contains Z].

If we embed L in $L \times L$ diagonally, we can consider in $L \times L$ the subgroup K_f generated by L_Δ and Γ_f. One has

$$H_f = \pi^{-1}(K_f),$$

$\pi: P \times P \to L \times L$ being the canonical projection.

Remark 1.1. Notice that H_f depends on f and not just on γ only in the point relative to Γ_f.

2 Equivariant cohomology

Let K be a topological group, X be a K-space. Then one can associate to X its equivariant cohomology ring, $H_K^*(X, \mathbb{Q}) = H^*(X \times_K EK, \mathbb{Q})$, $p: EK \to BK$ being the universal fibration for K. Notice that we have used rational coefficients, since this will always be the case, in what follows we shall denote $H_K^*(X, \mathbb{Q})$ simply by $H_K^*(X)$.

In the case in which, using the notations of Sect. 1, $K = G \times G$ and X is regular G embedding, one can apply the procedure given in [BDP] to compute $H_{G \times G}^*(X)$. We recall it briefly in our special case.

Given a regular fan F, a Stanley-Reisner, $(R-S)$-system $\mathscr{R}$ on F is the following set of data:

a) For any face f of F a commutative $\mathbb{Q}$-algebra with 1 R_f and a regular sequence $x_{f_1}^{(f)}, \ldots, x_{f_h}^{(f)}$, where $\{f_1, \ldots, f_h\}$ is the set of one-dimensional faces of f.

b) Given a face f of F, $f = \{f_1, \ldots, f_h\}$ and letting

$$f^{(i)} = \{f_1 \ldots \check{f}_i \ldots f_h\}$$

and algebra homomorphism

$$\phi_f^{f(i)}: R_{f^{(i)}} \to R_f/(x_{f_i}^{(f)})$$

such that

$$\phi_f^{f^{(i)}}(x_{f_i}^{f^{(i)}}) \equiv x_j^{(f)} \ \forall j \neq i.$$

To $\mathscr{R}$ one associates the algebra

$$R \subset \bigoplus_f R_f$$

defined as the algebra of sequences (a_f) such that

$$\psi_f^{f^{(i)}}(a_{f^{(i)}}) \equiv a_f$$

for any f and $f^{(i)}$ as above.

R is called the $(R-S)$-algebra of $\mathscr{R}$.

Example. One defines $\mathscr{C}$, the constant $(R-S)$-system, as follows. If $f = (f_1 \ldots f_h)$ we set $C_f = \mathbb{Q}[x_{f_1} \ldots x_{f_h}]$ and $\varrho_f^{f^{(i)}}$ has then an obvious definition.

If F is a regular fan in C and let X_F be the associate embedding, we can define such a $(S-R)$-system as follows. Given a face f of F and letting θ_f be the corresponding orbit, we set

$$R_f = H_{G \times G}(\theta_f).$$

If $f = \{f_1 \ldots f_h\}$, i.e. $\bar{\theta}_f$ is the intersection of the boundary divisors $D_{f_1} \ldots D_{f_h}$ which are the closures of the orbits $\theta_{f_1}, \ldots, \theta_{f_h}$ then the invertible sheaf $\theta(D_{f_i})$ has a canonical $G \times G$ linearization and so has its restriction to θ_f.

We then set $x_{f_i} = c_1(\theta(D_{f_i})/_{\theta_f}) \in H^2_{G \times G}(\theta_f)$. Finally, if $f^{(i)}$ and f are as above, then for the orbit $\theta_f, \bar{\theta}_f = D_{f_i} \cap \bar{\theta}_{f^{(i)}}$.

Now taking $K \subset G$ to be a maximal compact subgroup, we can find a $K \times K$ stable tubular neighborhood N of θ_f in $\theta_{f^{(i)}} \cup \theta_f$.

Let $N^* = N - \theta_f \subset \theta_{f^{(i)}}$.

Using the fact that for a $G \times G$-space, $K \times K$ equivariant cohomology is the same as $G \times G$-equivariant cohomology, and Gysing sequence, one sees that

$$H^*_{K \times K}(N^*) \cong H^*_{G \times G}(\theta_f)/_{x_{f_i}}$$

so the natural map

$$H^*_{G \times G}(\theta_{f^{(i)}}) \cong H^*_{K \times K}(\theta_{f^{(i)}}) \to H^*_{K \times K}(N^*)$$

given by inclusion gives the desired $\psi_f^{f^{(i)}}$. The main result in [BDP] in our case gives:

Theorem 2.1. *$H^*_{G \times G}(X)$ is the $(R-S)$-algebra of the $(R-S)$-system defined above.*

Thus in order to compute $H^*_{G \times G}(X)$ it is essentially enough to determine the $(R-S)$-system associated to X. This is what we are going to do in the next section.

3 The fundamental sheaf

Definition. Let F be a fan. A *sheaf* $\mathscr{A}$ on F is the set of the following data:

a) For any face f of F, a commutative ring A_f, with identity;

b) if f' is a face of f, an homomorphism

$$\phi_f^{f'}: A_{f'} \to A_f$$

such that if f' is a face of f_1 and f_2 and these are faces of f then

$$\phi_{f_1}^{f'} \phi_f^{f_1} = \phi_{f_2}^{f'} \phi_f^{f_2}.$$

Lemma 3.1. *Given a sheaf $\mathscr{A}$ on a regular fan, we can construct an associated $(R-S)$-system on F denoted by $\mathscr{A} \otimes \mathscr{C}$, the tensor product of $\mathscr{A}$ with the constant system.*

Proof. We construct $\mathscr{A} \otimes \mathscr{C}$ as follows. Given f in F, we set $R_f = A_f \otimes C_f = A_f[x_{f_1} \dots x_{f_h}]$ and as maps $\psi_f^{f'}$ we set

$$\psi_f^{f'} = \phi_f^{f'} \otimes \varrho_f^{f'}.$$

Our main technical result is

Theorem 3.2. *Let F be a regular fan in C, X_F be the associated regular embedding, $\mathscr{R}$ the associated $(R-S)$-system on F. Then $\mathscr{R} = \mathscr{A} \otimes \mathscr{C}$, where $\mathscr{A}$ is the following sheaf on F. Given a face f of F, letting γ be the smallest face of C containing f and P the associated parabolic subgroup, then*

$$A_f \cong H^*(BP, \mathbb{Q}).$$

If f' is a face of f, then $P' \supseteq P$ and we set

$$\phi_f^{f'}: H^*(BP', \mathbb{Q}) \to H^*(BP, \mathbb{Q})$$

to be the map induced by inclusion.

Proof. We start with the determination of R_f. By definition $R_f \cong H^*_{G \times G}(\theta_f, \mathbb{Q}) \cong H^*_{G \times G}(G/H_f \mathbb{Q}) \cong H^*(BH_f, \mathbb{Q})$.

Now consider the homomorphism $\pi: H_f \to K_f$ used in the definition of H_f. Its kernel is unipotent so that

$$H^*(BH_f, \mathbb{Q}) \cong H^*(BK_f, \mathbb{Q}).$$

On the other hand, the map $L \times \Gamma_f \to K_f$ given by product is well known to be an isogeny. Thus since we work with rational coefficients, it induces an isomorphism

$$H^*(BK_f, \mathbb{Q}) \cong H^*(B(L \times \Gamma_f), \mathbb{Q}) \cong H^*(BL, \mathbb{Q}) \otimes H^*(B\Gamma_f, \mathbb{Q}).$$

Since $\dim \Gamma_f = \dim f$ and Γ_f is a torus, we get that $H^*(B\Gamma_f, \mathbb{Q}) \cong \mathbb{Q}[x_1 \dots x_h]$, $h = \operatorname{cod} \theta_f$. On the other hand, we have that, if we consider the projection $p: P \to L$, this has a unipotent kernel, so that

$$H^*(BP, \mathbb{Q}) \cong H^*(BL, \mathbb{Q}).$$

Putting this together we get

$$H^*_{G \times G}(\theta_f) \cong H^*(BP, \mathbb{Q})[x_1 \dots x_h] = A_g \otimes \mathbb{Q}[x_1 \dots x_h].$$

We now have to determine the map $\psi_f^{f^{(i)}}$. We just sketch the argument which is a tedious verification. Let $f = \{f_1 \dots f_h\}$. Let L_i be the line bundle on θ_f associated to

$\theta(D_i)/\theta_f$, $L_i^* = L_i$-{zero section}. Then on one hand we clearly have

$$H^*_{G\times G}(L_i^*) \cong H^*_{K\times K}(N^*).$$

On the other hand, L_i^* is itself a $G\times G$ homogeneous space isomorphic to $G\times G/H_f^{(i)}$ where $H_f^{(i)}\subset H_f$ is the following subgroup. Consider in Γ_f the subgroup $\Gamma_f^{(i)}$ generated by those one parameter groups (λ, λ^{-1}) where λ is a point in $f^{(i)}\cap X_*(T)$ and let $K_f^{(i)}\subset K_f$ be the subgroup generated by L and $\Gamma_f^{(i)}$.

Then $H_f^{(i)} = \pi^{-1}(K_f^{(i)})$, $\pi: P\times P\to L\times L$ being the natural projection.

Now it is clear from the definition that one has an embedding $K_f^{(i)}\subset K_{f^{(i)}}$, so an induced map $H^*(BK_{f^{(i)}}, \mathbb{Q})\to H^*(BK_f^{(i)}, \mathbb{Q})$.

From this one readily sees that we get a commutative diagram

$$\begin{array}{ccc} H^*(BK_{f^{(i)}}, \mathbb{Q}) & \longrightarrow & H^*(BK_f^{(i)}, \mathbb{Q}) \\ \downarrow \wr & & \downarrow \wr \\ H^*_{G\times G}(\theta_{f^{(i)}}) & \xrightarrow{\psi_f^{f^{(i)}}} & H^*_{G\times G}(\theta_f)/(x_{f_i}) \cong H^*(BH_f^{(i)}, \mathbb{Q}). \end{array}$$

Using this, the maps $\psi_f^{f^{(i)}}$ are immediately computed, proving our claim. q.e.d.

Notice that if F' is a fan which is a partial decomposition of F i.e. such that every face of F' is contained in a face of F, and we have a sheaf $\mathscr{A}$ in F, we can obviously define the sheaf induced on F', by setting for a face f' of F' $A_{f'}$ equal to A_f, f being the smallest face of F containing f' and defining the maps accordingly.

From this we immediately get:

Proposition 3.3. *Let F be a fan in C, let X_F be the corresponding G-embedding, $\mathscr{R}$ the corresponding $(R-S)$-system, $\mathscr{R} = \mathscr{A}\otimes\mathscr{C}$, $\mathscr{A}$ is the sheaf defined in the theorem. Then $\mathscr{A}$ is induced by the following sheaf $\mathscr{A}_C$ on C.*

If γ is a face on C, $A_\gamma = H^*(BP, \mathbb{Q})$, P being the parabolic subgroup associated to γ and if γ' is a face of γ so that $P'\supset P$

$$\phi_\gamma^{\gamma'}: H^*(BP', \mathbb{Q})\to H^*(BP, \mathbb{Q})$$

is the map induced by inclusion.

Proof. Obvious from the theorem.

We want now to completely describe $\mathscr{A}_C$. Let $V^* = X^*(T)\otimes\mathbb{Q}$. For any parabolic $P\supset T$, choose the unique Levi factor $T\subset L\subset P$.

We have $H^*(BP, \mathbb{Q})\cong H^*(BL, \mathbb{Q}) = A_\gamma$ for a suitable face γ of C. On the other hand, $H^*(BT, \mathbb{Q})\cong\mathbb{Q}[V^*]$. Let $W_L = N_L(T)/T$. The Weyl group of L (if L is associated to a face γ this is just the subgroup of W generated by those simple reflections which fix γ): W_L acts on V^*.

Consider the map

$$H^*(BL, \mathbb{Q})\to H^*(BT, \mathbb{Q}) = \mathbb{Q}[V^*].$$

Then we know that $H^*(BL, \mathbb{Q})$ maps injectively onto $\mathbb{Q}[V^*]^{W_L}$.

Since these identifications are obviously compatible with the maps induced by inclusion, we get our main result.

Theorem 3.4. *The sheaf $\mathscr{A}_C$ on C admits the following combinatorial description. Given a face γ of C, let W_γ be the subgroup of W generated by those single reflections which fix γ. Then*

$$A_\gamma = \mathbb{Q}[V^*]^{W_\gamma}.$$

If γ' is a face of γ clearly, $W_{\gamma'} \supset W_\gamma$ and

$$\phi_\gamma^{\gamma'}: A_{\gamma'} \to A_\gamma$$

is the obvious inclusion.

In this way we have described the fundamental sheaf for G in purely combinatorial terms, so that the computation of $H^*_{G \times G}(X)$ is reduced for any regular G-embedding X to combinatorial computations in terms of $\mathscr{A}_C$ and of the fan associated to X.

We finish giving the example of type A_{n-1}, $n \geqq 2$. In this case $W = S^n$, the faces of C are induced by sequences $\underline{n} = (n_1, \ldots, n_h)$, with $n_1 + \ldots + n_h = n$ and the corresponding Weyl group is

$$S_{\underline{n}} = S^{n_1} \times S^{n_2} \times \ldots \times S^{n_h}$$

which is naturally embedded in S^n. We introduce a partial order $\leqq$ on sequences as above as the order generated by the following elementary steps:

$$(n_1, \ldots, n_k) \geqq (n_1, \ldots, n_{i-1}, n_{i+1}, n_{i+2}, \ldots, n_h)$$

for $1 \leqq i \leqq h-1$.

Then it is easy to see that if γ' and γ are faces of C with sequences n' and n, γ' is contained in $\gamma \Leftrightarrow \underline{n}' < \underline{n}$. Notice also that we have the obvious embedding $S_{\underline{n}} \hookrightarrow S_{\underline{n}'}$. In order to describe $\mathscr{A}_C$, it is better to describe a slightly larger sheaf.

Consider $V^* \oplus \mathbb{Q}$ with obvious $S^n = W$ action. Then we know that we can choose a basis of $V^* \oplus \mathbb{Q}$ so to identify it with the canonical n-dimensional permutation representation of S^n. Using this basis, we identify

$$\mathbb{Q}(V^* \oplus \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]$$

and then setting $A'_{\underline{n}} = \mathbb{Q}[x_1, \ldots, x_h]^{S_{\underline{n}}}$ i.e. the functions symmetric in the variables $x_1, \ldots, x_{n_1}; x_{n_1+1}, \ldots, x_{n_1+n_2}; \ldots; x_{n_1+\ldots+n_{h-1}+1}, \ldots, x_n$ separately, we get

$$A_{\underline{n}} = A'_{\underline{n}}/_{x_1+\ldots x_n}.$$

References

[BDP] Bifet, E., De Concini, C., Procesi, C.: Equivariant cohomology of regular embeddings. (Preprint)

[DP] De Concini, C., Procesi, C.: Complete symmetric varieties. II. Adv. Stud. Pure Math. 6, 481–513 (1984)

[O] Oda, T.: Torus embeddings and applications. Lect. Math. Phys., Math., Tata Inst. Fundam. Res. 57 (1978)

[S] Strickland, E.: Equivariant Betti numbers for symmetric varieties. J. Algebra (to appear)