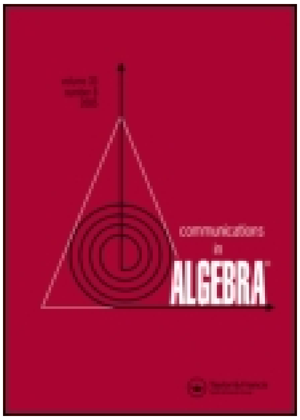

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E. Strickland [a]

[a] Dipartimento di Matematica, Universitá di Roma "Tor Vergata", Via della Ricerca Scientifica, Roma, 00133, Italy

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Projective Spaces in Flag Varieties

E. Strickland

Dipartimento di Matematica
Università di Roma "Tor Vergata",
Via della Ricerca Scientifica, 00133, Roma, Italy.

0. Introduction

Let G be a semisimple algebraic group over an algebraically closed field k of characteristic zero. Let $\mathcal{B}$ be the variety of Borel subgroups. Consider the the projective embedding $\mathcal{B} \to \mathbb{P}(H^0(\mathcal{B}, L_\rho)^*)$, where L_ρ is the line bundle associated to the Steinberg weight ρ, which is the half sum of positive roots. In this paper we show than any positive dimensional projective space contained in the image of $\mathcal{B}$ is necessarily of dimension one.

Furthermore we determine exactly these lines. Indeed we show that if $\ell \subset \mathcal{B}$ is such a line, then there exists a minimal parabolic subgroup $P \subset G$ such that ℓ is the set of Borel subgroups which are contained in P. In particular this implies that the possible homology classes of such lines correspond under the usual identification of the root lattice with $H_2(\mathcal{B}, \mathbb{Z})$ to the set of simple roots.

The result is obtained as an application of some properties of the intersection of Schubert cycles in the cohomology ring of $\mathcal{B}$.

1. Linear Spaces in $\mathcal{B}$

Given a semisimple simply connected algebraic group G we want to recall a few facts on the geometry of the projective variety, $\mathcal{B}$, of the Borel subgroups of G.

Let us choose a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$. Let $W = N(T)/T$ be the Weyl group. Let $\mathfrak{t} = \text{Lie}T$ be the corresponding Cartan subalgebra, $\Phi \subset \mathfrak{t}^*$ be the root system associated to Lie$T$ and Φ^+ be the set of positive roots corresponding to the choice of B and $\{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots. Let us denote by B^- the unique Borel subgroup such that $B \cap B^- = T$. Furthermore we shall denote by $s_\alpha \in W$ the simple ref ection corresponding to a root $\alpha \in \Phi$ and, for an element $w \in W$, $\ell(w)$ the length of w with respect to the generators given by the simple ref ections $s_i = s_{\alpha_i}$. As usual we shall denote by w_0 the longest element in W. We shall also consider the weight lattice $\Lambda = X^*(T)$, which is the dual of the root lattice Q, and denote by $\{\omega_1, \ldots, \omega_r\}$ the set of fundamental weights defined by $\langle\omega_i, \alpha_j\rangle = \delta_{i,j}$. We set $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = \sum_{i=1}^r \omega_i$.

Since all Borel subgroups in G are conjugate and a Borel subgroup equals its normalizer, we can identify $\mathcal{B}$ with G/B. Once this has been done, we get an explicit cellular decomposition by locally closed affine spaces of $\mathcal{B}$, whose cells, indexed by the elements $w \in W$, are the B orbits $C_w = Bn_wB/B$, $n_w \subset N(T)$ being a representative of w. The C_w's are called Schubert cells and their closures $\overline{C}_w$ Schubert varieties. One has dim $C_w = \ell(w)$. It follows from this [F], that the Chow ring $A^*(\mathcal{B})$ has a basis given by the classes $[\overline{C}_w]$, dual to the classes of the Schubert varieties and, if $k = \mathbb{C}$, coincides with the cohomology (doubling the degrees). We also have that if we consider the pairing on $A^*(\mathcal{B})$ given by

$$(x, y) = \int_{\mathcal{B}} xy$$

for $x, y \in A^*(\mathcal{B})$, then we have that $([\overline{C}_w], [\overline{C}_{w'}]) = \delta_{ww',w_0}$. This is easily seen by taking the cellular decomposition C_w^- corresponding to B^- and remarking that the intersection of C_w and $C_{w'}^-$ is empty unless $ww' = w_0$, while if $ww' = w_0$, C_w and $C_{w'}^-$ intersect transversally in a single point.

As far as A^1 is concerned, one can give another description. Namely, given a character $\lambda \in X^*(T)$, we can extend, using the natural homomorphism $B \to T$, λ to a character of B, and thus consider the corresponding one dimensional B module k_λ. We now define the line bundle L_λ on $\mathcal{B} = G/B$ by $G \times_B k_\lambda$. By taking the first Chern class $c(L_\lambda)$ we get a isomorphism between Λ and $A^1(\mathcal{B})$ which takes the fundamental weights ω_i to the classes $[\overline{C}_{w_0 s_i}]$.

Remark that, using the above considerations, we deduce that, if we identify $A^{\dim \mathcal{B}-1}(\mathcal{B})$ with the root lattice Q by associating to a simple root α_i the class $[\overline{C}_{s_i}]$, we get that for $x \in A^1(\mathcal{B})$ and $y \in A^{\dim \mathcal{B}-1}(\mathcal{B})$, $(x, y) = \langle x, y \rangle$.

Before we proceed, we need the following simple fact:

Proposition 1.1. *Let $X \subset \mathcal{B}$ be a subvariety. Let $[X] \in A^*(\mathcal{B})$ denote the corresponding dual class. Then $[X]$ is a linear combination of the Schubert classes $[\overline{C}_w]$ with non negative integer coefficients.*

Proof. We can also clearly assume that X is of pure dimension d. The only thing we need to prove is that, when we express $[X]$ as a linear combination of Schubert classes, the coefficients are non negative.

Indeed by a result of Kleiman (see [H] p. 268), since $\mathcal{B}$ is a homogeneous space, we can find an element $g \in G$ such that for every Schubert cell C_w of dimension complementary to that of X, $X \cap \overline{C}_w$ is contained in C_w and consists of a finite number of simple points. This clearly implies that, if we set $n_w = |X \cap \overline{C}_w|$, then $[X] = \sum_{\ell(w)=d} n_{w_0 w^{-1}} [\overline{C}_w]$, hence the claim. □

Corollary 1.2. *Let $a_1, \ldots, a_t$ be Schubert classes, then the product class $a_1 \cdots a_t$ is a linear combination with non negative integer coefficients of Schubert classes.*

Proof. By an easy induction, we can assume that $t = 2$. Let $a_1 = [\overline{C}_{w_1}]$, $a_2 = [\overline{C}_{w_2}]$. Then again, by the result of Kleiman quoted above, we can find $g \in G$ such that $a_1 a_2 = [\overline{C}_{w_1} \cap g\overline{C}_{w_2}]$ and the corollary follows from Proposition 1.1. □

Recall now that to any simple root α_i we can associate a minimal parabolic group $B \subset P_i \subset G$ such that the natural map

$$\pi_i : G/B \to G/P_i$$

is a $\mathbb{P}^1$-fibration whose fiber over [1] is exactly the Schubert variety $[\overline{C}_{s_i}]$. Also given a weight $\lambda \in \Lambda$, the corresponding line bundle L_λ is very ample if and only if $\langle \lambda, \alpha_i \rangle > 0$ for all $i = 1, \dots, r$. Furthermore L_λ is the pull back of a line bundle on G/P_i, which by abuse of notation we shall denote by the same name, if and only if $\langle \lambda, \alpha_i \rangle = 0$, and, if this is the case, L_λ is very ample on G/P_i if and only if $\langle \lambda, \alpha_j \rangle > 0$ for all $j \neq i$.

Let us now consider the weight $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^r \omega_i$. By the above discussion the line bundle L_ρ is very ample on $\mathcal{B}$ while the line bundle L_{ρ_i}, with $\rho_i = \rho - \omega_i$ is very ample on G/P_i.

Consider now $\mathcal{B}$ embedded in $\mathbb{P}(H^0(\mathcal{B}, L_\rho)^*)$ and $C^-_{w'}$.

We have the following

Theorem 1.3. *Let S be a positive dimensional linear subspace in $\mathbb{P}(H^0(\mathcal{B}, L_\rho)^*)$. Assume $S \subset \mathcal{B}$. Then*

(1) *dim S=1*

(2) *There is a Borel subgroup $B \subset G$ and a simple ref ection s_i, such that $S$is the Schubert variety $\overline{C}_{s_i}$relative to B i.e. $S = P/B$, P being the unique minimal parabolic subgroup containing B and associated to s_i.*

Proof. Set $t = \dim S$ and $N = |\Phi^+| = \dim \mathcal{B}$. Denote by c_i the Chern class of the line bundle L_{ω_i}, so that for any $\lambda = \sum_i n_i \omega_i$, $c(L_\lambda) = \sum_i n_i c_i$.

By definition we have that

$$\int_{\mathcal{B}} [S] \cap (c_1 + \cdots + c_r)^t = 1$$

Now, by Proposition 1.1 and Corollary 1.2, we immediately deduce that if M is any degree t monomial in $c_1, \dots, c_r$,

$$\int_{\mathcal{B}} [S] \cap M \geq 0.$$

It follows, using the expansion of $(c_1 + \cdots + c_r)^t$, that there exists a unique i such that

$$\int_B [S] \cap c_i^t = 1,$$

while, for any other degree t monomial in $c_1, \ldots, c_r$,

$$\int_B [S] \cap M = 0.$$

In particular we deduce that

$$\int_B [S] \cap (c_1 + \cdots \check{c}_i + c_r)^t = 0.$$

Consider now the projection $\pi_i B \to G/P_i$. As we have already remarked, the line bundle $L_{\rho-\omega_i}$ is very ample on G/P_i, so that

$$\int_B [S] \cap (c_1 + \cdots \check{c}_i + c_r)^t = \int_{G/P_i} \pi_{i*}[S] \cap (c_1 + \cdots \check{c}_i + c_r)^t = 0$$

implies that dim $\pi_i(S) < t$.

Now π_i is a $\mathbb{P}^1$-fibration, so dim $\pi_i(S) < t - 1$. Moreover $\pi_i^{-1}(\pi_i(S))$ is an irreducible variety containing S and of the same dimension of S. Hence $S = \pi_i^{-1}(\pi_i(S))$.

We have thus shown that our S is the total space of the $\mathbb{P}^1$-fibration $S \to \pi_i(S)$.

If we consider the line bundle L on S obtained by restricting L_ρ, we deduce that S equals $\mathbb{P}(\pi_{i*}(L))$. Thus $H^2(S) = H^2(\pi_i(S)) \oplus H^0(\pi_i(S))$. If $t > 1$, then rk$H^2(\pi_i(S)) \geq 1$, so we would deduce that rk$H^2(S) \geq 2$, a contradiction. Therefore the first part of our Theorem follows.

As for the second part, notice that we have proved that S is a fiber of the $\mathbb{P}^1$-fibration π_i. Thus, if we take a Borel subgroup $B \in S$, we deduce that $S = \overline{C}_{s_i}$.

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