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Lines in G/P

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1. Introduction

Let G be a semisimple algebraic group over an algebraically closed field k (char$k \neq 2$). Let $P \subset G$ denote a parabolic subgroup. Consider the projective homogeneous space G/P and a very ample line bundle L on G/P. In this paper we shall give an answer to the following two questions:

(1) For which L the image of G/P inside the projective space $\mathbb{P}((H^0(G/P, L))^*)$ contains a line.
(2) Supposing such a line exists, which is its class in $H_2(G/P, \mathbb{Z})$ and, for a fixed class, describe the subvariety of the Grassmannian of lines in $\mathbb{P}((H^0(G/P, L))^*)$ which lie in G/P and represent this class.

Our goal is reached as follows. Fix a maximal torus T in G and a Borel subgroup $B \supset T$, so that we can consider the corresponding root lattice Q, weight lattice Π, set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and set of fundamental weights $\Omega = \{\omega_1, \dots, \omega_n\}$. We then choose $P \supset B$ so that P corresponds to a subset S of Δ or equivalently Ω or also equivalently of the Dynkin diagram of G. Recall that the Picard group of G/P can be identified with a sublattice of Π. Taken an ample $L \in Pic(G/P)$, and setting $\lambda = \sum_i n_i\omega_i$ equal to the corresponding element in Π, we show that: 1) $G/P \subset \mathbb{P}((H^0(G/P, L))*)$ contains a line if and only if $n_i = 1$ for at least one i.

Suppose now this is the case, and recall that we can identify $H_2(G/P, \mathbb{Z})$ with a sublattice of the lattice ${}^\vee Q$ spanned by coroots. Then:

2) The homology class of a line in $G/P \subset \mathbb{P}((H^0(G/P, L))^*)$ is exactly equal to one of the ${}^\vee\alpha_i$ for which $n_i = 1$. The corresponding variety of lines in G/P, Z_{S_i}, then depends only on ${}^\vee\alpha_i$.

Once this has been shown, we describe the structure of our variety Z_{S_i}. In Sect. 3 we achieve this under some additional assumptions (see Lemma 1, in particular this works in full generality in the simply laced case). In this case once i has been fixed, we can define a new parabolic subgroup P' and consider the parabolic $Q = P \cap P'$. We then show that Z_{S_i} is G/P' and that the natural map $G/Q \to G/P'$ is the family of such lines. In the general case we show that Z_{S_i} has at most two orbits and we give a detailed description of the exceptional cases in which Z_{S_i} is not homogeneous in Sect. 4. In the case in which P is maximal, similar results to ours have been obtained in [CC] (see also [LM]).

2. Some lemmas on root systems

As in the introduction, let G be a semisimple algebraic group over an algebraically closed field k. We choose a maximal torus T and a Borel subgroup B and let $W = N(T)/T$ be the Weyl group. We denote by Q the corresponding root lattice, Π the weight lattice, ${}^\vee Q$ the lattice of coroots. We also let $\Phi \subset Q$ be the root system, $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots and $\Omega = \{\omega_1, \dots, \omega_n\}$ be the set of fundamental weights.

We choose a W invariant scalar product (,) on $Q \otimes \mathbb{Q}$. Given a root α, we define the corresponding coroot ${}^\vee\alpha$ by

$$\langle x, {}^\vee\alpha\rangle := \frac{2(x,\alpha)}{(\alpha,\alpha)}$$

We shall denote by $D := D_\Phi$ the Dynkin diagram of our root system. The vertex of D corresponding to a simple root α_j will be simply denoted by j.

The height of a positive root $\alpha = \sum_i n_i\alpha_i$, $n_i \geq 0$ for all i, is defined by $ht\alpha = \sum_i n_i$.

We set $A(j) = \{i | \, (\alpha_i, \alpha_j) \neq 0\}$. Recall that to j we can associate a maximal parabolic subgroup $P_j \supset B$ and to any subset $S \subset D$ a parabolic group $P_S = \cap_{j\in S} P_j$ so that $P_D = B$. Also we set $\Phi_S^+ = \{\alpha \in \Phi^+ | \text{ Supp } \alpha \cap S \neq \emptyset\}$. One has

$$\mathfrak{p}_S := \text{Lie } P_S = \mathfrak{b} \oplus (\oplus_{\alpha \notin \Phi_S^+} \mathfrak{g}_{-\alpha}).$$

Notice that d$im\ G/P_S = |\Phi_S^+|$.

We now want to prove two rather easy lemmas regarding root systems that will be useful in the sequel. Recall that, if the Dynkin diagram D is connected, there are only two root lengths. We shall say that a given simple root is long if it is a long root with respect to the connected component of D where it belongs. By convention, if there is only one root length i.e. our component is simply laced, every root is long.

Lemma 1. *Let S be a subset of the Dynkin diagram D. Let j be a vertex in S. Then the following conditions are equivalent for the pair (S, j):*
1) Either α_j is long or the connected component of $(D-S)\cup\{j\}$ containing the vertex j is simply laced.
2) For each root α supported outside S,

(*) $$|\langle \alpha, {}^\vee\alpha_j\rangle| \leq 1,$$

Proof. Indeed if condition 1) holds, then 2) is clear.

On the other hand suppose 2) holds and α_j is short. Assume by contradiction that there is a vertex t in the connected component of $(D-S)\cup\{j\}$ containing j such that α_t is long. Then either the roots α_j, α_t span a root system of type G_2 and $\langle \alpha_t, {}^\vee\alpha_j\rangle = -3$, or there is a subdiagram with vertices $(j, i_1, \ldots, i_r, t)$ of type $C_{r+2}, r \geq 0$. In this case a simple computation shows that the root $\beta = 2\sum_{h=1}^{r} \alpha_{i_h} + \alpha_t$ is such that $\langle \beta, {}^\vee\alpha_j\rangle = -2$. In both cases we get a contradiction.

Let us now take a subset S of the Dynkin diagram D. Set W_S equal to the Weyl group of P_S i.e. the subgroup of W generated by the simple reflections $s_i, i \notin S$. Choose $j \in S$. We then have,

Lemma 2. *1) For any positive root β having the same length as α_j and of the form $\beta = \alpha_j + \gamma$, γ being a vector supported in $D - S$, there exists $w \in W_S$ with $w\alpha_j = \beta$.*
2) If furthermore the conditions of lemma 1) are not satisfied, so that there is a root β supported in $D - S$ with $|\langle \beta, {}^\vee\alpha_j\rangle| = p > 1$, then any pair of long roots of the form $p\alpha_j + \gamma$, γ being a vector supported in $D - S$, are conjugate under the action of W_S.

Proof. 1) We proceed by induction on the height of β. If ht$\beta = 1$, then $\beta = \alpha_j$ and there is nothing to prove. Let now ht$\beta > 1$. We claim that there is a vertex $i \notin S$ such that $(\beta, \alpha_i) > 0$. Assume the contrary. Then clearly $(\beta, \alpha_i) \leq 0$ for each $i \neq j$. We deduce that $(\beta, \alpha_j) > 0$, otherwise $(\beta, \beta) \leq 0$. Also since $s_j\beta = \beta - \langle \beta, {}^\vee\alpha_j\rangle\alpha_j$ is a positive root, we must have $\langle \beta, {}^\vee\alpha_j\rangle = 1$, so that $s_j\beta = \gamma$. By our assumptions, we have $(\beta, \gamma) \leq 0$. But now, γ, β and hence α_j have the same length, so that we deduce that $\langle \gamma, {}^\vee\alpha_j\rangle = \langle \alpha_j, {}^\vee\gamma\rangle = -1$, and then $\langle \beta, {}^\vee\gamma\rangle = 1$, getting a contradiction.

Take now $i \notin S$ such that $\langle \beta, {}^\vee\alpha_i\rangle > 0$. We have that ht$s_i\beta <$ htβ and $s_i\beta$ satisfies all the assumptions of our lemma. It follows that there exists $w' \in W_j$ with $w'\alpha_j = s_i\beta$. Thus $s_iw'\alpha_j = \beta$, proving our claim.

2) The G_2 case i.e. the only case in which $p = 3$, is trivial. So let us assume $p = 2$. Consider, as in the proof of Lemma 1, the long simple root α_t and the subdiagram with vertices $(j, i_1, \dots, i_r, t)$ of type $C_{r+2}, r \geq 0$. Take the root $\beta = 2\sum_{h=1}^{p} \alpha_{i_h} + \alpha_t$. We have to show that any long root $\delta = 2\alpha_j + \gamma$, γ being a vector supported in $D - S$, is W_S- conjugate to $\alpha = 2\alpha_j + \beta$. Assume δ is such that $(\delta, \alpha_i) \leq 0$ for each $i \neq j$. We deduce that $(\delta, \alpha_j) > 0$, otherwise $(\delta, \delta) \leq 0$. Since α_j is short, we get $\langle \delta, {}^\vee\alpha_j \rangle = 2$, so $s_j\delta = \gamma$. Thus γ is a long root and

$$\langle \delta, {}^\vee\gamma \rangle = 2\langle \alpha_j, {}^\vee\gamma \rangle + \langle \gamma, {}^\vee\gamma \rangle = -2 + 2 = 0$$

This implies that $\langle \delta, {}^\vee\alpha_i \rangle = 0$ for all $i \neq j$ in suppδ. Thus δ is the unique highest root whose support is contained in suppδ and from this we immediately get that suppδ=$(j, i_1, \dots, i_r, t)$ and $\delta = \alpha$. Thus, if $\delta \neq \alpha$, there is $i \neq j$, necessarily in $D - S$, such that $(\delta, \alpha_i) > 0$, so $s_i\delta = 2\alpha_j + (\gamma - \langle \delta, {}^\vee\alpha_i \rangle \alpha_i)$ is of height smaller than δ. At this point the proof proceed as in part 1) (notice that our argument implies that α is the unique root of minimum height among the roots of the form $2\alpha_j + \gamma$, γ being a vector supported in $D - S$).

3. Lines in G/P_S

Given a k-vector space V, we denote by $\mathbb{P}(V)$ the projective space of one dimensional subspaces in V. If $X \subset \mathbb{P}(V)$ is a projective variety, we can consider the set $L(X)$ of lines in $\mathbb{P}(V)$ which lie in X. $L(X)$ is a projective subvariety of the Grassmannian $G(2, V)$ of lines in $\mathbb{P}(V)$. To see this consider the partial flag variety $F \subset \mathbb{P}(V) \times G(2, V)$ consisting of pairs (p, ℓ) with $p \in \ell$. We have two projections $p : F \to \mathbb{P}(V)$ and $q : F \to G(2, V)$, so we can take $q(p^{-1}(X))$. The fact that X is closed and q proper, clearly implies that both $p^{-1}(X)$ and $q(p^{-1}(X))$ are closed and hence projective. Also q restricts to a morphism $\overline{q} : p^{-1}(X) \to q(p^{-1}(X))$, and clearly $L(X)$ is the subset of points for which the fibers of $\overline{q}$ have positive dimension. The fact that $L(X)$ is closed, then follows from [Ha, pag. 95, ex. 3.22]. We shall consider the variety $L(X)$ with its reduced structure. Also we set $F(X) := q^{-1}(L(X)$ and call the $\mathbb{P}^1$-fibration $q : F(X) \to L(X)$ the family of lines in X.

We start by recalling a few facts about the complete homogeneous spaces G/P_S.

For each $S \subset D$, the Picard group of G/P_S can be identified with the sublattice of the weight lattice generated by the fundamental weights ω_i, with $i \in S$. Also, given a weight $\lambda = \sum_{i \in S} n_i\omega_i$, the corresponding line bundle L_λ is defined as follows. First we extend the character $e^{-\lambda} : T \to k^*$ to a character of P_S, so that we get a one dimensional P_S-module k_λ, and then we set $L_\lambda = G \times_{P_S} k_\lambda$. One knows that $H^0(G/P_S, L_\lambda) \neq 0$ if and only if $n_i \geq 0$, i.e. λ is dominant. Furthermore if $n_i > 0$ for each i, then L_λ is ample and automatically very ample [RR] . From now on we shall assume that λ is dominant. We have that $H^0(G/P_S, L_\lambda)$ is the dual of the Weyl module V_λ [RR] with highest weight λ. So if $n_i > 0$ for each i, we get an embedding of G/P_S into $\mathbb{P}(V_\lambda) = \mathbb{P}(H^0(G/P_S, L_\lambda)^*)$.

The Chow group $A^{m-1}(G/P_S)$, m = dim G/P_S, of 1-dimensional cycles in G/P_S (if $k = \mathbb{C}$, the complex numbers, we can take $H_2(G/P_S, \mathbb{Z})$), can be identified with the lattice generated by the coroots ${}^\vee\alpha_i$, $i \in S$, and given a class ${}^\vee\beta = \sum_{i \in S} m_i^\vee \alpha_i$ and a line bundle $L(\lambda)$, we have

$$\int_{^\vee\beta} c_1(L(\lambda)) = \sum_{i \in S} m_i \langle \lambda, {}^\vee\alpha_i \rangle.$$

In particular, if we consider the embedding of G/P_S into $\mathbb{P}(V_\lambda)$, and the corresponding variety of lines Z_λ in G/P_S, we get a locally constant map $Z_\lambda \to A^{m-1}(G/P_S)$. We shall say that a class in $A^{m-1}(G/P_S)$ can be represented by a line in G/P_S with respect to the projective embedding into $\mathbb{P}(V_\lambda)$, if it lies in the image of our map. Notice that the lines representing a given class are a union of connected components of Z_λ.

These considerations have the following consequence.

Lemma 3. *A class ${}^\vee\beta = \sum_{i \in S} m_i^\vee \alpha_i$ can be represented by a line with respect to the projective embedding given by the line bundle $L(\lambda)$ with $\lambda = \sum_{i \in S} n_i \omega_i$, $n_i > 0$ only if there exists a vertex $j \in S$ such that*

1. ${}^\vee\beta = {}^\vee\alpha_j$
2. $n_j = 1$.

Furthermore, if this is the case, the variety Z_S^j of lines of class ${}^\vee\alpha_j$ is independent from the choice of the λ satisfying condition (2).

Proof. The first part is clear since for such a line we must have

$$1 = \int_{^\vee\beta} c_1(L(\lambda)) = \sum_{i \in S} m_i n_i.$$

As for the second, it is clear that it suffices to show our claim for $\lambda = \sum_{i \in S} n_i \omega_i$, $n_i > 0$ and $\lambda' = \lambda + \omega_i$, $i \neq j$. For the time being, let us denote by Z (resp. Z') the variety of lines representing the class ${}^\vee\alpha_j$ in the projective embedding in $\mathbb{P}(V_\lambda)$ (resp. $\mathbb{P}(V_{\lambda'})$). Recall, [MR], that the multiplication map $H^0(G/P_S, L_\lambda) \otimes H^0(G/P_S, L_{\omega_i}) \to H^0(G/P_S, L_{\lambda'})$ is surjective, so that we get an embedding

$$\phi : \mathbb{P}(V_{\lambda'}) \to \mathbb{P}(V_\lambda \otimes V_{\omega_i}).$$

Take the Segre embedding

$$\psi : \mathbb{P}(V_\lambda) \times \mathbb{P}(V_{\omega_i}) \to \mathbb{P}(V_\lambda \otimes V_{\omega_i})$$

We have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(V_\lambda) \times \mathbb{P}(V_{\omega_i}) & \xrightarrow{\psi} & \mathbb{P}(V_\lambda \otimes V_{\omega_i}) \\ \uparrow{\scriptstyle h_\lambda \times h_{\omega_i}} & & \uparrow{\scriptstyle \phi} \\ G/P_S & \xrightarrow{h_{\lambda'}} & \mathbb{P}(V_{\lambda'}) \end{array}$$

where, for a dominant μ, h_μ is the morphism associated to the line bundle L_μ. Notice now that ϕ and ψ induce embeddings

$$\begin{array}{ccc} G(2, V_\lambda) \times \mathbb{P}(V_{\omega_i}) & \xrightarrow{\tilde{\psi}} & G(2, V_\lambda \otimes V_{\omega_i}) \\ & & \uparrow{\scriptstyle \tilde{\phi}} \\ & & G(2, V_{\lambda'}) \end{array}$$

Now remark that since

$$\int_{^\vee\alpha_j} c_1(L(\omega_i)) = 0,$$

each line in G/P_S representing the class ${}^\vee\alpha_j$ is mapped to a point by h_{ω_i}. We thus get a morphism $Z \to \mathbb{P}(V_{\omega_i})$ defined by mapping each line in Z to its image under h_{ω_i}. We can thus identify Z with the graph of this morphism in $G(2, V_\lambda) \times \mathbb{P}(V_{\omega_i})$. It is now clear that $\tilde{\psi}(Z) = \tilde{\phi}(Z')$, proving our claim.

We shall now look at the action of the maximal torus T on Z_S^j. We have

Proposition 1. *Every G orbit in Z_S^j contains a T fixpoint.*

Proof. Let ℓ be a line in Z_S^j. Set p equal to the class of P_S in G/P_S. By homogeneity, there is an element $g \in G$ with $p \in g\ell$. Now take another point $q \in g\ell$. q lies in a unique Bruhat cell $S(w) = BwP_S/P_S$, for a $w \in W$. Thus there is an element $h \in B$ such that $hq = wp$. Since $hp = p$, we get that $\ell' = hg\ell$ contains the two T fixpoints p and wp. But now if we choose λ as in Lemma 3 and embed G/P_S in $\mathbb{P}(V_\lambda)$, we get that ℓ' is a line in $\mathbb{P}(V_\lambda)$ which joins two T fixpoints. It is hence stable under T.

Let us now as usual start with our subset S of the Dynkin diagram. Given a vertex j in S, we set S_j equal to the new subsets

$$\overline{S}_j = S \cup \{i | \langle \alpha_i, ^\vee \alpha_j \rangle \neq 0\}$$

and

$$S_j = \overline{S}_j - \{j\}.$$

The following theorem gives a complete description of the varieties Z_S^j in the case in which the conditions of Lemma 1 are satisfied:

Theorem 1. *Let S be a subset of the Dynkin diagram D. Let j be a vertex in S. Assume that either α_j is long or the connected component of $(D-S)\cup\{j\}$ containing the vertex j is simply laced. We have:*
1) There is a natural isomorphism between Z_S^j and G/P_{S_j}.
2) The incidence variety $\overline{Z}_S^j = \{(\ell, q) \in Z_S^j \times G/P_S |\ q \in \ell\}$ has a natural identification with $G/P_{\overline{S}_j}$.
3) Under the identification in 2), the projection $\pi_1(\ell, q) = \ell$ can be identified with the projection $G/P_{\overline{S}_j} \to G/P_{S_j}$ and the projection $\pi_2(\ell, q) = q$ with the projection $G/P_{\overline{S}_j} \to G/P_S$.

If the conditions of lemma 1) are not satisfied, so that there is a root β supported in $D - S$ with $|\langle\beta, {}^\vee\alpha_j\rangle| = p > 1$, then setting $\alpha = p\alpha_j + \beta$, the line ℓ_α in $\mathbb{P}(V_\lambda)$ joining the points p and $s_\alpha p$ lies in Z_S^j, Z_S^j is the closure of the orbit $G\ell_\alpha$ and $Z_S^j - G\ell_\alpha = G/P_{S_j}$.

Proof. We shall assume that $\lambda = 2\sum_{i\in S}\omega_i - \omega_j$, so that Z_S^j coincides with the variety of all lines in $\mathbb{P}(V_\lambda)$ lying in G/P_S.

By the definition of $\overline{S}_j$ and S_j, we immediately get that the natural projection $\pi : G/P_{\overline{S}_j} \to G/P_{S_j}$ is a $\mathbb{P}^1$-fibration.

Consider now the projection $\xi : G/P_{\overline{S}_j} \to G/P_S$.

Each fiber of π is a mapped by ξ to a $\mathbb{P}^1$ which is embedded as a line in $\mathbb{P}(V_\lambda)$. We thus get a map

$$\gamma : G/P_{S_j} \to Z_S^j$$

Composing with the embedding of Z_S^j into $G(2, V_\lambda)$ and the Plücker embedding of $G(2, V_\lambda)$ into $\mathbb{P}(\bigwedge^2 V_\lambda)$, we get a map

$$\tilde{\gamma} : G/P_{S_j} \to \mathbb{P}(\bigwedge^2 V_\lambda).$$

Notice now that, if we take an highest weight vector $v \in V_\lambda$, the vector $v \wedge s_{\alpha_j}v \in \bigwedge^2 V_\lambda$ is a highest weight vector of weight $2\lambda - \alpha_j$, and $\tilde{\gamma}$ is given by taking the G orbit of the class of $v \wedge s_{\alpha_j}v$ in $\mathbb{P}(\bigwedge^2 V_\lambda)$. A simple computation shows that $\langle 2\lambda - \alpha_j, {}^\vee\alpha_i\rangle \neq 0$ if and only if $i \in S_j$. This clearly implies that $\tilde{\gamma}$ and hence γ are embeddings. We thus get a natural inclusion $G/P_{S_j} \subset Z_S^j$.

We want now to study the orbit structure in Z_S^j. Since by the above Proposition 1, each G orbit contains a T fixpoint, we start understanding the T fixpoints. Let ℓ be a T stable line. We clearly have that ℓ must be the line joining two T fixpoints $p_1 = w_1 p$ and $p_2 = w_2 p$, $w_1, w_2 \in W$. Applying w_1^{-1}, we can assume that ℓ is the line joining p and wp for some $w \in W$. We identify the tangent space to p in G/P_S with the Lie algebra

$$\mathfrak{n}_S^- = \sum_{\alpha \in \Phi_S^+} \mathfrak{g}_{-\alpha},$$

where as usual, for a given root β, $\mathfrak{g}_\beta \subset \mathfrak{g}$ denotes the root subspace of weight β.

The tangent direction to ℓ in p must be by T stability a T stable line in $\mathfrak{n}_S^-$. This means that it must be given by $\mathfrak{g}_{-\alpha}$ for some $\alpha \in \Phi_S^+$. Let Γ_α denote the $Sl(2)$ corresponding to such α. Clearly $\ell := \ell_\alpha = \Gamma_\alpha p$ and $p_2 = s_\alpha p$. It follows that $\langle \lambda, {}^\vee\alpha \rangle = 1$. Write $\alpha = m\alpha_j + \gamma$, where γ is supported in $D - \{j\}$, then

$$1 = \langle \lambda, {}^\vee\alpha \rangle = m\frac{(\alpha_j, \alpha_j)}{(\alpha, \alpha)} + 2\sum_{i \in S - \{j\}} \langle \omega_i, {}^\vee\gamma \rangle.$$

We deduce that $\langle \omega_i, {}^\vee\alpha \rangle = 0$ for all $i \in S - \{j\}$. Under our conditions, we then have that, if α_j is long, then necessarily also α is long and $m = 1$, while, if α_j is short, $(\alpha_j, \alpha_j) = (\alpha, \alpha)$ and again $m = 1$. Using the first part of Lemma 2, we then get that there is a $w \in W_S$ with $w\alpha_j = \alpha$. This clearly implies that $\ell_\alpha = w\ell_{\alpha_j} \in G/P_{S_j}$. From this we get that all the T fixpoints lie in a single G orbit. By the above Proposition 1, we know that each G orbit contains at least one T fixpoint, so that we deduce that Z_S^j is homogeneous. But $G/P_{S_j} \subset Z_S^j$, so $G/P_{S_j} = Z_S^j$ proving (1). Both (2) and (3) follow immediately from our previous considerations.

Suppose now that the conditions of Lemma 1 are not satisfied. Then one has two possibilities. Using the notations of Theorem 1, we have that either $m = 1$ and α is short or $m = 2$ (or $m = 3$ in the G_2 case) and α is long. Accordingly, using both parts of Lemma 2, we get two G orbits in Z_S^j. The orbit of $G\ell_{\alpha_j} = G/P_{S_j}$ is closed, so to finish our proof we need only to see that the orbit $G\ell_\alpha$ with α long is not closed and hence, since Z_S^j is a closed subvariety, necessarily contains G/P_{S_j} in its closure.

The G_2 case is trivial and we leave it to the reader.

Let us now suppose $m = 2$ and, using the notation of Lemma 2, take $\alpha = 2\alpha_j + \beta$ with $\beta = 2\sum_{h=1}^r \alpha_{i_h} + \alpha_t$. Consider the set $\Gamma \subset \Phi$ defined as $\Gamma = (\Phi_S^+ - \{\alpha\}) \cup s_\alpha(\Phi_S^+ - \{\alpha\})$. Now remark that, as a T-module, the tangent space to $G\ell_\alpha$ in ℓ_α equals the direct sum of the root spaces $\mathfrak{g}_{-\delta}$

with $\delta \in \Gamma$. Indeed, since under the embedding $G(2, V_\lambda) \to \mathbb{P}(\bigwedge^2 V_\lambda)$, the line ℓ_α is represented by the class of the vector $v \wedge n_\alpha v$, where $v \in V_\lambda$ is a highest weight vector and n_α a representative in $N(T)$ of the reflection s_α, it is clear that $\mathfrak{g}_{-\delta}$ contributes to the tangent space to $G\ell_\alpha$ in ℓ_α if and only if, given $x \in \mathfrak{g}_{-\delta} - \{0\}$,

$$xv \wedge n_\alpha v + v \wedge xn_\alpha v \neq 0.$$

If this is the case, then either xv or $xn_\alpha v$ is not zero, and $\delta \neq \pm\alpha$, i.e. $\delta \in \Gamma$. Now if $\delta \in (\Phi_S^+ - \{\alpha\}) - ((\Phi_S^+ - \{\alpha\}) \cap s_\alpha(\Phi_S^+ - \{\alpha\}))$ (resp. $\delta \in (s_\alpha(\Phi_S^+ - \{\alpha\}) - ((\Phi_S^+ - \{\alpha\}) \cap s_\alpha(\Phi_S^+ - \{\alpha\})))$, then $xv \wedge n_\alpha v \neq 0$, while $v \wedge xn_\alpha v = 0$ (resp. $v \wedge xn_\alpha v \neq 0$, while $xv \wedge n_\alpha v = 0$), so certainly $xv \wedge n_\alpha v + v \wedge xn_\alpha v \neq 0$. If $\delta \in (\Phi_S^+ - \{\alpha\}) \cap s_\alpha(\Phi_S^+ - \{\alpha\})$, then both $xv \wedge n_\alpha v$ and $v \wedge xn_\alpha v$ are non zero, but the 2-dimensional space spanned by xv and $n_\alpha v$ clearly does not contain v, so it is different from the space spanned by v and $xn_\alpha v$, thus proving our claim also in this case.

In order to show that $G\ell_\alpha$ is not closed, it suffices to see that the Lie algebra of the stabilizer of ℓ_α does not contain a Borel subalgebra. From what we have seen above, this follows once we show that there is a root $\delta \in \Gamma$ such that also $-\delta$ lies in Γ. For this take $\delta = \alpha_j + \sum_{h=1}^r \alpha_{i_h}$. Then one easily sees that $s_\alpha(\delta + \alpha_t) = -\delta$ and our claim follows.

Remark 1. Notice that our theorem applies in particular in the case of G/B, i.e. when S is the entire Dynkin diagram. In this case we deduce as a special case our result [S] stating that a class in $H_2(G/P_S, \mathbb{Z})$ can be represented by a line with respect to the projective embedding given by the line bundle $L(\rho)$ with $\rho = \sum_i \omega_i$,if and only if it equals ${}^\vee\alpha_j$ for some j. Furthermore the variety of these lines equals G/P_j, P_j being the minimal parabolic associated to the node j.

4. The exceptional cases

We shall now discuss the various cases in which the conditions of lemma 1 are not satisfied. This will be done case by case.

We start with G_2. In this case we have to take the maximal parabolic P corresponding to the short simple root. We let ω be the corresponding fundamental weight. Then it is well known and easy to see that $H^0(G/P, L_\omega)$ has dimension 7 and G/P is embedded as a non degenerate quadric in the 6 dimensional projective space $\mathbb{P}(H^0(G/P, L_\omega)^*)$. A quadric in $\mathbb{P}^6$ is a complete homogeneous space for the corresponding special orthogonal group $SO(7)$ which is of type B_3. If we consider our quadric as a homogeneous space for $SO(7)$, the conditions of lemma 1 are satisfied, so we get that the variety of lines in our quadric G/P is the variety of isotropic lines with respect to the symmetric bilinear form defining it i.e. the unique closed orbit for $SO(7)$ acting on the projectification of its adjoint representation.

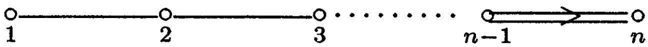
We now pass to the B_n case. We consider a vector space V of dimension 2n+1 with a non degenerate symmetric bilinear form and we can take as G the corresponding special orthogonal group.

As a preliminary step, let us embed our orthogonal space V in a $2n+2$ dimensional orthogonal space W and choose once and for all, one of the two $SO(W)$ orbits $\mathcal{O}$ in the variety of maximal isotropic subspaces of W. Notice that we can identify $\mathcal{O}$ with the variety $\mathcal{T}$ of maximal isotropic subspaces in V as follows. Given $U \in \mathcal{O}$, then clearly $U \cap V \in \mathcal{T}$, so that we get a map

$$c : \mathcal{O} \to \mathcal{T}.$$

On the other hand, if we fix $U' \in \mathcal{T}$, and we take a subspace $U \in \mathcal{O}$ containing it, we have that U/U' is an isotropic line in the plane $U'^{\perp}/U'$, $U'^{\perp}$ being the orthogonal space to U' in W. But there are exactly two such lines and, of the corresponding two maximal isotropic subspaces in W, only one lies in $\mathcal{O}$. We deduce that the map c is an isomorphism.

Let us now go back to our problem. If we index the vertices of the Dynkin diagram as follows



we get that for a subset $S = \{r_1 < \cdots < r_t\}$ of the Dynkin diagram, the variety G/P_S is the variety consisting of flags $(V_1 \subset V_2 \subset \cdots \subset V_t)$ with V_j isotropic and $\dim V_j = r_j$. Furthermore the conditions of lemma 1 are not satisfied for parabolic subgroups G/P_S with $r_t = n$, and $r_{t-1} < n-1$, and for the homology class represented by the simple coroot ${}^{\vee}\alpha_n$.

Let $S = \{r_1 < \cdots < r_t\}$ be a subset of the Dynkin diagram with $r_t = n$, and $r_{t-1} < n-1$. We have

Proposition 2. *1. The variety Z_S^n can be described as the variety of all flags $(V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H)$, H being an n-1 dimensional isotropic subspace in W and dim $V_i = r_i$, $1 \leq i \leq t-1$.*

2. The incidence variety $\overline{Z}_S^n = \{(\ell, q) \in Z_S^n \times G/P_S |\ q \in \ell\}$ has a natural identification with the variety of all flags $((V_1 \subset V_2 \cdots \subset V_{t-1} \subset H \subset U), (V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H)) \in Z_S^n$, $U \in \mathcal{O}$ via the map

$$T : \overline{Z}_S^n \to Z_S^n \times G/P_S$$

$$T((V_1 \subset V_2 \cdots \subset V_{t-1} \subset H \subset U)) = ((V_1 \subset V_2 \cdots \subset V_{t-1} \subset H), (V_1 \subset V_2 \cdots \subset V_{t-1} \subset c(U)).$$

Proof. Set $\tilde{Z}^n_S$ equal to the variety of all flags $(V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H)$, H being an n-1 dimensional isotropic subspace in W and dim $V_i = r_i$, $1 \leq i \leq t-1$.

If we take the projection $p_1 : \tilde{Z}^n_S \times G/P_S \to \tilde{Z}^n_S$ on the first factor and set $\pi = p_1 T$, we clearly get that π is a P^1 fibration.

Also each fiber of this fibration is mapped to a line in G/P_S and the obvious injectivity of T implies that two such lines are distinct.

To determine the homology class of these lines, let us set $\overline{S} = \{r_1 < \cdots < r_{t-1}\}$. Take the projection $p : G/P_S \to G/P_{\overline{S}}$. We have, since,

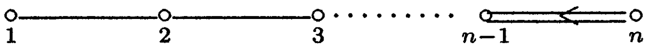
$$\left\langle \sum_{j=1}^{t-1} \omega_{i_j}, {}^\vee\alpha_n \right\rangle = 0,$$

that a line ℓ lies in Z^n_S if and only if $p(\ell)$ is a point. But it is clear by the definition that this property is satisfied for the fibers of π. We deduce that $\tilde{Z}^n_S$ embeds in Z^n_S. On the other hand, we have seen that Z^n_S consists of two G orbits, thus to show our claim, it suffices to see that $\tilde{Z}^n_S$ is not homogeneous. This is clear since $\tilde{Z}^n_S$ contains the closed orbit consisting of those flags $(V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H)$ with $H \subset V$.

Thus $\tilde{Z}^n_S = Z^n_S$ and all our claims follow at once.

We now pass to the C$_n$ case. We consider a vector space V of dimension 2n with a non degenerate symplectic bilinear form and we can take as G the corresponding symplectic group.

If we index the vertices of the Dynkin diagram as follows



we get that for a subset $S = \{r_1 < \cdots < r_t\}$ of the Dynkin diagram, the variety G/P_S is the variety consisting of flags $(V_1 \subset V_2 \subset \cdots \subset V_t)$ with V_j isotropic and dim$V_j = r_j$. Furthermore the conditions of lemma 1 are not satisfied for parabolic subgroups G/P_S with $r_t < n$, and for the homology class represented by the simple coroot ${}^\vee\alpha_{r_t}$.

Let $S = \{r_1 < \cdots < r_t\}$ be a subset of the Dynkin diagram with $r_t < n$. We have:

Proposition 3. *1) The variety $Z^{r_t}_S$ can be described as the variety of all flags $(V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H \subset K)$, with H being an $r_t - 1$ dimensional isotropic subspace in V, dim $V_i = r_i$, $1 \leq i \leq t-1$ (notice that if $r_t - 1 = r_{t-1}$, $H = V_{t-1}$), and dim $K = r_t + 1$.*
2) The incidence variety $\overline{Z}^n_S = \{(\ell, q) \in Z^n_S \times G/P_n |\, q \in \ell\}$ has a natural identification with the variety of all flags $(V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H \subset$

$U \subset K$), with $(V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H \subset K) \in Z_S^n$ and dim $U = r_t$, via the map

$$T : \overline{Z}_S^n \to Z_S^n \times G/P_S$$

$T((V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H \subset U)) = ((V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H \subset K), (V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset U \subset K))$. *(Notice that U is automatically isotropic, since the symplectic form has rank at most one, and hence zero, on this space).*

Proof. Set $\tilde{Z}_S^{r_t}$ equal to the variety of all flags $(V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H \subset K)$, with H being an $r_t - 1$ dimensional isotropic subspace in V, dim $V_i = r_i, 1 \leq i \leq t-1$.

If we take the projection $p_1 : \tilde{Z}_S^{r_t} \times G/P_S \to \tilde{Z}_S^{r_t}$ on the first factor and set $\pi = p_1 T$, we clearly get that π is a P^1 fibration.

Also each fiber of this fibration is mapped to a line in G/P_S and the obvious injectivity of T implies that two such lines are distinct.

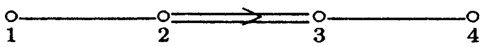
To determine the homology class of these lines, let us set $\overline{S} = \{r_1 < \cdots < r_{t-1}\}$. Take the projection $p : G/P_S \to G/P_{\overline{S}}$. We have since,

$$\left\langle \sum_{j=1}^{t-1} \omega_{i_j}, {}^{\vee}\alpha_{r_t} \right\rangle = 0,$$

that a line ℓ lies in $Z_S^{r_t}$ if and only if $p(\ell)$ is a point. But is clear by the definition that this property is satisfied for the fibers of π. We deduce that $\tilde{Z}_S^{r_t}$ embeds in $Z_S^{r_t}$. But we have seen that $Z_S^{r_t}$ consists of two G orbits, thus to show our claim, it suffices to see that $\tilde{Z}_S^{r_t}$ is not homogeneous. This is clear since $\tilde{Z}_S^{r_t}$ contains the closed orbit consisting of those flags $(V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset H \subset K)$ with K isotropic.

Thus $\tilde{Z}_S^{r_t} = Z_S^{r_t}$ and all our claims follow at once.

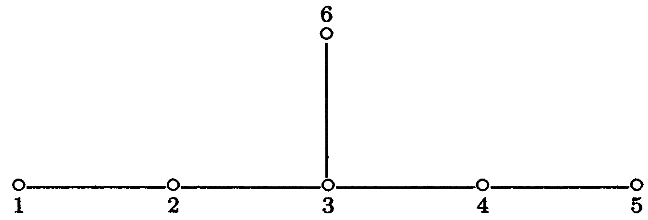
We finally pass to the case F_4. If we index the vertices of the Dynkin diagram as follows



we have that the conditions of lemma 1 are not satisfied for parabolic subgroups G/P_S with $S = \{3\}, \{4\}, \{13\}, \{14\}, \{34\}, \{134\}$. Also if $3 \in S$, then we need to consider the homology class represented by the simple coroot ${}^{\vee}\alpha_3$. Otherwise we need to consider the homology class represented by the simple coroot ${}^{\vee}\alpha_4$.

We start with the case $S = \{4\}$.

Let G' be a simply connected group of type E_6. Recall that once we have chosen a maximal torus in G' and a Borel subgroup containing it, we can uniquely associate an involution σ of G' to the non trivial automorphism of its Dynkin diagram



so that G is the subgroup of elements fixed by σ (see [He] page 518,table V, type EIV).

Notice that dim $G'/G = 26$. Denote by V' the 27 dimensional representation of G' whose highest weight is ω_1'. Denote by $P' \subset G'$ the corresponding maximal parabolic subgroup. The restriction of V' to G decomposes into the sum

$$V' = V_4 \oplus k$$

where V_4 is the irreducible representation of G whose highest weight is ω_4 and k is a trivial one dimensional module. One has that, if we consider the hyperplane $H = \mathbb{P}(V_4)$ of $\mathbb{P}(V')$ as a point in $\mathbb{P}(V'^*)$, then its orbit under G' is isogenous to G'/G and it is hence dense in $\mathbb{P}(V'^*)$. It follows from Bertini theorem ([Ha], II 8.18]) that, if we identify the orbit of the highest weight line in V' with G'/P', the intersection $G'/P' \cap H$ is smooth and irreducible. Since it contains G/P_4 and has dimension 15=dim G/P_4, we deduce that $G'/P' \cap H = G/P_4$. We then get from Theorem 1, that the variety Z_4^4 coincides with the intersection in the Grassmannian $G(2, V')$ of lines in $\mathbb{P}(V')$ of the varieties G'/P_2' and $G(2, H)$. Since we have seen that the G' orbit of H is dense, we can then use a result of Kleiman ([Ha], II 10.8) to deduce that Z_4^4 is smooth, irreducible and of codimension 2 in G'/P_2'.

We now pass to G/P_3. We have seen that $G/P_3 \subset Z_4^4 \subset G(2, V')$. Thus, using Theorem 1 in the case of $G(2, V')$, we deduce that a line in G/P_3 consists of a pencil of lines in $\mathbb{P}(V')$ contained in a plane and having a point p in common. Clearly $p \in G/P_4$, so that we get a G equivariant fibration

$$f : Z_3^3 \to G/P_4.$$

On the other hand, if we consider the incidence variety $Y = \{(\ell, p)|\ \ell \in G/P_3,\ p \in \ell\}$, we can identify Y with $G/P_{3,4}$ and the projection onto the second factor with the canonical G-equivariant fibration

$$p : G/P_{3,4} \to G/P_4.$$

We deduce that $f^{-1}([P_1])$ is the variety of lines in the variety $P_4/P_{3,4}$. Setting L equal to the adjoint Levi factor of P_4 i.e. the quotient of P_4 modulo its solvable radical, we can identify $P_4/P_{3,4}$ with $L/\overline{P}$ ($\overline{P}$ being the image of $P_{3,4}$ in L). L is of type B_3 and P is the maximal parabolic subgroup associated to its simple short root. Thus the set $\overline{Z}$ of lines in $L/\overline{P}$ has been completely described in Proposition 2. Furthermore the quotient map $P_4 \to L$ induces an action of P_4 on $\overline{Z}$ and we clearly have that $Z_3^3 = G \times_{P_4} \overline{Z}$.

We now briefly discuss the remaining cases, in which we take G/P_S with $|S| \geq 2$. If $3 \in S$, we consider the homology class represented by the simple coroot ${}^\vee\alpha_3$, while if $3 \notin S$, we consider the homology class represented by the simple coroot ${}^\vee\alpha_4$. We set $i = 3$ in the first case and $i = 4$ in the second and $S' = S - \{i\}$.

Denote by $p : G/P_S \to G/P_{S'}$ the canonical G-equivariant projection. Since

$$\left\langle \sum_{j\in S'} \omega_j, {}^\vee\alpha_i \right\rangle = 0,$$

we have that a line ℓ lies in Z_S^i if and only if $p(\ell)$ is a point. Thus $Z_S^i = G \times_{P_S} \overline{Z}$, where $\overline{Z}$ is the variety of lines in $P_{S'}/P_S$. This is a complete homogeneous space for a group of type B or C (the adjoint Levi factor of $P_{S'}$) and so $\overline{Z}$ has already been described above.

This completes our analysis of the cases in which the variety of lines in a complete homogeneous space G/P is not itself homogeneous.

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