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Quotients of flag varieties by a maximal torus

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1. Introduction

Let G be a semisimple, simply connected algebraic group over an algebraically closed field k and let $T \subset G$ be a maximal torus in G and $B \subset G$ a Borel subgroup containing T.

In two recent papers ([K1] and [K2] Senthamarai Kannan classified all parabolic subgroups $G \supset P \supset B$ with the property that there exists an ample line bundle L on G/P such that, with respect to the T linearization of L induced by the unique G linearization, the set $G/P(T)^{ss}$ of semistable points coincides with the set $G/P(T)^s$ of stable points.

In this note, we give a general characterization of those ample line bundles L on G/P. We then show how to recover in a very simple way Kannan's result from ours.

To state our result, we need to introduce some notations and recall a few facts. $X^*(T)$ will denote the character lattice of T and $X_*(T)$ its dual lattice, i.e. the lattice of one parameter subgroups in T.We shall denote by

$$\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$$

the duality pairing.

Let $\Phi \subset X^*(T)$ denote the root system associated to T and let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ denote the set of simple roots corresponding to the choice of B. Similarly let $\check{\Phi} \subset X_*(T)$ denote the set of coroots and $\check{\Delta} = \{\check{\alpha}_1, \ldots, \check{\alpha}_l\}$ denote the set of simple coroots corresponding to the choice of B. There is a canonical bijection between Δ and $\check{\Delta}$ and we assume that the root

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α_i corresponds to the coroot $\check{\alpha}_i$ under this bijection. Also, given a subset $\Gamma \subset \Delta$, we shall denote the corresponding subset in $\check{\Delta}$ by $\check{\Gamma}$. Finally we define the set of fundamental weights $\Omega = \{\omega_1, \dots, \omega_l\}$ by $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{i,j}$ and the set of fundamental coweights $\check{\Omega} = \{\check{\omega}_1, \dots, \check{\omega}_l\}$ by $\langle \alpha_i, \check{\omega}_j \rangle = \delta_{i,j}$ (notice that $\check{\Omega} \subset X_*(T) \otimes \mathbb{Q}$).

One knows that $\check{\Delta}$ is a basis for $X_*(T)$ and that we can identify the Picard group of G/B with $X^*(T)$. Also recall that there is a bijection between parabolic subgroups $P \supset B$ and subsets of Δ (or equivalently $\check{\Delta}$). Under this correspondence, if P corresponds to $\Gamma \subset \Delta$, $Pic(G/P)$ can be identified with the character group $X^*(P)$ where, restricting characters to T, we think of $X^*(P)$ as the subgroup of $X^*(T)$ consisting of those $\lambda \in X^*(T)$ such that $\langle \lambda, \check{\alpha} \rangle = 0$ for all $\check{\alpha} \in \check{\Gamma}$. Moreover one knows that $\lambda \in X^*(P)$ corresponds to an ample (and, using the results in [RR], automatically very ample) line bundle L_λ if and only if $\langle \lambda, \check{\alpha} \rangle > 0$ for all $\check{\alpha} \in \check{\Delta} - \check{\Gamma}$.

Finally set, as usual, $W = N(T)/T$, the Weyl group. W acts on $X^*(T)$ and it is generated by the simple reflections s_i $(i = 1, \dots, l)$ with respect to the hyperplanes orthogonal to the simple coroots $\check{\alpha}_i$.

We are now in the position to state our main result.

Theorem 1.1. *Let $P \subset G$ be a parabolic subgroup. Let $\lambda \in X^*(P)$ be such that L_λ is ample. Then, if we denote by $(G/P)^{ss}_\lambda$ (resp. $(G/P)^s_\lambda$), the set of semistable (resp. stable points) for the T action with respect to L_λ,*

$$(G/P)^{ss}_\lambda = (G/P)^s_\lambda$$

if and only if for all $w \in W$, $\check{\omega}_i \in \check{\Omega}$, one has $\langle \lambda, w\check{\omega}_i \rangle \neq 0$.

After this is proved, it is not hard to deduce Kannan's results, as we shall show below.

2. Quotients

Let us start by recalling a few facts about Geometric Invariant Theory (see [MFK] [Se]). Given a projective algebraic variety X over k on which a reductive group H acts and an H linearized very ample line bundle L, we can consider the ring

$$R = \oplus_{n \geq 0} H^0(X, L^n)$$

as an H-module and consider the ring R^H of H invariant elements. Since H acts on R in a degree preserving way, R^H is naturally graded and we can consider R^H_+, its part of positive degree. At this point one can define the set of semistable points as the set

$$X^{ss} = \{x \in X | \exists s \in R^H_+ \text{ with } s(x) \neq 0\}.$$

We define the set of stable points X^s as the subset of X^{ss} consisting of those points having finite stabilizer and whose orbit is closed.

This is not the place where to discuss properties of X^{ss} and X^s, let us just say that a good categorical quotient $X^{ss}//H$ exists and furthermore the image of X^s in $X^{ss}//H$ coincides with the set theoretical quotient X^s/H and has only finite quotient singularities (it is indeed smooth, if each point in X^s has trivial stabilizer).

Here we shall be only interested in the case $H = T$. In this special case we take a point $x \in X$, we take a representative $\tilde{x} \in H^0(X, L)$ for x and write

$$\tilde{x} = \sum_{\lambda \in X^*(T)} v_\lambda$$

where v_λ is a weight vector for $t \in T$ of weight λ. We set $M_x = \{\lambda \in X^*(T) | v_\lambda \neq 0\}$. It is clear that M_x does not depend on the choice of $\tilde{x}$. We define now, following [Se] Section 2, for every $\check{\chi} \in X_*(T)$,

$$\mu^L(x, \check{\chi}) = -\min_{\lambda \in M_x} \langle \lambda, \check{\chi} \rangle .$$

It is then not hard to see that x is semistable if and only if $\mu^L(x, \check{\chi}) \geq 0$ for all $\check{\chi} \in X_*(T)$, while x is stable if and only if $\mu^L(x, \check{\chi}) > 0$ for all $\check{\chi} \in X_*(T) - 0$.

If we furthermore suppose, as we shall do from now on, that $X = G/P$ and that $L = L_\lambda$, we can say a little bit more.

Let P correspond to a subset Γ of $\check{\Delta}$. Consider the subgroup $W_P \subset W$ generated by the reflections s_i for $\check{\alpha}_i \in \Gamma$. One knows [Bou], that every coset wW_P contains a unique element of shortest length so that we can identify W/W_P with a subset of W. Then Bruhat decomposition tells us that G/P is the disjoint union of the Schubert cells BwP/P, where w runs through W/W_p. If $x \in BwP/P$ and $\check{\chi}$ is such that $\langle \alpha_i, \check{\chi} \rangle \geq 0$ for all $\alpha_i \in \Delta$, then one has ([Se] Lemma 5.1)

$$\mu^L(x, \check{\chi}) = \langle w\lambda, \check{\chi} \rangle . \tag{2.1}$$

With these preliminaries in mind, we can now give the following:

Proof of Theorem 1.1. Let $L = L_\lambda$ be ample on G/P. Assume that for all $w \in W, \check{\omega}_i \in \check{\Omega}$, one has $\langle \lambda, w\check{\omega}_i \rangle \neq 0$.

Choose , once and for all for each $w \in W$, a representative $n_w \in N(T)$. Take $x \in (G/P)^{ss}_\lambda$. It is now clear from the definitions that $M_{n_w} x = wM_x$. Also, since the pairing $\langle\ ,\ \rangle$ is obviously W invariant, we deduce that for all $\check{\chi} \in X_*(T)$, $\mu^L(n_w x, w\check{\chi}) = \mu^L(x, \check{\chi})$. In particular we deduce that $n_w x$ is also semistable.

Fix $\check{\chi} \in X_*(T)$. Then there exists $w \in W$ such that $w\check{\chi}$ is dominant, i.e. $\langle \alpha_i, w\check{\chi} \rangle \geq 0$ for all $i = 1, \dots , l$.

Now assume that $n_w x$ lies in the Schubert variety BuP/P for a given $u \in W/W_p$. Then by (2.1) we have

$$\mu^L(x, \check{\chi}) = \mu^L(n_w x, w\check{\chi}) = \langle u\lambda, \check{\chi}\rangle\,. \tag{2.2}$$

Write $w\check{\chi} = \sum_i n_i \check{\omega}_i$ with $n_i \geq 0$ for each $i = 1, \ldots, l$. Since $n_w x \in (G/P)^{ss}_\lambda$, we deduce applying (1.2) to $\check{\omega}_i$ that $\langle u\lambda, \check{\omega}_i\rangle \geq 0$ for all $i = 1, \ldots l$. But $\langle u\lambda, \check{\omega}_i\rangle = \langle \lambda, u^{-1}\check{\omega}_i\rangle \neq 0$, so that $\langle u\lambda, \check{\omega}_i\rangle < 0$ for all $i = 1, \ldots l$. Substituting in (1.3), we deduce that if $\check{\chi} \neq 0$ so that not all n_i are zero. It follows that

$$\mu^L(x, \check{\chi}) = \langle u\lambda, \check{\chi}\rangle = \sum_i n_i \langle u\lambda, \check{\omega}_i\rangle < 0$$

so that $x \in (G/P)^s_\lambda$ as desired.

Let us now suppose that there is a fundamental coweight $\check{\omega}_i$ and an element $w \in W$ such that $\langle \lambda, w\check{\omega}_i\rangle = 0$. Multiply $w\check{\omega}_i$ by an integer m so that $mw\check{\omega}_i \in X_*(T)$ and it corresponds to a one parameter subgroup $\phi : G_m \to T$. Set $H \subset G$ equal to the centralizer of $\phi(G_M)$. Since $\check{\omega}_i$ is a fundamental coweight, H has semisimple rank $l-1$. Indeed it is the the Levi factor of the parabolic subgroup $n_w^{-1} Q n_w^{-1}$ where Q is the parabolic subgroup containing B corresponding to $\Delta - \{\alpha_i\}$. From this we deduce that BH is a Borel subgroup of H, hence $P_H = P \cap H$ is a parabolic subgroup of H and $H/P_H \subset G/P$ is a closed subvariety.

If we take the restriction L_H of L_λ to H/P_H, then the G linearization of L induces an H linearization of L_H. It is clear that the one parameter group $\phi(G_M)$ fixes H/P_H pointwise. Also $\phi(G_M)$ acts on the fiber of L_H over the point $[P_H]$ by the character

$$(-\lambda) \circ \phi(t) = t^{-m\langle \lambda, w\omega_i\rangle} = 1\,.$$

Hence $\phi(G_M)$ acts trivially on L_H and we get an $\overline{H} = H/\phi(G_M)$ on L_H. Take now a highest weight vector $s \in H^0(G/P, L_\lambda)$. It is clear that the restriction $\overline{s} \in H^0(H/P_H, L_H)$. Set $W_H = N(T) \cap H/T \subset W$, the Weyl group of H. Consider the section

$$z = \prod_{u \in W_H} (n_u s) \in H^0\left(G/P, L_\lambda^{|W_H|}\right)\,.$$

Then the restriction $\overline{z} \in H^0(H/P_H, L_H^{|W_H|})$ is non zero and it is a weight vector whose weight is W_H invariant and is trivial on $\phi(G_m)$. We deduce immediately that $\overline{z}$ and hence z is a T invariant vector. The fact that $\overline{z} \neq 0$ clearly means that there exists a point $x \in (G/P)^{ss} \cap H/P_H$. Since $\phi(G_m)$ fixes H/P_H pointwise, we deduce that $x \in (G/P)^{ss} - (G/P)^s$, as desired. □

We now want to analyze for which G and $P \subset G$ there exists a $\lambda \in X^*(P)$ such that L_λ is ample and $(G/P)^{ss} = (G/P)^s$. As we have seen this means that, if P corresponds to the subset $\Gamma \subset \Delta$, we have to find a character $\lambda \in X^*(T)$ such that

(1) $\langle \lambda, \check{\alpha}_i \rangle = 0$ for all $\check{\alpha}_i \in \check{\Gamma}$.
(2) $\langle \lambda, \check{\alpha}_i \rangle > 0$ for all $\check{\alpha}_i \notin \check{\Gamma}$.
(3) $\langle \lambda, w\check{\omega}_i \rangle \neq 0$ for all $w \in W, \check{\omega}_i \in \check{\Omega}$.

We first reduce to the case in which G is essentially simple. Recall that if $G = G_1 \times G_2$ then $T = T_1 \times T_2$ with $T_i = T \cap G_i$ $(i = 1,2)$ and for every parabolic subgroup $P = P_1 \times P_2$ with $P_i = P \cap G_i$ $(i = 1,2)$. Also $Pic(G/P) = Pic(G_1/P_2) \times Pic(G_2/P_2)$. We then have

Proposition 2.1. *Then there exist an ample line bundle L_λ on G/P such that $G/P^{ss}_\lambda = G/P^s_\lambda$ if and only if, writing $\lambda = (\lambda_1, \lambda_2)$, $G/P^{ss}_{\lambda_i} = G/P^s_{\lambda_i}$ for $i = 1,2$.*

Proof. The proof follows, since clearly the element $\lambda \in X^*(T)$ satisfies properties (1), (2) and (3) above if and only if the elements $\lambda \in X^*(T_i)$ also satisfy the same properties for $i = 1,2$. □

From now on we shall assume the our group G is essentially simple. We leave to the reader to formulate, using the above Proposition, results in the general case.

We have

Theorem [K2] 2.2. *Assume G is not of type A. Then if $P \subset G$ is a parabolic subgroup such that there is an ample line bundle L_λ on G/P with $G/P^{ss}_\lambda = G/P^s_\lambda$. Then $P = B$, a Borel subgroup.*

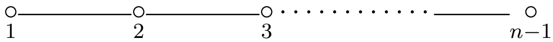
Proof. First of all it is clear that if $P = B$, there exists a line bundle L_λ on G/B such that $G/B^{ss}_\lambda = G/B^s_\lambda$, otherwise it would easily follow that $X^*(T)$ would be contained in the union of the finitely many hyperplanes orthogonal to the elements $w\check{\omega}_i$, with $w \in W$, $i = 1, \dots, l$.

Now remark that if G is not of type A_n, also the dual root system $\check{\Phi}$ is not of type A_n and one knows, see for example [Bou], that each coroot is W conjugate to a multiple of a fundamental coweight. Now let $P \supsetneq B$. Let $\lambda \in X^*(P)$. There exists a simple coroot $\check{\alpha}_i$ with $\langle \lambda, \check{\alpha}_i \rangle = 0$. Assume that $\check{\alpha}_i = mw\check{\omega}_j$. Then

$$0 = \frac{1}{m} \langle \lambda, \check{\alpha}_i \rangle = \langle \lambda, \check{\omega}_j \rangle$$

and our claim follows. □

It remains to analyze the case $G = SL(n)$ i.e. G is of type A_{n-1}. In this case the Dynkin diagram is



and we index the set of fundamental weights and simple roots accordingly. We have

Lemma 2.3. *For each $i, j = 1, \dots, n$, there exists an element $w \in W$ with $\langle \omega_j, w\check{\omega}_i \rangle = 0$ if and only if n divides ij.*

Proof. Recall that, if we consider R^n, with basis $e_1 \dots e_n$, and usual scalar product, then we can set $\alpha_i = \check{\alpha}_i = e_i - e_{i+1}$ and $\omega_i = \check{\omega}_i = \frac{n-i}{n}(e_1 + \cdots + e_i) - \frac{i}{n}(e_{i+1} + \cdots + e_n)$for $i = 1, \dots, n$. Recall that $W = S_n$ acting by permuting coordinates. Computing we get that

$$\langle \omega_i, w\check{\omega}_j \rangle = 0$$

if and only if the system

$$\begin{cases} jx + (n-j)y &= 0 \\ zx + (i-z)y &= 0 \end{cases} \tag{2.3}$$

admits a solution (x, y, z) with $x \neq 0$ and z an integer such that $0 < z < i$. Indeed the vector $v = x(e_1 + \cdots + e_j) + y(e_{j+1} + \cdots + e_n)$ is a non zero multiple of ω_j if and only if it is orthogonal to $e_1 + \cdots + e_n$, that is if $jx + (n-j)y = 0$ with x (and y) not equal to zero. On the other hand let w be a permutation and suppose that $z = |\{1, \dots i\} \cap w\{1, \dots i\}|$. Then a vector v, which as above is a multiple of ω_j, is orthogonal to $\check{\omega}_i$ if and only if it is orthogonal to $w(e_1 + \cdots e_i)$. That is. if and only if $zx + (i-z)y = 0$. Finally the fact that x and y are both not equal to zero implies that $0 < z < i$, proving our claim.

Now eliminate x from (2.3) getting $nz = ij$. This proves that n divides ij.

Suppose now that n divides ij. Then clearly the triple (x, y, z) with $z = \frac{ij}{n}$, $x = n - j$, $y = -j$ is a solution of the system (2.3) and hence taking as w any permutation such that $z = |\{1, \dots i\} \cap w\{1, \dots i\}|$ we get that $\langle \omega_i, w\check{\omega}_j \rangle = 0$ as desired. □

We have seen that a parabolic subgroup $P \supset B$ if associated to a subset $\Gamma \subset \Delta$. To Γ there corresponds the set $I = \{i | \alpha_i \notin \Gamma\}$ and we shall denote P by P_I. We have

Theorem 2.4 [K2](see also [K1]). *Let $G = SL(n)$. Let $I = \{i_1, \dots, i_r\}$ with $1 \leq i_1 \cdots \leq i_t < n$. Then there exists an ample line bundle L_λ on G/P_I such that $(G/P_I)^{ss}_\lambda = (G/P_I)^s_\lambda$ if and only if $GCD(n, i_1, \dots, i_r) = 1$*

Proof. From Lemma 2.3, we have that an L_λ with the above properties exists if and only if there is no $j < n$ with n dividing $i_s j$ for each $s = 1, \dots, r$.

Assume that $(n, i_1, \dots, i_h) = 1$, and that such a j exists. Let p be a prime such that p^t, $t > 0$ is the highest power of p dividing n. Then there must exist an index s such that p does not divide i_s. This implies that p^t divides j, hence n divides j, contrary to the fact that $n > j$.

Viceversa assume that p divides $(n, i_1, \dots, i_h)$. Then set $j = \frac{n}{p}$. We have $j i_s = n \frac{i_s}{p}$ as desired. □

References

[Bou] N. Bourbaki, Groupes et algèbres de Lie (chap. 4,5,6) Masson Paris 1981.

[MFK] D. Mumford, J. Fogarty, F. Kirwan, Geometric Invariant Theory (Third Edition), Springer - Verlag Berlin, Heidelberg, NewYork, 1993.

[K1] S. Senthamarai Kannan, Torus Quotients of Homogeneous spaces, Proceedings of the Indian Acad. Sci.(Math. Sci.) **108** 1998, 1–12.

[K2] S. Senthamarai Kannan, Torus Quotients of Homogeneous spaces II, Proceedings of the Indian Acad. Sci.(Math. Sci.) **to appear**.

[RR] S. Ramanan, A. Ramanathan, Projective normality of flag varieties and Schubert varieties, Invent. Math. **79** no. 2 1985, 217–224.

[Se] C.S. Seshadri, Quotient spaces modulo reductive algebraic groups, Ann. Math.**95**, 1972, 511–556.