DETERMINING SAFETY STOCK WITH BACKLOGGING AND DELIVERY SLACK TIME

M.E. Nenni [1], M.M. Schiraldi [2], S.L. Van de Velde [3]

[1] DPGI - “Federico II” University of Naples
Piazzale Tecchio 80 – 80125, Napoli, Italy

[2] DIM - “Tor Vergata” University of Rome
Via del Politecnico 1 – 00133, Roma, Italy

[3] Rotterdam School of Management - “Erasmus” University
P.O. Box 1738, 3000 DR Rotterdam, The Netherlands

Abstract

In this work a technique for evaluating the effect of backlogging and the influence on service level of the presence of slack time in the deliveries is presented. A supplier-retailer-customer supply chain is taken into consideration, so that it is not possible for the retailer – who holds the safety stock – to react to a sudden increase of the demand with any kind of expedited production/replenishment order. With this technique it is possible to search for the optimal trade-off between time and safety stock in order to reach a certain service level, as well as to determine the increase in the service level that is possible to gain having at disposal some slack time from the moment in which an order is received and the moment the ordered item must be loaded in the carrier or consigned in the hands of the customer. Thus, the concept of virtual safety stock is refined, along with a numerical investigation of its effectiveness versus the use of traditional physical safety stock.

Keywords:
Safety Stock, Service Level, Backlogging, Virtual Stock.

1 INTRODUCTION

Safety stocks are needed to hedge against demand and supplier delivery time uncertainty (Zinn, 1990). As a means to avoid stock-outs, safety stocks hence play an important role in achieving customer satisfaction and retention. However, safety stock holding cost may be very high and actually some safety stock may never be used; indeed, it is just-in-case inventory (Wijngaard,1989). Any investment in safety stock beyond what is absolutely required is therefore wasteful (Krupp, 1997). Traditional safety stock theory is based on the assumption of immediate delivery of the ordered products. However, this condition is often absent in practice. Ordered products, ready to be shipped, are often kept in the warehouse for one or more two days for various reasons: the supply chain may be very short; the logistic service provider only transports the products on certain days of the week; or the customer may be willing to accept late deliveries in exchange for a financial incentive. The possibility to recover a time slice from the time an order is received and the moment the product is delivered to the customer may play a critical role in the determination of safety stock, because service level is deeply influenced by this delay: an order which is received when no stock is available does not generate a stock-out condition indeed, if the delivery can be delayed until a new replenishment lot is arrived. Consequently, in order to reach a certain service level it may be appropriate to combine a certain safety stock level with the opportunity of delayed deliveries, which grant a sort of virtual safety stock.

In this paper, we analyze a supplier-retailer-customer supply chain taken in consideration, so that it is not possible for the retailer to react to a sudden increase of the demand with any kind of expedited production/replenishment order, but should only rely on his safety stocks or on the opportunity of delayed deliveries.

2 THE TRADE-OFF BETWEEN SAFETY STOCK AND TIME

The main idea which stands for the conceiving of the virtual safety stock is the trade-off between stock and time. Actually, the so-called technique of ”sandbagging” gives basically three alternatives to protect against uncertainty in industrial systems: reserving productive capacity, extending lead times when committing due dates for lots, and reserving a number of product output. The two last concepts are nothing more that the safety time and the safety stock that Whybark and Williams described in 1976 as the two main way of protecting inventory from uncertainty, provided that there are two main kinds of uncertainties in inventory control - demand-side uncertainty and supply-side uncertainty (Natarajan and Goyal, 1994). The trade-off between safety stock and safety time has been extensively discussed in the context of MRP systems; safety time is here defined as the extra time on top of the time a lot needs to go through a manufacturing stage, in order to obtain a conservative assumption for planning purposes. Whybark and Williams (1976) found, through simulations, that safety time is preferable to safety stock in case of uncertainty of timing of demand or replenishment, but that safety stock is preferable to safety time in case the uncertainty lies in the quantity of demand or supply. In contrast, Grasso and Taylor (1984) found that safety time was never preferable to safety stock; Buzacott and Shanthikumar (1994) pointed out that Grasso and Taylor’s findings might be due to their specific for safety time definition and that systems other safety time interpretations might have outperformed safety stock systems; they also concluded that safety time is usually preferable to safety stock only if the required shipments over the lead time can be forecasted accurately. Accordingly, we can safely conclude that the merit of increasing safety time versus increasing safety stock in MRP system is not yet well understood. On the contrary, what on we all may agree is the possibility of substituting safety stock with safety time. Chang (1985) attempted to evaluate this ”interchangeability” considering the maximum production during the safety time to be equivalent to a safety stock in that it represents the maximum extra demand that could have met; in a follow-up study, Chang and Hung (1999) proposed a method for determining safety stock levels when making available-to-promise statements considering both production lead time and production rate randomness. Both of these works are though referred to manufacturing systems where a production facility characterized by a certain - random or fixed - yield rate accumulates products in the inventory; but this systems is inappropriate to represent a retailing facility where inventory is filled up by periodic replenishment deliveries from a supplier; in this latter case the supplier reliability plays a much more critical role, because customer satisfaction only depends on the occurrence of a single event - the delivery of the new lot - which may be delayed or affected by other kind of variability, as for example the presence of defective products (Shih, 1990) or mismatch between amount received and quantity requisitioned (Noori, Keller, 1986). Chang's concept of safety time is based on the foreknowledge of the extra demand enough in advance to produce it during the safety time, but this cannot comply with the intrinsic concept of unpredictability which is embodied in the safety stock purpose: indeed, the classical demand-system formulation of safety stock (Zinn, 1990) is itself conceived to protect from those sudden variation of demand and delivery time which occur when it is anyway too late to react with an expedited order. At least, and that is basically our opinion, when possible it may be convenient to backlog the order in excess. To sum up, in this work the assumption of advance demand information is also given - in the form of orders that are received in advance of their due dates (Karaesemen, Liberopoulous, Dallery 2003) - but the possibility to react to the sudden increase of the demand is excluded, nor with production neither with expedited replenishment orders.

In a previous work (Schiraldi and Van de Velde, 2001) under the same hypotheses we attempted to quantify the effect of the presence of an extra-time for the delivery - called DST, Delivery Slack Time - on safety stock and service level, though using a simple heuristic and under restricting hypotheses. In that work we managed to demonstrate that the presence of DST acts on service level in the same way safety stock does, so the concept of Virtual Safety Stock was introduced. In this work we managed to solve the trade off between safety stock and time in its most general case; so a final method to compute service level starting from the simultaneous use of physical - traditional - safety stock and virtual safety stock - originating from DST - is here presented, along with numerical computation of its effectiveness in a number of different scenarios.

3 THE IMPACT OF PHYSICAL AND VIRTUAL SAFETY STOCK ON SERVICE LEVEL

Let's analyse a typical retailer situation: his supplier provides him with a replenishment quantity of Q products, which in the average case is delivered every period T. In the average case, this lot of size Q will suffice to satisfy all his customers' orders in period T. The retailer's performances are measured through his service level, defined as the ratio between the number of satisfied orders and the number of received orders. The retailer's service level is uncertain being the customers' demand (D) and the supplier's delivery time (DT) two stochastic variables. Thus, the retailer keeps safety stock to secure his service level from two kind of unexpected events: a delay in the supplier's replenishment delivery or an increase of the customer's demand. Specifically, safety stock level is determined under the assumption that there is no possibility to react to these unexpected events modifying the quantity/timing of the new replenishment lot; in other words, the acknowledge of the unexpected event occurs when it is already too late: no warning of the delivery delay is given, the shipment is already in transit or there is no way to recognize the demand increase. Note that these assumptions represent the fundamental principles of the dimensioning of safety stock level in both Re-Order Cycle (ROC) and Re-Order Level (ROL) standard inventory policies.

Adding safety stock on top of cycle stock increases the probability that, when the new replenishment lot arrives, enough stock was present in the retailer's warehouse to satisfy all the cumulated customer orders that has been received. However, a slightly different way to represent the same problem is the following: adding safety stock on top of cycle stock increases the probability that when a single order is received, enough stock is available in the retailer's warehouse to satisfy that customer. This definition shifts the problem focus from the demand variability which is recorder when the new lot arrives, to the time variability of when the stock get exhausted. The passage is not trivial and allows us to introduce the concept of Delivery Slack Time (DST) in the problem formulation. Shifting the focus from quantities to timings we make the additional simplifying assumption that the customer may order only one product at a time. In this way we avoid introducing another stochastic variable, the order quantity ; thus we can more simplier state that a customer order cannot be satisfied if both of the following two conditions are verified:

1. stock is exhausted within the arrival of the customer's order;
2. no replenishment will arrive within the date of the delivery.

Thus two moments need to be defined:

t_r is the moment in which an order is received;

t_d is the moment in which the ordered product needs to be delivered.

The physical presence of the ordered products is needed in t_d not in t_r. Clearly, being a stock-out condition in t_r, the bigger the interval between t_d and t_r, the higher the probability that a new replenishment lot will arrive and the customer can be satisfied. Thus we define *Delivery Slack Time* (*DST*) such as:

$$DST = (t_d - t_r)$$
$$DST \geq 0$$

So we have that if

$P_{ex}(t)$ is the probability that all the stock is exhausted within time *t ;*

$P_{rep}(t)$ is the probability that within time *t* no replenishment lot has arrived,

then the probability $P_{nsat}(t_r)$ *not* to satisfy an order which is received in time t_r can be determined through

$$P_{nsat}(t_r) = P_{ex}(t_r) \cdot P_{rep}(t_d) = P_{ex}(t_r) \cdot P_{rep}(t_r + DST) \qquad (1)$$

Note that $P_{ex}(t_r)$ depends on the customer demand variability and can be modified adding safety stock in the warehouse of the retailer, while $P_{rep}(t_d)$ depends on the supplier's delivery time variability and can be modified acting on *DST*.

Recall that *DST* is defined as the time interval between the moment in which the order is received and the moment in which the ordered product needs to be delivered. We discriminate three cases:

1. The *DST* time interval can be totally *inside* the retailer's order-to-delivery process: for example, it is known - and accepted by the customers - that the retailer performs the deliveries only on Friday

morning; thus, if an order is received on Monday evening, the retailer can rely on three working days of *DST*. If on Monday the retailer has no stocks in his warehouse, he should not refuse the order because he may retrieve the product by Friday, and the customer will not realize that his order found the retailer in a stock-out condition.

2. The *DST* time interval can be totally *outside* the retailer's order-to-delivery process: for example, a customer orders a product and expects an immediate delivery; the retailer negotiates a delayed delivery (backlog) of thee days, offering an economical compensation to the customer. In this case he will have a DST of three days too, though the customer will notice that the retailer was in stock-out when he lauched the order.
3. The *DST* time interval can be partially *inside* and partially *outside* the retailer's order-to-delivery process: for example, the order is received on Monday being a preliminary agreement that the delivery will be on Friday. Despite of this, the retailer negotiates other two days of backlog. In this way the total *DST* will be five days.

This discrimination will be useful when the concept of *virtual safety stock* will be introduced, later on. Preliminary, remind that a function becomes properly *virtual* when the performances of the system do not decrease with respect to the traditional *physical* configuration; in this case we will talk about pure *virtual stock* only when the customer will not notice the stock-out upon the receipt of his order.

Clearly, in the extreme case in which all customers accept their orders to be backlogged indefinitely, service level result to be 100% independently from $P_{ex}(t_r)$, because $P_{rep}(t_r + \infty) = 0$ and then $P_{nsat}(t_r) = 0$. Now, being (B_{max}) the maximum backlog time allowed by the customer, that is the maximum delay that a single customer may accept to wait for the delivery of his product, clearly we have that

$$DST \geq B_{max}$$

and $DST = B_{max}$ only if the *DST* is completely outside the retailer's order-to-delivery process.

In the traditional formulation of safety stock, it is not considered the case in which a customer may wait a certain delay from the launch of the order to his delivery; in other words, the safety stock is determined on the hypothesis of immediate delivery. But if backlogging is allowed or, more simply, if $DST > 0$ then the safety stock may be reduced.

At this point, two main problems arise: first, in order to quantitatively evaluate the influence of *DST* on service level, the probability distribution which describes the probability of exhausting the cycle stock in each time *t* inside *T* must be computed; second, we should not forget that the supplier delivery time is random as well, so it is not possible to completely rely on the arrival of the new replenishment lot at the end of period *T*.

Now $P_{ex}(t)$ will be determined assuming a ROL inventory policy; analogously, a similar method can be used with the ROC case. According to the *demand system* formulation with ROL inventory policy, safety stock is determined with the hypotheses that customer demand and supplier delivery time are independent random variables, so that:

1. the demand follows a Normal distribution, N($\overline{D}$;σ_D);
2. the supplier delivery follows a Normal distribution N($\overline{DT}$;σ_{DT}).

Hence the Safety Stock (*SS*) is defined as

$$SS = k\sqrt{\sigma_D^2 \cdot \overline{DT} + \sigma_{DT}^2 \cdot \overline{D}^2}$$

where *k* originates from the management specified service level (*SL*), that is

$$SL = erf(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{k} \exp\left(-\frac{z^2}{2}\right)dz \quad (2)$$

If the demand increase over what it was expected, the cycle stock will be exhausted in advance with respect to when it was planned, that is the end of period T where $\overline{D} \cdot T = Q$. Let us for instance call $t^* <$ T the time in which cycle stock is exhausted, in a specific case in which the demand had increased over the average: if it happens that no order are received during the interval [t^*;*T*], we could affirm that 100% service level has been – luckily – reached even without any safety stock. Similarly, in accordance to what has been previously described, if all the customers that launch orders in the interval [t^*;*T*] accept their orders to be backlogged of ($T – t^*$) time units - until the new replenishment lot *Q* is arrived - we will reach the 100% service level without safety stock as well.

To our knowledge, no contribution is present in literature on the frequency distribution which describes the probability of exhausting the cycle stock in each time t^* inside T, being the demand distributed in accordance to N($\overline{D}$;σ_D); a method to perform this computation is through a simple assumption, which though does not influence the correctness of the reasoning: Fig. 1 below shows the inventory level according to ROL policy for a generic retailer in a supply chain, along with the Normal distribution which describes the variability of the demand:

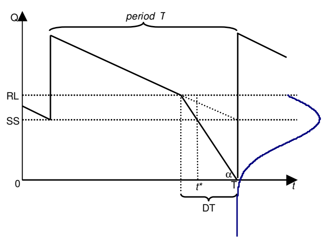

Figure 1: the saw-tooth diagram (variability of the demand)

As it has been previously stated, the result of a sudden increase in the demand – after the replenishment order has been launched when the Re-Order Level (*RL*) has been reached – is that the crossing point between the inventory line and the *SS* level will be in point t^* instead at the end of the period *T*. In the situation shown in the figure, the *SS* is anyway adequate to fulfill the demand while obviously a further increase in the demand would have resulted in a stock-out. Thus, if we make the assumption of linear demand during the delivery time *DT*, we can say that if our safety stock has been chosen to ensure a specific service level *SL**, we expect that we would not run out of cycle stock *before* time t^*.

Under this assumption we evaluate the probability of exhausting the cycle stock *exactly* in time t^*, that is the probability of exhausting the cycle stock in a interval [$t^*–\varepsilon$; $t^*+\varepsilon$] being ε an infinitesimal time amount.

$$P_{ex}[t^* - \varepsilon; t^* + \varepsilon] = \frac{1}{\sqrt{2\pi}}\int_{k(t^*-\varepsilon)}^{k(t^*+\varepsilon)} \exp\left(-\frac{z^2}{2}\right)dz \quad (3)$$

where

$$k(t) = (d \cdot DT - d \cdot t)\left(\frac{\sqrt{DT}}{\sigma_d \cdot t}\right) \quad (4)$$

Proof. the exhaustion of cycle stock in advance with respect to when it was planned is due only to variability of the demand, not to variability of delivery time. Thus, when safety stock if kept only to protect from variability of the demand, aiming at reaching service level *SL** where, in accordance to (2),

$$SL^* = erf\,(k^*)$$

then the safety stock *SS** should be provided, that is

$$SS^* = k^* \sqrt{\sigma_D^2 \cdot \overline{DT}}$$

with few simple geometrical consideration, given α the angle related to the maximum expected demand, it stands:

$$\begin{cases} (\overline{DT} - t^*) \cdot \tan(\alpha) = SS \\ \overline{DT} \cdot \tan(\alpha) = SS + \overline{D} \cdot \overline{DT} \end{cases}$$

hence,

$$k^* = (D \cdot \overline{DT} - D \cdot t^*) \cdot \left(\frac{\sqrt{\overline{DT}}}{t^* \cdot \sigma_D} \right)$$

so if

$$k^* = k(t^*)$$

the probability that cycle stock is exhausted *before t** is then the complement of *SL**

$$P_{ex}[-\infty; t^*] = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{k(t^*)} \exp\left(-\frac{z^2}{2} \right) dz$$

hence the probability that cycle stock is exhausted in the interval [*t**+ ε; *t**-ε] becomes

$$P_{ex}[t^* - \varepsilon; t^* + \varepsilon] = P_{ex}[-\infty; t^* + \varepsilon] - P_{ex}[-\infty; t^* - \varepsilon]$$

that is

$$P_{ex}[t^* - \varepsilon; t^* + \varepsilon] = -\frac{1}{\sqrt{2\pi}} \int_{k(t^* - \varepsilon)}^{k(t^* + \varepsilon)} \exp\left(-\frac{z^2}{2} \right) dz$$

Clearly, if on top of cycle stock some more *safety* stock is kept, the time *t** when all the stock is exhausted shifts forward. If we indicate with *PSS* an undefined quantity of *physical* safety stock to be added to the cycle stock, analogously to the previous case, we have that (4) becomes

$$k^*(t) = (PSS + d \cdot DT - d \cdot t) \left(\frac{\sqrt{DT}}{\sigma_d \cdot t} \right) \tag{5}$$

In this way, acting on the *PSS* value it is possible to modify the probability of exhausting the physical stock before a certain time. It should be noted that the name *PSS*, *physical* safety stock, originates from the contrast with the *virtual* safety stock that will be described further on.

Now $P_{ex}[t^*-\varepsilon;\ t^*-\varepsilon]$ has been determined and related to *PSS* which is a free parameter that managers shall set. In order not to be able to satisfy the order arrived at time *t**, on top of the absence of stock of the previous lot, the condition that the new replenishment lot has not yet arrived must be set. Thus, the probability P_{rep} that the replenishment is arriving in the time interval (*t**;∞] is

$$P_{rep}(t^*; \infty] = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\hat{k}(t^*)} \exp\left(-\frac{z^2}{2} \right) dz \tag{6}$$

where

$$\hat{k} = \frac{t - DT}{\sigma_{TA}} \tag{7}$$

Proof. A variation in the supplier delivery time is not related to the variability of the customer demand, thus, when safety stock if kept only to protect from variability of the delivery time, aiming at reaching service level *SL** where in accordance to (2),

$$SL^* = erf\,(\hat{k}^*)$$

then the safety stock *SS** should be provided, that is

$$SS^* = \hat{k}^* \sqrt{\sigma_{DT}^2 \cdot \overline{D}^2} = \sigma_{DT} \cdot \overline{D} \cdot \hat{k}^*$$

with few simple geometrical consideration it stands:

$$\frac{SS^*}{\overline{D}} = t^* - \overline{DT}$$

Hence

$$\hat{k}^* = \frac{t^* - \overline{DT}}{\sigma_{DT}}$$

so if

$$\hat{k}^* = \hat{k}(t^*)$$

hence the probability that the replenishment arrives later than *t** is then the complement of *SL**, that is

$$P_{rep}(t^*; \infty] = \frac{1}{\sqrt{2\pi}} \int_{\hat{k}(t^*)}^{\infty} \exp\left(-\frac{z^2}{2} \right) dz\ .$$

Clearly, if it is possible to satisfy the order even though the replenishment delays, a delay must be added and the probability that the replenishment arrives too late decreases; so, analogously, indicating the delay with *DST* we have that (7) becomes

$$\hat{k}^* = \frac{t^* - \overline{DT} + DST}{\sigma_{DT}} \tag{8}$$

In this way, acting on the *DST* value it is possible to modify the probability of receiving the replenishment when it is too late to satisfy the customer, and this influence the service level in the same way the *PSS* does. For this reason it is possible to interpret the *DST* as a *Virtual Safety Stock*. Now, given $P_{rep}(t^*;\infty]$ and the previously determined $P_{ex}[t^*-\varepsilon;\ t^*+\varepsilon]$ it is possible to evaluate the probability $P_{nsat}(t^*)$ of not being able to satisfy an order which arrives at time *t**, in relation to *PSS* and *DST*, that is, with (3) and (6),

$$P_{nsat}(t^*) = \left[-\frac{1}{\sqrt{2\pi}} \int_{k(t^* - \varepsilon)}^{k(t^* + \varepsilon)} \exp\left(-\frac{z^2}{2} \right) dz \right] \cdot \left[\frac{1}{\sqrt{2\pi}} \int_{\hat{k}(t^*)}^{\infty} \exp\left(-\frac{z^2}{2} \right) dz \right]$$

Being valid (5) and (8). In this way the probability of being able to satisfy all the order that arrive at *any* time *t*, which is the service level, through

$$SL = 1 - \int_0^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{\hat{k}(t^*)}^{\infty} \exp\left(-\frac{z^2}{2} \right) dz \right] \cdot \left[-\frac{1}{\sqrt{2\pi}} \int_{k(t^* - \varepsilon)}^{k(t^* + \varepsilon)} \exp\left(-\frac{z^2}{2} \right) dz \right] d(t^*)$$

Hence it is possible to determine the service level which is reached through any combination of *physical* and *virtual* safety stock acting on both *PSS* and *DST* simultaneously.

4 MODEL VALIDATION

A first element to notice is related to the size of ε in (3) and hence in (9); unfortunately, being present the cumulative Normal distribution integral form, it is not possible to solve (9) in a closed form; one way to evaluate (9) is through a numerical investigation: thus, clearly results depends on the size of ε, which in the analysis software shall be set to the smallest value possible in order to have a precise evaluation of the integral: for simplicity purposes, time t can be measured in days, thus in our numerical simulations ε has been chosen in the interval [0.001÷0.02] days. Hence, through (3) it is possible to build the graph in Fig. 2, where the purple line describes the probability of exhausting cycle stock in each time inside the *DT*.

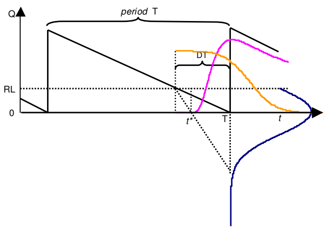

Figure 2: the saw-tooth diagram with P_{ex} and P_{rep} distributions (illustrative)

Fig. 2 has been built for illustrative purposes in an extremal case in which the demand distribution is *D*=N(1,1) and the delivery time distribution is *DT*=N(3,2), in order to put in evidence the asymmetry of the $P_{ex}(t^*)$ distribution. When the ratio $\bar{D}/\sigma_D$ increases, as well as when $\overline{DT}$ increases, the $P_{ex}(t^*)$ distribution tends to a symmetric distribution, where the average value of the $P_{ex}(t^*)$ distribution is in correspondence to the end of period T. Clearly, adding *PSS* to the cycle stock approximately results in shifting the $P_{ex}(t^*)$ to the right, while no influence has on the distribution nor the DT variability neither the presence of *VSS*.

A second element to be noticed is related to the $P_{rep}(t^*+DST;\infty]$ cumulative distribution, which is represented in Fig. 2 with the orange line. Recall that the P_{rep} $(t^*+DST;\infty]$distribution describes the probability that the replenishment delivery arrives later than $t^* + DST$; being $DST = 0$, thus in absence of *VSS*, clearly the probability of the replenishment occurring at the end of period T is exactly 50%. However, according to the classical theory of Safety Stock, the replenishment delivery times follows a Normal distribution, thus the probability that a replenishment occurs in a negative time (!) can assume non-zero values, as well as the probability that a replenishment occurs *before* the correspondent order has been launched, which is absurd. In our work, this simplifying hypothesis has been kept in order to compare the *VSS* effectiveness to the traditional *SS* effectiveness: but, at the same time, this has resulted in the fact that the P_{rep} value in $t = 0$ may not be 100%, which should not be considered totally correct. It is important to notice that this imperfectness, which occurs when σ_{DT} assumes values similar or greater than $\overline{DT}$, it is anyway present in the classical safety stock theory.

The evaluation of the *VSS* - *PSS* trade-off and that of its influence on the service level has been carried out in different scenarios, with different distribution of the demand and of the delivery times. Being the demand set to *D* = N(1;0.5), as delivery time increases in the average and standard deviation, the amount of *VSS* (measured in days of *DST*) has been evaluated in order to reach different service levels, and compared to the amount of the relative physical safety stock (measured in number of items) that can to be substituted: in this case the relative merit of backlogging or exploiting *DST* resulted to be negligible with respect of using physical safety stock, independently of the average value of the delivery time. This is maximally due to the value of average demand, which is set to one-item-per-day. The situation drastically changes when average demand increases, as it is shown in Table 1 below (every fractional number has been rounded up).

DST is measured in days, *PSS* in product units.
Delivery time: *DT*=N(10;1)

Target SL	*D*= N(1; 0.5)		*D*= N(100; 50)		*D*= N(1000; 500)	
	DST (*PSS* =0)	*PSS* (*DST* =0)	*DST* (*PSS* =0)	*PSS* (*DST* =0)	*DST* (*PSS* =0)	*PSS* (*DST* =0)
50.00%	0	0	0	0	0	0
55%	1	1	1	24	1	236
60%	1	1	1	48	1	474
65%	1	1	1	73	1	721
70%	1	1	1	99	1	982
75%	2	2	2	127	2	1262
80%	2	2	2	158	2	1575
85%	2	2	2	194	2	1939
90%	3	3	3	240	3	2398
95%	3	4	3	308	3	3078
99.99%	6	7	6	696	6	6958

Table 1: Trade-off between *VSS*(*DST*) and *PSS* as demand increases

This table shows the equivalence between *virtual* and *physical* safety stock; it is possible to see that, when i.e. *D*= N(1000; 500), having at disposal a 3-days delivery slack time, 95% service level is granted and thus a safety stock level of 3078 items can be eliminated.

In order to evaluate the behaviour of the *physical-virtual* safety stock trade-off when the demand and delivery time probability distribution change shape, Table 2 summarizes the results as demand standard deviation increases with respect to demand average.

DST is measured in days, *PSS* in product units.
Delivery time: *DT*=N(10;1)

Target SL	*D*= N(1; 0.1)		*D*= N(1; 2)		*D*= N(1; 10)	
	DST (*PSS* =0)	*PSS* (*DST* =0)	*DST* (*PSS* =0)	*PSS* (*DST* =0)	*DST* (*PSS* =0)	*PSS* (*DST* =0)
50.00%	0	0	0	0	0	0
55%	1	1	1	1	3	4
60%	1	1	2	2	5	9
65%	1	1	2	3	6	13
70%	1	1	3	4	7	17
75%	1	2	3	5	7	22
80%	1	2	4	6	8	27
85%	2	2	5	7	8	33
90%	2	3	5	9	9	41
95%	2	4	6	11	9	53
99.99%	4	7	9	24	12	118

Table 2: Trade-off between *VSS*(*DST*) and *PSS* as demand standard deviation increases.

It is possible to see that *VSS* is much more convenient that *PSS* when the standard deviation of the demand assumes high values, because the *VSS* increases less that the *PSS* does as demand standard deviation increases. On the contrary it resulted that when delivery time standard deviation increases with respect to delivery time average, the *VSS* behaves similarly to *PSS*.

5 A COST MODEL

Clearly since *PSS* is measured in items and *VSS*, that is *DST*, in days, the evaluation of the effectiveness of substituting stock with time is of relative importance; much more useful would be if these data were translated in economic values. A fair cost model related to backlogging opportunities to be used in the well-known saw-tooth inventory diagram can be derived by that described by Grubbstrom and Erdem (1999), thus introducing the following notation:

h inventory holding cost, per unit and time unit;

b backlog cost, per unit and time unit;

on top of these, we recall the symbols already introduced

B_{max} the maximum backlog time allowed by the customer.

The main difference, under an economic point of view, between protecting the inventory from uncertainty with safety stock of with backlogging is that while safety stock cost is bore continuously – provided that SS level is determined *in advance* with respect to the occurrence of the demand or the delivery time variation – backlogging is carried out *only* when it is necessary – that means only *if* the demand or the delivery time will actually vary. Hence, it is important to notice that the cost *b* could not be bore every time; moreover, until the *DST* is recovered into the order-to-delivery process inside the retailer procedures – as it has been stated in the previous paragraphs – backlog is not exploited. Differently, the customer may ask for an economic compensation for a delayed delivery, provided that the delay does not exceed a certain time limit B_{max}. As a consequence, in any economic confrontation *ex ante* between physical and virtual safety stock, one should recall that is comparing a *certain* cost (safety stock cost) with an *expected* cost (backlog cost)

However, in order to explicit the trade-off between *virtual* and *physical* safety stock we will here make the assumption that the *DST* is totally external to the retailer's order-to delivery process; in other words, in this cost model *DST* is only retrieved through backlog.

Hence, given the demand and the delivery time probability distributions, it is possible to evaluate an *upper bound* for the expected cost C_{tot} to reach a target service level over period T through the combined use of *VSS* and *PSS*, that is

$$C_{tot} = C_{PSS} + C_{VSS}$$

$$C_{tot} \le PSS \cdot h \cdot T + b \cdot \left[\left(k\sqrt{\sigma_{DT}^2 \cdot \overline{D}^2 + \sigma_D^2 \cdot \overline{DT}} \right) - PSS \right] \cdot \frac{B_{\max}}{2} \cdot [1 - erf(k)]$$

Proof. Clearly the physical safety stock holding cost over the period *T* is *PSS* · *h* ·*T* because is a deterministic value, while the backlogging cost can be evaluated as follows: provided that an order cannot be backlogged for a time greater than B_{max}, the cost of backlogging the order which arrives in time *t'* is upper bounded by

$$C(t') = b \cdot q' \cdot (B_{\max} - t')$$

where *q'* is the ordered quantity. Now, *q'* averages the maximum demand (per time unit) that we intended to satisfy aiming at the given service level, that is

$$q' \cong \frac{(SS - PSS)}{DST}$$

Under the previously specified assumptions, it becomes

$$q' \cong \frac{(SS - PSS)}{B_{\max}}$$

Recalling that

$$SS = k\sqrt{\sigma_D^2 \cdot \overline{DT} + \sigma_{DT}^2 \cdot \overline{D}^2}$$

hence, integrating over *t'* from 0 to *DST* we have

$$\int_c^{DST} C(t')dt' =$$

$$= b \cdot \int_0^{DST} \left[\frac{\left(k\sqrt{\sigma_{DT}^2 \cdot \overline{D}^2 + \sigma_D^2 \cdot \overline{DT}} \right) - PSS}{B_{\max}} \right] (B_{\max} - t')dt'$$

Now the probability of backlogging should be considered : a most correct way should be to evaluate it through (9), but for simplicity purposes an upperbound can be determined: knowing that *VSS* will be used in all cases in which *PSS* will not suffice, it is possible to say that at most *all* the *VSS* will be used in (1-*SL*) percentages of the cases: this is translated into the multiplication by [1-*erf*(*k*)] in accordance to (2). Thus with iterative calculations (10) can be minimized acting on *PSS.*

6 CONCLUSION

In this work a technique for evaluating the effect of backlogging and the influence of the presence of delivery slack time on the service level has been presented. With this technique it is possible to search for the optimal trade-off between time and safety stock in order to reach a certain service level, as well as to determine the increase in the service level that is possible to gain having at disposal some slack time from the moment in which an order is received and the moment the ordered item *must* be loaded in the carrier or consigned in the hands of the customer. The concept of increasing service level without safety stock but with only the exploitation of time had suggested the introduction of the *virtual safety stock* (*VSS*) concept, which is the translation, in product units, of the presence of the *delivery slack time*. For this reason, in this work the traditional safety stock [10] kept to protect from demand and supplier delivery time variability, has been called *physical safety stock* (*PSS*). A numerical investigation of the effectiveness of the *VSS* versus the *PSS* has been carried out, and the *VSS* has turned out to be much more effective than *PSS* when high demand rates or high variability of demand is present. A cost model to economically compare the two safety stock types has been described, where backlog cost and stockholding cost is considered.

7 REFERENCES

[1] J.A.G. Krupp (1997). Safety Stock Management. Prod. and Inventory Management Journal 38, 3

[2] J.A. Buzacott, J.G. Shanthikumar (1994). Safety Stock versus Safety time in MRP Controlled Production Systems. Management Science, 40/12.

[3] C. Chang (1985). The interchangeability of safety stock and safety time. Journal of Operations Management, 6 , 35-42.

[4] W. Zinn, H. Marmostein (1990). Comparing two alternative methods of determining safety stock leves: the Demand and the Forecast system. Journal of business logistic, 11/1.

[5] Y. Hung C. Chang (1999). Determining safety stocks for production planning in uncertain Manufacturing. International Journal of Production Economics, 58 , 199–208.

[6] E.T. Grasso B.W. Taylor (1984). A simulation-based investigation of supply / timing uncertainty in MRP systems. International Journal of Prod. Res., 22, 485-497.

[7] S.C. Graves (1988). Safety stocks in manufacturing systems. Journal of Manufacturing Operation Management, 1, 67-101.

[8] C. New (1975). Safety Stock for Requirements Planning. Production and Inventory Management, 2 , 1-18.

[9] R. Natarajan S.K. Goyal (1994). Safety stocks in JIT environments. International Journal of Operations and Production Management 14/10, 64–71.

[10] W. Shih. Optimal inventory policies when stockouts result from defective products. International Journal of Production Research 18/6 , 677–686.

[11] A.H. Noori G. Keller (1986). One-period order quantity strategy with uncertain match between the amount received and quantity requisitioned. INFOR 24/1, 1–11.

[12] F. Karaesmen, G. Liberopoulos, Y. Dallery (2003). Production/inventory control with advance demand information (in: stochastic modeling and optimization of manufacturing systems and supply chains). International series in Operations Research and Management Science. Kluwer Academic Publishing.

[13] D.C. Whybark, J.G. Williams (1976). MRP under uncertainty. Decision Science, 7, 595-606.

[14] J. Wjingaard (1989). Timing and lot-sizing in production control. Journal of Manufacturing Operations Management, 2, 35-51.

[15] J. Wjingaard (2002). The effect of foreknowledge of demand in case of a restricted capacity: the single-stage single-product case with lost sales.

[16] J. Wjingaard, J.C. Wortmann (1985). MRP and inventories. European Journal of Operational Research, 20, 281-293.

[17] R.W. Grubbstrom, A. Erdem. The EOQ with backlogging derived without derivatives. International Journal of Production Economics, 59, 529-530.

DST is measured in days, *PSS* in product units.
Delivery time: *DT*=N(10;1)

	D= N(1; 0.5)		*D*= N(100; 50)		*D*= N(1000; 500)	
Target SL	*DST* (*PSS* =0)	*PSS* (*DST* =0)	*DST* (*PSS* =0)	*PSS* (*DST* =0)	*DST* (*PSS* =0)	*PSS* (*DST* =0)
50.00%	0	0	0	0	0	0
55%	1	1	1	24	1	236
60%	1	1	1	48	1	474
65%	1	1	1	73	1	721
70%	1	1	1	99	1	982
75%	2	2	2	127	2	1262
80%	2	2	2	158	2	1575
85%	2	2	2	194	2	1939
90%	3	3	3	240	3	2398
95%	3	4	3	308	3	3078
99.99%	6	7	6	696	6	6958

DST is measured in days, *PSS* in product units.
Delivery time: *DT*=N(10;1)

	D= N(1; 0.1)		*D*= N(1; 2)		*D*= N(1; 10)	
Target SL	*DST* (*PSS* =0)	*PSS* (*DST* =0)	*DST* (*PSS* =0)	*PSS* (*DST* =0)	*DST* (*PSS* =0)	*PSS* (*DST* =0)
50.00%	0	0	0	0	0	0
55%	1	1	1	1	3	4
60%	1	1	2	2	5	9
65%	1	1	2	3	6	13
70%	1	1	3	4	7	17
75%	1	2	3	5	7	22
80%	1	2	4	6	8	27
85%	2	2	5	7	8	33
90%	2	3	5	9	9	41
95%	2	4	6	11	9	53
99.99%	4	7	9	24	12	118