Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Algebra 306 (2006) 610–621
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Equivariant cohomology of the wonderful group compactification

Elisabetta Strickland

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy

Received 17 November 2005

Available online 10 February 2006

Communicated by Peter Littelmann

Abstract

In this paper we compute the rational $G \times G$-equivariant cohomology ring of the so-called wonderful compactification of G (see [C. de Concini, C. Procesi, Complete symmetric varieties, in: Invariant Theory, Montecatini, 1982, in: Lecture Notes in Math., vol. 996, Springer-Verlag, Berlin, 1983, pp. 1–44]). This is obtained as an application of the results in [E. Bifet, C. de Concini, C. Procesi, Cohomology of regular embeddings, Adv. Math. 82 (1) (1990) 1–34; E. Strickland, Computing the equivariant cohomology of group compactifications, Math. Ann. 291 (2) (1991) 275–280] by a careful analysis of the relevant Stanley–Reisner systems.

1. Introduction

1.1. Let H be a connected affine algebraic group over the complex numbers $\mathbb{C}$. In [BDP] one introduces the notion of a regular embedding X of a H-homogeneous space and gives a recipe to compute the (rational) H-equivariant cohomology of X in terms of the (rational) H-equivariant cohomology of each of the H-orbits in X and of some combinatorial data associated to the incidence structure of orbit closures. A special case of

E-mail address: strickla@mat.uniroma2.it.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.01.021

this situation is given by the so-called symmetric varieties, as explained in [BDP]. An even more special case and the one which will be extensively treated here is the following.

Take a semisimple adjoint group G and let $H = G \times G$. We consider G as a $G \times G$-space with respect to the left and right multiplication. We can then construct, following [DP] or [St1], a canonical $G \times G$-equivariant compactification X of G. Our purpose here is to explicitly compute the rational $G \times G$-equivariant cohomology ring of X. We shall also determine the natural morphism

$$j : H^*_{G\times G}(pt, \mathbb{Q}) \to H^*_{G\times G}(X, \mathbb{Q})$$

induced by the map $X \to pt$, so that we will also obtain $H^*(X, \mathbb{Q})$ as the quotient $H^*_{G\times G}(X, \mathbb{Q})/I$, where I is the ideal generated by the image of the positive degree part of $H^*_{G\times G}(pt, \mathbb{Q})$ under j.

To state our results, let us first of all choose a maximal torus T in G with character group $X(T)$ and a Borel subgroup $T \subset B \subset G$. Associated to these choices, we obtain a root system $\Phi \subset X(T)$, and a basis of simple roots $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ for $X(T)$ and hence for the vector space $\mathfrak{h}_\mathbb{Q} := X(T)_* \otimes \mathbb{Q}$ (here $X_*(T)$ is the dual of $X(T)$, i.e., the lattice of one parameter subgroups in T).

Let us recall that the Weyl group $W = N(T)/T$ of G acts on $\mathfrak{h}_\mathbb{Q}$ as a group generated by reflections and let us denote by $\{s_1, \dots, s_\ell\}$ the set of simple reflections.

Given a subset $\Gamma \subset \{1, \dots, \ell\}$, we shall denote by W_Γ the subgroup of W generated by the reflections s_i with $i \in \Gamma$. In particular $W = W_{\{1,\dots,\ell\}}$.

Now consider the vector space $V = \mathfrak{h}_\mathbb{Q} \oplus \mathfrak{h}_\mathbb{Q}$. Using the basis of simple roots, we can then identify the ring of polynomial functions $\mathbb{Q}[V]$ on V with the polynomial ring $\mathbb{Q}[u_1, \dots, u_\ell, z_1, \dots, z_\ell]$. Also, we can make the change of variables

$$x_i = \frac{u_i - z_i}{2}, \qquad y_i = \frac{u_i + z_i}{2}, \quad i = 1, \dots, \ell,$$

and identify $\mathbb{Q}[V]$ with $\mathbb{Q}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]$.

Given a monomial $m = x_1^{n_1} \cdots x_\ell^{n_\ell}$, we set $\operatorname{supp} m = \{i \mid n_i \neq 0\} \subset \{1, \dots, \ell\}$.

We now take in $\mathbb{Q}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]$, the span A of the elements of the form

$$x_1^{n_1} \cdots x_\ell^{n_\ell} p(y_1, \dots, y_\ell)$$

such that, setting $\Gamma = \{1, \dots, \ell\} - \operatorname{supp} m$, $p(y_1, \dots, y_\ell) \in \mathbb{Q}[y_1, \dots, y_\ell]^{W_\Gamma}$, the ring of invariant polynomials with respect to the reflection group W_Γ.

It is easy to see that A is a graded subring of $\mathbb{Q}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]$, where we set $\deg x_i = \deg y_i = 2$ for $i = 1, \dots, \ell$.

We can now state our first main result:

Theorem 1.2. *As a graded ring A is naturally isomorphic to $H^*_{G\times G}(X, \mathbb{Q})$.*

In order to compute the image of the natural morphism

$$j : H^*_{G\times G}(pt, \mathbb{Q}) \to H^*_{G\times G}(X, \mathbb{Q}),$$

we go back to the old set of variables

$$u_i = x_i + y_i, \qquad z_i = y_i - x_i, \quad \text{for } i = 1, \dots, \ell.$$

W acts on the spaces U and Z spanned by the u_i's and z_i's, respectively. We thus get an action of $W \times W$ on our polynomial ring $\mathbb{Q}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]$ and we can then consider the subring B of invariant polynomials under this $W \times W$-action. Our second main result is then the following:

Theorem 1.3.

(1) $B \subset A$.
(2) *The morphism j is injective and, under the identification of A with $H^*_{G\times G}(X, \mathbb{Q})$ given in Theorem* 1.2, *B is identified with $j(H^*_{G\times G}(pt, \mathbb{Q}))$.*

We finish this introduction by recalling that in [St2] an algorithm was given to compute these cohomologies. In a sense, in this paper we complete that project.

In [DP1] (see also [LP]), a different approach is given to the computation of $H^*_{G\times G}(X, \mathbb{Q})$ as the ring of invariants of the $T \times T$-equivariant cohomology of the closure of the maximal torus T in X.

2. The (RS)-system associated to the wonderful embedding

Since we are going to apply the results of [BDP] only in the case in which the relevant regular fan is the positive quadrant $C = \{(a_1, \dots, a_\ell) \in \mathbb{R}^\ell \mid a_i \geqslant 0, \ \forall i = 1, \dots, \ell\}$, we shall directly assume that we are in this case and hence we shall not recall the definition of a regular fan here.

Definition 2.1. A Stanley–Reisner (RS)-system $\mathfrak{A}$ on C is the following set of data:

(1) For any subset $\Gamma = \{i_1, \dots, i_h\} \subset \{1, \dots, \ell\}$ or equivalently for the face C_Γ defined by $C_\Gamma = \{(z_1, \dots, z_\ell) \in C \mid z_i = 0, \ \forall i \notin \Gamma\}$, a graded commutative $\mathbb{Q}$-algebra with identity, A_Γ, together with a regular sequence of homogeneous elements $\underline{x}^\Gamma = x^\Gamma_{i_1}, \dots, x^\Gamma_{i_h}$.
(2) For all $j \in \Gamma$, setting $\Gamma_j := \Gamma - \{j\}$, a homomorphism of graded algebras

$$\phi^{\Gamma_j}_\Gamma : A_{\Gamma_j} \to A_\Gamma / \big(x^\Gamma_j\big)$$

such that

$$\phi^{\Gamma_j}_\Gamma \big(x^{\Gamma_j}_i\big) \equiv x^\Gamma_i \mod \big(x^\Gamma_j\big), \quad \forall i \in \Gamma_j.$$

Given such a (RS)-system $\mathfrak{A}$, we associate to it an algebra A, called the (RS)-algebra of $\mathfrak{A}$. This algebra is defined as the subalgebra $A \subset \bigoplus_\Gamma A_\Gamma$ consisting of the sequences (a_Γ), $a_\Gamma \in A_\Gamma$ such that

$$\phi_\Gamma^{\Gamma_j}(a_{\Gamma_j}) \equiv a_\Gamma \mod \left(x_j^\Gamma\right)$$

for all $\Gamma \subset \{1, \dots, \ell\}$ and for all $j \in \Gamma$.

We now want to recall how one can associate such a (RS)-system to the wonderful compactification X of a semisimple adjoint group G.

For this, let us briefly recall the combinatorial structure of the $G \times G$-orbits in X. If we consider the complement $D = X - G$, then D is a divisor with normal crossings and smooth irreducible components $D_1, \dots, D_\ell$.

For each subset $\Gamma \subset \{1, \dots, \ell\}$, the intersection

$$D_\Gamma = \bigcap_{j \in \Gamma} D_j$$

is irreducible and it is the closure of a unique $G \times G$-orbit $\mathcal{O}_\Gamma$ (of course $X = D_\emptyset$). Then the correspondence associating to each subset Γ of $\{1, \dots, \ell\}$ the orbit $\mathcal{O}_\Gamma$ is a bijection. In particular the orbit corresponding to $\{1, \dots, \ell\}$ is the unique closed orbit in X, which is isomorphic to $G/B \times G/B$, and we have that $\Gamma \subset \Gamma'$ if and only if $\overline{\mathcal{O}}_\Gamma \supset \mathcal{O}_{\Gamma'}$.

Also recall that every line bundle on X admits a canonical $\tilde{G} \times \tilde{G}$-linearization, $\tilde{G}$ being the universal cover of G. This implies that if $\mathrm{Pic}(X)$ is the Picard group of X, then, taking equivariant Chern classes, we get an isomorphism

$$\mathrm{Pic}(X) \otimes \mathbb{Q} \simeq H^2_{G \times G}(X, \mathbb{Q}). \tag{1}$$

Finally, denoting by Λ the weight lattice, i.e., the character group of the maximal torus $\tilde{T}$ which is the preimage of T in $\tilde{G}$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "h^*"] \arrow[d] & \mathrm{Pic}(\mathcal{O}_{\{1,\dots,\ell\}}) \arrow[d] \\ \Lambda \arrow[r, "a"] & \Lambda \times \Lambda \end{tikzcd} \tag{2}$$

where h^* is induced by inclusion and $a(\lambda) = (\lambda, 0) - (0, \lambda)$, while the vertical arrows are isomorphisms. Using this, one gets an identification of $\mathrm{Pic}(X)$ with the lattice Λ of weights for our root system Φ and, under this identification, $[\mathcal{O}(D_i)] = \alpha_i \in \mathrm{Pic}(X)$.

We are now going to recall the geometric structure of each orbit $\mathcal{O}_\Gamma$.

Take a subset $\Sigma \subset \{1, \dots, \ell\}$. Corresponding to Σ, we have the subset $\Delta_\Sigma \subset \Delta = \{\alpha_1, \dots, \alpha_\ell\}$ consisting of the α_i's with $i \in \Sigma$. Consider the root system Φ_Σ consisting of those roots in Φ which are linear combinations of roots in Δ_Σ.

For each root α denote by $\mathfrak{g}_\alpha \subset \mathfrak{g} := \mathrm{Lie}\, G$ the root subspace associated to α and by $X_\alpha = \exp \mathfrak{g}_\alpha$ the corresponding root subgroup in G.

We then can define the Levi factor L associated to Σ as the subgroup of G generated by T and by the X_α's with $\alpha \in \Phi_\Sigma$.

We also consider the two parabolic subgroups $P_\Sigma^+ \supset B$ and $P_\Sigma^- \supset B^-$, B^- being the opposite Borel subgroup to B relative to our chosen maximal torus T, with P_Σ^+ defined as the subgroup generated by B and L, P_Σ^- by B^- and L.

Finally, we denote by $\overline{L}$ the adjoint quotient of L. Notice that we have quotient homomorphisms

$$\pi_\Sigma^\pm : P_\Sigma^\pm \to \overline{L}.$$

We can then consider

$$\pi^+ \times \pi^- : P_\Sigma^+ \times P_\Sigma^- \to \overline{L} \times \overline{L}$$

and take the subgroup $Q_\Sigma \subset P_\Sigma^+ \times P_\Sigma^-$, which is defined as the preimage under $\pi^+ \times \pi^-$ of the diagonal subgroup in $\overline{L} \times \overline{L}$.

We then have, [DP],

Proposition 2.2. *For each $\Gamma \subset \{1, \dots, \ell\}$, set $\Sigma = \{1, \dots, \ell\} - \Gamma$. There is an isomorphism of $G \times G$-varieties between the orbit $\mathcal{O}_\Gamma$ and $G \times G / Q_\Sigma$.*

We can now, following [St2], give the definition of the (RS)-system $\mathfrak{R}_X$, associated to X. Take $\Gamma = \{i_1, \dots, i_h\} \subset \{1, \dots, \ell\}$. We set

$$R_\Gamma = H^*_{G\times G}(\mathcal{O}_\Gamma) = H^*_{Q_\Sigma}(pt, \mathbb{Q}).$$

To define the regular sequence $\underline{x}^\Gamma$, we consider the $G \times G$-equivariant divisors $D_1, \dots, D_l$ and we set for any $j \in \Gamma$, x_j^Γ equal to the first equivariant Chern class $c_1(\mathcal{O}(D_j)|_{\mathcal{O}(\Gamma)}) \in H^2_{G\times G}(\mathcal{O}_\Gamma)$.

In order to use the above to make explicit computations, let us recall a few well-known facts.

Given a connected linear algebraic group M, let U be its unipotent radical and set $M' = M/U$. Then take a maximal torus $T \subset M'$ and let $W = N(T)/T$ be the corresponding Weyl group. Set $\mathfrak{h}_\mathbb{Q} = X_*(T) \otimes \mathbb{Q}$, $X_*(T)$ being the lattice of one parameter subgroups of T. W acts on $\mathfrak{h}_\mathbb{Q}$ and on its coordinate ring $\mathbb{Q}[\mathfrak{h}_\mathbb{Q}]$.

Proposition 2.3.

$$H^*_M(pt, \mathbb{Q}) \simeq H^*_{M'}(pt, \mathbb{Q}) \simeq \mathbb{Q}[\mathfrak{h}_\mathbb{Q}]^W.$$

Since W acts on $\mathfrak{h}_\mathbb{Q}$ as a group generated by reflections, we get that $\mathbb{Q}[\mathfrak{h}_\mathbb{Q}]^W$ is a polynomial ring.

Let us go back to our G. Take the maximal torus $T \subset G$ and consider the maximal torus $T \times T \subset G \times G$. Clearly $X(T \times T) = X(T) \times X(T)$, the corresponding root system is just $\Phi \times \{0\} \cup \{0\} \times \Phi$, and $\Delta \times \{0\} \cup \{0\} \times \Delta$ is a set of simple roots and the Weyl group is just $W \times W$.

As before we define $\mathfrak{h}_{\mathbb{Q}} = X_*(T) \otimes \mathbb{Q}$ and we set for each $i = 1, \dots, \ell$,

$$x_i = \frac{(\alpha_i, 0) - (0, \alpha_i)}{2} \quad \text{and} \quad y_i = \frac{(\alpha_i, 0) + (0, \alpha_i)}{2}.$$

We then clearly have that we can identify $\mathbb{Q}[\mathfrak{h}_{\mathbb{Q}} \times \mathfrak{h}_{\mathbb{Q}}]$ with $\mathbb{Q}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]$. Notice that, if we restrict the natural action of $W \times W$ on $\mathbb{Q}[\mathfrak{h}_{\mathbb{Q}} \times \mathfrak{h}_{\mathbb{Q}}]$ to the diagonal subgroup, then clearly the subring $\mathbb{Q}[y_1, \dots, y_\ell]$ is stable under this action and can be identified in a W-equivariant way with $\mathbb{Q}[\mathfrak{h}_{\mathbb{Q}}]$.

Once these notations have been fixed, we can state the following:

Proposition 2.4. *For each $\Gamma \subset \{1, \dots, \ell\}$, set $\Sigma = \{1, \dots, \ell\} - \Gamma$. Consider the ring $\mathbb{Q}[x_1, \dots, x_\ell]/I_\Sigma \otimes \mathbb{Q}[y_1, \dots, y_\ell]^{W_\Sigma}$, where I_Σ is the ideal generated by the x_i, $i \in \Sigma$ and $W_\Sigma \subset W$ is the subgroup generated by the simple reflections s_i, $i \in \Sigma$. Then*:

(1) *$R_\Gamma \simeq \mathbb{Q}[x_1, \dots, x_\ell]/I_\Sigma \otimes \mathbb{Q}[y_1, \dots, y_\ell]^{W_\Sigma}$.*
(2) *For each $j \in \Gamma$ x_j^Γ is the image of x_j modulo I_Σ.*
(3) *If $j \in \Gamma$ and $\Gamma_j = \Gamma - \{j\}$, then*

$$\phi_\Gamma^{\Gamma_j} : R_{\Gamma_j} \to R_\Gamma / \bigl(x_j^\Gamma\bigr)$$

is the homomorphism $\mu_j \otimes \iota_j$ where μ_j is the identity of $\mathbb{Q}[x_1, \dots, x_\ell]/I_{\Sigma \cup \{j\}}$ and ι_j is the inclusion $\mathbb{Q}[y_1, \dots, y_\ell]^{W_{\Sigma \cup \{j\}}} \subset \mathbb{Q}[y_1, \dots, y_\ell]^{W_\Sigma}$.

Proof. (1) follows from Propositions 2.2 and 2.3, once we remark the following two facts.

First of all, denote by S_Σ the connected component of the identity of the subgroup of Q_Σ which is the intersection of our maximal torus $T \times T$ with Q_Σ. S_Σ is a maximal torus in Q_Σ. Furthermore S_Σ coincides with the connected component of the identity of the intersection of the kernels of the characters $(\alpha_i, 0) - (0, \alpha_i)$ with $i \in \Sigma$.

Secondly, the Weyl group of Q_Σ modulo its unipotent radical coincides with the subgroup of the diagonal subgroup of $W \times W$ generated by the reflection s_i, with $i \in \Sigma$.

(2) Using formulas (1) and (2) we get that, if we consider the unique closed orbit $\mathcal{O}_{\{1,\dots,\ell\}} \simeq G/B \times G/B$, then we have a commutative diagram

$$\begin{array}{ccc}
H^2_{G \times G}(X, \mathbb{Q}) & \xrightarrow{h^*} & H^2_{G \times G}(\mathcal{O}_{\{1,\dots,\ell\}}, \mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathfrak{h}^*_{\mathbb{Q}} & \xrightarrow{a} & \mathfrak{h}^*_{\mathbb{Q}} \times \mathfrak{h}^*_{\mathbb{Q}}
\end{array}$$

where h^* is induced by inclusion and $a(\lambda) = (\lambda, 0) - (0, \lambda)$, while the vertical arrows are isomorphisms. Also $[\mathcal{O}(D_i)] = \alpha_i \in \operatorname{Pic}(X)$. This and the definition of the x_j^Γ's clearly implies the claim.

(3) follows from the description given in [BDP] of the homomorphism $\phi_{\Gamma}^{\Gamma_j}$ and from the first two points.

Remark 2.5. Notice that the classes x_i are nothing else that the $G \times G$-equivariant classes of the boundary divisors in X.

Once we have established this proposition, we have clearly reduced the proof of our Theorem 1.2 to a purely algebraic statement. Indeed, by [BDP], we have that if R is the Stanley–Reisner algebra of the (RS)-system $\mathfrak{R}_X$, then we have an isomorphism of graded algebras

$$R \simeq H_{G\times G}(X, \mathbb{Q}). \tag{3}$$

So, let us give the following:

Proof of Theorem 1.2. Recall that the ring A has been defined as the span of the elements of the form

$$x_1^{n_1} \cdots x_\ell^{n_\ell} p(y_1, \dots, y_\ell)$$

such that setting $\Gamma = \{1, \dots, \ell\} - \mathrm{supp}(x_1^{n_1} \cdots x_\ell^{n_\ell})$, then $p(y_1, \dots, y_\ell) \in \mathbb{Q}[y_1, \dots, y_\ell]^{W_\Gamma}$, the ring of invariant polynomials with respect to the reflection group W_Γ.

On the other hand, the Stanley–Reisner ring R, which by formula (3) is isomorphic to the $G \times G$-equivariant cohomology of X, is the subring of the direct sum

$$\bigoplus_{\Gamma \subset \{1,\dots,\ell\}} R_\Gamma = \bigoplus_{\Gamma \subset \{1,\dots,\ell\}} \mathbb{Q}[x_1, \dots, x_\ell]/I_{\{1,\dots,\ell\}-\Gamma} \otimes \mathbb{Q}[y_1, \dots, y_\ell]^{W_{\{1,\dots,\ell\}-\Gamma}}$$

consisting of sequences (a_Γ), $a_\Gamma \in R_\Gamma$ such that

$$\phi_\Gamma^{\Gamma_j}(a_{\Gamma_j}) \equiv a_\Gamma \mod x_j^\Gamma$$

for all Γ and $j \in \Gamma$. Notice that since $\phi_\Gamma^{\Gamma_j}$ is clearly injective, we get that if $(a_\Gamma) \in R$ and $a_{\Gamma_j} \neq 0$ for some $j \in \Gamma$, then automatically we have that $a_\Gamma \neq 0$.

In particular we get that the homomorphism $\mu : R \to R_{\{1,\dots,\ell\}} = \mathbb{Q}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]$ defined by

$$\mu((a_\Gamma)) = a_{\{1,\dots,\ell\}}$$

is injective.

We are now going to show that its image coincides with our ring A, thus proving our claim. To see this, let us take $(a_\Gamma) \in R$ and let us write

$$a_{\{1,\dots,\ell\}} = \sum_{\Gamma \subset \{1,\dots,\ell\}} p_\Gamma \prod_{h \notin \Gamma} x_h$$

with $p_\Gamma \in \mathbb{Q}[y_1, \ldots, y_\ell][x_h]_{h \notin \Gamma} := S_\Gamma$. Now set $\psi_\Gamma : \mathbb{Q}[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell] \to S_\Gamma$ equal to the quotient homomorphism modulo (x_i), $i \in \Gamma$. We clearly have that

$$\psi_\Gamma(a_{\{1,\ldots,\ell\}}) = \sum_{\Gamma' \supset \Gamma} p_{\Gamma'} \prod_{h \notin \Gamma'} x_h.$$

On the other hand, considering R_Γ as a subring of S_Γ, we clearly get that $\psi_\Gamma(a_{\{1,\ldots,\ell\}}) = a_\Gamma$. This and Proposition 2.4 clearly imply that $a_{\{1,\ldots,\ell\}} \in A$ so that $A \supset \mu(R)$.

At this point, take $b \in A$ and write it as

$$b = \sum_{\Gamma \subset \{1,\ldots,\ell\}} q_\Gamma \prod_{h \notin \Gamma} x_h$$

with $q_\Gamma \in \mathbb{Q}[y_1, \ldots, y_\ell][x_h]_{h \notin \Gamma} = S_\Gamma$. Set for each $\Gamma \in \{1, \ldots, \ell\}$,

$$a_\Gamma := \sum_{\Gamma' \supset \Gamma} q_{\Gamma'} \prod_{h \notin \Gamma'} x_h.$$

It is immediate to verify that the sequence $(a_\Gamma) \in R$ and that $\psi((a_\Gamma)) = b$, so that $A \subset \mu(R)$ proving our claim. $\square$

It remains now to prove Theorem 1.3.

Proof of Theorem 1.3. Let us recall, see [DP], that X has a cellular decomposition by affine cells. In particular, this easily implies that the $G \times G$-equivariant cohomology of X is a free module over $H^*_{G\times G}(pt, \mathbb{Q})$. So, the homomorphism

$$j : H^*_{G\times G}(pt, \mathbb{Q}) \to H^*_{G\times G}(X, \mathbb{Q}),$$

is injective and

$$H^*(X, \mathbb{Q}) = H^*_{G\times G}(X, \mathbb{Q})/J,$$

where J is the ideal in $H^*_{G\times G}(X, \mathbb{Q})$ generated by the elements of positive degree in the image of $H^*_{G\times G}(pt, \mathbb{Q})$. Thus it only remains to determine the image of $H^*_{G\times G}(pt, \mathbb{Q})$ in $H^*_{G\times G}(X, \mathbb{Q})$.

Now notice that the inclusion of A into $\mathbb{Q}[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$ clearly coincides, under the identification of A with $H^*_{G\times G}(X, \mathbb{Q})$ and of $\mathbb{Q}[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$ with $H^*_{G\times G}(\mathcal{O}_{\{1,\ldots,\ell\}}, \mathbb{Q})$, with the homomorphism in equivariant cohomology induced by the inclusion of the closed orbit $\mathcal{O}_{\{1,\ldots,\ell\}} \simeq G/B \times G/B$ into X. Consider the maps

$$\mathcal{O}_{\{1,\ldots,\ell\}} \to X \to pt.$$

They are both equivariant, so we get that the image of $H^*_{G\times G}(pt, \mathbb{Q})$ into $H^*_{G\times G}(\mathcal{O}_{\{1,\ldots,\ell\}}, \mathbb{Q})$ coincides with the image of $H^*_{G\times G}(pt, \mathbb{Q})$ in $H^*_{G\times G}(X, \mathbb{Q})$.

To finish, recall that we have set

$$x_i = \frac{(\alpha_i, 0) - (0, \alpha_i)}{2} \quad \text{and} \quad y_i = \frac{(\alpha_i, 0) + (0, \alpha_i)}{2},$$

so by passing to the variables $u_i = (\alpha_i, 0)$ and $v_i = (0, \alpha_i)$, we have an identification of $\mathbb{Q}[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$ with $\mathbb{Q}[u_1, \ldots, u_\ell] \otimes \mathbb{Q}[v_1, \ldots, v_\ell]$ and of the image of $H^*_{G\times G}(pt, \mathbb{Q})$ with $\mathbb{Q}[u_1, \ldots, u_\ell]^W \otimes \mathbb{Q}[v_1, \ldots, v_\ell]^W$. This proves Theorem 1.3. $\square$

We finish giving in detail the example $G = SL(2)$. In this case, we have that $\ell = 1$ and $R_{\{1\}} = \mathbb{Q}[x, y]$. Also $W = \mathbb{Z}/2\mathbb{Z} = \{e, \varepsilon\}$ acts on $\mathbb{Q}[y]$ by $\varepsilon(y) = -y$. It follows that A is the ring of polynomials in x and y of the form $f(y^2) + xg(x, y)$.

It is easy to see that A is generated by the three elements $z_1 = y^2$, $z_2 = x$, $z_3 = xy$ subject to the relation $z_3^2 = z_1 z_2^2$.

Also, setting $u = x + y$, $v = y - x$, we get that the ideal J is generated by the elements xy and $x^2 + y^2$, so in terms of z_1, z_2, z_3, by z_3 and $z_1 + z_2^2$. In particular we get that $H^*(X, \mathbb{Q}) = \mathbb{Q}[z_2]/z_2^4$, in accord with the fact that in this case X is the three-dimensional projective space.

3. Further properties

Let us now consider in $\mathbb{Q}[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$ the subring

$$C := \mathbb{Q}[x_1, \ldots, x_\ell] \otimes \mathbb{Q}[y_1, \ldots, y_\ell]^W.$$

It is clear by our description of $B = H^*_{G\times G}(X, \mathbb{Q})$ that $C \subset B$ so we can consider B as a C-module. Let us denote by $\mathcal{S} \subset X(T)$ the semigroup of positive linear combinations of the simple roots Δ. We define a $\mathcal{S} \times \mathbb{N}$-multigrading on $\mathbb{Q}[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$ by setting

$$d(x_i) = \alpha_i, \qquad d(y_i) = 1, \quad i = 1, \ldots, \ell.$$

Notice that with this multigrading, both B and C are multigraded subrings.

Consider now $\mathbb{Q}[y_1, \ldots, y_\ell]$ as a module over $\mathbb{Q}[y_1, \ldots, y_\ell]^W$. We need to recall some results from [BGG,De]. One defines, for each simple root α_i, the operator $\Delta_i : \mathbb{Q}[y_1, \ldots, y_\ell] \to \mathbb{Q}[y_1, \ldots, y_\ell]$ by

$$\Delta_i(f) = \frac{f - s_i f}{y_i},$$

$s_i \in W$ being the simple reflection with respect to the hyperplane orthogonal to α_i. Given $w \in W$, $w = s_{i_1} \cdots s_{i_k}$ and $k = l(w)$, then the operator $\Delta_w = \Delta_{i_1} \cdots \Delta_{i_k}$ depends only on $w \in W$ and one defines the polynomials

$$u_w := \Delta_{ww_0}(u_{w_0}),$$

where $w_0 \in W$ is the longest element and u_{w_0} is the Weyl denominator polynomial, i.e., the product of the elements $\sum_i n_i y_i$, for $\sum_i n_i \alpha_i$ a positive root, divided by $|W|$.

The following facts hold:

(1) u_w is a polynomial of degree $\ell(w)$.
(2) For any subset $\Gamma \subset \{1, \dots, \ell\}$, $u_w \in \mathbb{Q}[y_1, \dots, y_\ell]^{W_\Gamma}$ if and only if $\ell(s_i w) > \ell(w)$ for each $i \in \Gamma$.
(3) Given $w \in W$ set $L_w = \{i \mid \ell(s_i w) > \ell(w)\}$. The polynomials u_w with $L_W \supset \Gamma$ are a basis of $\mathbb{Q}[y_1, \dots, y_\ell]^{W_\Gamma}$ as a module over $\mathbb{Q}[y_1, \dots, y_\ell]^W$.

Let us now go back to our ring B. In B we have the polinomial $U_{w_0} = x_1 \cdots x_\ell u_{w_0}$ of multidegree $(\alpha_1 + \cdots + \alpha_\ell, \ell(w_0))$. For any $w \in W$, we define the polynomial

$$U_w := \frac{\Delta_{w w_0}}{\prod_{i \in L_w} x_i}(U_{w_0}).$$

We clearly have that $U_w \in B$ and has multidegree $(\sum_{i \notin L_w} \alpha_i, \ell(w))$. We have:

Theorem 3.1.

(1) *The polynomials U_w, $w \in W$, are a basis of B as a C-module. In particular B is a free C-module.*
(2) *If for any $(\gamma, m) \in \mathcal{S} \times \mathbb{N}$ we denote by $B_{(\gamma,m)}$ the component of B of multidegree (γ, m), we have*

$$\sum_{(\gamma,m) \in \mathcal{S} \times \mathbb{N}} \dim B_{(\gamma,m)} e^{(\gamma+m)} = \frac{\sum_{w \in W} e^{(\sum_{i \notin L_w} \alpha_i + \ell(w))}}{\prod_{i=1}^{\ell}(1 - e^{\alpha_i}) \prod_{i=1}^{\ell}(1 - e^{d_i})} \tag{4}$$

where $d_1, \dots, d_\ell$ are the degrees of W and we write $e^{(\gamma+m)}$ for $e^{(\gamma,m)}$.

Proof. (2) is an immediate consequence of (1), so let us prove (1). We have already remarked that the U_w lie in B and we also have that since the u_w are linearly independent on $\mathbb{Q}[y_1, \dots, y_\ell]^W$, they are also linearly independent over C. This immediately implies that the U_w's are linearly independent over C.

It remains to see that the U_w's span B over C. Take any element $f \in B$. Consider it as a polynomial in $\mathbb{Q}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]$. Set $L_f = \{i \mid f \notin (x_i)\mathbb{Q}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]\}$. Then we have that

$$f = \prod_{i \notin L_f} x_i F,$$

where $F \in \mathbb{Q}[x_1, \ldots, x_\ell] \otimes \mathbb{Q}[y_1, \ldots, y_\ell]^{W_{L_f}}$. By what we have recalled above we deduce that we can write

$$F = \sum_{w \mid L_w \supset L_f} d_w u_w$$

with $d_w \in C$ for each w. Now notice that if $L_w \supset L_f$,

$$\prod_{i \notin L_f} x_i u_w = \prod_{i \notin L_f,\, i \in L_w} x_i U_w.$$

It follows that f is a linear combination of the U_w's with coefficients in C proving our claim. □

Remark 3.2. Notice that, if we specialize all the α_i's to 1 in formula (4), and we use the well-known formula

$$(1-e)^\ell = \left(\sum_{w \in W} e^{\ell(w)}\right) \prod_{i=1}^{\ell} \bigl(1 - e^{d_i}\bigr),$$

we get that the right-hand side of formula (4) specializes to

$$\frac{(\sum_{w \in W} e^{\ell(w)})(\sum_{w \in W} e^{\ell - |L_w| + \ell(w)})}{\prod_{i=1}^{\ell} (1 - e^{d_i})^2}.$$

In view of Theorem 1.3, by taking the numerator, we get back the expression for the Poincaré polynomial of X given in [DP].

Acknowledgment

The author thanks the referee for pointing out that, after this paper had been submitted, the interesting preprint [U] by V. Uma has appeared. In this preprint similar results for integral equivariant K-theory of a regular $G \times G$-embedding of G are proved. The inclusion of Section 3 has been suggested by the referee taking into account [U].

References

[BGG] I.N. Bernšteĭn, I.M. Gelfand, S.I. Gelfand, Schubert cells, and the cohomology of the spaces G/P, Uspekhi Mat. Nauk 28 (3(171)) (1973) 3–26.

[BDP] E. Bifet, C. de Concini, C. Procesi, Cohomology of regular embeddings, Adv. Math. 82 (1) (1990) 1–34.

[DP] C. de Concini, C. Procesi, Complete symmetric varieties, in: Invariant Theory, Montecatini, 1982, in: Lecture Notes in Math., vol. 996, Springer-Verlag, Berlin, 1983, pp. 1–44.

[DP1] C. de Concini, C. Procesi, Cohomology of compactifications of algebraic groups, Duke Math. J. 53 (3) (1986) 585–594.

[De] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, Invent. Math. 21 (1973) 287–301.
[LP] P. Littelmann, C. Procesi, Equivariant cohomology of wonderful compactifications, in: Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Paris, 1989, in: Progr. Math., vol. 92, Birkhäuser Boston, Boston, MA, 1990, pp. 219–262.
[St1] E. Strickland, A vanishing theorem for group compactifications, Math. Ann. 277 (1) (1987) 165–171.
[St2] E. Strickland, Computing the equivariant cohomology of group compactifications, Math. Ann. 291 (2) (1991) 275–280.
[U] V. Uma, Equivariant K-theory of compactifications of algebraic groups, arXiv: math.AG/0512187.