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THE CRYSTAL DUALITY PRINCIPLE: FROM GENERAL SYMMETRIES TO GEOMETRICAL SYMMETRIES

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Abstract. We give functorial recipes to get, out of any Hopf algebra over a field, two pairs of Hopf algebras bearing some geometrical content. If the ground field has zero characteristic, the first pair is made of a function algebra $F[G_+]$ over a connected Poisson group and a universal enveloping algebra $U(\mathfrak{g}_-)$ over a Lie bialgebra $\mathfrak{g}_-$: in addition, the Poisson group as a variety is an affine space, and the Lie bialgebra as a Lie algebra is graded; apart for these last details, the second pair is of the same type, namely $\big(F[G_-], U(\mathfrak{g}_+)\big)$ for some Poisson group G_- and some Lie bialgebra $\mathfrak{g}_+$. When the Hopf algebra H we start from is already of geometric type the result involves Poisson duality: the first Lie bialgebra associated to $H = F[G]$ is $\mathfrak{g}^*$ (with $\mathfrak{g} := \mathit{Lie}(G)$), and the first Poisson group associated to $H = U(\mathfrak{g})$ is of type G^*, i.e. it has $\mathfrak{g}$ as cotangent Lie bialgebra. If the ground field has positive characteristic, then the same recipes give similar results, but for the fact that the Poisson groups obtained have dimension 0 and height 1, and *restricted* universal enveloping algebras are obtained. We show how all these "geometrical" Hopf algebras are linked to the original one via 1-parameter deformations, and explain how these results follow from quantum group theory.

"Yet these crystals are to Hopf algebras
but as is the body to the Children of Rees:
the house of its inner fire, that is within it
and yet in all parts of it, and is its life"

N. Barbecue, "Scholia"

Introduction

Among all Hopf algebras over a field $\Bbbk$, there are two special families which are of relevant interest for their geometrical meaning. The function algebras $F[G]$ of algebraic groups G and the universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$, if $\mathit{Char}(\Bbbk) = 0$, or the restricted universal enveloping algebras $\mathbf{u}(\mathfrak{g})$ of restricted Lie algebras $\mathfrak{g}$, if $\mathit{Char}(\Bbbk) > 0$; to

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX be short, we call both the latters "enveloping algebras" and denote them by $\mathcal{U}(\mathfrak{g})$, and similarly by "restricted Lie algebra" when $\mathit{Char}(\Bbbk) = 0$ we shall simply mean "Lie algebra". Function algebras are exactly those Hopf algebras which are commutative, and enveloping algebras those which are connected, cocommutative and generated by their primitives.

In this paper we give functorial recipes to get, out of any Hopf algebra, two pairs of Hopf algebras of geometrical type, namely one pair $\big(F[G_+], \mathcal{U}(\mathfrak{g}_-)\big)$ and a second pair $\big(F[K_+], \mathcal{U}(\mathfrak{k}_-)\big)$. In addition, the algebraic groups thus obtained are *Poisson groups*, and the (restricted) Lie algebras are (restricted) *Lie bialgebras*. Therefore, to each Hopf algebra, encoding a general notion of "symmetry", we can associate in a functorial way some symmetries — "global" ones when taking an algebraic group, "infinitesimal" when considering a Lie algebra — of geometrical type, where the geometry involved is in fact Poisson geometry. Moreover, the groups concerned are always connected, and if $\mathit{Char}(\Bbbk) > 0$ they have dimension 0 and height 1, which makes them pretty interesting from the point of view of arithmetic geometry (hence in number theory).

The construction of the pair $(G_+, \mathfrak{g}_-)$ uses pretty classical (as opposite to "quantum") methods: in fact, it might part of be the content of any basic textbook on Hopf algebras (and, surprisingly enough, it is not!). Instead, to make out the pair $(K_+, \mathfrak{k}_-)$ one relies on the construction of the first pair, and make use of the theory of quantum groups.

Let's describe our results in some detail. Let $J := \mathit{Ker}(\epsilon_H)$ be the augmentation ideal of H (where ϵ_H is the counit of H), and let $\underline{J} := \big\{J^n\big\}_{n \in \mathbb{N}}$ be the associated J–adic filtration, $\widehat{H} := G_{\underline{J}}(H)$ the associated graded vector space and $H^\vee := H \Big/ \bigcap_{n \in \mathbb{N}} J^n$. One proves that $\underline{J}$ is a Hopf algebra filtration, hence $\widehat{H}$ is a graded Hopf algebra: the latter happens to be connected, cocommutative and generated by its primitives, so $\widehat{H} \cong \mathcal{U}(\mathfrak{g}_-)$ for some (restricted) Lie algebra $\mathfrak{g}_-$; in addition, since $\widehat{H}$ is graded also $\mathfrak{g}_-$ itself is graded (as a restricted Lie algebra). The fact that $\widehat{H}$ be cocommutative allows to define on it a Poisson cobracket (from the natural Poisson cobracket $\nabla := \Delta - \Delta^{\mathrm{op}}$ on H) which makes $\widehat{H}$ into a graded *co-Poisson* Hopf algebra, and eventually this implies that $\mathfrak{g}_-$ is a *Lie bialgebra*. So the right-hand side half of the first pair of "Poisson geometrical" Hopf algebras is just $\widehat{H}$.

On the other hand, one consider a second filtration — increasing, whereas $\underline{J}$ is decreasing — namely $\underline{D}$ which is defined in a dual manner to $\underline{J}$: for each $n \in \mathbb{N}$, let δ_n the composition of the n–fold iterated coproduct followed by the projection onto $J^{\otimes n}$ (note that $H = \Bbbk \cdot 1_H \oplus J$); then $\underline{D} := \big\{D_n := \mathit{Ker}(\delta_{n+1})\big\}_{n \in \mathbb{N}}$. Let now $\widetilde{H} := G_{\underline{D}}(H)$ be the associated graded vector space and $H' := \bigcup_{n \in \mathbb{N}} D_n$. Again, one shows that $\underline{D}$ is a Hopf algebra filtration, hence $\widetilde{H}$ is a graded Hopf algebra: moreover, the latter is commutative, so $\widetilde{H} = F[G_+]$ for some algebraic group G_+. One proves also that $\widetilde{H} = F[G_+]$ has no non-trivial idempotents, thus G_+ is connected; a deeper analysis shows that in the positive characteristic case G_+ has dimension 0 and height 1; in addition, since $\widetilde{H}$ is graded, G_+ as a variety is just an affine space. The fact that $\widetilde{H}$ be commutative allows to define on it a Poisson bracket (from the natural Poisson bracket on H given by the commutator) which makes $\widetilde{H}$ into a graded *Poisson* Hopf algebra: this means G_+ is an algebraic *Poisson group*. So the left-hand side half of the first pair of "Poisson geometrical" Hopf algebras is just $\widetilde{H}$.

The relationship among H and the "geometrical" Hopf algebras $\widehat{H}$ and $\widetilde{H}$ can be expressed in terms of "reduction steps" and regular 1-parameter deformations, namely

$$\widetilde{H} \xleftarrow[\mathcal{R}^t_{\underline{D}}(H)]{0 \leftarrow t \rightarrow 1} \!\!\rightarrow H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftarrow[\mathcal{R}^t_{\underline{J}}(H^\vee)]{1 \leftarrow t \rightarrow 0} \!\!\rightarrow \widehat{H} \qquad (\bigstar)$$

where the "one-way" arrows are Hopf algebra morphisms and the "two-ways" arrows are 1-parameter regular deformations of Hopf algebras, realized through the *Rees* Hopf algebras $\mathcal{R}^t_{\underline{D}}(H)$ and $\mathcal{R}^t_{\underline{J}}(H^\vee)$ associated to the filtration $\underline{D}$ of H and to the filtration $\underline{J}$ of $H^\vee$.

The construction of the pair $(K_+, \mathfrak{k}_-)$ uses quantum group theory, the basic ingredients being $\mathcal{R}^t_{\underline{D}}(H)$ and $\mathcal{R}^t_{\underline{J}}(H^\vee)$. In the present context, by quantum group we mean, loosely speaking, a Hopf $\Bbbk[t]$–algebra (t an indeterminate) H_t such that either *(a)* $H_t \big/ t\,H_t \cong F[G]$ for some connected Poisson group G — then we say H_t is a QFA — or *(b)* $H_t \big/ t\,H_t \cong \mathcal{U}(\mathfrak{g})$, for some restricted Lie bialgebra $\mathfrak{g}$ — then we say H_t is a QrUEA. Formula $(\bigstar)$ says that $H'_t := \mathcal{R}^t_{\underline{D}}(H)$ is a QFA, with $H'_t \big/ t\,H'_t \cong \widetilde{H} = F[G_+]$, and also that $H^\vee_t := \mathcal{R}^t_{\underline{J}}(H)$ is a QrUEA, with $H^\vee_t \big/ t\,H^\vee_t \cong \widehat{H} \cong \mathcal{U}(\mathfrak{g}_-)$. Now, a general result — the "Global Quantum Duality Principle", in short GQDP — teaches us how to construct from the QFA H'_t a QrUEA, call it $\big(H'_t\big)^\vee$, and how to build out of the QrUEA $H^\vee_t$ a QFA, say $\big(H^\vee_t\big)'$; then $\big(H'_t\big)^\vee \big/ t\,\big(H'_t\big)^\vee \cong \mathcal{U}(\mathfrak{k}_-)$ for some (restricted) Lie bialgebra $\mathfrak{k}_-$, and $\big(H^\vee_t\big)' \big/ t\,\big(H^\vee_t\big)' \cong F[K_+]$ for some connected Poisson group K_+. This provides the pair $(K_+, \mathfrak{k}_-)$. The very construction implies that $\big(H'_t\big)^\vee$ and $\big(H^\vee_t\big)'$ yield another frame of regular 1-parameter deformations for H' and $H^\vee$, namely

$$\mathcal{U}(\mathfrak{k}_-) \xleftarrow[(H'_t)^\vee]{0 \leftarrow t \rightarrow 1} \!\!\rightarrow H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0} \!\!\rightarrow F[K_+] \qquad (✠)$$

which is the analogue of $(\bigstar)$. In addition, when $\mathit{Char}(\Bbbk) = 0$ the GQDP also claims that the two pairs $(G_+, \mathfrak{g}_-)$ and $(K_+, \mathfrak{k}_-)$ are related by Poisson duality: namely, $\mathfrak{k}_-$ *is the cotangent Lie bialgebra to* G_+, and $\mathfrak{g}_-$ *is the cotangent Lie bialgebra* of K_+ (in short, we write $\mathfrak{k}_- = \mathfrak{g}^\times$ and $K_+ = G^\star_-$). Therefore the four "Poisson symmetries" G_+, $\mathfrak{g}_-$, K_+ and $\mathfrak{k}_-$, attached to H are actually encoded simply by the pair (G_+, K_+).

In particular, when $H' = H = H^\vee$ from $(\bigstar)$ and $(✠)$ together we find

$$\begin{array}{ccccc}
F[G_+] & \xleftarrow[H'_t]{0 \leftarrow t \rightarrow 1}\!\!\rightarrow & H' & \xleftarrow[(H'_t)^\vee]{1 \leftarrow t \rightarrow 0}\!\!\rightarrow & \mathcal{U}(\mathfrak{k}_-) \quad \Big(= U\big(\mathfrak{g}_+^\times\big) \text{ if } \mathit{Char}(\Bbbk) = 0 \Big) \\
 & & \| & & \\
 & & H & & \\
 & & \| & & \\
\mathcal{U}(\mathfrak{g}_-) & \xleftarrow[H^\vee_t]{0 \leftarrow t \rightarrow 1}\!\!\rightarrow & H^\vee & \xleftarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0}\!\!\rightarrow & F[K_+] \quad \Big(= F\big[G^\star_-\big] \text{ if } \mathit{Char}(\Bbbk) = 0 \Big)
\end{array}$$

which gives *four* different regular 1-parameter deformations from H to Hopf algebras encoding geometrical objects of Poisson type (i.e. Lie bialgebras or Poisson algebraic groups).

When the Hopf algebra H we start from is already of geometric type, the result involves Poisson duality. Namely, if $\mathit{Char}(\Bbbk) = 0$ and $H = F[G]$ then $\mathfrak{g}_- = \mathfrak{g}^*$ (where $\mathfrak{g} := \mathit{Lie}(G)$), and if $H = \mathcal{U}(\mathfrak{g}) = U(\mathfrak{g})$ then $\mathit{Lie}(G_+) = \mathfrak{g}^*$, i.e. G_+ has $\mathfrak{g}$ as cotangent Lie bialgebra. If instead $\mathit{Char}(\Bbbk) > 0$ we have only a slight variation on this result.

The construction of $\widehat{H}$ and $\widetilde{H}$ needs only "half the notion" of a Hopf bialgebra: in fact, we construct $\widehat{A}$ for any "augmented algebra" A (i.e., roughly, an algebra with an augmentation, or counit, that is a character), and we construct $\widetilde{C}$ for any "coaugmented coalgebra" C (i.e. a coalgebra with a coaugmentation, or "unit", that is a coalgebra morphism from $\Bbbk$ to C). In particular this applies to bialgebras, for which both $\widehat{B}$ and $\widetilde{B}$ are (graded) Hopf algebras; we can also perform a second construction as above using $\big(B_t'\big)^\vee$ and $\big(B_t^\vee\big)'$ (thanks to a stronger version of the GQDP), and get from these by specialization at $t = 0$ a second pair of bialgebras $\Big(\big(B_t'\big)^\vee\Big|_{t=0}, \big(B_t^\vee\big)'\Big|_{t=0}\Big)$: then again $\big(B_t'\big)^\vee\Big|_{t=0} \cong \mathcal{U}(\mathfrak{k}_-)$ for some restricted Lie bialgebra $\mathfrak{k}_-$, but on the other hand $\big(B_t^\vee\big)'\Big|_{t=0}$ is commutative and with no non-trivial idempotents, but it's not, in general, a Hopf algebra! Thus the spectrum of $\big(B_t^\vee\big)'\Big|_{t=0}$ is an algebraic Poisson monoid, irreducible as an algebraic variety, but it is not necessarily a Poisson group.

It is worth stressing that everything in fact follows from the GQDP, which — in the stronger formulation — deals with augmented algebras and coaugmented coalgebras over 1-dimensional domain. All the content of this paper can in fact be obtained as a corollary of the GQDP as follows: pick any augmented algebra or coaugmented coalgebra over $\Bbbk$, and take its scalar extension from $\Bbbk$ to $\Bbbk[t]$; the latter ring is a 1-dimensional domain, hence we can apply the GQDP, and every result in the present paper will follow.

This note is the written version of the author's talk at the international workshop *"Contemporary Geometry and Related Topics"*, held in Belgrade in May 15-21, 2002. We dwell somewhat in detail upon the very constructions under study, but we skip proofs and other technicalities, which are postponed to a forthcoming article (namely [Ga3]).

Finally, a few words about the organization of the paper. In §1 we collect a bunch of definitions, and some standard, technical results. In §2 we introduce the "connecting functors" $A \mapsto A^\vee$ (on augmented algebras) and $C \mapsto C'$ (on coaugmented coalgebras), and the (associated) "crystal functors" $A \mapsto \widehat{A}$ and $C \mapsto \widetilde{C}$; we also explain the relationship between these two pairs of functors with respect to Hopf duality. §3 cope with the effect of connecting and crystal functors on bialgebras and Hopf algebras. §4 considers the deformations provided by Rees modules, while §5 treats deformations arising from the previous ones via quantum group theory, introducing the "Drinfeld-like functors". In §6 we look at function algebras and enveloping algebras, we collect all our results in the "Crystal Duality Principle", and explain how this result can be also proved via quantum group theory.

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§ 1 Notation and terminology

1.1 Algebras, coalgebras, and the whole zoo. Let $\Bbbk$ be a field, which will stand fixed throughout. In this paper we deal with (unital associative) $\Bbbk$–algebras and (counital coassociative) $\Bbbk$–coalgebras in the standard sense, cf. [Sw] or [Ab]; in particular we shall use notations as in [*loc. cit.*]. For any given (counital coassociative) $\Bbbk$–coalgebra C we denote by $coRad(C)$ its coradical, and by $G(C) := \big\{ c \in C \,\big|\, \Delta(c) = c \otimes c \big\}$ its set of group-like elements; we say C is *monic* if $\big|G(C)\big| = 1$; we say C is *connected* if $coRad(C)$ is one-dimensional: of course "connected" implies "monic".

We call *augmented algebra* the datum of a unital associative $\Bbbk$–algebra A together with a distinguished unital algebra morphism $\underline{\epsilon} : A \longrightarrow \Bbbk$ (so the unit $u : \Bbbk \longrightarrow A$ is a section of $\underline{\epsilon}$): these form a category in the obvious way. We call *indecomposable elements* of an augmented algebra A the elements of the set $Q(A) := J_A \big/ J_A^{\,2}$ with $J_A := Ker\big(\underline{\epsilon} : A \longrightarrow \Bbbk\big)$. We denote $\mathcal{A}^+$ the category of all augmented $\Bbbk$–algebras.

We call *coaugmented coalgebra* the datum of a counital coassociative $\Bbbk$–coalgebra C together with a distinguished counital coalgebra morphism $\underline{u} : \Bbbk \longrightarrow C$ (so $\epsilon : C \longrightarrow \Bbbk$ is a section of $\underline{u}$), and let $\underline{1} := \underline{u}(1)$, a group-like element in C: these form a category in the obvious way. For such a C we call *primitive* the elements of the set $P(C) := \big\{ c \in C \mid \Delta(c) = c \otimes \underline{1} + \underline{1} \otimes c \big\}$. We denote $\mathcal{C}^+$ the category of all coaugmented $\Bbbk$–coalgebras.

We denote $\mathcal{B}$ the category of all $\Bbbk$–bialgebras; clearly each bialgebra B can be seen both as an augmented algebra, w.r.t. $\underline{\epsilon} = \epsilon \equiv \epsilon_B$ (the counit of B) and as a coaugmented coalgebra, w.r.t. $\underline{u} = u \equiv u_B$ (the unit map of B), so that $\underline{1} = 1 = 1_B$: then $Q(B)$ is naturally a Lie coalgebra and $P(B)$ a Lie algebra over R. In the following we'll do such an interpretation throughout, looking at objects of $\mathcal{B}$ as objects of $\mathcal{A}^+$ and of $\mathcal{C}^+$. We call $\mathcal{HA}$ the category of all Hopf $\Bbbk$–algebras; this naturally identifies with a subcategory of $\mathcal{B}$.

We call *Poisson algebra* any (unital) commutative algebra A endowed with a Lie bracket $\{\ ,\ \} : A \otimes A \longrightarrow A$ (i.e., $\big(A, \{\ ,\ \}\big)$ is a Lie algebra) such that the *Leibnitz identities* $\{a\,b, c\} = \{a, c\}\, b + a\, \{b, c\}$, $\{a, b\,c\} = \{a, b\}\, c + b\, \{a, c\}$, hold (for all $a, b, c \in A$). We call *Poisson bialgebra*, or *Poisson Hopf algebra*, any bialgebra, or Hopf algebra, say H, which is also a Poisson algebra (w.r.t. the same product) enjoying $\Delta\big(\{a,b\}\big) = \big\{\Delta(a), \Delta(b)\big\}$, $\epsilon\big(\{a,b\}\big) = 0$, $S\big(\{a,b\}\big) = \big\{S(b), S(a)\big\}$ for all $a, b, c \in H$, the condition on the antipode S being required in the Hopf algebra case, where the (Poisson) bracket on $H \otimes H$ is defined by $\{a \otimes b, c \otimes d\} := \{a,b\} \otimes c\,d + a\,b \otimes \{c,d\}$ (for all $a, b, c, d \in H$).

We call *co-Poisson coalgebra* any (counital) cocommutative coalgebra C with a Lie cobracket $\delta : C \longrightarrow C \otimes C$ (i.e., $\big(C,\delta\big)$ is a Lie coalgebra) such that the *co-Leibnitz identity* $\big(id \otimes \Delta\big) \circ \big(\delta(a)\big) = \sum_{(a)} \Big(\delta\big(a_{(1)}\big) \otimes a_{(2)} + \sigma_{1,2}\big(a_{(1)} \otimes \delta\big(a_{(2)}\big)\big)\Big)$ holds for all $a \in C$, where $\sigma_{1,2} : C^{\otimes 3} \longrightarrow C^{\otimes 3}$ is given by $x_1 \otimes x_2 \otimes x_3 \mapsto x_2 \otimes x_1 \otimes x_3$. We call *co-Poisson bialgebra*, or *co-Poisson Hopf algebra*, any bialgebra, or Hopf algebra, say H, which is also a co-Poisson coalgebra (w.r.t. the same coproduct) enjoying $\delta(a\,b) = \delta(a)\,\Delta(b) + \Delta(a)\,\delta(b)$, $(\epsilon \otimes \epsilon)\big(\delta(a)\big) = 0$, $\delta\big(S(a)\big) = \big(S \otimes S\big)\big(\delta(a)\big)$ for all $a, b \in H$, the condition on the antipode S being required in the Hopf algebra case. Finally, we call *bi-Poisson bialgebra*, or *bi-Poisson Hopf algebra*, any bialgebra, or Hopf algebra, say H, which is simultaneously a Poisson and co-Poisson bialgebra, or Hopf algebra, for some Poisson bracket and cobracket enjoying $\delta\big(\{a,b\}\big) = \big\{\delta(a), \Delta(b)\big\} + \big\{\Delta(a), \delta(b)\big\}$ for all $a, b \in H$ (see [CP], [KT] and references therein for further details on the above notions).

A *graded algebra* is an algebra A which is $\mathbb{Z}$–graded as a vector space and whose structure maps m and u are morphisms of degree zero in the category of graded vector spaces, where $A \otimes A$ has the standard grading inherited from A and $\Bbbk$ has the trivial grading. Similarly we define the graded versions of coalgebras, bialgebras and Hopf algebras, and also the graded versions of Poisson algebras, co-Poisson coalgebras, Poisson/co-Poisson/bi-Poisson bialgebras, and Poisson/co-Poisson/bi-Poisson Hopf algebras, but for the fact that the Poisson bracket, resp. cobracket, must be a morphism (of graded spaces) of degree -1, resp. $+1$. We write $V = \oplus_{z \in \mathbb{Z}} V_z$ for the degree splitting of any graded vector space V.

1.2 Function algebras. According to standard theory, the category of commutative Hopf algebras is antiequivalent to the category of algebraic groups (over $\Bbbk$): then we call *Spec*(H) (*spectrum of* H) the image of a Hopf algebra H in this antiequivalence, and conversely we call *function algebra* or *algebra of regular functions* the preimage $F[G]$ of an algebraic group G. Note that we do *not* require algebraic groups to be reduced (i.e. $F[G]$ to have trivial nilradical) and we do *not* make any restrictions on dimensions: in particular we deal with *pro-affine* as well as *affine* algebraic groups. We say that G is *connected* if $F[G]$ is i.p.-free; this is equivalent to the classical topological notion when $dim(G)$ is finite.

Given an algebraic group G, let $J_G := Ker(\epsilon_{F[G]})$ be the augmentation ideal of $F[G]$; the *cotangent space of* G (at its unity) is $\mathfrak{g}^\times := J_G \Big/ J_G^{\,2} = Q\big(F[G]\big)$, endowed with its weak topology; *the tangent space of* G *(at its unity)* is the topological dual $\mathfrak{g} := \big(\mathfrak{g}^\times\big)^\star$ of $\mathfrak{g}^\times$: this is a Lie algebra, the *tangent Lie algebra of* G. If $Char(\Bbbk) = p > 0$, then $\mathfrak{g}$ is a *restricted* Lie algebra (also called "p–Lie algebra").

We say that G is an algebraic *Poisson* group if $F[G]$ is a Poisson Hopf algebra. Then the tangent Lie algebra $\mathfrak{g}$ of G is a *Lie bialgebra*, and the same holds for $\mathfrak{g}^\times$. If $Char(\Bbbk) = p > 0$, then $\mathfrak{g}$ and $\mathfrak{g}^\times$ are *restricted* Lie bialgebras, the p–operation on $\mathfrak{g}^\times$ being trivial.

1.3 Enveloping algebras and symmetric algebras. Given a Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g})$ its *universal enveloping algebra*. If $Char(\Bbbk) = p > 0$ and $\mathfrak{g}$ is a *restricted* Lie algebra, we denote by $\mathbf{u}(\mathfrak{g}) = U(\mathfrak{g}) \Big/ \big(\{ x^p - x^{[p]} \,|\, x \in \mathfrak{g} \} \big)$ its *restricted universal enveloping algebra*. If $Char(\Bbbk) = 0$, then $P\big(U(\mathfrak{g})\big) = \mathfrak{g}$. If instead $Char(\Bbbk) = p > 0$, then $P\big(U(\mathfrak{g})\big) = \mathfrak{g}^\infty := Span\Big(\big\{x^{p^n} \,\big|\, n \in \mathbb{N}\big\}\Big)$, the latter carrying a natural structure of restricted Lie algebra with $X^{[p]} := X^p$. Note then that $U(\mathfrak{g}) = \mathbf{u}(\mathfrak{g}^\infty)$ for any Lie algebra $\mathfrak{g}$, so any universal enveloping algebra can be thought of as a restricted universal enveloping algebra. Both $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$ are cocommutative connected Hopf algebras, generated by $\mathfrak{g}$ itself. Conversely, if $Char(\Bbbk) = 0$ then each cocommutative connected Hopf algebra is the universal enveloping algebra of some Lie algebra, and if $Char(\Bbbk) = p > 0$ then each cocommutative connected Hopf algebra H which is generated by $P(H)$ is the restricted universal enveloping algebra of some restricted Lie algebra (cf. [Mo], Theorem 5.6.5, and references therein). Therefore, in order to unify the terminology and notation we call both universal enveloping algebras (when $Char(\Bbbk) = 0$) and restricted universal enveloping algebras "enveloping algebras", and denote them by $\mathcal{U}(\mathfrak{g})$; similarly, with a slight abuse of terminology we shall talk of "restricted Lie algebra" even when $Char(\Bbbk) = 0$ simply meaning "Lie algebra". Thus enveloping algebras are simply the objects of the category of cocommutative connected Hopf algebras generated by their primitive elements, regardless of the characteristic of the ground field.

If a cocommutative connected Hopf algebra generated by its primitive elements is also *co-Poisson*, then the restricted Lie algebra $\mathfrak{g}$ such that $H = \mathcal{U}(\mathfrak{g})$ is indeed a (restricted) Lie *bialgebra*. Conversely, if a (restricted) Lie algebra $\mathfrak{g}$ is also a Lie bialgebra then $\mathcal{U}(\mathfrak{g})$ is a cocommutative connected *co-Poisson* Hopf algebra (cf. [CP]).

Let V be a vector space: then the symmetric algebra $S(V)$ has a natural structure of Hopf algebra, given by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$ for all $x \in V$. If $\mathfrak{g}$ is a Lie algebra, then $S(\mathfrak{g})$ is also a Poisson Hopf algebra w.r.t. the Poisson bracket given by $\{x, y\}_{S(\mathfrak{g})} = [x, y]_{\mathfrak{g}}$ for all $x, y \in \mathfrak{g}$. If $\mathfrak{g}$ is a Lie coalgebra, then $S(\mathfrak{g})$ is also a co-Poisson Hopf algebra w.r.t. the Poisson cobracket determined by $\delta_{S(\mathfrak{g})}(x) = \delta_{\mathfrak{g}}(x)$ for all $x \in \mathfrak{g}$. Finally, if $\mathfrak{g}$ is a Lie bialgebra, then $S(\mathfrak{g})$ is a bi-Poisson Hopf algebra with respect to the previous Poisson bracket and cobracket (cf. [KT] and references therein).

1.4 Filtrations. Let $\big\{F_z\big\}_{z\in\mathbb{Z}} =: \underline{F} : \big(\{0\} \subseteq \big) \cdots \subseteq F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \big(\subseteq V\big)$ be a filtration of a vector space V. We denote by $G_{\underline{F}}(V) := \bigoplus_{z\in\mathbb{Z}} F_z \big/ F_{z-1}$ the associated graded vector space. We say that $\underline{F}$ is *exhaustive* if $V^{\underline{F}} := \bigcup_{z\in\mathbb{Z}} F_z = V$; we say it is *separating* if $V_\downarrow := \bigcap_{z\in\mathbb{Z}} F_z = \{0\}$. We say that a filtered vector space is *exhausted* if the filtration is exhaustive; we say that it is *separated* if the filtration is separating.

A filtration $\underline{F} = \big\{F_z\big\}_{z\in\mathbb{Z}}$ in an algebra A is said to be an *algebra filtration* iff $m\big(F_\ell \otimes F_m\big) \subseteq F_{\ell+m}$ for all $\ell, m, n \in \mathbb{Z}$. Similarly, a filtration $\underline{F} = \big\{F_z\big\}_{z\in\mathbb{Z}}$ in a coalgebra C is said to be a *coalgebra filtration* iff $\Delta\big(F_z\big) \subseteq \sum_{r+s=z} F_r \otimes F_s$ for all $z \in \mathbb{Z}$. Finally, a filtration $\underline{F} = \big\{F_z\big\}_{z\in\mathbb{Z}}$ in a bialgebra, or in a Hopf algebra, H is said to be a *bialgebra filtration*, or a *Hopf (algebra) filtration*, iff it is both an algebra and a coalgebra filtration and — in the Hopf case — in addition $S\big(F_z\big) \subseteq F_z$ for all $z \in \mathbb{Z}$. The notions of *exhausted* and *separated* for filtered algebras, coalgebras, bialgebras and Hopf algebras are defined as for vector spaces with respect to the proper type of filtrations.

Lemma 1.5. *Let $\underline{F}$ be an algebra filtration of an algebra A. Then $G_{\underline{F}}(A)$ is a graded algebra; if, in addition, it is commutative, then it is a commutative graded Poisson algebra. If E is another algebra with algebra filtration $\underline{\varPhi}$ and $\phi : A \longrightarrow E$ is a morphism of algebras such that $\phi(F_z) \subseteq \varPhi_z$ for all $z \in \mathbb{Z}$, then the morphism $G(\phi) : G_{\underline{F}}(A) \longrightarrow G_{\underline{\varPhi}}(E)$ associated to ϕ is a morphism of graded algebras. In addition, if $G_{\underline{F}}(A)$ and $G_{\underline{\varPhi}}(E)$ are commutative, then $G(\phi)$ is a morphism of graded commutative Poisson algebras.*

The analogous statement holds replacing "algebra" with "coalgebra", "commutative" with "cocommutative" and "Poisson" with "co-Poisson". In addition, if we start from bialgebras, or Hopf algebras, with bialgebra filtrations, or Hopf filtrations, then we end up with graded commutative Poisson bialgebras, or Poisson Hopf algebras, and graded cocommutative co-Poisson bialgebras, or co-Poisson Hopf algebras, respectively.

Sketch of proof. The only non-trivial part is about the Poisson structure on $G_{\underline{F}}(A)$ and the co-Poisson structure on $G_{\underline{F}}(C)$ (for a coalgebra C). Indeed, let $\underline{F} := \big\{F_z\big\}_{z\in\mathbb{Z}}$ be an algebra filtration of A. For any $\overline{x} \in F_z\big/F_{z-1}$, $\overline{y} \in F_\zeta\big/F_{\zeta-1}$ ($z, \zeta \in \mathbb{Z}$), let $x \in F_z$, resp. $y \in F_\zeta$, be a lift of x, resp. of y: then $[x,y] := (xy - yx) \in F_{z+\zeta-1}$ because $G_{\underline{F}}(A)$ is commutative; therefore we define

$$\big\{\overline{x}, \overline{y}\,\big\} \;:=\; \overline{[x,y]} \;\equiv\; [x,y] \mod F_{z+\zeta-2} \;\in\; F_{z+\zeta-1}\big/F_{z+\zeta-2} \;.$$

This gives a Poisson bracket on $G_{\underline{F}}(A)$ making it into a graded commutative *Poisson* algebra. Similarly, if $\underline{F} := \big\{F_z\big\}_{z\in\mathbb{Z}}$ is a coalgebra filtration of C, for any $\overline{x} \in F_z\big/F_{z-1}$ ($z \in \mathbb{Z}$), let $x \in F_z$ be a lift of x: then $\nabla(x) := \big(\Delta(x) - \Delta^{\mathrm{op}}(x)\big) \in \sum_{r+s=z-1} F_r \otimes F_s$, for $G_{\underline{F}}(C)$ is cocommutative, so the formula

$$\delta(\overline{x}\,) \;:=\; \overline{\nabla(x)} \;\equiv\; \nabla(x) \mod \sum_{r+s=z-1} F_r \otimes F_s \;\in\; \sum_{r+s=z} \big(F_r\big/F_{r-1}\big) \otimes \big(F_s\big/F_{s-1}\big)$$

defines a structure of graded cocommutative *co-Poisson* coalgebra onto $G_{\underline{F}}(C)$. $\square$

Lemma 1.6. *Let C be a coalgebra. If $\underline{F} := \big\{F_z\big\}_{z\in\mathbb{Z}}$ is a coalgebra filtration, then $C^{\underline{F}} := \bigcup_{z\in\mathbb{Z}} F_z$ is a coalgebra, which injects into C, and $C_{\underline{F}} := C\Big/\bigcap_{z\in\mathbb{Z}} F_z$ is a coalgebra, which C surjects onto. The same holds for algebras, bialgebras, Hopf algebras, with algebra, bialgebra, Hopf algebra filtrations respectively.* $\square$

§ 2 Connecting functors on (co)augmented (co)algebras

2.1 The $\underline{\epsilon}$– filtration $\underline{J}$ on augmented algebras. Let A be an augmented algebra (cf. §1.1). Let $J := Ker(\underline{\epsilon})$: then $\underline{J} := \big\{ J_{-n} := J^n \big\}_{n \in \mathbb{N}}$ is clearly an algebra filtration of A, called *the $\underline{\epsilon}$–filtration* of A; hereafter we shall consider it as a $\mathbb{Z}$–indexed filtration, by trivially completing it ($J_z := A \; \forall \, z \in -\mathbb{N}_+$). We say that A is *$\underline{\epsilon}$–separated* if $\underline{J}$ is separating, i.e. $J^\infty := \bigcap_{n \in \mathbb{N}} J^n = \{0\}$. Next (trivial) lemma points out some properties of $\underline{J}$:

Lemma 2.2.

(a) $\underline{J}$ is an algebra filtration of A, which contains the radical filtration of A, that is $J^n \supseteq Rad(A)^n$ for all $n \in \mathbb{N}$ where $Rad(A)$ is the (Jacobson) radical of A.

(b) If A is $\underline{\epsilon}$–separated, then it is i.p.-free.

(c) $A^\vee := A \Big/ \bigcap_{n \in \mathbb{N}} J^n$ is a quotient augmented algebra of A, which is $\underline{\epsilon}$–separated. □

Now we come to the first, somewhat relevant result:

Proposition 2.3. *Mapping $A \mapsto A^\vee := A \Big/ \bigcap_{n \in \mathbb{N}} J^n$ gives a well-defined functor from the category of augmented algebras to the subcategory of $\underline{\epsilon}$–separated augmented algebras. Also, the augmented algebras A of the latter subcategory are characterized by $A^\vee = A$.* □

Remark 2.4: It is worth mentioning a special example of $\underline{\epsilon}$–separated augmented algebras, namely the *graded* ones: these are those augmented algebras with an algebra grading such that the augmentation is a morphism of graded algebras w.r.t. the trivial grading on the ground field. Then one easily proves that

Every graded augmented algebra A is ϵ–separated, or equivalently $A = A^\vee$.

2.5 Drinfeld's $\delta_\bullet$– maps. Let C be a coaugmented coalgebra (cf. §1.1). For every $n \in \mathbb{N}$, define $\Delta^n \colon H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := \mathrm{id}_C$, and $\Delta^n := \big(\Delta \otimes \mathrm{id}_C^{\otimes (n-2)} \big) \circ \Delta^{n-1}$ if $n \geq 2$. For any ordered subset $\Phi = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \dots < i_k$, define the linear map $j_\Phi : H^{\otimes k} \longrightarrow H^{\otimes n}$ by $j_\Phi(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := \underline{1}$ if $i \notin \Phi$ and $b_{i_m} := a_m$ for $1 \leq m \leq k$; then set $\Delta_\Phi := j_\Phi \circ \Delta^k$, $\Delta_\emptyset := \Delta^0$, and $\delta_\Phi := \sum_{\Psi \subseteq \Phi} (-1)^{n - |\Psi|} \Delta_\Psi$, $\delta_\emptyset := \epsilon$. By the inclusion-exclusion principle, this definition admits the inverse formula $\Delta_\Phi = \sum_{\Psi \subseteq \Phi} \delta_\Psi$. We shall also make use of the shorthand notation $\delta_0 := \delta_\emptyset$, $\delta_n := \delta_{\{1,2,\dots,n\}}$. Next lemma is a trivial, technical one:

Lemma 2.6. *Let $a, b \in C$. Then*

(a) $\delta_n \, = \, \big(\mathrm{id}_C - \underline{u} \circ \epsilon \big)^{\otimes n} \circ \Delta^n$ *for all* $n \in \mathbb{N}_+$ *;*

(b) The maps δ_n (and similarly the δ_Φ's, for all finite $\Phi \subseteq \mathbb{N}$) are coassociative, i.e.

$$\big(\mathrm{id}_C^{\otimes s} \otimes \delta_\ell \otimes \mathrm{id}_C^{\otimes (n-1-s)} \big) \circ \delta_n \, = \, \delta_{n+\ell-1} \quad \textit{for all} \quad n, \ell, s \in \mathbb{N} , \; 0 \leq s \leq n-1 \, . \quad \square$$

2.7 The $\delta_\bullet$– filtration $\underline{D}$ on coaugmented coalgebras. Let C be as above, and take notations of §2.5. For all $n \in \mathbb{N}$, let $D_n := \mathit{Ker}(\delta_{n+1})$: then $\underline{D} := \big\{D_n\big\}_{n\in\mathbb{N}}$ is clearly an ascending filtration of C, *the $\delta_\bullet$–filtration* of C; hereafter we shall consider it as a $\mathbb{Z}$–indexed filtration, by trivially completing it ($D_z := \{0\}\ \forall\, z \in -\mathbb{N}_+$). We say that C is *$\delta_\bullet$–exhausted* if $\underline{D}$ is exhaustive, i.e. $\bigcup_{n\in\mathbb{N}} D_n = C$. Next lemma highlights some properties of $\underline{D}$, in particular it shows that it is a refinement of the coradical filtration. We use the notion of "wedge" product, namely $X \bigwedge Y := \Delta^{-1}\big(C \otimes Y + X \otimes C\big)$ for all subspaces X, Y of C, with $\bigwedge^1 X := X$ and $\bigwedge^{n+1} X := \big(\bigwedge^n X\big) \bigwedge X$ for all $n \in \mathbb{N}_+$.

Lemma 2.8.

(a) $D_0 = \Bbbk \cdot \underline{1}$, *and* $D_n = \Delta^{-1}\big(C \otimes D_{n-1} + D_0 \otimes C\big) = \bigwedge^{n+1} D_0$ *for all* $n \in \mathbb{N}$.

(b) $\underline{D}$ *is a coalgebra filtration of* C, *which is contained in the coradical filtration of* C, *that is* $D_n \subseteq C_n$ *if* $\underline{C} := \big\{C_n\big\}_{n\in\mathbb{N}}$ *is the coradical filtration of* C.

(c) C *is $\delta_\bullet$–exhausted* $\iff$ C *is connected* $\iff$ $\underline{D} = \underline{C}$.

(d) $C' := \bigcup_{n\in\mathbb{N}} D_n$ *is a subcoalgebra of* C: *more precisely, it is the irreducible (hence connected) component of* C *containing* $\underline{1}$. $\square$

Here is the second relevant result, the (dual) analogue of Proposition 2.3:

Proposition 2.9. *Mapping* $C \mapsto C' := \bigcup_{n\in\mathbb{N}} D_n$ *gives a well-defined functor from the category of coaugmented coalgebras to the subcategory of $\delta_\bullet$–exhausted (=connected) coaugmented coalgebras. Moreover, the coaugmented coalgebras C of the latter subcategory are characterized by* $C' = C$. $\square$

Remark 2.10: the characterization of connected coalgebras in terms of the $\delta_\bullet$–filtration given in Lemma 2.8*(c)* yields easy, alternative proofs of two well-known facts.

(a) *Every graded coaugmented coalgebra C is connected;*

(b) *Every connected coaugmented coalgebra is monic.*

2.11 Connecting functors and Hopf duality. Let's start from $A \in \mathcal{A}^+$, with $J := \mathit{Ker}(\underline{\epsilon})$: the n–th piece of its $\underline{\epsilon}$–filtration is $J^n := \mathit{Im}\Big(J^{\otimes n} \hookrightarrow H^{\otimes n} \xrightarrow{\mu^n} H\Big)$ where the left hand side arrow is the natural embedding induced by $J \hookrightarrow H$ and μ^n is the n–fold iterated multiplication of H. Similarly, let $C \in \mathcal{C}^+$: the s–th piece of its $\delta_\bullet$–filtration is $D_s := \mathit{Ker}\Big(H \xhookrightarrow{\Delta^{s+1}} H^{\otimes(s+1)} \twoheadrightarrow J^{\otimes(s+1)}\Big)$ where the right hand side arrow is the natural projection induced by $\big(\mathrm{id}_C - \underline{u} \circ \epsilon\big) : H \twoheadrightarrow J$ and Δ^{s+1} is the $(s+1)$–fold iterated comultiplication of C. In categorial terms, *Ker* is dual to *Im*, the iterated comultiplication is dual to the iterated multiplication, and the above embedding $J^{\otimes r} \hookrightarrow H^{\otimes n}$ is dual to the projection $H^{\otimes r} \twoheadrightarrow J^{\otimes r}$ (for $r \in \mathbb{N}$). Therefore

(a) *The notions of $\underline{\epsilon}$–filtration and of $\delta_\bullet$–filtration are dual to each other;*

(b) *The notions of* $A^\vee := A \Big/ \bigcap_{n\in\mathbb{N}} J^n$ *and of* $C' := \bigcup_{n\in\mathbb{N}} D_n$ *are dual to each other;*

(c) The notions of $\underline{\epsilon}$–separated (for an augmented algebra) and $\delta_\bullet$–exhausted (for a coaugmented coalgebra) are dual to each other.

Remark 2.12: *(a) $(\ )^\vee$ and $(\ )'$ as "connecting" functors.* We now explain in what sense both $A^\vee$ and C' are "connected" objects. Indeed, C' is truly connected, in the sense of coalgebra theory (cf. Lemma 2.8*(d)*). On the other hand, we might expect that $A^\vee$ be (or correspond to) a "connected" object by duality. In fact, when A is commutative then it is the algebra of regular functions $F[\mathcal{V}]$ of an algebraic variety $\mathcal{V}$; the augmentation on A is a character, hence corresponds to the choice of a point $P_0 \in \mathcal{V}$: thus the augmented algebra A does correspond to the *pointed variety* $(\mathcal{V}, P_0)$. Then $A^\vee = F[\mathcal{V}_0]$ where $\mathcal{V}_0$ is the connected component of $\mathcal{V}$ containing P_0! This follows at once because $A^\vee$ is i.p.-free (cf. Lemma 2.2). More in general, for any augmented algebra A, if $A^\vee$ is commutative then its spectrum is a *connected* algebraic variety. For these reasons, we shall use the name "connecting functors" for both functors $A \mapsto A^\vee$ and $C \mapsto C'$.

(b) Asymmetry of connecting functors on bialgebras. Let B be a bialgebra. As the notion of $B^\vee$ is dual to that of B', and since $B = B'$ implies that B is a *Hopf* algebra (cf. Corollary 3.4*(b)*), one might dually conjecture that $B = B^\vee$ *implies that B is a Hopf algebra*. Actually, *this is false*, the bialgebra $B := F\big[Mat(n,\Bbbk)\big]$ yielding a counter-example: $F\big[Mat(n,\Bbbk)\big] = F\big[Mat(n,\Bbbk)\big]^\vee$, and yet $F\big[Mat(n,\Bbbk)\big]$ is *not* a Hopf algebra.

(c) Hopf duality and augmented pairings. The most precise description of the relationship between connecting functors of the two types uses the notion of "augmented pairing":

Definition 2.13. *Let $A \in \mathcal{A}^+$, $C \in \mathcal{C}^+$. We call augmented pairing between A and C any bilinear mapping $\langle\ ,\ \rangle : A \times C \longrightarrow \Bbbk$ such that, for all $x, x_1, x_2 \in A$ and $y \in C$,*

$$\big\langle x_1 \cdot x_2, y \big\rangle = \big\langle x_1 \otimes x_2, \Delta(y) \big\rangle := \textstyle\sum_{(y)} \big\langle x_1, y_{(1)} \big\rangle \cdot \big\langle x_2, y_{(2)} \big\rangle\,, \quad \langle 1, y \rangle = \epsilon(y)\,, \quad \langle x\,, \underline{1}\,\rangle = \underline{\epsilon}(x)\,.$$

For any $B, P \in \mathcal{B}$ we call bialgebra pairing between B and P any augmented pairing between the latters (thought of as augmented algebra and coaugmented coalgebras as explained in §1.1) such that, for all $x \in B$, $y_1, y_2 \in P$, we have also, symmetrically,

$$\big\langle x, y_1 \cdot y_2 \big\rangle = \big\langle \Delta(x), y_1 \otimes y_2 \big\rangle := \textstyle\sum_{(x)} \big\langle x_{(1)}, y_1 \big\rangle \cdot \big\langle x_{(2)}, y_2 \big\rangle\,.$$

For any $H, K \in \mathcal{HA}$ we call Hopf algebra pairing (or Hopf pairing) between H and K any bialgebra pairing such that, in addition, $\big\langle S(x), y \big\rangle = \big\langle x, S(y) \big\rangle$ for all $x \in H$, $y \in K$.

We say that a pairing as above is perfect on the left (right) if its left (right) kernel is trivial; we say it is perfect if it is both left and right perfect.

Theorem 2.14. *Let $A \in \mathcal{A}^+$, $C \in \mathcal{C}^+$ and let $\pi\colon A \times C \longrightarrow \Bbbk$ be an augmented pairing. Then π induce a filtered augmented pairing $\pi_f : A^\vee \times C' \longrightarrow \Bbbk$ and a graded augmented pairing $\pi_G : G_{\underline{J}}(A) \times G_{\underline{D}}(C) \longrightarrow \Bbbk$ (notation of §1.4), both perfect on the right. If in addition π is perfect then π_f and π_G are perfect as well.* $\square$

§ 3 Connecting and crystal functors on bialgebras and Hopf algebras

3.1 The program. Our purpose in this section is to see the effect of connecting functors on the categories of bialgebras and of Hopf algebras. Then we shall move one step further, and look at the graded objects associated to the filtrations $\underline{J}$ and $\underline{D}$ in a bialgebra: these will eventually lead to the "crystal functors", the main achievement of this section.

From now on, every bialgebra B will be considered as a coaugmented coalgebra w.r.t. its unit map, hence w.r.t the group-like element 1 (the unit of B), and the corresponding maps δ_n $(n \in \mathbb{N})$ and $\delta_\bullet$–filtration $\underline{D}$ will be taken into account. Similarly, B will be considered as an augmented algebra w.r.t. the distinguished algebra morphism $\underline{\epsilon} = \epsilon$ (the counit of B), and the corresponding $\underline{\epsilon}$–filtration (also called ϵ–filtration) $\underline{J}$ will be considered.

We begin with a technical result about the "multiplicative" properties of the maps δ_n .

Lemma 3.2. *([KT], Lemma 3.2) Let $B \in \mathcal{B}$, $a, b \in B$, and $\Phi \subseteq \mathbb{N}$, with Φ finite. Then*

(a) $\quad \delta_\Phi(ab) \, = \, \sum_{\Lambda \cup Y = \Phi} \delta_\Lambda(a) \, \delta_Y(b)$ *;*

(b) *if $\Phi \neq \emptyset$, then* $\quad \delta_\Phi(ab - ba) \, = \, \sum_{\substack{\Lambda \cup Y = \Phi \\ \Lambda \cap Y \neq \emptyset}} \big(\delta_\Lambda(a) \, \delta_Y(b) - \delta_Y(b) \, \delta_\Lambda(a) \big)$. $\square$

Lemma 3.3. *Let B be a bialgebra. Then $\underline{J}$ and $\underline{D}$ are bialgebra filtrations. If B is also a Hopf algebra, then $\underline{J}$ and $\underline{D}$ are Hopf algebra filtrations.* $\square$

Corollary 3.4. *Let B be a bialgebra. Then*

(a) $B^\vee := B \Big/ \bigcap_{n \in \mathbb{N}} J^n$ is an ϵ–separated (i.p.-free) bialgebra, which B surjects onto.

(b) $B' := \bigcup_{n \in \mathbb{N}} D_n$ is a $\delta_\bullet$–exhausted (connected) Hopf algebra, which injects into B : more precisely, it is the irreducible (actually, connected) component of B containing 1.

(c) If in addition $B = H$ is a Hopf algebra, then $H^\vee$ is a Hopf algebra quotient of H and H' is a Hopf subalgebra of H. $\square$

Theorem 3.5. *Let B be a bialgebra, $\underline{J}$, $\underline{D}$ its ϵ–filtration and $\delta_\bullet$–filtration respectively.*

(a) $\widehat{B} := G_{\underline{J}}(B)$ is a graded cocommutative co-Poisson Hopf algebra generated by $P\big(G_{\underline{J}}(B)\big)$, the set of its primitive elements. Therefore $\widehat{B} \cong \mathcal{U}(\mathfrak{g})$ as graded co-Poisson Hopf algebras, for some restricted Lie bialgebra $\mathfrak{g}$ which is graded as a Lie algebra. In particular, if $\mathrm{Char}(\Bbbk) = 0$ and $\dim(B) \in \mathbb{N}$ then $\widehat{B} = \Bbbk \cdot 1$ and $\mathfrak{g} = \{0\}$.

(b) $\widetilde{B} := G_{\underline{D}}(B)$ is a graded commutative Poisson Hopf algebra. Therefore, $\widetilde{B} \cong F[G]$ for some connected algebraic Poisson group G which, as a variety, is a (pro)affine space. If $\mathrm{Char}(\Bbbk) = 0$ then $\widetilde{B} \cong F[G]$ is a polynomial algebra, i.e. $F[G] = \Bbbk\big[\{x_i\}_{i \in \mathcal{I}}\big]$ (for some set $\mathcal{I}$); in particular, if $\mathrm{Char}(\Bbbk) = 0$ and $\dim(B) \in \mathbb{N}$ then $\widetilde{B} = \Bbbk \cdot 1$ and $G = \{1\}$. If $\mathrm{Char}(\Bbbk) = p > 0$ then G has dimension 0 and height 1, and if $\Bbbk$ is perfect then $\widetilde{B} \cong F[G]$ is a truncated polynomial algebra, i.e. $F[G] = \Bbbk\big[\{x_i\}_{i \in \mathcal{I}}\big] \Big/ \big(\{x_i^p\}_{i \in \mathcal{I}}\big)$ (for some set $\mathcal{I}$). $\square$

3.6 The crystal functors. It is clear from the very construction that mapping $A \mapsto \widehat{A} := G_{\underline{J}}(A)$ (for all $A \in \mathcal{A}^+$) defines a functor from $\mathcal{A}^+$ to the category of graded, augmented $\Bbbk$–algebras; similarly, mapping $C \mapsto \widetilde{C} := G_{\underline{D}}(C)$ (for all $C \in \mathcal{C}^+$) defines a functor from $\mathcal{C}^+$ to the category of graded, coaugmented $\Bbbk$–coalgebras. The first functor factors through the functor $A \mapsto A^\vee$, and the second through the functor $C \mapsto C'$.

The analysis in the present section shows that when restricted to $\Bbbk$–bialgebras the output of the previous functors are objects of Poisson-geometric type (Lie bialgebras and Poisson groups): therefore, the functors $B \mapsto \widehat{B}$ and $B \mapsto \widetilde{B}$ (for $B \in \mathcal{B}$) on $\Bbbk$–bialgebras are "geometrification functors", in that they sort out of the generalized symmetry encoded by B some *geometrical* symmetries; we'll show in §4 that each of them can be seen as a "crystallization process" (in the sense, loosely speaking, of Kashiwara's motivation for the term "crystal basis" in quantum group theory: we move from one fiber to another, very peculiar one, within a 1-parameter family of algebraic objects), so we call them *"crystal functors"*. It is worth stressing that, by their very construction, applying either crystal functor one looses some information about the starting object, while at the same time still keeping something. So $\widehat{B}$ tells nothing about the coalgebra structure of $B^\vee$ (for all enveloping algebras — like $\widehat{B}$ — roughly look the same from the coalgebra point of view), yet it grasps some information on its algebra structure; on the other hand, conversely, $\widetilde{B}$ gives no information about the algebra structure of B' (in that the latter is simply a polynomial algebra), but instead it tells something non-trivial about its coalgebra structure.

We finish this section with the obvious improvement of Theorem 2.14:

Theorem 3.7. *Let $B, P \in \mathcal{B}$ and let $\pi : B \times P \longrightarrow \Bbbk$ be a bialgebra pairing. Then π induces filtered bialgebra pairings $\pi_f : B^\vee \times P' \longrightarrow \Bbbk$, $\pi^f : B' \times P^\vee \longrightarrow \Bbbk$, and graded bialgebra pairings $\pi_G : \widehat{B} \times \widetilde{P} \longrightarrow \Bbbk$, $\pi^G : \widetilde{B} \times \widehat{P} \longrightarrow \Bbbk$; π_f and π_G are perfect on the right, π^f and π^G on the left. If in addition π is perfect then all these induced pairings are perfect as well. If in particular $B, P \in \mathcal{HA}$ are Hopf algebras and π is a Hopf algebra pairing, then all the induced pairings are (filtered or graded) Hopf algebra pairings.* $\square$

§ 4 Deformations I — Rees algebras, Rees coalgebras, etc.

4.1 Filtrations and "Rees objects". Let V be a vector space over $\Bbbk$, and let $\{F_z\}_{z\in\mathbb{Z}} := \underline{F} : \big(\{0\} \subseteq \big) \cdots \subseteq F_{-m} \subseteq \cdots \subseteq F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots \big(\subseteq V \big)$ be a filtration of V by vector subspaces F_z ($z \in \mathbb{Z}$). First we define the associated *blowing space* to be the $\Bbbk$–subspace $\mathcal{B}_{\underline{F}}(V)$ of $V\big[t, t^{-1}\big]$ (where t is any indeterminate) given by $\mathcal{B}_{\underline{F}}(V) := \sum_{z\in\mathbb{Z}} t^z F_z$; this is isomorphic to the *first graded module*[1] *associated to* $\big(V, \underline{F}\big)$,

[1] I pick the terminology about (associated) graded modules from Serge Lang's textbook *"Algebra"*.

i.e. $\bigoplus_{z\in\mathbb{Z}} F_z$. Second, we let the associated *Rees module* be the $\Bbbk[t]$–submodule $\mathcal{R}^t_{\underline{F}}(V)$ of $V\big[t,t^{-1}\big]$ generated by $\mathcal{B}_{\underline{F}}(V)$; easy computations give $\Bbbk$–vector space isomorphisms

$$\mathcal{R}^t_{\underline{F}}(V)\Big/(t-1)\,\mathcal{R}^t_{\underline{F}}(V) \;\cong\; \bigcup_{z\in\mathbb{Z}} F_z \;=:\; V^{\underline{F}} \quad , \qquad \mathcal{R}^t_{\underline{F}}(V)\Big/t\,\mathcal{R}^t_{\underline{F}}(V) \;\cong\; G_{\underline{F}}(V)$$

where $G_{\underline{F}}(V) := \bigoplus_{z\in Z} F_z\big/F_{z-1}$ is the *second graded module associated to* $\big(V,\underline{F}\big)$. In other words, $\mathcal{R}^t_{\underline{F}}(V)$ is a $\Bbbk[t]$–module which specializes to $\bigcup_{z\in\mathbb{Z}} F_z$ for $t=1$ and specializes to $G_{\underline{F}}(V)$ for $t=0$; therefore the $\Bbbk$–vector spaces $\bigcup_{z\in\mathbb{Z}} F_z$ and $G_{\underline{F}}(V)$ can be seen as 1-parameter (polynomial) deformations of each other via the 1-parameter family of $\Bbbk$–vector spaces given by $\mathcal{R}^t_{\underline{F}}(V)$, in short $V^{\underline{F}} := \bigcup_{z\in\mathbb{Z}} F_z \xleftrightarrow[\mathcal{R}^t_{\underline{F}}(V)]{1\leftarrow t\rightarrow 0} G_{\underline{F}}(V)$.

We can repeat this construction within the category of algebras, coalgebras, bialgebras or Hopf algebras over $\Bbbk$ with a filtration in the proper sense (by subalgebras, subcoalgebras, etc.): then we'll end up with corresponding objects $\mathcal{B}_{\underline{F}}(V)$, $\mathcal{R}^t_{\underline{F}}(V)$, etc. of the like type (algebras, coalgebras, etc.). In particular we'll cope with Rees bialgebras.

4.2 Connecting functors and Rees modules. Let $A\in\mathcal{A}^+$ be an augmented algebra. By Lemma 2.2 the ϵ–filtration $\underline{J}$ of A is an algebra filtration: therefore we can build out of it the associated Rees *algebra* $\mathcal{R}^t_{\underline{J}}(A)$. By the previous analysis, this yields a 1-parameter deformation $A\cong \mathcal{R}^t_{\underline{J}}(A)\Big|_{t=1} \xleftrightarrow[\mathcal{R}^t_{\underline{J}}(A)]{1\leftarrow t\rightarrow 0} \mathcal{R}^t_{\underline{J}}(A)\Big|_{t=0} \cong G_{\underline{J}}(A) =: \widehat{A}$, where hereafter we use notation $M\Big|_{t=c} := M\Big/(t-c)M$ for any $\Bbbk[t]$–module M and any $c\in\Bbbk$. Note that all fibers in this deformation are isomorphic as vector spaces *but* perhaps for the special fiber at $t=0$, i.e. exactly $\widehat{A}$, for at that fiber the subspace J^∞ is "shrunk to zero". This is settled passing from A to $A^\vee$, for which we do have a *regular* 1-parameter deformation, i.e. one in which *all* fibers are pairwise isomorphic (as vector spaces), namely

$$A^\vee := A\Big/J^\infty \;\cong\; \mathcal{R}^t_{\underline{J}}(A^\vee)\Big|_{t=1} \xleftrightarrow[\mathcal{R}^t_{\underline{J}}(A^\vee)]{1\leftarrow t\rightarrow 0} \mathcal{R}^t_{\underline{J}}(A)\Big|_{t=0} \cong G_{\underline{J}}(A) =: \widehat{A} \tag{4.1}$$

where we implicitly used the identities $\widehat{A^\vee} = \mathcal{R}^t_{\underline{J}}(A^\vee)\Big|_{t=0} = \mathcal{R}^t_{\underline{J}}(A)\Big|_{t=0} = \widehat{A}$.

The situation is (dually!) similar for the connecting functor on coaugmented coalgebras. Indeed, let $C\in\mathcal{C}^+$ be a coaugmented coalgebra. Then by Lemma 2.8 the $\delta_\bullet$–filtration $\underline{D}$ of C is a coalgebra filtration, thus we can build out of it the associated Rees *coalgebra* $\mathcal{R}^t_{\underline{D}}(C)$. By the previous analysis, the latter provides a 1-parameter deformation

$$C' := \bigcup_{n\in\mathbb{N}} D_n \;\cong\; \mathcal{R}^t_{\underline{D}}(C)\Big|_{t=1} \xleftrightarrow[\mathcal{R}^t_{\underline{D}}(C)]{1\leftarrow t\rightarrow 0} \mathcal{R}^t_{\underline{D}}(C)\Big|_{t=0} \cong G_{\underline{D}}(C) =: \widetilde{C} \; . \tag{4.2}$$

Note that all fibers in this deformation are pairwise isomorphic vector spaces, so this is a *regular* 1-parameter deformation.

4.3 The bialgebra and Hopf algebra case. We now consider a bialgebra $B \in \mathcal{B}$. In this case, the results of §4 ensure that $B^\vee$ is a bialgebra, $\widehat{B}$ is a (graded, etc.) Hopf algebra, and also that $\mathcal{R}^t_{\underline{J}}(B^\vee)$ is a $\Bbbk[t]$–bialgebra (because $\underline{J}$ is a bialgebra filtration). Therefore, using Theorem 3.5, formula (4.1) becomes

$$B^\vee \xleftarrow[\mathcal{R}^t_{\underline{J}}(B^\vee)]{1 \leftarrow t \rightarrow 0} \widehat{B} \cong \mathcal{U}(\mathfrak{g}_-) \tag{4.3}$$

for some restricted Lie bialgebra $\mathfrak{g}_-$ as was $\mathfrak{g}$ in Theorem 3.5*(a)* (for later purposes we need to change symbol). Similarly, by Corollary 3.4 we know that B' is a Hopf algebra, $\widetilde{B}$ is a (graded, etc.) Hopf algebra and that $\mathcal{R}^t_{\underline{D}}(B')$ is a Hopf $\Bbbk[t]$–algebra (because $\underline{D}(B')$ is a Hopf algebra filtration of B'): thus, again by Theorem 3.5, formula (4.2) becomes

$$B' \xleftarrow[\mathcal{R}^t_{\underline{D}}(B)]{1 \leftarrow t \rightarrow 0} \widetilde{B} \cong F[G_+] \tag{4.4}$$

(noting that $\mathcal{R}^t_{\underline{D}}(B') = \mathcal{R}^t_{\underline{D}}(B)$ because $\underline{D}(B') = \underline{D}(B)$) for some Poisson algebraic group G_+ as was G in Theorem 3.5*(b)*. "Splicing together" these two pictures one gets the following scheme:

$$F[G_+] \cong \widetilde{B} \xleftarrow[\mathcal{R}^t_{\underline{D}}(B)]{0 \leftarrow t \rightarrow 1} B' \hookrightarrow B \twoheadrightarrow B^\vee \xleftarrow[\mathcal{R}^t_{\underline{J}}(B^\vee)]{1 \leftarrow t \rightarrow 0} \widehat{B} \cong \mathcal{U}(\mathfrak{g}_-) \tag{4.5}$$

This drawing shows how the bialgebra B gives rise to two Hopf algebras of Poisson geometrical type, namely $F[G_+]$ on the left-hand side and $\mathcal{U}(\mathfrak{g}_-)$ on the right-hand side, through bialgebra morphisms and regular bialgebra deformations. Namely, in both cases one has first a "reduction step", i.e. $B \mapsto B'$ or $B \mapsto B^\vee$, (yielding "connected" objects, cf. Remark 2.12*(a)*), then a regular 1-parameter deformation via Rees bialgebras.

Finally, if $H \in \mathcal{HA}$ is a Hopf algebra then all objects in (4.5) are Hopf algebras too, i.e. also $H^\vee$ — over $\Bbbk$ — and $\mathcal{R}^t_{\underline{J}}(B^\vee)$ — over $\Bbbk[t]$. Therefore (4.5) reads

$$F[G_+] \cong \widetilde{H} \xleftarrow[\mathcal{R}^t_{\underline{D}}(H)]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftarrow[\mathcal{R}^t_{\underline{J}}(H^\vee)]{1 \leftarrow t \rightarrow 0} \widehat{H} \cong \mathcal{U}(\mathfrak{g}_-) \tag{4.6}$$

with the one-way arrows being morphisms *of Hopf algebras* and the two-ways arrows being 1-parameter regular deformations *of Hopf algebras*. In the special case when H is connected, i.e. $H = H'$, and "coconnected", that is $H = H^\vee$, formula (4.6) looks simply

$$F[G_+] \cong \widetilde{H} \xleftarrow[\mathcal{R}^t_{\underline{D}}(H)]{0 \leftarrow t \rightarrow 1} H \xleftarrow[\mathcal{R}^t_{\underline{J}}(H)]{1 \leftarrow t \rightarrow 0} \widehat{H} \cong \mathcal{U}(\mathfrak{g}_-) \tag{4.7}$$

which means we can (regularly) deform H itself to "Poisson geometrical" Hopf algebras.

<u>*Remarks:*</u> *(a)* There is no simple relationship, *a priori*, between the Poisson group G_+ and the Lie bialgebra $\mathfrak{g}_-$ in (4.5) or (4.6), or even (4.7): examples do show that; in particular, either G_+ or $\mathfrak{g}_-$ may be trivial while the other is not.

(b) The Hopf duality relationship between connecting functors of the two types explained in §§2.11–14 extend to the deformations built upon them by means of Rees modules.

Indeed, by the very constructions one sees that there is a neat category-theoretical duality between the definition of $\mathcal{R}_{\underline{J}}^t(A)$ and of $\mathcal{R}_{\underline{D}}^t(C)$ (for $A \in \mathcal{A}^+$ and $C \in \mathcal{C}^+$). Even more, Theorem 2.14 "extends" (in a sense) to the following result:

Theorem 4.4. *Let $A \in \mathcal{A}^+$, $C \in \mathcal{C}^+$ and let $\pi \colon A \times C \longrightarrow \Bbbk$ be an augmented pairing. Then π induces an augmented pairing $\pi_{\mathcal{R}} \colon \mathcal{R}_{\underline{J}}^t(A) \times \mathcal{R}_{\underline{D}}^t(C) \longrightarrow \Bbbk[t]$ which is perfect on the right. If in addition π is perfect then $\mathcal{R}_{\underline{J}}^t(A) = \Big\{ \eta \in A(t) \,\Big|\, \pi_t\big(\eta,\kappa\big) \in \Bbbk[t]\,,\ \forall\, \kappa \in \mathcal{R}_{\underline{D}}^t(C) \Big\}$ and $\mathcal{R}_{\underline{D}}^t(C) = \Big\{ \kappa \in C(t) \,\Big|\, \pi_t\big(\eta,\kappa\big) \in \Bbbk[t]\,,\ \forall\, \eta \in \mathcal{R}_{\underline{J}}^t(A) \Big\}$, where $S(t) := \Bbbk(t) \otimes_{\Bbbk} S$ for $S \in \{A, C\}$ and $\pi_t \colon A(t) \times C(t) \longrightarrow \Bbbk(t)$ is the obvious $\Bbbk(t)$–linear pairing induced by π, and $\pi_{\mathcal{R}}$ is perfect as well. If A and C are bialgebras, resp. Hopf algebras, then everything holds with bialgebra, resp. Hopf algebra, pairings instead of augmented pairings.* $\square$

§ 5 Deformations II — from Rees bialgebras to quantum groups

5.1 From Rees bialgebras to quantum groups via the GQDP. In this section we show how, for any $\Bbbk$–bialgebra B, we can get another deformation scheme like (4.5). In fact, this will be built upon the latter, applying (part of) the "Global Quantum Duality Principle" explained in [Ga1–2], in its stronger version about bialgebras.

Indeed, the deformations in (4.5) were realized through Rees bialgebras, namely $\mathcal{R}_{\underline{J}}^t(B)$ and $\mathcal{R}_{\underline{D}}^t(B)$: these are torsion-free (actually, free) as $\Bbbk[t]$–modules, hence one can apply the construction made in [*loc. cit.*] via the so-called Drinfeld's functors to get some new torsion-free $\Bbbk[t]$–bialgebras. The latters (just like the Rees bialgebras we start from) again specialize to special bialgebras at $t = 0$; in particular, if B is a Hopf algebra the new bialgebras are Hopf algebras too, and precisely "quantum groups" in the sense of [*loc. cit.*].

The construction goes as follows. To begin with, set $B_t^\vee := \mathcal{R}_{\underline{J}}^t(B)$: this is a free, hence torsion-free, $\Bbbk[t]$–bialgebra. We define

$$\big(B_t^\vee\big)' := \Big\{\, b \in B_t^\vee \,\Big|\, \delta_n(b) \in t^n \big(B_t^\vee\big)^{\otimes n}\,,\ \forall\ n \in \mathbb{N} \,\Big\}\,.$$

On the other hand, let $B_t' := \mathcal{R}_{\underline{D}}^t(B)$: this is again a free, hence torsion-free, $\Bbbk[t]$–bialgebra. Using notation $J' := Ker\big(\epsilon \colon B_t' \longrightarrow \Bbbk[t]\big)$ and also $B'(t) := \Bbbk(t) \otimes_{\Bbbk[t]} B_t' = \Bbbk(t) \otimes_{\Bbbk} B'$, we define

$$\big(B_t'\big)^\vee := \textstyle\sum_{n \geq 0} t^{-n} \big(J'\big)^n = \sum_{n \geq 0} \big(t^{-1} J'\big)^n \qquad \big(\subseteq B'(t)\big)\,.$$

The first important point is the following:

Proposition 5.2. *Both $\big(B_t^\vee\big)'$ and $\big(B_t'\big)^\vee$ are free (hence torsion-free) $\Bbbk[t]$–bialgebras; moreover, the mappings $B \mapsto \big(B_t^\vee\big)'$ and $B \mapsto \big(B_t'\big)^\vee$ are functorial. The analogous results hold for Hopf $\Bbbk$–algebras, replacing "bialgebra(s)" with "Hopf algebra(s)" throughout.* $\square$

Proposition 5.3.

(a) $\left.\big(B_t^\vee\big)'\right|_{t=1} := \big(B_t^\vee\big)' \Big/ (t-1)\big(B_t^\vee\big)' \cong B^\vee$ *as* $\Bbbk$*–bialgebras.*

(b) $\left.\big(B_t'\big)^\vee\right|_{t=1} := \big(B_t'\big)^\vee \Big/ (t-1)\big(B_t'\big)^\vee \cong B'$ *as* $\Bbbk$*–bialgebras.* $\square$

Theorem 5.4.

(a) $\left.\big(B_t^\vee\big)'\right|_{t=0} := \big(B_t^\vee\big)' \Big/ t\,\big(B_t^\vee\big)'$ *is a commutative* $\Bbbk$*–bialgebra with no non-trivial idempotent elements. Furthermore, if* $p := \mathit{Char}(\Bbbk) > 0$ *then each non-zero element of* $\mathit{Ker}\Big(\epsilon\colon \left.\big(B_t^\vee\big)'\right|_{t=0} \longrightarrow \Bbbk\Big)$ *has nilpotency order* p*. Therefore* $\left.\big(B_t^\vee\big)'\right|_{t=0}$ *is the function algebra* $F[M]$ *of some connected Poisson algebraic monoid* M*, and if* $\mathit{Char}(\Bbbk) > 0$ *then* M *has dimension 0 and height 1. If in addition* $B = H \in \mathcal{HA}$ *then* $\left.\big(H_t^\vee\big)'\right|_{t=0}$ *is a Hopf* $\Bbbk$*–algebra, and* $K_+ := \mathit{Spec}\Big(\left.\big(H_t^\vee\big)'\right|_{t=0}\Big) = M$ *is a (connected) algebraic Poisson group.*

(b) $\left.\big(B_t'\big)^\vee\right|_{t=0} := \big(B_t'\big)^\vee \Big/ t\,\big(B_t'\big)^\vee$ *is a connected cocommutative Hopf* $\Bbbk$*–algebra generated by* $P\Big(\left.\big(B_t'\big)^\vee\right|_{t=0}\Big)$*. Therefore* $\left.\big(B_t'\big)^\vee\right|_{t=0} = \mathcal{U}(\mathfrak{k}_-)$ *for some Lie bialgebra* $\mathfrak{k}_-$ *.*

(c) If $\mathit{Char}(\Bbbk) = 0$ *and* $B = H \in \mathcal{HA}$ *is a Hopf* $\Bbbk$*–algebra, let* $\widehat{H} = U(\mathfrak{g}_-)$ *and* $\widetilde{H} = F[G_+]$ *as in Theorem 3.5: then (notation of (a) and (b))* $K_+ = G_-^\star$ *and* $\mathfrak{k}_- = \mathfrak{g}_+^\times$ *, that is* $\mathit{coLie}(K_+) = \mathfrak{g}_-$ *and* $\mathfrak{k}_- = \mathit{coLie}(G_+)$ *as Lie bialgebras.* $\square$

5.5 Deformations through Drinfeld's fuctors. The outcome of the previous analysis is that for each $\Bbbk$–bialgebra $B \in \mathcal{B}$ a second scheme — besides (4.5) — is available, yielding regular 1-parameter deformations, namely (letting $p := \mathit{Char}(\Bbbk)$)

$$\mathcal{U}(\mathfrak{k}_-) \xleftarrow[\;(B_t')^\vee\;]{0 \leftarrow t \rightarrow 1} B' \hookrightarrow B \twoheadrightarrow B^\vee \xleftarrow[\;(B_t^\vee)'\;]{1 \leftarrow t \rightarrow 0} F[M] \tag{5.1}$$

This provides another recipe, besides (4.5), to make two other bialgebras of Poisson geometrical type, namely $F[M]$ and $\mathcal{U}(\mathfrak{k}_-)$, out of the bialgebra B, through bialgebra morphisms and regular bialgebra deformations. Like for (4.5), in both cases there is first the "reduction step" $B \mapsto B'$ or $B \mapsto B^\vee$ and then a regular 1-parameter deformation via $\Bbbk[t]$–bialgebras. However, this time on the right-hand side we have in general only a bialgebra, not a Hopf algebra. When $B = H \in \mathcal{HA}$ is a Hopf $\Bbbk$–algebra, then (5.1) improves, in that all objects therein are Hopf algebras too, and morphisms and deformations are ones of Hopf algebras. In particular $M = K_+$ is a (connected Poisson algebraic) group, not only a monoid: at a glance, letting $p := \mathit{Char}(\Bbbk) \geq 0$, we have

$$\left.\begin{array}{ll} \text{if } p = 0\,, & U(\mathfrak{g}_+^\times) = \mathcal{U}(\mathfrak{k}_-) \\ \text{if } p > 0\,, & \mathcal{U}(\mathfrak{k}_-) \end{array}\right\} \xleftarrow[\;(H_t')^\vee\;]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftarrow[\;(H_t^\vee)'\;]{1 \leftarrow t \rightarrow 0} \left\{\begin{array}{l} F[K_+] = F[G_+^\star] \\ F[K_+] \end{array}\right. \tag{5.2}$$

This yields another recipe, besides (4.7), to make two new Hopf algebras of Poisson geometrical type, i.e. $F[K_+]$ and $\mathcal{U}(\mathfrak{k}_-)$, out of the Hopf algebra H, through Hopf algebra morphisms and regular Hopf algebra deformations. Again we have first the "reduction step" $H \mapsto H'$ or $H \mapsto H^\vee$, then a regular 1-parameter deformation via Hopf $\Bbbk[t]$–algebras.

In the special case when H is connected, that is $H = H'$, and "coconnected", that is $H = H^\vee$, formula (5.2) takes the simpler form, the analogue of (4.7),

$$\left.\begin{array}{ll} \text{if } p=0\,, & U(\mathfrak{g}_+^\times) = \mathcal{U}(\mathfrak{k}_-) \\ \text{if } p>0\,, & \mathcal{U}(\mathfrak{k}_-) \end{array}\right\} \xleftarrow[(H'_t)^\vee]{0 \leftarrow t \rightarrow 1}\!\!\rightarrow H \xleftarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0}\!\!\rightarrow \left\{\begin{array}{l} F[K_+] = F[G_+^\star] \\ F[K_+] \end{array}\right. \tag{5.3}$$

which means we can (regularly) deform H itself to Poisson geometrical Hopf algebras.

In particular, when $H' = H = H^\vee$ patching together (4.7) and (5.3) we find

$$\begin{array}{ccccc} F[G_+] & \xleftarrow[H'_t]{0 \leftarrow t \rightarrow 1}\!\!\rightarrow & H' & \xleftarrow[(H'_t)^\vee]{1 \leftarrow t \rightarrow 0}\!\!\rightarrow & \mathcal{U}(\mathfrak{k}_-) \quad \Big(= U\big(\mathfrak{g}_+^\times\big) \text{ if } \mathit{Char}(\Bbbk) = 0 \Big) \\ & & \| & & \\ & & H & & \\ & & \| & & \\ \mathcal{U}(\mathfrak{g}_-) & \xleftarrow[H^\vee_t]{0 \leftarrow t \rightarrow 1}\!\!\rightarrow & H^\vee & \xleftarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0}\!\!\rightarrow & F[K_+] \quad \Big(= F\big[G_-^\star\big] \text{ if } \mathit{Char}(\Bbbk) = 0 \Big) \end{array}$$

which gives *four* different regular 1-parameter deformations from H to Hopf algebras encoding geometrical objects of Poisson type (i.e. Lie bialgebras or Poisson algebraic groups).

5.6 Drinfeld-like functors. The constructions in the present section show that mapping $B \mapsto \big(B'_t\big)^\vee\big|_{t=0}$ and mapping $B \mapsto \big(B^\vee_t\big)'\big|_{t=0}$ (for all $B \in \mathcal{B}$) define two endofunctors of $\mathcal{B}$. The output of these endofunctors describe objects of Poisson-geometric type, namely Lie bialgebras and connected Poisson algebraic monoids: therefore, both $B \mapsto \big(B'_t\big)^\vee\big|_{t=0}$ and $B \mapsto \big(B^\vee_t\big)'\big|_{t=0}$ (for $B \in \mathcal{B}$) are *geometrification functors* on $\Bbbk$–bialgebras, just like the ones in §3.6, which we call "Drinfeld-like functors", because they are defined through the use of Drinfeld functors (cf. [Ga1–2]) for quantum groups. Thus we have four functorial recipes — our four *geometrification functors* — to sort out of the generalized symmetry B some *geometrical* symmetries. Hereafter we explain the duality relationship between Drinfeld-like functors and the associated deformations:

Theorem 5.7. *Let $B, P \in \mathcal{B}$, and let $\pi : B \times P \longrightarrow \Bbbk$ be a $\Bbbk$–bialgebra pairing. Then π induces a $\Bbbk[t]$–bialgebra pairing $\pi'_\vee : \big(B^\vee_t\big)' \times \big(P'_t\big)^\vee \longrightarrow \Bbbk[t]$ and a $\Bbbk$–bialgebra pairing $\pi'_\vee\big|_{t=0} : \big(B^\vee_t\big)'\big|_{t=0} \times \big(P'_t\big)^\vee\big|_{t=0} \longrightarrow \Bbbk$. If π is perfect and $\mathit{Char}(\Bbbk) = 0$, then $\big(B^\vee_t\big)' = \Big\{ \eta \in B(t) \,\Big|\, \pi_t(\eta, \kappa) \in \Bbbk[t]\,,\ \forall\, \kappa \in \big(P'_t\big)^\vee \Big\}$ and $\big(P'_t\big)^\vee = \Big\{ \kappa \in P(t) \,\Big|\, \pi_t(\eta, \kappa) \in \Bbbk[t]\,,\ \forall\, \eta \in \big(B^\vee_t\big)' \Big\}$, where $S(t) := \Bbbk(t) \otimes_\Bbbk S$ for $S \in \{B, P\}$ and $\pi_t : B(t) \times P(t) \longrightarrow \Bbbk(t)$ is the obvious $\Bbbk(t)$–linear pairing induced by π, and the induced pairings are perfect as well. If $B, P \in \mathcal{HA}$ are Hopf $\Bbbk$–algebras then everything changes accordingly.* $\square$

§ 6 Poisson duality and the Crystal Duality Principle

6.1 Crystal functors and Poisson duality. In this section we show that when connecting functors are applied to Hopf algebras encoding a classical symmetry — an algebraic (maybe Poisson) group, or a universal enveloping algebra of a Lie (bi)algebra (maybe restricted, if $\mathit{Char}(\Bbbk) > 0$) — we know in advance the result of applying some connecting or crystal functors. Namely, in case of function algebras of algebraic Poisson groups or (restricted) universal enveloping algebras of Lie bialgebras the outcome is explicitly expressed in terms of the *dual* Lie bialgebra or Poisson group. These cases are general, as we can always give an algebraic group the trivial Poisson group structure, and any (restricted) Lie algebra the trivial Lie cobracket to make it into a Lie bialgebra.

Theorem 6.2. *Let $H = F[G]$ be the function algebra of an algebraic Poisson group. Then $\widehat{F[G]}$ is isomorphic to a bi-Poisson Hopf algebra, namely (with $p := \mathit{Char}(\Bbbk)$)*

$$\widehat{F[G]} \;\cong\; S(\mathfrak{g}^\times) \;\text{ if }\; p=0\,, \qquad \widehat{F[G]} \;\cong\; S(\mathfrak{g}^\times) \Big/ \Big(\Big\{ \,\overline{x}^{\,p^{n(x)}} \,\Big|\, x \in \mathcal{N}\big(F[G]\big) \Big\} \Big) \;\text{ if }\; p>0$$

(notation of §1) where $\mathcal{N}\big(F[G]\big)$ is the nilradical of $F[G]$, $p^{n(x)}$ is the order of nilpotency of $x \in \mathcal{N}\big(F[G]\big)$ and the bi-Poisson Hopf structure of $S(\mathfrak{g}^\times) \Big/ \Big(\Big\{ \,\overline{x}^{\,p^{n(x)}} \,\Big|\, x \in \mathcal{N}\big(F[G]\big) \Big\} \Big)$ is the quotient one from $S(\mathfrak{g}^\times)$. In particular, if G is reduced then $\widehat{F[G]} \cong S(\mathfrak{g}^\times)$.

Sketch of proof. This follows essentially from definitions of $\widehat{F[G]}$ and of $\mathfrak{g}^\times$ (cf. §1). □

Theorem 6.3.

(a) Let $\mathit{Char}(\Bbbk) = 0$. Let $\mathfrak{g}$ be a Lie bialgebra. Then $\widetilde{U(\mathfrak{g})}$ is a bi-Poisson Hopf algebra, namely $\widetilde{U(\mathfrak{g})} \cong S(\mathfrak{g}) = F[\mathfrak{g}^\times]$ (notation of §1), where the bi-Poisson Hopf structure on $S(\mathfrak{g})$ is the canonical one.

(b) Let $\mathit{Char}(\Bbbk) = p > 0$. Let $\mathfrak{g}$ be a restricted Lie bialgebra. Then $\widetilde{\mathbf{u}(\mathfrak{g})}$ is a bi-Poisson Hopf algebra, namely $\widetilde{\mathbf{u}(\mathfrak{g})} \cong S(\mathfrak{g}) \big/ \big(\{ x^p \,|\, x \in \mathfrak{g} \} \big) = F[G^\star]$ (notation of §1) where the bi-Poisson Hopf structure on $S(\mathfrak{g}) \big/ \big(\{ x^p \,|\, x \in \mathfrak{g} \} \big)$ is induced by the canonical one on $S(\mathfrak{g})$ and $G^\star$ denotes a connected algebraic Poisson group of dimension 0 and height 1 whose cotangent Lie bialgebra is $\mathfrak{g}$.

Sketch of proof. By its very definition, the filtration $\underline{D}$ of $U(\mathfrak{g})$ or $\mathbf{u}(\mathfrak{g})$ is just the natural filtration given by the order of differential operators. From this the claim follows. □

6.4 The Crystal Duality Principle. To sum up, we can finally tide up — once more — all results presented above in a single formulation, the "Crystal Duality Principle". In short, we provided functorial recipes to get, out of *any* Hopf algebra H, four Hopf algebras of Poisson-geometrical type (arranged in two couples), hence four associated Poisson-geometrical symmetries: this is the "Principle", say. The word "Crystal" reminds the fact that the first couple — out of which the second one is sorted too — of special Hopf algebras, namely $\big(\widehat{H}, \widetilde{H}\,\big)$, is obtained via a *crystallization process* (cf. §3.6). Finally, the word "Duality" witnesses that if $\mathit{Char}(\Bbbk) = 0$ then Poisson duality is the link between the two couples of special Hopf algebras (thus only two are the *relevant* Poisson geometries associated to H) and that if H is of Poisson-geometrical type then the crystal functor yielding a Hopf algebra of *Hopf-dual* type is ruled by *Poisson duality* (in any characteristic).

6.5 The CDP as corollary of the GQDP. The construction of Drinfeld-like functors passes through the application of the Global Quantum Duality Principle (=GQDP in the sequel): thus part of the Crystal Duality Principle (=CDP in the sequel) is a direct consequence of the GQDP. In this section we briefly outline how *the whole CDP can be obtained as a corollary of the GQDP* (but for some minor details); see also [Ga1–2], §5.

For any $H \in \mathcal{HA}$, let $H_t := H[t] \equiv \Bbbk[t] \otimes_{\Bbbk} H$. Then H_t is a torsionless Hopf algebra over $\Bbbk[t]$, hence one of those to which the constructions in [Ga1–2] can be applied: in particular, we can act on it with Drinfeld's functors considered therein, which provide quantum groups, namely a quantized (restricted) universal enveloping algebra (=QrUEA) and a quantized function algebra (=QFA). Now, straightforward computation shows that the QrUEA is nothing but $H_t^\vee := \mathcal{R}_{\underline{J}}^t(H)$, and the QFA is just $H_t' := \mathcal{R}_{\underline{D}}^t(H)$, with $\widehat{H} \cong H_t^\vee\Big|_{t=0}$ and $\widetilde{H} \cong H_t'\Big|_{t=0}$. It follows that all properties of $\widehat{H}$ and $\widetilde{H}$ spring out as special cases of the results proved in [Ga1–2] for Drinfeld's functors, but for their being graded. Similarly, the fact that H' be a Hopf subalgebra of H follows from the fact that H_t' itself is a Hopf algebra (over $\Bbbk[t]$) and $H' = H_t'\Big|_{t=1}$; instead, $H^\vee$ is a quotient Hopf algebra of H because $H_t^\vee$ is a Hopf algebra (over $\Bbbk[t]$), hence $\overline{H_t^\vee} := H_t^\vee \Big/ \bigcap_{n\in\mathbb{N}} t^n H_t^\vee$ is a Hopf algebra, and finally $H^\vee = \overline{H_t^\vee}\Big|_{t=1}$. The fact that H_t' and $H_t^\vee$ be regular 1-parameter deformations respectively of H' and $H^\vee$ is then clear by construction. Finally, the parts of the CDP dealing with Poisson duality also are direct consequences of the like items in the GQDP applied to H_t' and to $H_t^\vee$ (but for Theorem 6.3*(b)*). The cases of (co)augmented (co)algebras or bialgebras can be easily treated the same, up to minor changes.

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