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The ring of conditions of a semisimple group

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ABSTRACT

In this paper we are going to give an explicit description of the so called ring of conditions of an adjoint semisimple algebraic group G, originally introduced in [C. De Concini, C. Procesi, Complete symmetric varieties. II. Intersection theory, in: Algebraic Groups and Related Topics, Kyoto/Nagoya, 1983, in: Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, pp. 481–513] in order to study intersection theory in G.

1. Introduction

Let G be an adjoint semisimple algebraic group over the field $\mathbb{C}$ of complex numbers. We can consider G as a $G \times G$-homogeneous space under left and right action. Thus G is identified with the quotient of $G \times G$ modulo the diagonal copy G_d of G, that is the subgroup of elements fixed by the order two automorphism obtained by exchanging the factors. In view of this we can apply the results of [DP2] and consider the so called ring of conditions $R(G)$ of G. This ring has been introduced in order to study some classical problems in enumerative geometry and has been recently used to give explicit formulae for intersection indices and Euler characteristic of hypersurfaces in G (see [K1,K2]).

The purpose of this paper is to give an explicit description of $R(G,\mathbb{Q}) := R(G) \otimes \mathbb{Q}$. In order to achieve this, we are first going to generalize the definition of $R(G)$ and introduce a $G \times G$-equivariant version $R_{G\times G}(G)$ of it. $R(G)$ will then be the quotient of $R_{G\times G}(G)$ modulo an explicit ideal.

$R_{G\times G}(G)$ is the direct limit of the $G \times G$-equivariant cohomology of embeddings of G as a $G \times G$-homogeneous space, so in order to describe it one needs to first describe, for any such embedding Y, the equivariant cohomology ring $H^*_{G\times G}(Y,\mathbb{Q})$. This is achieved using results in [BDP,St2] and the methods in [St3] (see also [U]). Once this is obtained, one has to describe the cohomology homomorphism induced by a $G \times G$-equivariant morphism $f : Y_1 \to Y_2$ between two regular embeddings Y_1 and Y_2.

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A different approach for the study of the cohomology of the wonderful compactification has been developed in [DP1] and [LP]. This has been generalized in [BJ] to the case of wonderful varieties of minimal rank. Furthermore in that paper equivariant Chern classes have been computed in terms of the associated toric variety.

The paper is organized as follows. Section 2 contains a brief digest of the main definitions and properties related to equivariant cohomology which we are going to use in our work.

In Section 3 we are going to recall the definition of regular toroidal embeddings, their construction and classification, in terms of an associated toric variety given in [DP2].

In Section 4, given a regular toroidal embedding Y, we are going to describe its equivariant cohomology. Our first step will be the determination of the $G \times G$-orbits in Y obtained using the orbit structure of the associated toric variety. Once this has been done, we are going to use this result to describe, following [St3], $H^*_{G\times G}(Y, \mathbb{Q})$.

Finally in Sections 5 and 6, we are going to deduce rather easily, as a consequence of our previous work, the main results describing both $R_{G\times G}(G)$ and $R(G)$.

It should be noted that although our description of $R_{G\times G}(G)$ is rather explicit as a ring of functions on the product of the reflection representation of the Weyl group of G times a fundamental Weyl chamber, a presentation of this ring in terms of generators and relations is far from being obtained. We believe that this would be quite useful and we hope to be able to tackle this problem in the future.

Finally we should remark the analogy of our results with those of McMullen [McM] describing the polytopal algebra which can be thought as a toric version of our $R_{G\times G}(G)$.

2. A brief digest of equivariant cohomology

In this section we are going to define equivariant cohomology and recall a few of its properties.

Let K be a topological group. Consider the universal fibration $p : EK \to BK$, where EK is contractible with free K-action and $BK = EK/K$ is the classifying space of K.

Take a K-space X. The equivariant cohomology of X with coefficients in a commutative ring A is the cohomology ring $H^*_K(X, A) := H^*(X_K, A)$ of the Borel construction [Bo2] $X_K := EK \times_K X$. We denote by $\pi : X_K \to BK$ the fibration over BK with fiber X.

In what follows K will always be a complex algebraic group and X an algebraic variety. A will be the field $\mathbb{Q}$ of rational numbers.

Since the Borel construction is clearly functorial for K-spaces, we deduce that, given a K-equivariant map $f : X \to Y$ of K-spaces, we get a homomorphism $f^* : H^*_K(Y, A) \to H^*_K(X, A)$. In particular the projection to a point $q : X \to \mathrm{pt}$ induces on $H^*_K(X, \mathbb{Q})$ the structure of an algebra over $H^*_K(\mathrm{pt}, \mathbb{Q}) = H^*(BK, \mathbb{Q})$.

Also assume that $q : V \to X$ is a K-equivariant complex vector bundle on X, namely a vector bundle with a K-action compatible with the projection q and such that K acts linearly on the fibers. Take the Borel construction V_K and consider the induced projection $q_K : V_K \to X_K$. We get a complex vector bundle on X_K whose Chern classes in $H^*_K(X, \mathbb{Q})$ are called the equivariant Chern classes of V.

Suppose K is connected. Let U be the unipotent radical of K. Using the fact that U is contractible, we deduce that $H^*_K(\mathrm{pt}) \simeq H^*_{K/U}(\mathrm{pt})$. Thus we can assume that K is reductive. If this is the case, choose a maximal torus T in K. Set $\mathfrak{t} = \mathrm{Lie}\, T$ and $W = N(T)/T$ the Weyl group. Then one knows [Bo1] that $H^*_K(\mathrm{pt}, \mathbb{Q}) \simeq \mathbb{Q}[\mathfrak{t}^*]^W$, where the elements of $\mathfrak{t}^*$ have degree 2. This is well known to be a polynomial ring, since W is generated by reflections.

Following for example [GKM], we define a space X to be equivariantly formal if, setting $I \subset H^*_K(\mathrm{pt})$ equal to the ideal of elements of positive degree, then

(1) $H^*_K(X)$ is a free $H^*_K(\mathrm{pt})$-module of finite rank,
(2) $H^*(X) = H^*_K(X)/IH^*_K(X)$.

It is known, see [Bo2], that if X has only cohomology in even degrees, then X is equivariantly formal. Now if X is a smooth projective K-variety with finitely many K-orbits, one knows as a consequence of a result by Białynicki-Birula [Bia] (see also [BDP] for a discussion), that X can be paved by a finite number of locally closed affine spaces and hence has only cohomology in even degrees. It follows that X is equivariantly formal, thus the knowledge of the equivariant cohomology can be in principle used to determine the cohomology of X.

3. Regular embeddings

We are going to work over the field $\mathbb{C}$ of complex numbers, but everything we are going to state will work verbatim over an arbitrary algebraically closed field of characteristic zero if we use, instead of cohomology (respectively equivariant cohomology), the Chow ring (respectively the equivariant Chow ring).

Given an algebraic group H, by a H-embedding we mean a pair (Y, j_Y) where Y is a normal $H \times H$-variety and $j_Y : H \to Y$ is a $H \times H$-equivariant embedding ($H \times H$ acts on H by left and right multiplication), whose image is a dense open set in Y. Given two H-embeddings (Y, j_Y) and $(Y', j_{Y'})$, a morphism of embeddings is a morphism $\pi : Y' \to Y$ such that $j_Y = \pi j_{Y'}$. In this case we shall say that Y' lies over Y. Notice that if such a morphism π exists, it is clearly unique.

Also notice that if H is commutative, for example a torus, since the right and left multiplications coincide, the $H \times H$-action on Y reduces to an H-action. In particular if H is a torus, an H-embedding is a toric variety with torus H.

Let us recall that when G is an adjoint semisimple group, one has the so called wonderful compactification X of G [DP,St1]. This is a G-embedding (X, j) with $j : G \to X$.

From now on we assume that we have chosen once and for all a maximal torus T in G and a Borel subgroup $T \subset B \subset G$ and we let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be the corresponding set of simple roots for the associated root system Φ.

Let us briefly recall the combinatorial structure of the $G \times G$-orbits in X. If we consider the complement $D = X - G$, then D is a divisor with normal crossings and smooth irreducible components $D_1, \dots, D_\ell$.

For each subset $\Gamma \subset \Delta$, the intersection

$$D_\Gamma = \bigcap_{\alpha_j \notin \Gamma} D_j$$

is irreducible and it is the closure of a unique $G \times G$-orbit $\mathcal{O}_\Gamma$ (of course $X = D_\Delta$). Then the correspondence associating to each subset Γ of Δ the orbit $\mathcal{O}_\Gamma$ is a bijection. In particular the orbit corresponding to the empty set is the unique closed orbit in X, which is isomorphic to $G/B \times G/B$, and we have that $\Gamma \supset \Gamma'$ if and only if $\overline{\mathcal{O}_\Gamma} \supset \mathcal{O}_{\Gamma'}$ (the reader should be careful enough to notice that our indexing for orbits is "complementary" to the one used in our previous paper [St3] in the sense that there the orbit $\mathcal{O}_\Gamma$ was denoted by $\mathcal{O}_{\Delta \setminus \Gamma}$).

Recall that every line bundle on X admits a canonical $\tilde{G} \times \tilde{G}$-linearization, $\tilde{G}$ being the universal cover of G, since $\tilde{G}$ is simply connected (see [DP]). This implies that if $\mathrm{Pic}(X)$ is the Picard group of X, then, taking equivariant Chern classes, we get an isomorphism

$$\mathrm{Pic}(X) \otimes \mathbb{Q} \simeq H^2_{G \times G}(X, \mathbb{Q}). \tag{1}$$

Finally, denoting by Λ the weight lattice, i.e. the character group of the maximal torus $\tilde{T}$ which is the pre-image of T in $\tilde{G}$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "h^*"] \arrow[d] & \mathrm{Pic}(\mathcal{O}_\emptyset) \arrow[d] \\ \Lambda \arrow[r, "a"] & \Lambda \times \Lambda \end{tikzcd} \tag{2}$$

where h^* is induced by inclusion and $a(\lambda) = (\lambda, 0) - (0, \lambda)$, while the vertical arrows are isomorphisms. Using this, one gets an identification of $\mathrm{Pic}(X)$ with the lattice Λ of weights for our root system Φ and, under this identification, $[\mathcal{O}(D_i)] = \alpha_i \in \mathrm{Pic}(X)$.

Denote by L_i the line bundle corresponding to the invertible sheaf $\mathcal{O}(D_i)$ and by $s_i \in H^0(X, L_i)$ a section (unique up to a non-zero scalar) whose divisor is D_i. This section is $\tilde{G} \times \tilde{G}$-invariant.

Consider now the vector bundle $V = \bigoplus_{i=1}^{\ell} L_i$. It has the $\tilde{G} \times \tilde{G}$-invariant section $s = \bigoplus_{i=1}^{\ell} s_i$.

Set now

$$\mathcal{T} = V - \{(v_1, \dots, v_\ell) \mid v_i = 0 \text{ for some } i\}.$$

$\mathcal{T}$ is a principal G_m^ℓ-bundle with a $\tilde{G} \times \tilde{G}$-action commuting with the G_m^ℓ-action. If we take any G_m^ℓ-variety Z, we may consider $\mathcal{T}_Z = Z \times_{G_m^\ell} \mathcal{T}$. In particular we get that $V = \mathcal{T}_{\mathbb{A}^\ell}$ where $\mathbb{A}^\ell$ is the ℓ-dimensional affine space with the tautological G_m^ℓ-action.

Let us fix now a G_m^ℓ-embedding Z with the property that Z is smooth and there is a proper G_m^ℓ-equivariant morphism $\psi_Z : Z \to \mathbb{A}^\ell$. We then get a proper $G \times G$-equivariant morphism $\tilde{\psi}_Z : \mathcal{T}_Z \to V$.

We can now define Y_Z as the cartesian product

$$\begin{tikzcd} Y_Z \arrow[r, "s_Z"] \arrow[d, "\pi_Z"'] & \mathcal{T}_Z \arrow[d, "\tilde{\psi}_Z"] \\ X \arrow[r, "s"] & V. \end{tikzcd}$$

Since $s(j(G)) \subset \mathcal{T}$, we get that $s \circ j$ lifts to a map $h_Z : G \to \mathcal{T}_Z$. So, by the definition of Y_Z, we get an embedding $j_Z : G \to Y_Z$ such that $\pi_z \circ j_Z = j$.

Notice that if Z' is another smooth G_m^ℓ-equivariant embedding proper over $\mathbb{A}^\ell$ which lies over Z, we clearly get a morphism $\psi_{Z'}^{Z} : Y_{Z'} \to Y_Z$ such that $j_Z = \psi_{Z'}^{Z} \circ j_{Z'}$. In [DP2] it is shown the following:

Theorem 3.1. *Let Y_Z be defined as above. Then*:

(1) *(Y_Z, j_Z) is a smooth complete G-embedding lying over X.*
(2) *Given a G_m^ℓ-orbit $\mathcal{N} \subset Z$, consider $\mathcal{T}_\mathcal{N} \subset \mathcal{T}_Z$. Then $s_Z^{-1}(\mathcal{T}_\mathcal{N})$ is a $G \times G$-orbit in Y_Z. Moreover the map which associates to the G_m^ℓ-orbit $\mathcal{N}$ the $G \times G$-orbit $s_Z^{-1}(\mathcal{T}_\mathcal{N})$ is a bijection between the set of G_m^ℓ-orbits in Z and the set of $G \times G$-orbits in Y_Z.*

An embedding of the form Y_Z will be called a regular compactification of G. We now introduce the poset $\mathfrak{C}$ as follows. The elements of $\mathfrak{C}$ are the regular compactifications of G. Moreover we say that $Y_{Z'} \geqslant Y_Z$ if there is a (necessarily unique) morphism of embeddings $\psi_{Z,Z'} : Y_{Z'} \to Y_Z$ (notice that by abuse of notation, when considering the embedding (Y_Z, j_Z), we have omitted the map j_Z).

Correspondingly we get a directed system of rings $H^*(Y_Z, \mathbb{Q})$, with homomorphisms, if $Y_{Z'} \geqslant Y_Z$, $\psi_{Z,Z'}^* : H^*(Y_Z, \mathbb{Q}) \to H^*(Y_{Z'}, \mathbb{Q})$. The main result in [DP2] gives us a working definition for the ring of conditions $R(G)$. Namely:

Theorem 3.2. *(See [DP2].) We have a natural isomorphism of rings*

$$R(G) \simeq \varinjlim H^*(Y_Z, \mathbb{Q}).$$

The above result, among other things, can be used to give a definition of the equivariant ring of conditions. Namely for each regular compactification Y_Z, consider the $G \times G$-equivariant cohomology ring $H_{G\times G}^*(Y_Z, \mathbb{Q})$. As before if $Y_{Z'} \geqslant Y_Z$, we get an induced homomorphism (which we denote by the same letter as before) $\psi_{Z,Z'}^* : H_{G\times G}^*(Y_Z, \mathbb{Q}) \to H_{G\times G}^*(Y_{Z'}, \mathbb{Q})$. Hence we get a directed system of equivariant cohomology rings $H_{G\times G}^*(Y_Z, \mathbb{Q})$ and we can give the following:

Definition 1. The equivariant ring of conditions of the semisimple adjoint group G is the ring

$$R_{G\times G}(G) := \varinjlim H^*_{G\times G}(Y_Z, \mathbb{Q}).$$

4. Orbits in Y_Z

Definition 1 tells us that, in order to compute $R_{G\times G}(G)$, we first need to compute $H^*_{G\times G}(Y_Z, \mathbb{Q})$ for every Y_Z. In order to do this we can use the results in [BDP]. For this we need first to compute the equivariant cohomology of every $G \times G$-orbit in Y_Z.

The set $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of simple roots is a basis for the character group $X(T)$ of T and gives an explicit isomorphism of T with G_m^ℓ. Using this isomorphism we will, from now on, think of Z as a T-embedding. Now it is well known (see for example [Tor]) that any T-embedding corresponds to a fan in the real vector space $U = \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{R})$, rational with respect to the lattice $X(T)^\vee$ of coweights. In particular under our identifications, $\mathbb{A}^\ell$ corresponds to the fan given by the fundamental Weyl chamber C of positive linear combinations of fundamental coweights and its faces.

Our assumption that Z is smooth and proper over $\mathbb{A}^\ell$ tells us that the fan $\mathcal{F}_Z$ corresponding to Z gives a polyhedral decomposition of C by simplicial cones each spanned by a basis of $X(T)^\vee$.

Moreover, given a face F of $\mathcal{F}_Z$, we associate to F the torus $T_F \subset T$ defined as the common kernel of the characters $\chi \in X(T)$ vanishing on F (notice that the fact that T is a torus follows immediately, since F is a face of a simplicial cone spanned by a basis of $X(T)^\vee$).

One knows by the theory of toric varieties that we have a bijection between faces in $\mathcal{F}_Z$ and T-orbits in Z with the following properties:

(1) Given a face $F \in \mathcal{F}_Z$, the corresponding orbit $\mathcal{N}_F$ is T-equivariantly isomorphic to T/T_F.
(2) Given two faces $F_1, F_2 \in \mathcal{F}_Z$, $\overline{F}_1 \supseteq F_2$ if and only if $\overline{\mathcal{N}}_{F_2} \supseteq \mathcal{N}_{F_1}$.

We are now ready to study the structure of the $G \times G$-orbit $\mathcal{M}_F := s_Z^{-1}(\mathcal{T}_{\mathcal{N}_F}) \subset Y_Z$, for any face of the fan $\mathcal{F}_Z$.

Let Γ be the subset of the set of simple roots vanishing on F. Let $P_\Gamma \supset B$ denote the parabolic subgroup associated to Γ and P_Γ^- the opposite parabolic subgroup. Let $L_\Gamma := P_\Gamma \cap P_\Gamma^-$ denote the corresponding Levi factor and let $Z(L_\Gamma)$ be the center of L_Γ. Define the subgroup $H_\Gamma \subset L_\Gamma \times L_\Gamma$ by

$$H_\Gamma = \left\{(g_1, g_2) \in L_\Gamma \times L_\Gamma \;\middle|\; g_1 \cdot g_2^{-1} \in Z(L_\Gamma)\right\}.$$

Since $Z(L_\Gamma) \subset T$ (indeed $Z(L_\Gamma)$ is the common kernel of the α_i lying in Γ), we get the homomorphism

$$p_\Gamma : H_\Gamma \to T$$

defined by $p_\Gamma((g_1, g_2)) = g_1 \cdot g_2^{-1}$. We set $H_F := p_\Gamma^{-1}(T_F)$.

Consider now the quotient homomorphism

$$\mu_\Gamma : P_\Gamma \times P_\Gamma^- \to L_\Gamma \times L_\Gamma.$$

Let $G_F := \mu_\Gamma^{-1}(H_F)$. Notice that since clearly $T_F \subseteq Z(L_\Gamma)$, $\phi_F := p_\Gamma \circ \mu_\Gamma$ defines a surjective homomorphism of G_F onto T_F.

We have:

Proposition 4.1. *There is a $G \times G$-equivariant isomorphism between the orbit $\mathcal{M}_F$ and the homogeneous space $G \times G/G_F$.*

Proof. Consider the projection $\pi_Z : Y_Z \to X$. If Γ is the subset in Δ corresponding to the face F of $\mathcal{F}_Z$, then it is clear that $\pi_Z(\mathcal{M}_F) = \mathcal{O}_\Gamma$. On the other hand one knows (see [DP]) that as a $G \times G$-homogeneous space, $\mathcal{O}_\Gamma$ is isomorphic to $G \times G/G_\Gamma$, with $G_\Gamma = \mu_\Gamma^{-1}(H_\Gamma)$.

Consider now for any $i = \{1, \dots, \ell\}$ the line bundle L_i. If $\alpha_i \in \Gamma$, then the section s_i is nowhere zero on $\mathcal{O}_\Gamma$, so that the restriction of L_i to $\mathcal{O}_\Gamma$ is canonically trivialized.

If on the other hand $\alpha_i \notin \Gamma$, consider the homomorphism $e^{\alpha_i} \circ p_\Gamma \circ \mu_\Gamma : G_\Gamma \to G_m$ and let $G_{\Gamma,i}$ be its kernel. Then the $G \times G$-homogeneous G_m-principal bundle L_i^*, namely the complement of the zero section in L_i, is equal as a $G \times G$-homogeneous space to $G \times G/G_{\Gamma,i}$.

Using the fact that $\mathcal{M}_F = s_Z^{-1}(\mathcal{T}_{\mathcal{N}_F})$ and the structure of $\mathcal{N}_F$, our claim is straightforward. $\square$

Proposition 4.1 allows us to describe the equivariant cohomology ring $H^*_{G\times G}(\mathcal{M}_F, \mathbb{Q})$. By the very definition of equivariant cohomology this ring equals the ring $H^*_{G_F}(\{\mathrm{pt}\}, \mathbb{Q})$.

To compute $H^*_{G_F}(\{\mathrm{pt}\}, \mathbb{Q})$, notice that first of all the kernel of the homomorphism $\mu_\Gamma : G_F \to H_F$ is a unipotent group, so that $H^*_{G_F}(\{\mathrm{pt}\}, \mathbb{Q}) \cong H^*_{H_F}(\{\mathrm{pt}\}, \mathbb{Q})$.

Now in order to compute $H^*_{H_F}(\{\mathrm{pt}\}, \mathbb{Q})$, consider the homomorphism $p_\Gamma : H_F \to T_F$. The kernel of p_Γ is the Levi subgroup L_Γ diagonally embedded in $L_\Gamma \times L_\Gamma$. Let W_Γ denote the Weyl group of L_Γ, that is to say the group of linear transformations in the space $X(T) \otimes \mathbb{Q}$ generated by the simple reflections s_i with respect to the simple roots $\alpha_i \in \Gamma$ and one knows that $H^*_{L_\Gamma}(\{\mathrm{pt}\}, \mathbb{Q}) = \mathbb{Q}[U_\mathbb{Q}]^{W_\Gamma}$ with $U_\mathbb{Q} = \mathrm{Hom}_\mathbb{Z}(X(T), \mathbb{Q})$. On the other hand $H^*_{T_F}(\{\mathrm{pt}\}, \mathbb{Q}) = \mathbb{Q}[U_\mathbb{Q}]/I_F$, where I_F is the ideal of functions vanishing on F.

Since we work over the rationals, we then have

$$H^*_{H_F}(\{\mathrm{pt}\}, \mathbb{Q}) \simeq H^*_{L_\Gamma}(\{\mathrm{pt}\}, \mathbb{Q}) \otimes_\mathbb{Q} H^*_{T_F}(\{\mathrm{pt}\}, \mathbb{Q}). \tag{3}$$

Consider now the vector space $U_\mathbb{Q} \times U_\mathbb{Q}$ with the Weyl group W acting on the left factor. Since $W_\Gamma \subset W$, we also get an action of W_Γ on $U_\mathbb{Q} \times U_\mathbb{Q}$. We can rephrase the existence of the isomorphism (3) as:

Proposition 4.2. *The $G \times G$-equivariant cohomology ring of the orbit $\mathcal{M}_F$ with rational coefficients, is isomorphic to the ring $\mathbb{Q}[U_\mathbb{Q} \times F]^{W_\Gamma}$ of polynomial maps on $U_\mathbb{Q} \times F \subset U_\mathbb{Q} \times U_\mathbb{Q}$ invariant under the action of W_Γ.*

At this point we are ready to use the results in [BDP] to compute the $G \times G$-equivariant cohomology of Y_Z.

Let us recall some facts from [BDP]. Consider two faces F_1, F_2 in $\mathcal{F}_Z$ and assume that F_2 is a codimension one face in $\overline{F_1}$. This implies that $\mathcal{M}_{F_1}$ is a divisor in $\overline{\mathcal{M}_{F_2}}$. Take the normal line bundle L of $\mathcal{M}_{F_1}$ in $\overline{\mathcal{M}_{F_2}}$. Consider the principal G_m-bundle L^* defined as the complement of the zero section in L.

We know that $T_{F_2} \subset T_{F_1}$. Moreover recall that we have a surjective homomorphism $\phi_{F_1} : G_{F_1} \to T_{F_1}$. Thus we can consider the subgroup $G_{F_1}^{F_2} = \phi_{F_1}^{-1}(T_{F_2})$.

By reasoning as in the proof of Proposition 4.1, we deduce that L^* is a $G \times G$-homogeneous space isomorphic to $G \times G/G_{F_1}^{F_2}$.

Notice that $G_{F_1}^{F_2} \subset G_{F_1}$ and also $G_{F_1}^{F_2} \subset G_{F_2}$. Moreover if the set of simple roots vanishing on F_1 coincides with the set of simple roots vanishing on F_2, $G_{F_1}^{F_2} = G_{F_2}$.

In any case we get the following two homomorphisms:

$$T_{F_1}^{F_2} : H^*_{G\times G}(\mathcal{M}_{F_1}, \mathbb{Q}) \to H^*_{G\times G}(L^*, \mathbb{Q}); \tag{4}$$

and

$$S_{F_2}^{F_1} : H^*_{G\times G}(\mathcal{M}_{F_2}, \mathbb{Q}) \to H^*_{G\times G}(L^*, \mathbb{Q}). \tag{5}$$

Let us now take the ring $\bigoplus_{F\in\mathcal{F}_Z} H^*_{G\times G}(\mathcal{M}_F,\mathbb{Q})$ with componentwise multiplication.

We define $R_Z \subset \bigoplus_{F\in\mathcal{F}_Z} H^*_{G\times G}(\mathcal{M}_F,\mathbb{Q})$ as follows. R_Z consists of the sequences $(a_F)_{F\in\mathcal{F}_Z}$ with $a_F \in H^*_{G\times G}(\mathcal{M}_F,\mathbb{Q})$ such that for each pair (F_1,F_2) of faces in $\mathcal{F}_Z$ with F_2 being a codimension one face in $\overline{F_1}$, we have

$$T^{F_2}_{F_1}(a_{F_1}) = S^{F_1}_{F_2}(a_{F_2}). \tag{6}$$

It is clear that R_Z is a subring of $\bigoplus_{F\in\mathcal{F}_Z} H^*_{G\times G}(\mathcal{M}_F,\mathbb{Q})$.

At this point for any $G\times G$-orbit $\mathcal{M}_F$ in Y_Z, let

$$\gamma_F : H^*_{G\times G}(Y_Z,\mathbb{Q}) \to H^*_{G\times G}(\mathcal{M}_F,\mathbb{Q})$$

denote the homomorphism induced by inclusion.

The main result in [BDP] then gives:

Theorem 4.3. *Let Y_Z be a regular compactification of G. Then the ring homomorphism*

$$\gamma : H^*_{G\times G}(Y_Z,\mathbb{Q}) \to \bigoplus_{F\in\mathcal{F}_Z} H^*_{G\times G}(\mathcal{M}_F,\mathbb{Q}) \tag{7}$$

defined by $\gamma(a) = (\gamma_F(a))_{F\in\mathcal{F}_Z}$ for any $a \in H^*_{G\times G}(Y_Z,\mathbb{Q})$, is injective and its image is the ring R_Z.

Using Theorem 4.3, we are going to identify $H^*_{G\times G}(Y_Z,\mathbb{Q})$ with R_Z and, using this, we are going to describe it explicitly.

Definition 2. A function f on the space $U_\mathbb{Q}\times C$ is admissible with respect to the fan $\mathcal{F}_Z$ if:

(1) For every (closed) face F of $\mathcal{F}_Z$ the restriction of f to $U_\mathbb{Q}\times F$ is a polynomial function.
(2) Let Γ be a subset of the set of simple roots Δ and let C_Γ be the face of C defined by the vanishing of the roots in Γ. Then the restriction of f to $U_\mathbb{Q}\times C_\Gamma$ is invariant under the action of W_Γ on $U_\mathbb{Q}$.

Let us denote by P_Z the space of functions admissible with respect to the fan $\mathcal{F}_Z$. By the second condition in Definition 2 and Proposition 4.2, for each face F in $\mathcal{F}_Z$ we have a homomorphism $p_F : P_Z \to H^*_{G\times G}(\mathcal{M}_F,\mathbb{Q})$ and thus a homomorphism

$$p : P_Z \to \bigoplus_{F\in\mathcal{F}_Z} H^*_{G\times G}(\mathcal{M}_F,\mathbb{Q}) \tag{8}$$

defined by $p(f) = (p_F(f))_{F\in\mathcal{F}_Z}$ for any $f\in P_Z$.

Proposition 4.4. *The homomorphism p maps P_Z isomorphically onto R_Z.*

Proof. The injectivity is clear by the definition of P_Z as a ring of functions.

Now remark that if we take two faces F_1, F_2 in $\mathcal{F}_Z$, where F_2 is a codimension one face in $\overline{F_1}$, then the homomorphism

$$S^{F_1}_{F_2} : H^*_{G\times G}(\mathcal{M}_{F_2},\mathbb{Q}) \to H^*_{G\times G}\big(L^*,\mathbb{Q}\big)$$

is injective.

Indeed let Γ_1 and Γ_2 be the two subsets of Δ corresponding to F_1 and F_2. Clearly $\Gamma_1 \subseteq \Gamma_2$. As we have seen, we can identify $H^*_{G\times G}(\mathcal{M}_{F_1},\mathbb{Q})$ with $\mathbb{Q}[U_\mathbb{Q}\times F_1]^{W_{\Gamma_1}}$ and $H^*_{G\times G}(\mathcal{M}_{F_2},\mathbb{Q})$

with $\mathbb{Q}[U_{\mathbb{Q}} \times F_2]^{W_{\Gamma_2}}$. On the other hand, by the description of L^*, we immediately deduce that $H^*_{G\times G}(L^*, \mathbb{Q})$ can be identified with $\mathbb{Q}[U_{\mathbb{Q}} \times F_2]^{W_{\Gamma_1}}$. Using all these identifications, we obtain that $S^{F_1}_{F_2}$ is just given by the inclusion.

Similarly we then get that the homomorphism

$$T^{F_2}_{F_1} : H^*_{G\times G}(\mathcal{M}_{F_1}, \mathbb{Q}) \to H^*_{G\times G}\big(L^*, \mathbb{Q}\big)$$

can be identified with the restriction homomorphism $\mathbb{Q}[U_{\mathbb{Q}} \times F_1]^{W_{\Gamma_1}} \to \mathbb{Q}[U_{\mathbb{Q}} \times F_2]^{W_{\Gamma_1}}$. This clearly implies that the image of P_Z lies in R_Z.

On the order hand the compatibility conditions we have just explained imply that a sequence in R_Z can be clearly patched to give a well defined admissible function on $U_{\mathbb{Q}} \times C$ with respect to $\mathcal{F}_Z$, thus proving our claim. □

Theorem 4.3 together with Proposition 4.4 implies the following:

Theorem 4.5. *Let Y_Z be a regular compactification of G. Then there is a natural ring isomorphism between $H^*_{G\times G}(Y_Z, \mathbb{Q})$ and the ring P_Z of admissible functions with respect to the fan $\mathcal{F}_Z$.*

Notice that in the special case in which Y_Z is the wonderful compactification X, our result gives the following version of the main result in [St3]:

Corollary 4.6. *The ring $H^*_{G\times G}(Y_Z, \mathbb{Q})$ is naturally isomorphic to the ring P of polynomial functions f on $U_{\mathbb{Q}} \times C$ such that for each face C_Γ of C the restriction of f to $U_{\mathbb{Q}} \times C_\Gamma$ is invariant under W_Γ.*

5. The equivariant ring of conditions

At this point we are ready to perform the computation of the ring $R_{G\times G}(G)$ as a ring of functions.

Let us recall that by [DP2], given two regular embeddings Y_Z, $Y_{Z'}$, we have a (necessarily unique) $G \times G$-equivariant morphism

$$F^Z_{Z'} : Y_Z \to Y_{Z'}$$

if and only if we have a T-equivariant morphism $Z \to Z'$, that is if and only if the fan $\mathcal{F}_Z$ is a decomposition of the fan $\mathcal{F}_{Z'}$, in the sense that each face F in $\mathcal{F}_{Z'}$ is the union of faces in $\mathcal{F}_Z$. If this is the case, it is clear that each admissible function with respect to the fan $\mathcal{F}_{Z'}$ is also admissible with respect to the fan $\mathcal{F}_Z$. Thus we get a natural injection of $P_{Z'}$ into P_Z. Also it is immediate to verify, using the construction of the isomorphism between $H^*_{G\times G}(Y_Z, \mathbb{Q})$ and P_Z, that the homomorphism

$$\big(F^Z_{Z'}\big)^* : H^*_{G\times G}(Y_{Z'}, \mathbb{Q}) \to H^*_{G\times G}(Y_Z, \mathbb{Q})$$

can be identified with the injection of $P_{Z'}$ into P_Z.

It is now natural to give the following:

Definition 3. A function f on the space $U_{\mathbb{Q}} \times C$ is admissible if there exists a fan $\mathcal{F}_Z$ giving a polyhedral decomposition of C by simplicial cones each spanned by a basis of $X(T)^\vee$, such that f is admissible with respect to $\mathcal{F}_Z$.

Notice that since fans $\mathcal{F}_{Z'}$ and $\mathcal{F}_Z$ with the above properties have a common decomposition (see for example [Tor] or [O]), it immediately follows that the space $\mathcal{R}$ of admissible functions is a ring. By our previous considerations we then have:

Theorem 5.1. *The equivariant ring of conditions $R_{G\times G}(G)$ is naturally isomorphic to the ring $\mathcal{R}$ of admissible functions.*

Remark 5.2. The reader might have noticed that all our rings are graded and that all our isomorphisms are isomorphisms of graded rings. We have not mentioned this explicitly in the paper in order not to make the presentation heavier.

6. The cohomology of Y_Z and the ring of conditions

We are now going to explain how to deduce the computation of $H^*(Y_Z,\mathbb{Q})$ from the one of the equivariant cohomology $H^*_{G\times G}(Y_Z,\mathbb{Q})$. In this way we shall also obtain the computation of $R(G)$ from the one of $R_{G\times G}(G)$.

For this we have to describe the structure of $H^*_{G\times G}(Y_Z,\mathbb{Q})$ as a module over the ring $H^*_{G\times G}(\{\mathrm{pt}\},\mathbb{Q})$ induced by the projection $Y_Z\to\{\mathrm{pt}\}$.

Recall that, by [DP], Y_Z has a paving by affine spaces. It follows by what we have recalled in Section 2, that $H^*_{G\times G}(Y_Z,\mathbb{Q})$ is a free $H^*_{G\times G}(\{\mathrm{pt}\},\mathbb{Q})$-module and $H^*(Y_Z,\mathbb{Q})$ is the quotient of $H^*_{G\times G}(Y_Z,\mathbb{Q})$ over the ideal generated by the image of the positive degree part of $H^*_{G\times G}(\{\mathrm{pt}\},\mathbb{Q})$.

Since the projection $Y_Z\to\{\mathrm{pt}\}$ factors through the equivariant map $F_Z:Y_Z\to X$ (to be precise according to the notations of the previous section our F_Z should be denoted by $F_Z^{\mathbb{A}^\ell}$, but the simplified notation does not create any confusion), we can use the description of the $H^*_{G\times G}(\{\mathrm{pt}\},\mathbb{Q})$-module structure of $H^*_{G\times G}(X,\mathbb{Q})$ given in [St3].

This goes as follows. As before, let us identify $H^*_{G\times G}(X,\mathbb{Q})$ with a subring of the ring $P_{\mathbb{A}_\ell}$ of polynomial functions on $U_{\mathbb{Q}}\times C$. We can also identify $H^*_{G\times G}(\{\mathrm{pt}\},\mathbb{Q})$ with the ring $\mathbb{Q}[U_{\mathbb{Q}}\times U_{\mathbb{Q}}]^{W\times W}$. Consider now the map

$$q:U_{\mathbb{Q}}\times C\to U_{\mathbb{Q}}\times U_{\mathbb{Q}} \tag{9}$$

given by $q((u,c))=(u+c,u-c)$. Then take the induced map of functions $q^*:\mathbb{Q}[U_{\mathbb{Q}}\times U_{\mathbb{Q}}]\to\mathbb{Q}[U_{\mathbb{Q}}\times C]=\mathbb{Q}[U_{\mathbb{Q}}\times U_{\mathbb{Q}}]$. In [St3] it was shown that in fact $q^*(\mathbb{Q}[U_{\mathbb{Q}}\times U_{\mathbb{Q}}]^{W\times W})\subset P_{\mathbb{A}_\ell}$ and that under the above identifications, the $H^*_{G\times G}(\{\mathrm{pt}\},\mathbb{Q})$-module structure of $H^*_{G\times G}(X,\mathbb{Q})$ is given by q^*.

Recall that $\mathbb{Q}[U_{\mathbb{Q}}]^W$ is a polynomial ring freely generated by ℓ homogeneous elements $F_1,\dots,F_\ell$. Our previous considerations imply the following:

Theorem 6.1. *Let Y_Z be a regular compactification of G.*

Identify $H^*_{G\times G}(Y_Z,\mathbb{Q})$ with the ring P_Z of admissible functions with respect to the fan $\mathcal{F}_Z$. Then we get an identification of $H^*(Y_Z,\mathbb{Q})$ with the ring P_Z/J_Z where J_Z is the ideal generated by the regular sequence

$$F_1(u+c),\dots,F_\ell(u+c),F_1(u-c),\dots,F_\ell(u-c)$$

for $u\in U_{\mathbb{Q}}$, $c\in C$.

Similarly for the ring of conditions we get:

Theorem 6.2. *If we identify the equivariant ring of conditions $R_{G\times G}(G)$ with the ring $\mathcal{R}$ of admissible functions, we obtain an identification of the ring of conditions $R(G)$ with the ring $R_{G\times G}(G)/J$, where J is the ideal generated by the regular sequence*

$$F_1(u+c),\dots,F_\ell(u+c),F_1(u-c),\dots,F_\ell(u-c)$$

for $u\in U_{\mathbb{Q}}$, $c\in C$.

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