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PRESENTATION BY BOREL SUBALGEBRAS AND CHEVALLEY GENERATORS FOR QUANTUM ENVELOPING ALGEBRAS

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Abstract. We provide an alternative approach to the Faddeev-Reshetikhin-Takhtajan presentation of the quantum group $U_q(\mathfrak{g})$, with L–operators as generators and relations ruled by an R–matrix. We look at $U_q(\mathfrak{g})$ as being generated by the quantum Borel subalgebras $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$, and use the standard presentation of the latters as quantum function algebras. When $\mathfrak{g} = \mathfrak{gl}_n$ these Borel quantum function algebras are generated by the entries of a triangular q–matrix, thus eventually $U_q(\mathfrak{gl}_n)$ is generated by the entries of an upper triangular and a lower triangular q–matrix, which share the same diagonal. The same elements generate over $\Bbbk\big[q,q^{-1}\big]$ the unrestricted $\Bbbk\big[q,q^{-1}\big]$–integer form of $U_q(\mathfrak{gl}_n)$ of De Concini and Procesi, which we present explicitly, together with a neat description of the associated quantum Frobenius morphisms at roots of 1. All this holds, *mutatis mutandis*, for $\mathfrak{g} = \mathfrak{sl}_n$ too.

Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra over a field $\Bbbk$. Classically, it has two standard presentations: Serre's one, which uses a minimal set of generators, and Chevalley's one, using a linear basis as generating set. If $\mathfrak{g}$ instead is reductive a presentation is obtained by that of its semisimple quotient by adding the center. When $\mathfrak{g} = \mathfrak{gl}_n$, Chevalley's generators are the elementary matrices, and Serre's ones form a distinguished subset of them; the general case of any classical matrix Lie algebra $\mathfrak{g}$ is a slight variation on this theme. Finally, both presentations yield also presentations of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

At the quantum level, one has correspondingly a Serre-like and a Chevalley-like presentation of $U_q(\mathfrak{g})$, the quantized universal enveloping algebra associated to $\mathfrak{g}$ after Jimbo and Lusztig (i.e. defined over the field $\Bbbk(q)$, where q is an indeterminate). The first presentation is used by Jimbo (cf. [Ji1]) and Lusztig (see [Lu2]) and, *mutatis mutandis*, by Drinfeld too;

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in this case the generators are q–analogues of the Serre's generators, and starting from them one builds quantum root vectors via two different methods: iterated quantum brackets, as in [Ji2] — and maybe others — or braid group action, like in [Lu2]; see [Ga2] for a comparison between these two methods. The second presentation was introduced by Faddeev, Reshetikhin and Takhtajan (in [FRT]): the generators in this case, called L–operators, are q–analogues of the classical Chevalley generators; in particular, they are quantum root vectors themselves. An explicit comparison between quantum Serre-like generators and L–operators appears in [FRT], §2, for the cases of *classical* $\mathfrak{g}$; on the other hand, in [No], §1.2, a similar comparison is made for $\mathfrak{g} = \mathfrak{gl}_n$ between L–operators and quantum root vectors (for *any* root) built out of Serre's generators.

The first purpose of this note is to provide an alternative approach to the FRT presentation of $U_q(\mathfrak{g})$: it amounts to a series of elementary steps, yet the final outcome seems noteworthy. As a second, deeper result, we give an explicit presentation of the $\Bbbk\big[q,q^{-1}\big]$–subalgebra of $U_q(\mathfrak{g})$ generated by L–operators, call it $\widetilde{U}_q(\mathfrak{g})$. By construction, this is nothing but the *unrestricted* $\Bbbk\big[q,q^{-1}\big]$–integer form of $U_q(\mathfrak{g})$, defined by De Concini and Procesi (see [DP]), whose semiclassical limit is $\widetilde{U}_q(\mathfrak{g}) \Big/ (q-1)\,\widetilde{U}_q(\mathfrak{g}) \cong F\big[G^*\big]$, where G^* is a connected Poisson algebraic group dual to $\mathfrak{g}$ (cf. [DP], [Ga1] and [Ga3], §7.3 and §7.9): our explicit presentation of $\widetilde{U}_q(\mathfrak{g})$ yields another, independent (and much easier) proof of this fact. Third, by [DP] we know that *quantum Frobenius morphisms* exist, which embed $F\big[G^*\big]$ into the specializations of $\widetilde{U}_q(\mathfrak{g})$ at roots of 1: then our presentation of $\widetilde{U}_q(\mathfrak{g})$ provides an explicit description of them.

This analysis shows that the two presentations of $U_q(\mathfrak{g})$ correspond to different behaviors w.r.t. to specializations. Indeed, let $\widehat{U}_q(\mathfrak{g})$ be the $\Bbbk\big[q,q^{-1}\big]$–algebra given by Jimbo-Lusztig presentation *over* $\Bbbk\big[q,q^{-1}\big]$. Its specialization at $q=1$ is $\widehat{U}_q(\mathfrak{g}) \Big/ (q-1)\,\widehat{U}_q(\mathfrak{g}) \cong U(\mathfrak{g})$ (up to technicalities), with $\mathfrak{g}$ inheriting a Lie bialgebra structure (see [Ji1], [Lu2], [DL]). On the other hand, the integer form $\widetilde{U}_q(\mathfrak{g})$ mentioned above specializes to $F\big[G^*\big]$, the Poisson structure on G^* being exactly the one dual to the Lie bialgebra structure on $\mathfrak{g}$. So the existence of two different presentations of $U_q(\mathfrak{g})$ reflects the deep fact that $U_q(\mathfrak{g})$ provides, taking suitable integer forms, quantizations of two different semiclassical objects (this is a general fact, see [Ga3–4]). To the author's knowledge, this was not known so far, as the FRT presentation of $U_q(\mathfrak{g})$ was never used to study the integer form $\widetilde{U}_q(\mathfrak{g})$.

Let's sketch in short the path we follow. First, we note that $U_q(\mathfrak{g})$ is generated by the quantum Borel subgroups $U_q(\mathfrak{b}_-)$ and $U_q(\mathfrak{b}_+)$ (where $\mathfrak{b}_-$ and $\mathfrak{b}_+$ are opposite Borel subalgebras of $\mathfrak{g}$), which share a common copy of the quantum Cartan subgroup $U_q(\mathfrak{t})$. Second, there exist Hopf algebra isomorphisms $U_q(\mathfrak{b}_-) \cong F_q\big[B_-\big]$ and $U_q(\mathfrak{b}_+) \cong F_q\big[B_+\big]$, where $F_q\big[B_-\big]$ and $F_q\big[B_+\big]$ are the quantum function algebras associated to $\mathfrak{b}_-$ and $\mathfrak{b}_+$ respectively. Third, when $\mathfrak{g}$ is classical we resume the explicit presentation by generators and relations of $F_q\big[B_-\big]$ and $F_q\big[B_+\big]$, as given in [FRT], §1. Fourth, from the above we argue a presentation of $U_q(\mathfrak{g})$ where the generators are those of $F_q\big[B_-\big]$ and $F_q\big[B_+\big]$, the toral ones being taken only once, and relations are those of these quantum function algebras plus some additional relations between generators of opposite quantum Borel subgroups. We perform this last step in full detail for $\mathfrak{g} = \mathfrak{gl}_n$ and, with slight changes, for $\mathfrak{g} = \mathfrak{sl}_n$ as well. Fifth, we refine the last step to provide a presentation of $\widetilde{U}_q(\mathfrak{g})$.

As an application, our results apply also (with few changes) to the Drinfeld-like quantum groups $U_\hbar(\mathfrak{g})$: in particular we get a presentation of an $\hbar$–deformation of $F[G^*]$, say $\widetilde{U}_\hbar(\mathfrak{g}) =: F_\hbar[G^*]$. An explicit gauge equivalence between this $F_\hbar[G^*]$ and the $\hbar$–deformation provided by Kontsevitch' recipe is given in [FG].

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§ 1 The general case

1.1 Quantized universal enveloping algebras. Let $\Bbbk$ be a fixed field of zero characteristic, let q be an indeterminate, and let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk$. Let $U_q(\mathfrak{g})$ be the quantum group à la Jimbo-Lusztig defined over $\Bbbk(q)$: we define it after the conventions in [DP], or [DL], or [Ga1] (for $\varphi = 0$). Actually, we can define a quantum group like that for each lattice M between the root lattice Q and the weight lattice of P of $\mathfrak{g}$, thus we shall write $U_q^M(\mathfrak{g})$. Roughly, $U_q^M(\mathfrak{g})$ is the unital $\Bbbk(q)$–algebra with generators F_i, $\Lambda_i^{\pm 1}$, E_i for $i = 1, \dots, r =: \mathit{rank}(\mathfrak{g})$ and relations as in [DP], [Ga1], which depend on the Cartan datum of $\mathfrak{g}$ and on the choice of the lattice M; in particular, the Λ_i's are "toral" generators, roughly q–exponentials of the elements of a $\mathbb{Z}$–basis of M. Here we only recall the relation

$$E_i F_j - F_j E_i \;=\; \delta_{ij} \, \frac{K_i - K_i^{-1}}{q - q^{-1}} \qquad \forall \; i, j = 1, \dots, r \tag{1.1}$$

where K_i is a q–analogue of the coroot corresponding to the i-th node of the Dynkin diagram of $\mathfrak{g}$ (in fact, it is a suitable product of $\Lambda_k^{\pm 1}$'s). Also, we consider on $U_q^M(\mathfrak{g})$ the Hopf algebra structure given in [DP] or [Ga1].

The quantum Borel subalgebra $U_q^M(\mathfrak{b}_+)$ is simply the unital $\Bbbk(q)$–subalgebra of $U_q^M(\mathfrak{g})$ generated by $\Lambda_1^{\pm 1}, \dots, \Lambda_r^{\pm 1}, E_1, \dots, E_r$, and $U_q^M(\mathfrak{b}_-)$ the one generated by $F_1, \dots, F_r, \Lambda_1^{\pm 1}, \dots, \Lambda_r^{\pm 1}$. In fact, both of these are Hopf $\Bbbk(q)$–subalgebras of $U_q^M(\mathfrak{g})$. It follows that $U_q^M(\mathfrak{g})$ is generated by $U_q^M(\mathfrak{b}_+)$ and $U_q^M(\mathfrak{b}_-)$, and every possible commutation relation between these two subalgebras is a consequence of (1.1) and the commutation relations between the $\Lambda_i^{\pm 1}$'s and the F_j's or the E_j's. Finally, we call $U_q^M(\mathfrak{t})$ the unital $\Bbbk(q)$–subalgebra of $U_q^M(\mathfrak{g})$ (and of $U_q^M(\mathfrak{b}_\pm)$) generated by all the Λ_i's ($i = 1, \dots, n$), which also is a Hopf subalgebra.

Mapping $F_i \mapsto E_i$, $\Lambda_i^{\pm 1} \mapsto \Lambda_i^{\mp 1}$ and $E_i \mapsto F_i$ (for all $i = 1, \dots, n$) uniquely defines an algebra automorphism and coalgebra antiautomorphism of $U_q^M(\mathfrak{g})$, that is a Hopf algebra isomorphism $\Theta \colon U_q^M(\mathfrak{g}) \xrightarrow{\cong} U_q^M(\mathfrak{g})^{\text{op}}$, where hereafter given any Hopf algebra H we denote by H^{op} the same Hopf algebra as H but for taking the opposite coproduct. Restricting Θ to quantum Borel subalgebras gives Hopf algebra isomorphisms $U_q^M(\mathfrak{b}_\pm) \cong U_q^M(\mathfrak{b}_\mp)^{\text{op}}$.

1.2 Quantum function algebras. Let M be a lattice between Q and P as in §1.1, and define $M' := \big\{ \psi \in P \,\big|\, \langle \psi, \mu \rangle \in \mathbb{Z} \, , \, \forall \, \mu \in \mathbb{Z} \big\}$ where $\langle \ , \ \rangle$ is the $\mathbb{Q}$–valued scalar product on P induced by scalar extension from the natural $\mathbb{Z}$–valued pairing between Q and P. Such M' is again a lattice, said to be *dual* to M. Conversely, M is dual to M', i.e. $M = M''$.

We define quantum function algebras after Lusztig. To start with, letting M and M' be mutually dual lattices as above, we define $F_q^{M'}\big[G\big]$ as the unital $\Bbbk(q)$–algebra of all matrix coefficients of finite dimensional $U_q^M(\mathfrak{g})$–modules which have a basis of eigenvectors for all the Λ_i's ($i = 1, \dots, n$) with eigenvalues powers of q . Starting from $U_q^M(\mathfrak{b}_+)$ or $U_q^M(\mathfrak{b}_-)$ instead of $U_q^M(\mathfrak{g})$ the same recipe defines the Borel quantum function algebras $F_q^{M'}\big[B_+\big]$ and $F_q^{M'}\big[B_-\big]$ respectively. All these quantum function algebras are in fact Hopf algebras too.

Finally, the Hopf algebra monomorphisms $j_\pm : U_q^M(\mathfrak{b}_\pm) \hookrightarrow U_q^M(\mathfrak{g})$ induce Hopf algebra epimorphisms $\pi_\pm : F_q^{M'}\big[G\big] \twoheadrightarrow F_q^{M'}\big[B_\pm\big]$. See [DL] and [Ga1] for details.

1.3 Isomorphisms between QUEA's and QFA's over Borel subgroups. Let M and M' be mutually dual lattices as in §1.2. According to Tanisaki (cf. [Ta]) there exist perfect (i.e. non degenerate) Hopf pairings $U_q^M(\mathfrak{b}_+)^{\text{op}} \otimes U_q^{M'}(\mathfrak{b}_-) \longrightarrow \Bbbk(q)$, $U_q^M(\mathfrak{b}_-)^{\text{op}} \otimes U_q^{M'}(\mathfrak{b}_+) \longrightarrow \Bbbk(q)$; this implies $U_q^M(\mathfrak{b}_+)^{\text{op}} \cong F_q^M\big[B_-\big]$ and $U_q^M(\mathfrak{b}_-)^{\text{op}} \cong F_q^M\big[B_+\big]$. Composing the latters with the isomorphisms $U_q^M(\mathfrak{b}_+) \cong U_q^M(\mathfrak{b}_-)^{\text{op}}$ and $U_q^M(\mathfrak{b}_-) \cong U_q^M(\mathfrak{b}_+)^{\text{op}}$ in §1.1 it follows that $U_q^M(\mathfrak{b}_+) \cong F_q^M\big[B_+\big]$ and $U_q^M(\mathfrak{b}_-) \cong F_q^M\big[B_-\big]$ as Hopf $\Bbbk(q)$–algebras.

1.4 Generation of $U_q^M(\mathfrak{g})$ by quantum function algebras. We said in §1.1 that $U_q^M(\mathfrak{g})$ is generated by $U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_+)$, whose mutual commutation is a consequence of (1.1). In particular, we have a $\Bbbk(q)$–vector space isomorphism $U_q^M(\mathfrak{g}) = \big(U_q^M(\mathfrak{b}_-) \otimes U_q^M(\mathfrak{b}_+) \big) \Big/ J$, where J is the two-sided ideal of $U_q^M(\mathfrak{b}_-) \otimes U_q^M(\mathfrak{b}_+)$ — with the standard tensor product structure — generated by $\big(\{ K_\mu \otimes 1 - 1 \otimes K_\mu \}_{\mu \in M} \big)$, while the multiplication is a consequence of the internal commutation rules of $U_q^M(\mathfrak{b}_\pm)$ and by (1.1). Now, thanks to the isomorphisms in §1.3, we describe $U_q^M(\mathfrak{g})$ as being generated by $F_q^M\big[B_-\big]$ and $F_q^M\big[B_+\big]$, with mutual commutation being a consequence of the commutation formulas corresponding to (1.1) under those isomorphisms. So we have a $\Bbbk(q)$–vector space isomorphism $U_q^M(\mathfrak{g}) \cong \big(F_q^M\big[B_-\big] \otimes F_q^M\big[B_+\big] \big) \Big/ I$, where I is the ideal of $F_q^M\big[B_-\big] \otimes F_q^M\big[B_+\big]$ corresponding to J, while commutation rules are the internal ones of $F_q^M\big[B_\pm\big]$ and those corresponding to (1.1).

1.5 Relation with L–operators. Tracking carefully the construction of $U_q^M(\mathfrak{g})$ proposed in §1.4 above one realizes that this is just an alternative way to introduce $U_q^M(\mathfrak{g})$ via

L–operators as made in [FRT]. Such a comparison is essentially the meaning — or a possible interpretation — of the analysis carried on in [Mo]. Moreover, the latter analysis also shows that L–operators in [FRT] do correspond to suitable matrix coefficients in $F_q^M\big[B_-\big]$ and $F_q^M\big[B_+\big]$ (embedded inside $F_q^M\big[G\big]$); such matrix coefficients then correspond to quantum root vectors in $U_q^M(\mathfrak{b}_+)^{\text{op}}$ and $U_q^M(\mathfrak{b}_-)^{\text{op}}$ via the isomorphisms $F_q^M\big[B_-\big] \cong U_q^M(\mathfrak{b}_+)^{\text{op}}$ and $F_q^M\big[B_+\big] \cong U_q^M(\mathfrak{b}_-)^{\text{op}}$ in §1.3, and finally to quantum root vectors in $U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_+)$ via the isomorphisms $U_q^M(\mathfrak{b}_+)^{\text{op}} \cong U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_-)^{\text{op}} \cong U_q^M(\mathfrak{b}_+)$ in §1.1.

1.6 Integer $\Bbbk\big[q,q^{-1}\big]$–forms, specializations, quantum Frobenius morphisms. In order to look at "specializations of a quantum group at special values of the parameter q", one needs the given quantum group to be defined over a subring of $\Bbbk(q)$ whose elements are regular, i.e. have no poles, at such special values. As it is typical, we choose as ground ring the Laurent polynomial ring $\Bbbk\big[q,q^{-1}\big]$. Then instead of $U_q^M(\mathfrak{g})$ we must consider integer forms of $U_q^M(\mathfrak{g})$ over $\Bbbk\big[q,q^{-1}\big]$, i.e. Hopf $\Bbbk\big[q,q^{-1}\big]$–subalgebras of $U_q^M(\mathfrak{g})$ which give back all of $U_q^M(\mathfrak{g})$ by scalar extension from $\Bbbk\big[q,q^{-1}\big]$ to $\Bbbk(q)$: if $\overline{U}_q^{\,M}(\mathfrak{g})$ is such a $\Bbbk\big[q,q^{-1}\big]$–form, its *specialization* at $q=c\in\Bbbk$ is the quotient Hopf $\Bbbk$–algebra $\overline{U}_c^{\,M}(\mathfrak{g}) := \overline{U}_q^{\,M}(\mathfrak{g})\Big/(q-c)\,\overline{U}_q^{\,M}(\mathfrak{g})$.

There are essentially two main types of $\Bbbk\big[q,q^{-1}\big]$–integer forms: one is $\widehat{U}_q^M(\mathfrak{g})$ (the quantum analogue of Kostant's $\mathbb{Z}$–integer form of $\mathfrak{g}$) introduced by Lusztig in [Lu1], generated by q–binomial coefficients and q–divided powers); the second one is $\widetilde{U}_q^M(\mathfrak{g})$, introduced by De Concini and Procesi in [DP], generated by rescaled quantum root vectors; see [Ga1] for details. When q is specialized to any value in $\Bbbk$ *which is not a root of 1*, the choice of either of these two integer forms is irrelevant, because the corresponding specialized Hopf $\Bbbk$–algebras are mutually isomorphic. If instead q is specialized to $\varepsilon\in\Bbbk$ *which is a root of 1*, then the specialized algebra changes according to the choice of integer form.

Indeed, the behavior of $\widehat{U}_q^M(\mathfrak{g})$ and $\widetilde{U}_q^M(\mathfrak{g})$ w.r.t. specializations at roots of 1 is pretty different, even opposite. In particular, one has semiclassical limits $\widehat{U}_1^M(\mathfrak{g}) \cong U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, and $\widetilde{U}_1^M(\mathfrak{g}) \cong F\big[G_M^*\big]$, the regular function algebra of G_M^*, where G_M^* is a connected Poisson algebraic group with fundamental group isomorphic to $P\big/M$ and dual to $\mathfrak{g}$, the latter endowed with a structure of Lie bialgebra, inherited from $\widehat{U}_q^M(\mathfrak{g})$. Moreover, specializations of an integer form of either type at a root of 1, say $\varepsilon\in\Bbbk$, are linked to its semiclassical limit by the so-called *quantum Frobenius morphisms*

$$\widehat{U}_\varepsilon^M(\mathfrak{g}) \twoheadrightarrow \widehat{U}_1^M(\mathfrak{g}) \cong U(\mathfrak{g}) \quad , \qquad F\big[G_M^*\big] \cong \widetilde{U}_1^M(\mathfrak{g}) \hookrightarrow \widetilde{U}_\varepsilon^M(\mathfrak{g}) \quad . \tag{1.2}$$

Such a situation occurs exactly the same — *mutatis mutandis* — for the quantum Borel subalgebras $U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_+)$. In short, one has two types of $\Bbbk\big[q,q^{-1}\big]$–integer forms $\widehat{U}_q^M(\mathfrak{b}_\pm)$ and $\widetilde{U}_q^M(\mathfrak{b}_\pm)$, and quantum Frobenius morphisms

$$\widehat{U}_\varepsilon^M(\mathfrak{b}_\pm) \twoheadrightarrow \widehat{U}_1^M(\mathfrak{b}_\pm) \cong U(\mathfrak{b}_\pm) \quad , \qquad F\big[B_\pm^*\big] \cong \widetilde{U}_1^M(\mathfrak{b}_\pm) \hookrightarrow \widetilde{U}_\varepsilon^M(\mathfrak{b}_\pm) \quad . \tag{1.3}$$

By construction, $\widehat{U}_q^M(\mathfrak{g})$ is generated by $\widehat{U}_q^M(\mathfrak{b}_+)$ and $\widehat{U}_q^M(\mathfrak{b}_-)$, and similarly $\widetilde{U}_q^M(\mathfrak{g})$ is generated by $\widetilde{U}_q^M(\mathfrak{b}_+)$ and $\widetilde{U}_q^M(\mathfrak{b}_-)$. It follows that the morphisms in (1.3) can also be obtained from (1.2) by restriction to quantum Borel subalgebras; conversely, the quantum Frobenius morphisms in (1.2) are uniquely determined — and described — by those in (1.3).

By duality, the like happens also for quantum function algebras: in particular, there exist two $\Bbbk\big[q,q^{-1}\big]$–integer forms $\widehat{F}_q^M\big[G\big]$ and $\widetilde{F}_q^M\big[G\big]$ of $F_q^M\big[G\big]$, which are dual respectively to $\widehat{U}_q^M(\mathfrak{g})$ and $\widetilde{U}_q^M(\mathfrak{g})$ in Hopf theoretical sense, for which the dual of (1.2) holds, namely

$$F\big[G\big] \cong \widehat{F}_1^M\big[G\big] \hookrightarrow \widehat{F}_\varepsilon^M\big[G\big] \quad , \qquad \widetilde{F}_\varepsilon^M\big[G\big] \twoheadrightarrow \widetilde{F}_1^M\big[G\big] \cong U(\mathfrak{g}^*) \quad . \tag{1.4}$$

Similarly, the dual of (1.3) holds for quantum function algebras of Borel subgroups, namely

$$F\big[B_\pm\big] \cong \widehat{F}_1^M\big[B_\pm\big] \hookrightarrow \widehat{F}_\varepsilon^M\big[B_\pm\big] \quad , \qquad \widetilde{F}_\varepsilon^M\big[B_\pm\big] \twoheadrightarrow \widetilde{F}_1^M\big[B_\pm\big] \cong U(\mathfrak{b}_\pm^*) \quad , \tag{1.5}$$

which follow from (1.4) via the maps $F_q^M\big[G\big] \overset{\pi_\pm}{\twoheadrightarrow} F_q^M\big[B_\pm\big]$ in §1.2. See [Ga1] for details.

The point we want to stress now is the relation between the isomorphisms of Hopf $\Bbbk(q)$–algebras $U_q^M(\mathfrak{b}_+) \cong F_q^M\big[B_+\big]$ and $U_q^M(\mathfrak{b}_-) \cong F_q^M\big[B_-\big]$ in §1.3 and the $\Bbbk\big[q,q^{-1}\big]$–integer forms on both sides. The key fact is that the previous isomorphisms restrict to isomorphisms of Hopf $\Bbbk\big[q,q^{-1}\big]$–algebras $\widehat{U}_q^M(\mathfrak{b}_\pm) \cong \widetilde{F}_q^M\big[B_\pm\big]$ and $\widetilde{U}_q^M(\mathfrak{b}_\pm) \cong \widehat{F}_q^M\big[B_\pm\big]$. Therefore, looking at $U_q^M(\mathfrak{g})$ as generated by $F_q^M\big[B_-\big]$ and $F_q^M\big[B_+\big]$ as explained in §1.4 one argues that the *first*, resp. the *second*, quantum Frobenius morphisms in (1.2) are uniquely determined (and described) by the *second* ones, resp. the *first* ones, in (1.5).

§ 2 The case of $\mathfrak{gl}_n$

2.1 q–matrices. Let $\big\{t_{ij}\,\big|\,i,j=1,\dots,n\big\}$ be a set of elements in any $\Bbbk(q)$–algebra A, ideally displayed inside an $(n\times n)$–matrix they are the entries of. We'll say that $T := \big(t_{ij}\big)_{i,j=1,\dots,n}$ *is a q–matrix* if the t_{ij}'s enjoy the following relations

$$\begin{aligned} t_{ij}\,t_{ik} = q\,t_{ik}\,t_{ij} \ , \qquad t_{ik}\,t_{hk} = q\,t_{hk}\,t_{ik} \qquad\qquad & \forall \quad j<k\,,\,i<h\,, \\ t_{il}\,t_{jk} = t_{jk}\,t_{il} \ , \qquad t_{ik}\,t_{jl} - t_{jl}\,t_{ik} = \big(q-q^{-1}\big)\,t_{il}\,t_{jk} \qquad & \forall \quad i<j\,,\,k<l\,. \end{aligned}$$

in the algebra A. *In this case, the so-called "quantum determinant", defined as*

$$det_q\left(\big(t_{k,\ell}\big)_{k,\ell=1,\dots,n}\right) := \textstyle\sum_{\sigma\in\mathcal{S}_n} (-q)^{l(\sigma)}\, t_{1,\sigma(1)}\, t_{2,\sigma(2)} \cdots t_{n,\sigma(n)}$$

commutes with all the $t_{i,j}$'s. If in addition A is a $\Bbbk(q)$–bialgebra, we shall also require that

$$\Delta(t_{ij}) \;=\; \sum_{k=1}^n t_{ik}\otimes t_{kj} \ , \qquad \epsilon(t_{ij}) \;=\; \delta_{ij} \qquad\qquad \forall \;\, i,j=1,\dots,n \ .$$

In this case, the quantum determinant is group-like, that is $\Delta(det_q) = det_q \otimes det_q$ and $\epsilon(det_q) = 1$. Finally, if A is a Hopf algebra we call *Hopf q–matrix* any q–matrix like above whose entries are such that det_q is invertible in A; then $S\big(det_q^{\pm 1}\big) = det_q^{\mp 1}$.

For later use we also recall the following compact notation. Let $T_1 := T \otimes I$, $T_2 := I \otimes T \in A \otimes Mat_n\big(\Bbbk(q)\big)^{\otimes 2} \cong A \otimes Mat_{n^2}\big(\Bbbk(q)\big)$, where I is the identity matrix, and $T := \big(t_{ij}\big)_{i,j=1,\dots,n}$ is thought of as an element of $Mat_n(A) \cong A \otimes Mat_n\big(\Bbbk(q)\big)$; consider

$$R := \textstyle\sum_{i,j=1}^n q^{\delta_{ij}} e_{ii} \otimes e_{jj} + \big(q - q^{-1}\big) \sum_{1 \leq i < j \leq n} e_{ij} \otimes e_{ji} \in Mat_{n^2}\big(\Bbbk(q)\big)$$

where $e_{ij} := \big(\delta_{ih}\,\delta_{jk}\big)_{h,k=1}^n$ is the (i,j)–th elementary matrix. Then T is a q–matrix if and only if the identity $R\,T_2\,T_1 = T_1\,T_2\,R$ holds true in $A \otimes Mat_{n^2}\big(\Bbbk(q)\big)$; in detail, for the matrix entry in position $\big((i,j),(kl)\big)$ this reads $\sum\limits_{m,p=1}^n R_{ij,mp}\, t_{pk}\, t_{ml} = \sum\limits_{m,p=1}^n t_{im}\, t_{jp}\, R_{mp,kl}$. In the bialgebra case T is a q–matrix if in addition $\Delta(T) = T \dot{\otimes} T$, $\epsilon(T) = I$, and in the Hopf algebra case also $T\,S(T) = I = S(T)\,T$, i.e. $S(T) = T^{-1}$; see [FRT] and [No] for notations — we use assumptions and normalizations of the latter — and further details.

2.2 Presentation of $F_q^P[G]$, $F_q^P[B_-]$ and $F_q^P[B_+]$ for $G = GL_n$. Let's look at $G = GL_n$. After [APW], Appendix, we know that $F_q^P[GL_n]$ has the following presentation: it is the unital associative $\Bbbk(q)$–algebra with generators the elements of $\big\{ t_{ij} \,\big|\, i,j = 1,\dots,n \big\} \bigcup \big\{ det_q^{-1} \big\}$ and relations encoded by the requirement that $\big(t_{i,j}\big)_{i,j=1,\dots,n}$ be a q–matrix; in particular, $det_q^{\pm 1}$ belongs to the centre of $F_q^P[GL_n]$. Moreover, $F_q^P[GL_n]$ has the unique Hopf algebra structure such that $\big(t_{i,j}\big)_{i,j=1,\dots,n}$ be a Hopf q–matrix.

Similarly, $F_q^P[B_-]$ and $F_q^P[B_+]$ are defined in the same way *but* with the additional relations $t_{i,j} = 0\,(i,j = 1,\dots,n; i > j)$ for $F_q^P[B_-]$ and $t_{i,j} = 0\,(i,j = 1,\dots,n; i < j)$ for $F_q^P[B_+]$. Otherwise, we can say that $F_q^P[B_-]$, respectively $F_q^P[B_+]$, is generated by the entries of the q–matrix

$$\begin{pmatrix} t_{1,1} & 0 & \cdots & 0 & 0 \\ t_{2,1} & t_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{n-1,1} & t_{n-1,2} & \cdots & t_{n-1,n-1} & 0 \\ t_{n,1} & t_{n,2} & \cdots & t_{n,n-1} & t_{n,n} \end{pmatrix}, \text{ resp. } \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n-1} & t_{1,n} \\ 0 & t_{2,2} & \cdots & t_{2,n-1} & t_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{n-1,n-1} & t_{n-1,n} \\ 0 & 0 & \cdots & 0 & t_{n,n} \end{pmatrix}$$

and by the additional element $\big(t_{1,1}\, t_{2,2} \cdots t_{n,n}\big)^{-1}$. Moreover, both $F_q^P[B_-]$ and $F_q^P[B_+]$ are Hopf algebras, the Hopf structure being given by the assumption that their generating matrices be *Hopf q*–matrices. See also [PW] for all these definitions.

By the very definitions, the Hopf algebra epimorphisms $\pi_+ : F_q^P[GL_n] \twoheadrightarrow F_q^P[B_+]$ and $\pi_- : F_q^P[GL_n] \twoheadrightarrow F_q^P[B_-]$ mentioned in §1.2 are given by $\pi_+ : t_{ij} \mapsto t_{ij}\ (i \leq j)$, $t_{ij} \mapsto 0\ (i > j)$ and $\pi_- : t_{ij} \mapsto t_{ij}\ (i \geq j)$, $t_{ij} \mapsto 0\ (i < j)$ respectively.

2.3. The quantum algebras $U_q^M(\mathfrak{g})$, $U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_+)$ for $\mathfrak{g} = \mathfrak{gl}_n$, $M \in \{P,Q\}$. We recall (cf. for instance [GL]) the definition of the quantized universal enveloping algebra $U_q^P(\mathfrak{gl}_n)$: it is the associative algebra with 1 over $\Bbbk(q)$ with generators

$$F_1\,,\,F_2\,,\,\dots\,,\,F_{n-1}\,,\,G_1^{\pm 1}\,,\,G_2^{\pm 1}\,,\,\dots\,,\,G_{n-1}^{\pm 1}\,,\,G_n^{\pm 1}\,,\,E_1\,,\,E_2\,,\,\dots\,,\,E_{n-1}$$

and relations

$$\begin{aligned}
G_i G_i^{-1} = 1 = G_i^{-1} G_i\,, \qquad G_i^{\pm 1} G_j^{\pm 1} &= G_j^{\pm 1} G_i^{\pm 1} && \forall\, i,j \\
G_i F_j G_i^{-1} = q^{\delta_{i,j+1}-\delta_{i,j}} F_j\,, \qquad G_i E_j G_i^{-1} &= q^{\delta_{i,j}-\delta_{i,j+1}} E_j && \forall\, i,j \\
E_i F_j - F_j E_i &= \delta_{i,j} \frac{G_i G_{i+1}^{-1} - G_i^{-1} G_{i+1}}{q-q^{-1}} && \forall\, i,j
\end{aligned}$$

$$\begin{aligned}
E_i E_j = E_j E_i\,, \qquad F_i F_j &= F_j F_i && \forall\, i,j : |i-j|>1 \\
E_i^2 E_j - [2]_q E_i E_j E_i + E_j E_i^2 = 0\,, \qquad F_i^2 F_j - [2]_q F_i F_j F_i + F_j F_i^2 &= 0 && \forall\, i,j : |i-j|=1
\end{aligned}$$

with $[2]_q := q + q^{-1}$. Moreover, $U_q^P(\mathfrak{gl}_n)$ has a Hopf algebra structure given by

$$\begin{aligned}
&\Delta(F_i) = F_i \otimes G_i^{-1} G_{i+1} + 1 \otimes F_i\,, && S(F_i) = -F_i G_i G_{i+1}^{-1}\,, && \epsilon(F_i) = 0 && \forall\, i \\
&\Delta\big(G_i^{\pm 1}\big) = G_i^{\pm 1} \otimes G_i^{\pm 1}\,, && S\big(G_i^{\pm 1}\big) = G_i^{\mp 1}\,, && \epsilon\big(G_i^{\pm 1}\big) = 1 && \forall\, i \\
&\Delta(E_i) = E_i \otimes 1 + G_i G_{i+1}^{-1} \otimes E_i\,, && S(E_i) = -G_i^{-1} G_{i+1} E_i\,, && \epsilon(E_i) = 0 && \forall\, i\,.
\end{aligned}$$

The algebra $U_q^Q(\mathfrak{gl}_n)$ — defined as in [Ga1], §3 — can be realized as a Hopf subalgebra. Namely, define $L_i := G_1 G_2 \cdots G_i$, $K_j := G_j G_{j+1}^{-1}$ for all $i = 1, \dots, n$, $j = 1, \dots, n-1$. Then $U_q^Q(\mathfrak{gl}_n)$ is the $\Bbbk(q)$–subalgebra of $U_q^P(\mathfrak{gl}_n)$ generated by $\big\{F_1, \dots, F_{n-1}, K_1^{\pm 1}, \dots, K_{n-1}^{\pm 1}, L_n^{\pm 1}, E_1, \dots, E_{n-1}\big\}$. The quantum Borel subalgebra $U_q^P(\mathfrak{b}_+)$, resp. $U_q^P(\mathfrak{b}_-)$, is the subalgebra of $U_q^P(\mathfrak{gl}_n)$ generated by $\big\{G_1^{\pm 1}, \dots, G_n^{\pm 1}\big\} \cup \big\{E_1, \dots, E_{n-1}\big\}$, resp. by $\big\{G_1^{\pm 1}, \dots, G_n^{\pm 1}\big\} \cup \big\{F_1, \dots, F_{n-1}\big\}$. Similar definitions hold for $U_q^Q(\mathfrak{b}_\pm)$, but with the set $\big\{K_1^{\pm 1}, \dots, K_{n-1}^{\pm 1}, L_n^{\pm 1}\big\}$ instead of $\big\{G_1^{\pm 1}, \dots, G_n^{\pm 1}\big\}$. All these are in fact *Hopf* subalgebras.

2.4. The Hopf isomorphisms $\zeta_- \colon U_q^P(\mathfrak{b}_-) \cong F_q^P\big[B_-\big]$, $\zeta_+ \colon U_q^P(\mathfrak{b}_+) \cong F_q^P\big[B_+\big]$. The Hopf algebra isomorphisms of §1.3 are given explicitly by ($i = 1, \dots, n; j = 1, \dots, n-1$)

$$\begin{aligned}
&\zeta_- : U_q^P(\mathfrak{b}_-) \xrightarrow{\;\cong\;} F_q^P\big[B_-\big]\,, && G_i^{\pm 1} \mapsto t_{i,i}^{\mp 1}\,, && F_j \mapsto +\big(q-q^{-1}\big)^{-1} t_{j+1,j+1}^{-1}\, t_{j+1,j} \\
&\zeta_+ : U_q^P(\mathfrak{b}_+) \xrightarrow{\;\cong\;} F_q^P\big[B_+\big]\,, && G_i^{\pm 1} \mapsto t_{i,i}^{\pm 1}\,, && E_j \mapsto -\big(q-q^{-1}\big)^{-1} t_{j,j+1}\, t_{j+1,j+1}^{-1}
\end{aligned}$$

and their inverse are uniquely determined by

$$\begin{aligned}
&\zeta_-^{-1} : F_q^P\big[B_-\big] \xrightarrow{\;\cong\;} U_q^P(\mathfrak{b}_-)\,, && t_{i,i}^{\pm 1} \mapsto G_i^{\mp 1}\,, && t_{j+1,j} \mapsto +\big(q-q^{-1}\big)\, G_{j+1}^{-1}\, F_j \\
&\zeta_+^{-1} : F_q^P\big[B_+\big] \xrightarrow{\;\cong\;} U_q^P(\mathfrak{b}_+)\,, && t_{i,i}^{\pm 1} \mapsto G_i^{\pm 1}\,, && t_{j,j+1} \mapsto -\big(q-q^{-1}\big)\, E_j\, G_{j+1}^{+1}\,.
\end{aligned}$$

A straightforward computation shows that all the above are isomorphisms as claimed.

Theorem 2.5. *("short" FRT-like presentation of $U_q^{P}(\mathfrak{gl}_n)$)*

$U_q^{P}(\mathfrak{gl}_n)$ is the unital associative $\Bbbk(q)$–algebra with generators the elements of the set $\{\beta_{i,j}\}_{1\leq i\leq j\leq n} \bigcup \{\gamma_{j,i}\}_{1\leq i\leq j\leq n}$ and relations

$$\begin{aligned}\beta_{i,i+1}\,\gamma_{j+1,j} \;-\; \gamma_{j+1,j}\,\beta_{i,i+1} \;&=\; \big(\delta_{i,j+1}\left(1-q^{-1}\right) \,+\, \delta_{i,j-1}\left(1-q\right)\big)\,\beta_{i,i+1}\,\gamma_{j+1,j} \;- \\ &\qquad -\;\delta_{ij}\left(q-q^{-1}\right)\left(\alpha_i\,\alpha_{i+1}^{-1} \;-\; \alpha_i^{-1}\,\alpha_{i+1}\right)\end{aligned} \tag{2.1}$$

$$\beta_{k,k}\,\gamma_{k,k} = 1 \tag{2.2}$$

(for all $i,j=1,\dots,n-1$, $k=1,\dots,n$) plus the relations encoded in the requirement that the triangular matrices $B := \left(\beta_{ij}\right)_{i,j=1}^{n}$ and $\Gamma := \left(\gamma_{ij}\right)_{i,j=1}^{n}$ be q–matrices. Moreover, this algebra has the unique Hopf algebra structure such that these are Hopf q–matrices.

Proof. This follows directly from §1.4 and the isomorphisms in §2.4. Indeed, in the given presentation the $\beta_{h,k}$'s generate a copy of $F_q^{P}\big[B_+\big]$, with $\beta_{h,k} \cong t_{h,k}$, isomorphic to $U_q^{P}(\mathfrak{b}_+)$ via §2.4; similarly, the $\gamma_{r,s}$'s generate a copy of $F_q^{P}\big[B_-\big]$, with $\gamma_{r,s} \cong t_{r,s}$, isomorphic to $U_q^{P}(\mathfrak{b}_-)$. The additional set of "mixed" relations (2.1) involving simultaneously the $\beta_{i,i+1}$'s and the $\gamma_{j+1,j}$'s then correspond to the set of relations (1.1) — or to the third line of the set of relations in §2.3 — via the isomorphisms $\zeta_\pm$ of §2.4; indeed, these isomorphisms give $\beta_{i,i+1} \cong -\big(q-q^{-1}\big)E_i\,G_{i+1}^{+1}$, $\beta_{k,k} \cong G_k$, and $\gamma_{j+1,j} \cong +\big(q-q^{-1}\big)G_{j+1}^{-1}\,F_j$, $\gamma_{k,k} \cong G_k^{-1}$, whence computing $-\big(q-q^{-1}\big)^2\,\big[E_i\,G_{i+1}^{+1}\,, G_{j+1}^{-1}\,F_j\big]$ in $U_q^{P}(\mathfrak{gl}_n)$ we get formula (2.1). As to the Hopf structure, it is determined by that of the Hopf subalgebras $U_q^{P}(\mathfrak{b}_+)$ and $U_q^{P}(\mathfrak{b}_-)$: thus the claim follows from the previous discussion. □

2.6 Remark: note that any other commutation relation between a generator $\beta_{h,k}$ $(h<k)$ and a generator $\gamma_{r,s}$ $(r>s)$ can be deduced from the ones between the $\beta_{i,i+1}$'s and the $\gamma_{j+1,j}$'s using repeatedly the relations

$$\beta_{i,j} \;=\; \left(q-q^{-1}\right)^{-1}\left(\beta_{i,k}\,\beta_{k,j} \;-\; \beta_{k,j}\,\beta_{i,k}\right)\beta_{k,k}^{-1} \qquad\quad (\forall\ i<k<j\,)$$

which spring out of the relations $\beta_{i,k}\,\beta_{k,j}-\beta_{k,j}\,\beta_{i,k} \;=\; \big(q-q^{-1}\big)\,\beta_{k,k}\,\beta_{i,j}$ for the q–matrix B , and the relations

$$\gamma_{j,i} \;=\; \left(q-q^{-1}\right)^{-1}\left(\gamma_{k,i}\,\gamma_{j,k} \;-\; \gamma_{j,k}\,\gamma_{k,i}\right)\gamma_{k,k}^{+1} \qquad\quad (\forall\ j>k>i\,)$$

which arise from the relations $\gamma_{k,i}\,\gamma_{j,k}-\gamma_{j,k}\,\gamma_{k,i} \;=\; \big(q-q^{-1}\big)\,\gamma_{k,k}\,\gamma_{j,i}$ for the q–matrix Γ .

2.7 Quantum root vectors and L–operators. In this subsection we describe the generators of $U_q^{P}(\mathfrak{gl}_n)$ considered in Theorem 2.5 in terms of generators of the Faddeev-Reshetikhin-Takhtajan (FRT in short) presentation, the so-called L–operators, — in [FRT].

Our comparison "factors through" that with quantum root vectors built upon the Jimbo-Lusztig generators given in §2.3. For any x, y, a, let $[w,y]_a := x\,y - a\,y\,x$. Define

$$E^\pm_{i,i+1} := E_i \; , \qquad E^\pm_{i,j} := \big[\, E^\pm_{i,k} \, , E^\pm_{k,j} \,\big]_{q^{\pm 1}} \qquad \forall\; i<k<j$$
$$F^\pm_{i+1,i} := F_i \; , \qquad F^\pm_{j,i} := \big[\, F^\pm_{j,k} \, , F^\pm_{k,i} \,\big]_{q^{\mp 1}} \qquad \forall\; j>k>i$$

as in [Ji]: all these are quantum root vectors, in that in the semiclassical limit at $q=1$ they specialize to root vectors for $\mathfrak{gl}_n$, namely the elementary matrices e_{ij} with $i \neq j$. As a matter of notation, set also $\dot{E}^\pm_{i,j} := \big(q-q^{-1}\big)\, E^\pm_{i,j}$ and $\dot{F}^\pm_{j,i} := \big(q-q^{-1}\big)\, F^\pm_{j,i}$ for all $i<j$.

For the L–operators, introduced in [FRT], we recall from [No], §1.2, the formulas

$$\begin{array}{llll} L^+_{ii} := G_i^{+1} \; , & L^+_{ij} := +G_i^{+1}\, \dot{F}^+_{j,i} \; , & L^+_{j,i} := 0 & \forall\;\; i<j \\ L^-_{ii} := G_i^{-1} \; , & L^-_{ji} := -\dot{E}^+_{i,j}\, G_i^{-1} \; , & L^-_{i,j} := 0 & \forall\;\; i<j \end{array} \tag{2.3}$$

to define them; setting $L^+ := \big(L^+_{ij}\big)_{i,j=1}^n$ and $L^- := \big(L^-_{ij}\big)_{i,j=1}^n$, the relations

$$R\, L^+_1\, L^+_2 \;=\; L^+_2\, L^+_1\, R \; , \qquad R\, L^-_1\, L^-_2 \;=\; L^-_2\, L^-_1\, R \; , \qquad R\, L^+_1\, L^-_2 \;=\; L^-_2\, L^+_1\, R \tag{2.4}$$

express in compact form their mutual commutation properties (with notation as in §2.1). Indeed, the FRT presentation amounts exactly to claim that $U_q^P(\mathfrak{gl}_n)$ is the unital associative $\Bbbk(q)$–algebra with generators $L^\pm_{i,j}$ (for all $i,j=1,\dots,n$) and relations (2.4) and

$$L^+_{k,k}\, L^-_{k,k} \;=\; 1 \;=\; L^-_{k,k}\, L^+_{k,k} \qquad \forall\; k=1,\dots,n \tag{2.5}$$

and it has the unique Hopf algebra structure such that

$$\Delta\big(L^\varepsilon\big) \;=\; L^\varepsilon \,\dot{\otimes}\, L^\varepsilon \; , \qquad \epsilon\big(L^\varepsilon\big) \;=\; I \; , \qquad S\big(L^\varepsilon\big) = \big(L^\varepsilon\big)^{-1} \qquad \forall\;\; \varepsilon \in \{+\,,-\} \tag{2.6}$$

where L^+ and L^- are the upper or lower triangular matrices whose non-zero entries are the $L^+_{i,j}$'s and the $L^-_{j,i}$'s respectively, I is the $(n \times n)$–identity matrix and we use standard compact notation as in [FRT] or [CP].

Now, using the identifications $\zeta_+^{\pm 1}$ we get identities

$$\beta_{i,i} \;=\; G_i^{+1} \; , \qquad \beta_{i,j} \;=\; +(-q)^{j-i}\, G_j^{+1}\, \dot{E}^-_{i,j} \qquad\qquad \forall\;\; i<j\;. \tag{2.7}$$

Indeed, the identities $\beta_{ii} = G_i^{+1}$ and $\beta_{i,j} = -q\, G_j^{+1}\, \dot{E}^-_{i,j} = -\,\dot{E}^-_{i,j}\, G_j^{+1}$ for $j=i+1$ come out directly from the description of ζ_+^{-1} and the identifications $\beta_{i,i} \cong t_{i,i}$, $\beta_{i,i+1} \cong t_{i,i+1}$. In the other cases the result follows easily by induction on $j-i$, using the relations $\beta_{i,j} \;=\; \big(q-q^{-1}\big)^{-1} \big(\beta_{i,k}\,\beta_{k,j} \,-\, \beta_{k,j}\,\beta_{i,k}\big)\, \beta_{k,k}^{-1}$ (for $i<k<j$) given in §2.6.

Formulas (2.7) tell that the $\beta_{i,j}$'s are quantum root vectors too, for positive roots. Similarly, for negative roots the $\gamma_{j,i}$'s are involved. Namely, the identifications $\zeta_-^{\pm 1}$ yield

$$\gamma_{i,i} \;=\; G_i^{-1} \; , \qquad \gamma_{j,i} \;=\; -(-q)^{i-j}\, \dot{F}^-_{j,i}\, G_j^{-1} \qquad\qquad \forall\;\; i<j \tag{2.8}$$

which are the analogues of (2.7). Again this is proved by induction on $j-i$: the cases $j-i \leq 1$ are direct consequence of the description of ζ_-^{-1} and the identifications $\gamma_{i,i} \cong t_{i,i}$, $\gamma_{i+1,i} \cong t_{i+1,i}$, while the inductive step follow easily by means of the relations $\gamma_{j,i} = \big(q-q^{-1}\big)^{-1} \big(\gamma_{k,i}\,\gamma_{j,k} - \gamma_{j,k}\,\gamma_{k,i}\big)\,\gamma_{k,k}^{+1}$ (for $j>k>i$) given in §2.6.

In order to compare (2.3) with (2.7) and (2.8) we must be able to compare quantum root vectors with opposite superscripts. The tool is the unique $\Bbbk(q)$–algebra antiautomorphism

$$\varPsi : U_q^P(\mathfrak{gl}_n) \overset{\cong}{\hookrightarrow\!\!\!\rightarrow} U_q^P(\mathfrak{gl}_n)\,, \quad E_i \mapsto E_i\,, \quad F_i \mapsto F_i\,, \quad G_j^{\pm 1} \mapsto G_j^{\mp 1} \qquad \forall\ i,j$$

which is clearly an involution; a straightforward computation shows that

$$\varPsi\big(E_{i,j}^{\pm}\big) = (-q)^{\mp(i-j+1)}\,E_{i,j}^{\mp}\,, \qquad \varPsi\big(F_{j,i}^{\pm}\big) = (-q)^{\pm(i-j+1)}\,F_{j,i}^{\mp} \qquad \forall\ i<j\,. \tag{2.9}$$

Now comparing (2.3) with (2.7) and (2.8) using (2.9) we get

$$L_{ij}^{+} = \varPsi\big(\gamma_{j,j}^{-1}\,\gamma_{j,i}\,\gamma_{i,i}^{+1}\big)\,, \qquad L_{j\,i}^{-} = \varPsi\big(\beta_{i,i}^{+1}\,\beta_{i,j}\,\beta_{j,j}^{-1}\big) \qquad \forall\ i \leq j\,, \tag{2.10}$$

$$\gamma_{j,i} = \varPsi\big((L_{ii}^{+})^{-1}\,L_{ij}^{+}\,L_{jj}^{+}\big)\,, \qquad \beta_{i,j} = \varPsi\big(L_{jj}^{-}\,L_{ji}^{-}\,(L_{ii}^{-})^{-1}\big) \qquad \forall\ i \leq j\,. \tag{2.11}$$

2.8 Presentation of $\widetilde{U}_q^P(\mathfrak{g})$. Let again $G := GL_n$. It is well known that the $\Bbbk\big[q,q^{-1}\big]$–integer form $\widehat{F}_q^P\big[G\big]$ has the same presentation as $F_q^P\big[G\big]$, but over $\Bbbk\big[q,q^{-1}\big]$ instead of $\Bbbk(q)$. The same holds similarly for $\widehat{F}_q^P\big[B_+\big]$ and $\widehat{F}_q^P\big[B_-\big]$. In addition, $\widehat{F}_q^P\big[B_\pm\big] \cong \widetilde{U}_q^P(\mathfrak{b}_\pm)$ and $\widetilde{U}_q^P(\mathfrak{g})$ is generated by $\widetilde{U}_q^P(\mathfrak{b}_+)$ and $\widetilde{U}_q^P(\mathfrak{b}_-)$. Therefore, the previous analysis implies that $\widetilde{U}_q^P(\mathfrak{g})$ as a $\Bbbk\big[q,q^{-1}\big]$–algebra is generated by the entries of the q–matrices B and $\varGamma$ of Theorem 2.5. The latter provides explicitly some relations (over $\Bbbk\big[q,q^{-1}\big]$, that is inside $\widetilde{U}_q^P(\mathfrak{g})$ itself) among such generators, but these do *not* form a *complete* set of relations: the general mixed ones among $\beta_{i,j}$'s and $\gamma_{r,s}$'s are missing, as the ones in §2.6 do not make sense inside $\widetilde{U}_q^P(\mathfrak{g})$. However, since we know the relationship between these generators and L–operators and we do know all relations among the latter, we can eventually write down a complete set of relations for the given generators! This leads to the following presentation:

Theorem 2.9. *(FRT-like presentation of $\widetilde{U}_q^P(\mathfrak{gl}_n)$)*

$\widetilde{U}_q^P(\mathfrak{gl}_n)$ is the unital $\Bbbk\big[q,q^{-1}\big]$–algebra with generators the entries of the triangular matrices $B := \big(\beta_{ij}\big)_{i,j=1}^{n}$ and $\varGamma := \big(\gamma_{ij}\big)_{i,j=1}^{n}$ and relations

$$R\,B_2\,B_1 = B_1\,B_2\,R\,, \qquad\qquad R\,\varGamma_2\,\varGamma_1 = \varGamma_1\,\varGamma_2\,R \tag{2.12}$$

$$R^{\mathrm{op}}\,\varGamma_1^{D}\,B_2^{D} = B_2^{D}\,\varGamma_1^{D}\,R^{\mathrm{op}}\,, \qquad D_\beta \cdot D_\gamma = I = D_\gamma \cdot D_\beta \tag{2.13}$$

where $R := \sum_{i,j=1}^{n} q^{\delta_{ij}}\,e_{ii} \otimes e_{jj} + \big(q-q^{-1}\big)\sum_{1 \leq i<j \leq n} e_{ij} \otimes e_{ji}$, $X_1 := X \otimes I$, $X_2 := I \otimes X$ (like in §2.1), $R^{\mathrm{op}} := \sum_{i,j=1}^{n} q^{\delta_{ij}}\,e_{ii} \otimes e_{jj} + \big(q-q^{-1}\big)\sum_{1 \leq i<j \leq n} e_{ji} \otimes e_{ij}$, and $D_\beta := diag\big(\beta_{1,1},\dots,\beta_{n,n}\big)$, $D_\gamma := diag\big(\gamma_{1,1},\dots,\gamma_{n,n}\big)$, $B^{D} := D_\beta^{+1} \cdot B \cdot D_\beta^{-1}$, $\varGamma^{D} := D_\gamma^{-1} \cdot \varGamma \cdot D_\gamma^{+1}$.

The first (compact) relation in (2.13) above is also equivalent to

$$\sum_{i,k=1}^{n} q^{\delta_{i,k}}\,(e_{i,i}{\otimes}I)\Big(R^{\mathrm{op}}\,\varGamma_1^-\,B_2^+\Big)(I{\otimes}e_{k,k}) = \sum_{j,s=1}^{n} q^{\delta_{j,s}}\,(e_{j,j}{\otimes}I)\Big(B_2^-\,\varGamma_1^+\,R^{\mathrm{op}}\Big)(I{\otimes}e_{s,s}) \qquad (2.14)$$

where $X^{\pm} := \big(q^{\pm\delta_{h,k}}\chi_{h,k}\big)$ *for all* $X \in \{B,\varGamma\}$ *(and* $\chi \in \{\beta,\gamma\}$*), and in explicit, expanded form it is equivalent to the set of relations (for all* $i,k,j,s = 1,\dots,n$*)*

$$\begin{aligned} q^{\delta_{i,j}}\,\gamma_{i,k}\,\beta_{j,s} \;+\; \delta_{i>j}\,\big(q-q^{-1}\big)\,q^{\delta_{i,s}-\delta_{jk}}\,\gamma_{j,k}\,\beta_{i,s} \;\;&= \\ =\;\; q^{\delta_{k,s}}\,\beta_{j,s}\,\gamma_{i,k} \;+\; \delta_{s>k}\,\big(q-q^{-1}\big)\,q^{\delta_{i,s}-\delta_{jk}}&\beta_{j,k}\,\gamma_{i,s} \end{aligned} \qquad (2.15)$$

where obviously $\delta_{h>k} := 1$ *if* $h>k$ *and* $\delta_{h>k} := 0$ *if* $h \not> k$.

Furthermore, $\widetilde{U}_q^{P}(\mathfrak{gl}_n)$ *has the unique Hopf algebra structure given by*

$$\Delta(X) \,=\, X\,\dot{\otimes}\,X\;, \qquad \epsilon(X) \,=\, I\;, \qquad S(X) = X^{-1} \qquad \forall\; X \in \big\{B,\varGamma\big\}\;. \qquad (2.16)$$

Proof. The commutation formulas in (2.12) and the Hopf formulas in (2.16) are just the compact way to say that B and $\varGamma$ are Hopf q–matrices. The second half of (2.13) instead is nothing but another way of writing (2.2).

Moreover, the first half of (2.13) arises from the similar compact relation for L–operators and the link between the latter and the present generators. Indeed, merging (2.10) in the last identity in (2.4) we get

$$R\cdot\varPsi\big(D_\gamma^{-1}\varGamma^{T}D_\gamma^{+1}\big)_1\cdot\varPsi\big(D_\beta^{+1}B^{T}D_\beta^{-1}\big)_2 \;=\; \varPsi\big(D_\beta^{+1}B^{T}D_\beta^{-1}\big)_2\cdot\varPsi\big(D_\gamma^{-1}\varGamma^{T}D_\gamma^{+1}\big)_1\cdot R$$

(where a superscript T means "transpose"). Using the fact that $\varPsi$ is an algebra antiautomorphism and extending its action to $\varPsi(R) = R$ we then argue

$$\varPsi\Big(\big(D_\beta^{+1}B\,D_\beta^{-1}\big)_2\cdot\big(D_\gamma^{-1}\varGamma\,D_\gamma^{+1}\big)_1\cdot R^{\mathrm{op}}\Big) \;=\; \varPsi\Big(R^{\mathrm{op}}\cdot\big(D_\gamma^{-1}\varGamma\,D_\gamma^{+1}\big)_1\cdot\big(D_\beta^{+1}B\,D_\beta^{-1}\big)_2\Big)$$

whence eventually (2.13) follows at once because $\varPsi^2 = id$.

Finally, expanding (2.13) one gets explicitly (for all $i,k,j,s = 1,\dots,n$)

$$\begin{aligned} q^{\delta_{i,j}}\,\gamma_{i,i}^{-1}\,\gamma_{i,k}\,\gamma_{k,k}^{+1}\,\beta_{j,j}^{+1}\,\beta_{j,s}\,\beta_{s,s}^{-1} \;+\; \delta_{i>j}\,\big(q-q^{-1}\big)\,\gamma_{j,j}^{-1}\,\gamma_{j,k}\,\gamma_{k,k}^{+1}\,\beta_{i,i}^{+1}\,\beta_{i,s}\,\beta_{s,s}^{-1} \;\;= \\ =\;\; q^{\delta_{k,s}}\,\beta_{j,j}^{+1}\,\beta_{j,s}\,\beta_{s,s}^{-1}\,\gamma_{i,i}^{-1}\,\gamma_{i,k}\,\gamma_{k,k}^{+1} \;+\; \delta_{s>k}\,\big(q-q^{-1}\big)\,\beta_{j,j}^{+1}\,\beta_{j,k}\,\beta_{k,k}^{-1}\,\gamma_{i,i}^{-1}\,\gamma_{i,s}\,\gamma_{s,s}^{+1}\;. \end{aligned}$$

From this, making repeated use of all the relations encoded in (2.12) and in the second half of (2.13) one can cancel out all "diagonal" factors, i.e. those of type $\beta_{\ell,\ell}$ or $\gamma_{\ell,\ell}$. The outcome is (for all $i,k,j,s = 1,\dots,n$)

$$\begin{aligned} q^{\delta_{i,j}}\,\gamma_{i,k}\,\beta_{j,s} \;+\; \delta_{i>j}\,\big(q-q^{-1}\big)\,q^{\delta_{i,s}-\delta_{jk}}\,\gamma_{j,k}\,\beta_{i,s} \;\;&= \\ =\;\; q^{\delta_{k,s}}\,\beta_{j,s}\,\gamma_{i,k} \;+\; \delta_{s>k}\,\big(q-q^{-1}\big)\,q^{\delta_{i,s}-\delta_{jk}}&\beta_{j,k}\,\gamma_{i,s} \end{aligned}$$

that is exactly the set of relations (2.15). As a last step, manipulating a bit the exponents of q one gets (for $i,k,j,s=1,\dots,n$)

$$\begin{aligned} & q^{2\,\delta_{i,k}}\Big(q^{\delta_{i,j}}\left(q^{-\delta_{i,k}}\,\gamma_{i,k}\right)\left(q^{+\delta_{j,s}}\beta_{j,s}\right) + \delta_{i>j}\left(q-q^{-1}\right)\left(q^{-\delta_{j,k}}\,\gamma_{j,k}\right)\left(q^{+\delta_{i,s}}\beta_{i,s}\right)\Big) = \\ & = q^{2\,\delta_{j,s}}\Big(q^{\delta_{k,s}}\left(q^{-\delta_{j,s}}\beta_{j,s}\right)\left(q^{+\delta_{i,k}}\,\gamma_{i,k}\right) + \delta_{s>k}\left(q-q^{-1}\right)\left(q^{-\delta_{j,k}}\beta_{j,k}\right)\left(q^{+\delta_{i,s}}\,\gamma_{i,s}\right)\Big) \end{aligned} \tag{2.15}$$

when written in compact form yields exactly (2.14). □

Remark: the argument used to argue formulas (2.13) from the last identity in (2.4) may be also applied to the first two identities therein. This yields relations among the β_{ij}'s and among the γ_{ji}'s which are different from, but equivalent to, formulas (2.12).

Corollary 2.10. *The Poisson-Hopf $\Bbbk$–algebra $\widetilde{U}_1^{P}(\mathfrak{gl}_n)$ is the polynomial, Laurent-polynomial algebra in the variables $\big\{\overline{\beta}_{i,j}\big\}_{1\leq i\leq j\leq n}\bigcup\big\{\overline{\gamma}_{j,i}\big\}_{1\leq i\leq j\leq n}$, the $\beta_{\ell\ell}$'s and γ_{ii}'s being invertible, with relations $\beta_{ii}^{\pm 1}=\gamma_{ii}^{\mp 1}$ ($\forall\, i$), Hopf structure given (in compact notation) by*

$$\Delta\big(\overline{X}\big) = \overline{X}\dot{\otimes}\overline{X}\ , \qquad \epsilon\big(\overline{X}\big) = I\ , \qquad S\big(\overline{X}\big) = \overline{X}^{\,-1} \qquad \forall\ X\in\{B,\varGamma\}$$

(with B and $\varGamma$ as in Theorem 2.9) and with the unique Poisson structure such that

$$\begin{aligned} &\big\{\overline{x}_{i,h}\,,\overline{x}_{i,\ell}\big\} = \overline{x}_{i,h}\,\overline{x}_{i,\ell}\ ,\quad \big\{\overline{x}_{h,j}\,,\overline{x}_{\ell,j}\big\} = \overline{x}_{h,j}\,\overline{x}_{\ell,j}\ ,\quad \big\{\overline{x}_{h,h}\,,\overline{x}_{\ell,\ell}\big\} = 0 \quad (\,h<\ell\,) \\ &\big\{\overline{x}_{i,j},\overline{x}_{h,k}\big\} = 0 \quad (\,i<h\,,j>k\,)\,,\qquad \big\{\overline{x}_{i,j}\,,\overline{x}_{h,k}\big\} = 2\,\overline{x}_{i,k}\,\overline{x}_{h,j} \quad (\,i<h\,,j<k\,) \end{aligned} \tag{2.17}$$

with either all $x_{p\,q}$'s being $\beta_{p\,q}$'s (and $\beta_{p\,q}:=0$ for all $p>q$) or all $x_{p\,q}$'s being $\gamma_{p\,q}$'s (and $\gamma_{p\,q}:=0$ for all $p<q$), and

$$\big\{\overline{\beta}_{j,s}\,,\overline{\gamma}_{i,k}\big\} = \left(\delta_{i,j}-\delta_{k,s}\right)\cdot\overline{\beta}_{j,s}\,\overline{\gamma}_{i,k} + 2\,\delta_{i>j}\cdot\overline{\gamma}_{j,k}\,\overline{\beta}_{i,s} - 2\,\delta_{s>k}\cdot\overline{\beta}_{j,k}\,\overline{\gamma}_{i,s} \quad . \tag{2.18}$$

In particular $\widetilde{U}_1^{P}(\mathfrak{gl}_n)\cong F\big[(GL_n)_P^*\big]$ as Poisson Hopf algebras, where $(GL_n)_P^*$ is the algebraic group of pairs of matrices $\big(\varGamma,B\big)$ where $\varGamma$, resp. B, is a lower triangular, resp. upper triangular, invertible matrix, and the diagonals of $\varGamma$ and B are inverse to each other, with the Poisson structure dual to the Lie bialgebra of $\mathfrak{gl}_n$.

Proof. If we write $\overline{x}:=x \mod (q-1)\,\widetilde{U}_q^{P}(\mathfrak{gl}_n)$ for every $x\in\widetilde{U}_q^{P}(\mathfrak{gl}_n)$, then setting $q=1$ in the presentation of $\widetilde{U}_q^{P}(\mathfrak{gl}_n)$ of Theorem 2.9 yields a presentation for $\widetilde{U}_1^{P}(\mathfrak{gl}_n)$. The latter is a commutative, polynomial Laurent-polynomial algebra as claimed, whence $\widetilde{U}_1^{P}(\mathfrak{gl}_n)\cong F\big[(GL_n)_P^*\big]$ as algebras, via an isomorphism which for all $i\leq j$ maps $\overline{\beta_{ij}}:=\beta_{ij} \mod (q-1)\,\widetilde{U}_q^{P}(\mathfrak{gl}_n)$ to the matrix coefficient corresponding to the (i,j)–th entry of the matrix B in a pair $(\varGamma,B)$ as in the claim, and maps $\overline{\gamma_{ji}}:=\gamma_{ji} \mod (q-1)\,\widetilde{U}_q^{P}(\mathfrak{gl}_n)$ to the matrix coefficient corresponding to the (j,i)–th entry of the matrix $\varGamma$ in a pair $(\varGamma,B)$. The formulas for the Hopf structure in $\widetilde{U}_q^{P}(\mathfrak{gl}_n)$ imply that this is also an isomorphism of

Hopf algebras, for the Hopf structure on right hand side induced by the group structure of $(GL_n)_P^*$.

Since $\widetilde{U}_1^P(\mathfrak{gl}_n)$ is commutative, it inherits from $\widetilde{U}_q^P(\mathfrak{gl}_n)$ the unique Poisson bracket given by the rule $\{\overline{x}, \overline{y}\} := \dfrac{x\,y - y\,x}{q-1} \mod (q-1)\,\widetilde{U}_q^P(\mathfrak{gl}_n)$, for all $x, y \in \widetilde{U}_q^P(\mathfrak{gl}_n)$. Then the Poisson brackets in (2.18) come directly from (2.15), while all those in (2.17) spring out of the commutation formulas among the β_{ij}'s and among the γ_{ji}'s in (2.11).

Finally, checking that this Poisson structure on the algebraic group $(GL_n)_P^*$ is exactly the one dual to the Lie bialgebra structure of $\mathfrak{gl}_n$ is just a matter of bookkeeping. $\square$

2.11 The quantum Frobenius morphisms $F\big[(GL_n)_P^*\big] \cong \widetilde{U}_1^P(\mathfrak{gl}_n) \longhookrightarrow \widetilde{U}_\varepsilon^P(\mathfrak{gl}_n)$. Let $\Bbbk_\varepsilon$ be the extension of $\Bbbk$ by a primitive ℓ–th root of 1, say ε. Since $\widetilde{U}_q^P(\mathfrak{gl}_n)$ is generated by copies of $\widetilde{U}_q^P(\mathfrak{b}_+) \cong \widehat{F}_q^P\big[B_+\big]$ and $\widetilde{U}_q^P(\mathfrak{b}_-) \cong \widehat{F}_q^P\big[B_-\big]$, taking specializations the same is true for $\widetilde{U}_\varepsilon^P(\mathfrak{gl}_n)$; in particular the latter is presented like in Theorem 2.9 but with $q = \varepsilon$.

In addition, the quantum Frobenius morphisms $F\big[GL_n\big] \cong \widehat{F}_1^P\big[GL_n\big] \longhookrightarrow \widehat{F}_\varepsilon^P\big[GL_n\big]$ and $F\big[B_\pm\big] \cong \widehat{F}_1^P\big[B_\pm\big] \longhookrightarrow \widehat{F}_\varepsilon^P\big[B_\pm\big]$ have a pretty neat description, as they are given by $t_{i,j} \mapsto t_{i,j}^{\,\ell}$ (hereafter we denote by the same symbol an element in a quantum algebra and its corresponding coset after any specialization); see, for instance, [PW] for details. As we mentioned in §1.6, the morphism $F\big[(GL_n)_P^*\big] \cong \widetilde{U}_1^P(\mathfrak{gl}_n) \longhookrightarrow \widetilde{U}_\varepsilon^P(\mathfrak{gl}_n)$ is determined by its restriction to the quantum Borel subalgebras, hence to the copies of $\widehat{F}_1^P\big[B_+\big]$ and $\widehat{F}_1^P\big[B_-\big]$ which generate $\widetilde{U}_1^P(\mathfrak{gl}_n)$. When reformulated in light of Theorem 2.9, this implies

Theorem 2.12. *The quantum Frobenius morphism $F\big[(GL_n)_P^*\big] \cong \widetilde{U}_1^P(\mathfrak{gl}_n) \longhookrightarrow \widetilde{U}_\varepsilon^P(\mathfrak{gl}_n)$ is given by $\beta_{i,j} \mapsto \beta_{i,j}^{\,\ell}$, $\gamma_{j,i} \mapsto \gamma_{j,i}^{\,\ell}$, for all $i \leq j$.* $\square$

§ 3 The case of $\mathfrak{sl}_n$

3.1 From $\mathfrak{gl}_n$ to $\mathfrak{sl}_n$. In this section, we consider $\mathfrak{g} = \mathfrak{sl}_n$ and $G = SL_n$. The constructions and results of §2 about $\mathfrak{gl}_n$ essentially duplicate into the like for $\mathfrak{sl}_n$, up to minor details. In this section we shall draw these results, shortly explaining the changes in order.

First, the ideal generated by $(L_n - 1)$ in $U_q^P(\mathfrak{gl}_n)$ is a *Hopf ideal*: then we define $U_q^P(\mathfrak{sl}_n)$ as the quotient Hopf $\Bbbk(q)$–algebra $U_q^P(\mathfrak{sl}_n) := U_q^P(\mathfrak{gl}_n) \Big/ \big(L_n - 1\big)$. With like notation (see §2.3) for generators of $U_q^P(\mathfrak{gl}_n)$ and their images in $U_q^P(\mathfrak{sl}_n)$, we define $U_q^Q(\mathfrak{sl}_n)$ as the $\Bbbk(q)$–subalgebra of $U_q^P(\mathfrak{sl}_n)$ generated by $\big\{F_i, K_i^{\pm 1}, E_i\big\}_{i=1,\dots,n-1}$; this is also the image of $U_q^Q(\mathfrak{gl}_n)$ when mapping $U_q^P(\mathfrak{gl}_n)$ onto $U_q^P(\mathfrak{sl}_n)$. In this setting, $U_q^P(\mathfrak{b}_+)$, resp. $U_q^P(\mathfrak{b}_-)$, is the $\Bbbk(q)$–subalgebra of $U_q^P(\mathfrak{sl}_n)$ generated by $\big\{L_i^{\pm 1}, E_i\big\}_{i=1,\dots,n-1}$, resp. $\big\{F_i, L_i^{\pm 1}\big\}_{i=1,\dots,n-1}$, whereas $U_q^Q(\mathfrak{b}_+)$, resp. $U_q^Q(\mathfrak{b}_-)$, instead is the $\Bbbk(q)$–subalgebra of $U_q^Q(\mathfrak{sl}_n)$ generated by $\big\{K_i^{\pm 1}, E_i\big\}_{i=1,\dots,n-1}$, resp. $\big\{F_i, K_i\big\}_{i=1,\dots,n-1}$. All these are *Hopf* subalgebras of $U_q^P(\mathfrak{sl}_n)$

and $U_q^Q(\mathfrak{sl}_n)$, and Hopf algebra quotients of the similar quantum Borel algebras for $\mathfrak{gl}_n$.

In this context, we can repeat step by step the construction made for $\mathfrak{gl}_n$, up to minimal details (namely, taking into account everywhere the relation $L_n = 1$); in particular, in quantum function algebras the additional relation $t_{1,1}\, t_{2,2} \cdots t_{n,n} = 1$ has to be taken into account. Otherwise, the results for the $\mathfrak{sl}_n$ case can be immediately argued from the corresponding results for $\mathfrak{gl}_n$. The first of these results — analogue to Theorem 2.5 — is

Theorem 3.2. *("short" FRT-like presentation of $U_q^P(\mathfrak{sl}_n)$)*

$U_q^P(\mathfrak{sl}_n)$ is the quotient algebra of $U_q^P(\mathfrak{gl}_n)$ modulo the two-sided ideal I generated by $\left(\prod_{i=1}^n \beta_{ii} - 1\right)$ (or by $\left(\prod_{j=1}^n \gamma_{jj} - 1\right)$, which gives the same). Moreover, I is a Hopf ideal of $U_q^P(\mathfrak{gl}_n)$, therefore $U_q^P(\mathfrak{sl}_n)$ inherits from $U_q^P(\mathfrak{gl}_n)$ a structure of quotient Hopf algebra, given by formulas like in Theorem 2.5 (with the obvious, additional simplifications). In particular, $U_q^P(\mathfrak{sl}_n)$ has the same presentation as $U_q^P(\mathfrak{gl}_n)$ in Theorem 2.5 plus the additional relation $\beta_{1,1}\, \beta_{2,2} \cdots \beta_{n,n} = 1$, or $\gamma_{1,1}\, \gamma_{2,2} \cdots \gamma_{n,n} = 1$. □

3.3 Quantum root vectors, L–operators and new generators for $\widetilde{U}_q^P(\mathfrak{sl}_n)$. Definitions imply that the Hopf algebra epimorphism $U_q^P(\mathfrak{gl}_n) \twoheadrightarrow U_q^P(\mathfrak{sl}_n)$ maps any quantum root vector — say $E_{i,j}$ or $F_{j,i}$ — in $U_q^P(\mathfrak{gl}_n)$ onto a corresponding quantum root vector in $U_q^P(\mathfrak{sl}_n)$, for which we use the like notation. A similar result clearly holds for each L–operator — in $U_q^P(\mathfrak{gl}_n)$ — too, whose image in $U_q^P(\mathfrak{sl}_n)$ we still denote by the same symbol. The discussion in §§2.7–9 can then be repeated *verbatim*, in particular formulas (2.3) through (2.11) hold true within $U_q^P(\mathfrak{sl}_n)$ as well. The outcome then is the analogue of Theorem 2.9 — which can also be deduced directly from the latter, since $\widetilde{U}_q^P(\mathfrak{gl}_n)$ maps onto $\widetilde{U}_q^P(\mathfrak{sl}_n)$ — and its immediate corollary, namely

Theorem 3.4. *(FRT-like presentation of $\widetilde{U}_q^P(\mathfrak{sl}_n)$)*

$\widetilde{U}_q^P(\mathfrak{sl}_n)$ is the unital $\Bbbk\big[q,q^{-1}\big]$–algebra with generators the entries of the triangular matrices $B := \big(\beta_{ij}\big)_{i,j=1}^n$ and $\varGamma := \big(\gamma_{ij}\big)_{i,j=1}^n$ and relations (notations as in Theorem 2.9)

$$R\, B_2\, B_1 \; = \; B_1\, B_2\, R \;\; , \qquad\qquad R\, \varGamma_2\, \varGamma_1 \; = \; \varGamma_1\, \varGamma_2\, R \tag{3.1}$$

$$R^{\mathrm{op}}\, \varGamma_1^{D}\, B_2^{D} \; = \; B_2^{D}\, \varGamma_1^{D}\, R^{\mathrm{op}} \;\; , \qquad D_\beta \cdot D_\gamma \; = \; I \; = \; D_\gamma \cdot D_\beta \tag{3.2}$$

$$det\,(D_\beta) \; = \; 1 \; = \; det\,(D_\gamma) \tag{3.3}$$

The first (compact) relation in (2.13) above is equivalent to

$$\sum_{i,k=1}^n q^{\delta_{i,k}}\, (e_{i,i} \otimes I) \Big(R^{\mathrm{op}}\, \varGamma_1^-\, B_2^+ \Big) (I \otimes e_{k,k}) = \sum_{j,s=1}^n q^{\delta_{j,s}}\, (e_{j,j} \otimes I) \Big(B_2^-\, \varGamma_1^+\, R^{\mathrm{op}} \Big) (I \otimes e_{s,s}) \tag{3.4}$$

and in expanded form it is equivalent to the set of relations (for all $i,k,j,s = 1,\dots,n$)

$$\begin{aligned} q^{\delta_{i,j}}\, \gamma_{i,k}\, \beta_{j,s} \; + \; \delta_{i>j}\, \big(q - q^{-1}\big)\, q^{\delta_{i,s} - \delta_{jk}}\, \gamma_{j,k}\, \beta_{i,s} \;\; &= \\ = \;\; q^{\delta_{k,s}}\, \beta_{j,s}\, \gamma_{i,k} \; + \; \delta_{s>k}\, \big(q - q^{-1}\big)\, q^{\delta_{i,s} - \delta_{jk}} \beta_{j,k}\, \gamma_{i,s} \quad . \end{aligned} \tag{3.5}$$

Furthermore, $\widetilde{U}_q^P(\mathfrak{sl}_n)$ *has the unique Hopf algebra structure given by*

$$\Delta(X) = X \dot{\otimes} X \; , \qquad \epsilon(X) = I \; , \qquad S(X) = X^{-1} \qquad \forall \; X \in \{B, \Gamma\} \; . \quad \square \tag{3.6}$$

Corollary 3.5. *The Poisson-Hopf* $\Bbbk$*–algebra* $\widetilde{U}_1^P(\mathfrak{sl}_n)$ *is the polynomial algebra in the variables* $\{\overline{\beta}_{i,j}\}_{1 \leq i \leq j \leq n} \bigcup \{\overline{\gamma}_{j,i}\}_{1 \leq i \leq j \leq n}$ *modulo the relations* $\overline{\beta}_{1,1} \overline{\beta}_{2,2} \cdots \overline{\beta}_{n,n} = 1$, $\overline{\gamma}_{1,1} \overline{\gamma}_{2,2} \cdots \overline{\gamma}_{n,n} = 1$, $\overline{\beta}_{i,i} \overline{\gamma}_{i,1i} = 1$ *(for all* $i = 1, \dots n$*), Hopf structure given by*

$$\Delta\big(\overline{X}\big) = \overline{X} \dot{\otimes} \overline{X} \; , \qquad \epsilon\big(\overline{X}\big) = I \; , \qquad S\big(\overline{X}\big) = \overline{X}^{\,-1} \qquad \forall \; X \in \{B, \Gamma\}$$

(with B *and* Γ *as in Theorem 3.4) and with the unique Poisson structure such that*

$$\begin{aligned} &\{\overline{x}_{i,h} \,, \overline{x}_{i,\ell}\} = \overline{x}_{i,h} \, \overline{x}_{i,\ell} \; , \quad \{\overline{x}_{h,j} \,, \overline{x}_{\ell,j}\} = \overline{x}_{h,j} \, \overline{x}_{\ell,j} \; , \quad \{\overline{x}_{h,h} \,, \overline{x}_{\ell,\ell}\} = 0 \quad (h < \ell) \\ &\{\overline{x}_{i,j}, \overline{x}_{h,k}\} = 0 \quad (i < h \,, j > k) \; , \qquad \{\overline{x}_{i,j} \,, \overline{x}_{h,k}\} = 2 \, \overline{x}_{i,k} \, \overline{x}_{h,j} \quad (i < h \,, j < k) \end{aligned} \tag{3.7}$$

with either all $x_{p\,q}$*'s being* $\beta_{p\,q}$*'s (and* $\beta_{p\,q} := 0$ *for all* $p > q$*) or all* $x_{p\,q}$*'s being* $\gamma_{p\,q}$*'s (and* $\gamma_{p\,q} := 0$ *for all* $p < q$*), and*

$$\{\overline{\beta}_{j,s} \,, \overline{\gamma}_{i,k}\} = (\delta_{i,j} - \delta_{k,s}) \cdot \overline{\beta}_{j,s} \, \overline{\gamma}_{i,k} + 2 \, \delta_{i>j} \cdot \overline{\gamma}_{j,k} \, \overline{\beta}_{i,s} - 2 \, \delta_{s>k} \cdot \overline{\beta}_{j,k} \, \overline{\gamma}_{i,s} \quad . \tag{3.8}$$

In particular $\widetilde{U}_1^P(\mathfrak{sl}_n) \cong F\big[(SL_n)_P^*\big]$ *as Poisson Hopf algebras, where* $(SL_n)_P^*$ *is the algebraic group of pairs of matrices* $\big(\Gamma, B\big)$ *where* Γ*, resp.* B*, is a lower, resp. upper, triangular matrix with determinant equal to 1, and the diagonals of* Γ *and* B *are inverse to each other, with the Poisson structure dual to the Lie bialgebra of* $\mathfrak{sl}_n$ *.* $\square$

3.7 The quantum Frobenius morphisms $F\big[(SL_n)_P^*\big] \cong \widetilde{U}_1^P(\mathfrak{sl}_n) \hookrightarrow \widetilde{U}_\varepsilon^P(\mathfrak{sl}_n)$ **.** Once again, for quantum Frobenius morphisms one can repeat *verbatim* the discussion made for $U_q^P(\mathfrak{gl}_n)$ for the case of $U_q^P(\mathfrak{sl}_n)$ as well, via minimal changes where needed. Otherwise, the results in the $\mathfrak{gl}_n$ case induce similar results in the $\mathfrak{sl}_n$ case via the defining epimorphism $U_q^P(\mathfrak{gl}_n) \twoheadrightarrow U_q^P(\mathfrak{sl}_n)$. Indeed, the latter is clearly compatible (in the obvious sense) with specializations at roots of 1; therefore, the specializations of the epimorphism itself yield the following commutative diagram

$$\begin{array}{ccc} F\big[(GL_n)_P^*\big] \cong \widetilde{U}_1^P(\mathfrak{gl}_n) & \longrightarrow & \widetilde{U}_\varepsilon^P(\mathfrak{gl}_n) \\ \big\downarrow & & \big\downarrow \\ F\big[(SL_n)_P^*\big] \cong \widetilde{U}_1^P(\mathfrak{sl}_n) & \longrightarrow & \widetilde{U}_\varepsilon^P(\mathfrak{sl}_n) \end{array}$$

(for ε any root of 1) in which the vertical arrows are the above mentioned specialized epimorphisms and the horizontal ones are the quantum Frobenius (mono)morphisms.

This yields at once the following analogue of Theorem 2.12:

Theorem 3.8. *The quantum Frobenius morphism* $F\big[(SL_n)^*_P\big] \cong \widetilde{U}^P_1(\mathfrak{sl}_n) \lhook\joinrel\longrightarrow \widetilde{U}^P_\varepsilon(\mathfrak{sl}_n)$ *is given by* $\beta_{i,j} \mapsto \beta_{i,j}^{\,\ell}$, $\gamma_{j,i} \mapsto \gamma_{j,i}^{\,\ell}$, *for all* $i \leq j$. $\square$

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