

## POISSON GEOMETRICAL SYMMETRIES ASSOCIATED TO NON-COMMUTATIVE FORMAL DIFFEOMORPHISMS

FABIO GAVARINI

Università degli Studi di Roma “Tor Vergata” — Dipartimento di Matematica  
Via della Ricerca Scientifica 1, I-00133 Roma — ITALY

ABSTRACT. Let  $\mathcal{G}^{\text{dif}}$  be the group of all formal power series starting with  $x$  with coefficients in a field  $\mathbb{k}$  of zero characteristic (with the composition product), and let  $F[\mathcal{G}^{\text{dif}}]$  be its function algebra. In [BF] a non-commutative, non-cocommutative graded Hopf algebra  $\mathcal{H}^{\text{dif}}$  was introduced via a direct process of “disabelianisation” of  $F[\mathcal{G}^{\text{dif}}]$ , taking the like presentation of the latter as an algebra but dropping the commutativity constraint. In this paper we apply a general method to provide four one-parameters deformations of  $\mathcal{H}^{\text{dif}}$ , which are quantum groups whose semiclassical limits are Poisson geometrical symmetries such as Poisson groups or Lie bialgebras, namely two quantum function algebras and two quantum universal enveloping algebras. In particular the two Poisson groups are extensions of  $\mathcal{G}^{\text{dif}}$ , isomorphic as proalgebraic Poisson varieties but not as proalgebraic groups.

*“A series of outlaws joined and formed the Nottingham group,  
whose renowned chieftain was the famous Robin Hopf”*

*N. Barbecue, “Robin Hopf”*

### Introduction

The most general notion of “symmetry” in mathematics is encoded in the notion of Hopf algebra. Then, among all Hopf algebras (over a field  $\mathbb{k}$ ), there are two special families which are of relevant interest for their geometrical meaning: assuming for simplicity that  $\mathbb{k}$  have zero characteristic, these are the function algebras  $F[G]$  of algebraic groups  $G$  and the universal enveloping algebras  $U(\mathfrak{g})$  of Lie algebras  $\mathfrak{g}$ . Function algebras are exactly those Hopf algebras which are commutative, and enveloping algebras those which are connected (in the general sense of Hopf algebra theory) and cocommutative.

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Given a Hopf algebra  $H$ , encoding some generalized symmetry, one can ask whether there are any other Hopf algebras “close” to  $H$ , which are of either one of the above mentioned *geometrical types*, hence encoding geometrical symmetries associated to  $H$ . The answer is affirmative: namely (see [Ga4]), it is possible to give functorial recipes to get out of any Hopf algebra  $H$  two pairs of Hopf algebras of geometrical type, say  $(F[G_+], U(\mathfrak{g}_-))$  and  $(F[K_+], U(\mathfrak{k}_-))$ . Moreover, the algebraic groups thus obtained are connected *Poisson groups*, and the Lie algebras are *Lie bialgebras*; therefore in both cases *Poisson geometry* is involved. In addition, the two pairs above are related to each other by Poisson duality (see below), thus only either one of them is truly relevant. Finally, these four “geometrical” Hopf algebras are “close” to  $H$  in that they are 1-parameter deformations (with pairwise isomorphic fibers) of a quotient or a subalgebra of  $H$ .

The method above to associate Poisson geometrical Hopf algebras to general Hopf algebras, called “Crystal Duality Principle” (CDP in short), is explained in detail in [Ga4]. It is a special instance of a more general result, the “Global Quantum Duality Principle” (GQDP in short), explained in [Ga2–3], which in turn is a generalization of the “Quantum Duality Principle” due to Drinfeld (cf. [Dr], §7, and see [Ga1] for a proof).

Drinfeld’s QDP deals with quantum universal enveloping algebras (QUEAs in short) and quantum formal series Hopf algebras (QFSHAs in short) over the ring of formal power series  $\mathbb{k}[[\hbar]]$ . A QUEA is any topologically free, topological Hopf  $\mathbb{k}[[\hbar]]$ -algebra whose quotient modulo  $\hbar$  is the universal enveloping algebra  $U(\mathfrak{g})$  of some Lie algebra  $\mathfrak{g}$ ; in this case we denote the QUEA by  $U_{\hbar}(\mathfrak{g})$ . Instead, a QFSHA is any topological Hopf  $\mathbb{k}[[\hbar]]$ -algebra of type  $\mathbb{k}[[\hbar]]^S$  (as a  $\mathbb{k}[[\hbar]]$ -module,  $S$  being a set) whose quotient modulo  $\hbar$  is the function algebra  $F[[G]]$  of some formal algebraic group  $G$ ; then we denote the QFSHA by  $F_{\hbar}[[G]]$ .

The QDP claims that the category of all QUEAs and the category of all QFSHAs are equivalent, and provides an equivalence in either direction. From QFSHAs to QUEAs it goes as follows: given a QFSHA, say  $F_{\hbar}[[G]]$ , let  $J$  be its augmentation ideal (the kernel of its counit map) and set  $F_{\hbar}[[G]]^{\vee} := \sum_{n \geq 0} \hbar^{-n} J^n$ . Then  $F_{\hbar}[[G]] \mapsto F_{\hbar}[[G]]^{\vee}$  defines (on objects) a functor from QFSHAs to QUEAs. To go the other way round, i.e. from QUEAs to QFSHAs, one uses a perfectly dual recipe. Namely, given a QUEA, say  $U_{\hbar}(\mathfrak{g})$ , let again  $J$  be its augmentation ideal; for each  $n \in \mathbb{N}$ , let  $\delta_n$  be the composition of the  $n$ -fold iterated coproduct followed by the projection onto  $J^{\otimes n}$  (this makes sense since  $U_{\hbar}(\mathfrak{g}) = \mathbb{k}[[\hbar]] \cdot 1_{U_{\hbar}(\mathfrak{g})} \oplus J$ ): then set  $U_{\hbar}(\mathfrak{g})' := \bigcap_{n \geq 0} \delta_n^{-1}(\hbar^n U_{\hbar}(\mathfrak{g})^{\otimes n})$ , or more explicitly  $U_{\hbar}(\mathfrak{g})' := \{ \eta \in U_{\hbar}(\mathfrak{g}) \mid \delta_n(\eta) \in \hbar^n U_{\hbar}(\mathfrak{g})^{\otimes n}, \forall n \in \mathbb{N} \}$ . Then  $U_{\hbar}(\mathfrak{g}) \mapsto U_{\hbar}(\mathfrak{g})'$  defines (on objects) a functor from QUEAs to QFSHAs. The functors  $( )^{\vee}$  and  $( )'$  are inverse to each other, hence they provide the claimed equivalence.

Note that the objects (QUEAs and QFSHAs) involved in the QDP are *quantum groups*; their semiclassical limits then are endowed with Poisson structures: namely, every  $U(\mathfrak{g})$  is in fact a co-Poisson Hopf algebra and every  $F[[G]]$  is a (topological) Poisson Hopf algebra.

The geometrical structures they describe are then *Lie bialgebras* and *Poisson groups*. The QDP then brings further information: namely, the semiclassical limit of the image of a given quantum group is *Poisson dual* to the Poisson geometrical object we start from. In short

$$F_{\hbar}[[G]]^{\vee} / \hbar F_{\hbar}[[G]]^{\vee} = U(\mathfrak{g}^{\times}), \quad \text{i.e. (roughly)} \quad F_{\hbar}[[G]]^{\vee} = U_{\hbar}(\mathfrak{g}^{\times}) \quad (\text{I.1})$$

where  $\mathfrak{g}^{\times}$  is the *cotangent Lie bialgebra* of the Poisson group  $G$ , and

$$U_{\hbar}(\mathfrak{g})' / \hbar U_{\hbar}(\mathfrak{g})' = F[[G^{\star}]], \quad \text{i.e. (roughly)} \quad U_{\hbar}(\mathfrak{g})' = F_{\hbar}[[G^{\star}]] \quad (\text{I.2})$$

where  $G^{\star}$  is a connected Poisson group with cotangent Lie bialgebra  $\mathfrak{g}$ . So the QDP involves both Hopf duality (switching enveloping and function algebras) and Poisson duality.

The generalization from QDP to GQDP stems from a simple observation: the construction of Drinfeld’s functors needs not to start from quantum groups! Indeed, in order to define either  $H^{\vee}$  or  $H'$  one only needs that  $H$  be a torsion-free Hopf algebra over some 1-dimensional domain  $R$  and  $\hbar \in R$  be *any non-zero prime* (actually, even less is truly necessary, see [Ga2–3]). On the other hand, the outcome still is, in both cases, a “quantum group”, now meant in a new sense. Namely, a QUEA now will be any torsion-free Hopf algebra  $H$  over  $R$  such that  $H / \hbar H \cong U(\mathfrak{g})$ , for some Lie (bi)algebra  $\mathfrak{g}$ . Also, instead of QFSHAs we consider “quantum function algebras”, QFAs in short: here a QFA will be any torsion-free Hopf algebra  $H$  over  $R$  such that  $H / \hbar H \cong F[G]$  (plus one additional technical condition) for some connected (Poisson) group  $G$ . In this new framework Drinfeld’s recipes give that  $H^{\vee}$  is a QUEA and  $H'$  is a QFA, whatever is the torsion-free Hopf  $R$ -algebra  $H$  one starts from. Moreover, when restricted to quantum groups Drinfeld’s functors  $( )^{\vee}$  and  $( )'$  again provide equivalences of quantum group categories, respectively from QFAs to QUEAs and viceversa; then Poisson duality is involved once more, like in (I.1–2).

Therefore, the generalization process from the QDP to the GQDP spreads over several concerns. Arithmetically, one can take as  $(\hbar)$  any non-generic point of the spectrum of  $R$ , and define Drinfeld’s functors and specializations accordingly; in particular, the corresponding quotient field  $\mathbb{k}_{\hbar} := R / \hbar R$  might have *positive* characteristic. Geometrically, one considers algebraic groups rather than formal groups, i.e. *global vs. local* objects. Algebraically, one drops any topological worry ( $\hbar$ -adic completeness, etc.), and deals with *general* Hopf algebras rather than with quantum groups. This last point is the one of most concern to us now, in that it means that we have (functorial) recipes to get several quantum groups, hence — taking semiclassical limits — Poisson geometrical symmetries, springing out of the “generalized symmetry” encoded by a torsion-free Hopf algebra  $H$  over  $R$ : namely, for each non-trivial point of the spectrum of  $R$ , the quantum groups  $H^{\vee}$  and  $H'$  given by the corresponding Drinfeld’s functors. Note, however, that *a priori* nothing prevents any of these  $H^{\vee}$  or  $H'$  or their semiclassical limits to be (essentially) trivial.

The CDP comes out when looking at Hopf algebras over a field  $\mathbb{k}$ , and then applying the GQDP to their scalar extensions  $H[\hbar] := \mathbb{k}[\hbar] \otimes_{\mathbb{k}} H$  with  $R := \mathbb{k}[\hbar]$  (and  $\hbar := \hbar$  itself). A first application of Drinfeld's functors to  $H_{\hbar} := H[\hbar]$  followed by specialization at  $\hbar = 0$  provides the pair  $(F[G_+], U(\mathfrak{g}_-))$  mentioned above: in a nutshell,  $(F[G_+], U(\mathfrak{g}_-)) = (H_{\hbar}'|_{\hbar=0}, H_{\hbar}^{\vee}|_{\hbar=0})$ , where hereafter  $X|_{\hbar=0} := X/\hbar X$ . Then applying once more Drinfeld's functors to  $H_{\hbar}^{\vee}$  and to  $H_{\hbar}'$  and specializing at  $\hbar = 0$  yields the pair  $(F[K_+], U(\mathfrak{k}_-))$ , namely  $(F[K_+], U(\mathfrak{k}_-)) = ((H_{\hbar}^{\vee})'|_{\hbar=0}, (H_{\hbar}')^{\vee}|_{\hbar=0})$ . Finally, the very last part of the GQDP explained before implies that  $K_+ = G_+^*$  and  $\mathfrak{k}_- = \mathfrak{g}_+^{\times}$ .

While in the second step above one really needs the full strength of the GQDP, for the first step instead it turns out that the construction of Drinfeld's functors on  $H[\hbar]$ , can be fully "tracked through" and described at the "classical level", i.e. in terms of  $H$  alone. In addition, the exact relationship among  $H$  and the pair  $(F[G_+], U(\mathfrak{g}_-))$  can be made quite clear, and more information is available about this pair. We now sketch it in some detail.

Let  $J$  be the augmentation ideal of  $H$ , let  $\underline{J} := \{J^n\}_{n \in \mathbb{N}}$  be the associated (decreasing)  $J$ -adic filtration,  $\widehat{H} := G_{\underline{J}}(H)$  the associated graded vector space and  $H^{\vee} := H / \bigcap_{n \in \mathbb{N}} J^n$ . One can prove that  $\underline{J}$  is a Hopf algebra filtration, hence  $\widehat{H}$  is a graded Hopf algebra. The latter happens to be connected and cocommutative, so  $\widehat{H} \cong U(\mathfrak{g}_-)$  for some Lie algebra  $\mathfrak{g}_-$ ; in addition, since  $\widehat{H}$  is graded also  $\mathfrak{g}_-$  itself is graded as a Lie algebra. The fact that  $\widehat{H}$  be cocommutative allows to define on it a Poisson cobracket which makes  $\widehat{H}$  into a graded *co-Poisson* Hopf algebra; eventually, this implies that  $\mathfrak{g}_-$  is a *Lie bialgebra*. The outcome is that our  $U(\mathfrak{g}_-)$  is just  $\widehat{H}$ .

On the other hand, one considers a second (increasing) filtration defined in a dual manner to  $\underline{J}$ , namely  $\underline{D} := \{D_n := \text{Ker}(\delta_{n+1})\}_{n \in \mathbb{N}}$ . Let now  $\widetilde{H} := G_{\underline{D}}(H)$  be the associated graded vector space and  $H' := \bigcup_{n \in \mathbb{N}} D_n$ . Again, one shows that  $\underline{D}$  is a Hopf algebra filtration, hence  $\widetilde{H}$  is a graded Hopf algebra. Moreover, the latter is commutative, so  $\widetilde{H} = F[G_+]$  for some algebraic group  $G_+$ . One proves also that  $\widetilde{H} = F[G_+]$  has no non-trivial idempotents, thus  $G_+$  is connected; in addition, since  $\widetilde{H}$  is graded,  $G_+$  as a variety is just an affine space. The fact that  $\widetilde{H}$  be commutative allows to define on it a Poisson bracket which makes  $\widetilde{H}$  into a graded *Poisson* Hopf algebra: this means that  $G_+$  is an algebraic *Poisson group*. Thus eventually  $F[G_+]$  is just  $\widetilde{H}$ .

The relationship among  $H$  and the "geometrical" Hopf algebras  $\widehat{H}$  and  $\widetilde{H}$  can be expressed in terms of "reduction steps" and regular 1-parameter deformations, namely

$$\widetilde{H} \xleftarrow[0 \leftarrow \hbar \rightarrow 1]{\mathcal{R}_{\underline{D}}^{\hbar}(H)} H' \longleftrightarrow H \longrightarrow H^{\vee} \xleftarrow[1 \leftarrow \hbar \rightarrow 0]{\mathcal{R}_{\underline{J}}^{\hbar}(H^{\vee})} \widehat{H} \quad (\text{I.3})$$

where one-way arrows are Hopf algebra morphisms and two-ways arrows are regular 1-parameter deformations of Hopf algebras, realized through the Rees Hopf algebras  $\mathcal{R}_{\underline{D}}^{\hbar}(H)$  and  $\mathcal{R}_{\underline{J}}^{\hbar}(H^{\vee})$  associated to the filtration  $\underline{D}$  of  $H$  and to the filtration  $\underline{J}$  of  $H^{\vee}$ . Hereafter "regular" for a deformation means that all its fibers are pairwise isomorphic as vector spaces.

In classical terms, (I.3) comes directly from the construction above; on the other hand, in terms of the GQDP it comes from the fact that  $\mathcal{R}_{\underline{D}}^{\hbar}(H) = H_{\hbar}'$  and  $\mathcal{R}_{\underline{J}}^{\hbar}(H^{\vee}) = H_{\hbar}^{\vee}$ .

As we mentioned above, next step is the “application” of (suitable) Drinfeld’s functors to the Rees algebras  $\mathcal{R}_{\underline{D}}^{\hbar}(H) = H_{\hbar}'$  and  $\mathcal{R}_{\underline{J}}^{\hbar}(H^{\vee}) = H_{\hbar}^{\vee}$  occurring in (I.3). The outcome is a second frame of regular 1-parameter deformations for  $H'$  and  $H^{\vee}$ , namely

$$U(\mathfrak{g}_+^{\times}) = U(\mathfrak{k}_-) \xleftarrow[(H_{\hbar}')^{\vee}]{0 \leftarrow \hbar \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^{\vee} \xleftarrow[(H_{\hbar}^{\vee})']{1 \leftarrow \hbar \rightarrow 0} F[K_+] = F[G_-^{\star}] \quad (\text{I.4})$$

which is the analogue of (I.3). In particular, when  $H^{\vee} = H = H'$  from (I.3) and (I.4) together we find  $H$  as the mid-point of four deformation families, whose “external points” are Hopf algebras of “Poisson geometrical” type, namely

$$\begin{array}{ccccc} U(\mathfrak{g}_-) & \xleftarrow[H_{\hbar}^{\vee}]{0 \leftarrow \hbar \rightarrow 1} & H & \xleftarrow[(H_{\hbar}^{\vee})']{1 \leftarrow \hbar \rightarrow 0} & F[G_-^{\star}] \\ & & \parallel & & \\ F[G_+] & \xleftarrow[H_{\hbar}']{0 \leftarrow \hbar \rightarrow 1} & H & \xleftarrow[(H_{\hbar}')^{\vee}]{1 \leftarrow \hbar \rightarrow 0} & U(\mathfrak{g}_+^{\times}) \end{array} \quad (\text{I.5})$$

which gives *four* different regular 1-parameter deformations from  $H$  to Hopf algebras encoding Poisson geometrical objects. Then each of these four Hopf algebras may be thought of as a semiclassical geometrical counterpart of the “generalized symmetry” encoded by  $H$ .

The purpose of the present paper is to show the effectiveness of the CDP, applying it to a key example, the Hopf algebra of non-commutative formal diffeomorphisms of the line. Indeed, the interest of the latter, besides its own reasons, grows bigger as we can see it as a toy model for a broad family of Hopf algebras of great concern in mathematical physics, non-commutative geometry and beyond. Now I go and present the results of this paper.

Let  $\mathcal{G}^{\text{dif}}$  be the set of all formal power series starting with  $x$  with coefficients in a field  $\mathbb{k}$  of zero characteristic. Endowed with the composition product, this is an infinite dimensional pronipotent proalgebraic group — known as the “(normalised) Nottingham group” among group-theorists and the “(normalised) group of formal diffeomorphisms of the line” among mathematical physicists — whose tangent Lie algebra is a special subalgebra of the one-sided Witt algebra. The function algebra  $F[\mathcal{G}^{\text{dif}}]$  is a graded, commutative Hopf algebra with countably many generators, which admits a neat combinatorial description.

In [BF] a non-commutative version of  $F[\mathcal{G}^{\text{dif}}]$  is introduced: this is a non-commutative non-cocommutative Hopf algebra  $\mathcal{H}^{\text{dif}}$  which is presented exactly like  $F[\mathcal{G}^{\text{dif}}]$  but dropping commutativity, i.e. taking the presentation as one of a unital *associative* — and *not* commutative — algebra; in other words,  $\mathcal{H}^{\text{dif}}$  is the outcome of applying to  $F[\mathcal{G}^{\text{dif}}]$  a raw “disabelianization” process. In particular,  $H = \mathcal{H}^{\text{dif}}$  is *graded* and verifies  $H^{\vee} = H = H'$ , hence the scheme (I.5) makes sense and yields four Poisson symmetries associated to  $\mathcal{H}^{\text{dif}}$ .

Note that in each line in (I.5) there is essentially only *one* Poisson geometry involved, since Poisson duality relates mutually opposite sides; thus any classical symmetry on the

same line carries as much information as the other one (but for global-to-local differences). Nevertheless, in the case of  $H = \mathcal{H}^{\text{dif}}$  we shall prove that the pieces of information from either line in  $(\boxtimes)$  are complementary, because  $G_+$  and  $G_-^*$  happen to be isomorphic as proalgebraic Poisson varieties but *not* as groups. In particular, we find that the Lie bialgebras  $\mathfrak{g}_-$  and  $\mathfrak{g}_+^\times$  are both isomorphic as Lie algebras to the free Lie algebra  $\mathcal{L}(\mathbb{N}_+)$  over a countable set, but they have different, non-isomorphic Lie coalgebra structures. Moreover,  $G_-^* \cong \mathcal{G}^{\text{dif}} \times \mathcal{N} \cong G_+$  as Poisson varieties, where  $\mathcal{N}$  is a proaffine Poisson variety whose coordinate functions are in bijection with a basis of the derived subalgebra  $\mathcal{L}(\mathbb{N}_+)$ ; indeed, the latter are obtained by iterated Poisson brackets of coordinate functions on  $\mathcal{G}^{\text{dif}}$ , in short because both  $F[G_-^*]$  and  $F[G_+]$  are freely generated as Poisson algebras by a copy of  $F[\mathcal{G}^{\text{dif}}]$ . For  $G_-^*$  we have a more precise result, namely  $G_-^* \cong \mathcal{G}^{\text{dif}} \ltimes \mathcal{N}$  (a semidirect product) as *proalgebraic groups*: thus in a sense  $G_-^*$  is the *free Poisson group over  $\mathcal{G}^{\text{dif}}$* , which geometrically speaking is obtained by “pasting” to  $\mathcal{G}^{\text{dif}}$  all 1-parameter subgroups freely obtained via iterated Poisson brackets of those of  $\mathcal{G}^{\text{dif}}$ ; in particular, these Poisson brackets iteratively yield 1-parameter subgroups which generate  $\mathcal{N}$ .

We perform the same analysis simultaneously for  $\mathcal{G}^{\text{dif}}$ , for its subgroup of *odd* formal diffeomorphisms and for all the groups  $\mathcal{G}_\nu$  of *truncated* (at order  $\nu \in \mathbb{N}_+$ ) formal diffeomorphisms, whose projective limit is  $\mathcal{G}^{\text{dif}}$  itself; *mutatis mutandis*, the results are the like.

The case of  $\mathcal{H}^{\text{dif}}$  is just one of many samples of the same type: indeed, several cases of Hopf algebras built out of combinatorial data — graphs, trees, Feynman diagrams, etc. — have been introduced in (co)homological theories (see e.g. [LR] and [Fo1–2], and references therein) and in renormalization studies (see [CK1–3]); in most cases these algebras — or their (graded) duals — are commutative polynomial, like  $F[\mathcal{G}^{\text{dif}}]$ , and admit non-commutative analogues (thanks to [Fo1–2]), so our discussion apply almost *verbatim* to them too, with like results. Thus the given analysis of the “toy model” Hopf algebra  $\mathcal{H}^{\text{dif}}$  can be taken as a general pattern for all those cases.

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## § 1 Notation and terminology

**1.1 The classical data.** Let  $\mathbb{k}$  be a fixed field of zero characteristic.

Consider the set  $\mathcal{G}^{\text{dif}} := \{ x + \sum_{n \geq 1} a_n x^{n+1} \mid a_n \in \mathbb{k} \forall n \in \mathbb{N}_+ \}$  of all formal series starting with  $x$ : endowed with the composition product, this is a group, which can be seen as the group of all “formal diffeomorphisms”  $f: \mathbb{k} \rightarrow \mathbb{k}$  such that  $f(0) = 0$  and  $f'(0) = 1$  (i.e. tangent to the identity), also known as the *Nottingham group* (see, e.g., [Ca] and references therein). In fact,  $\mathcal{G}^{\text{dif}}$  is an infinite dimensional (pro)affine algebraic group, whose

function algebra  $F[\mathcal{G}^{\text{dif}}]$  is generated by the coordinate functions  $a_n$  ( $n \in \mathbb{N}_+$ ). Giving to each  $a_n$  the weight<sup>1</sup>  $\partial(a_n) := n$ , we have that  $F[\mathcal{G}^{\text{dif}}]$  is an  $\mathbb{N}$ -graded Hopf algebra, with polynomial structure  $F[\mathcal{G}^{\text{dif}}] = \mathbb{k}[a_1, a_2, \dots, a_n, \dots]$  and Hopf algebra structure given by

$$\begin{aligned} \Delta(a_n) &= a_n \otimes 1 + 1 \otimes a_n + \sum_{m=1}^{n-1} a_m \otimes Q_{n-m}^m(a_*), & \epsilon(a_n) &= 0 \\ S(a_n) &= -a_n - \sum_{m=1}^{n-1} a_m S(Q_{n-m}^m(a_*)) = -a_n - \sum_{m=1}^{n-1} S(a_m) Q_{n-m}^m(a_*) \end{aligned}$$

where  $Q_t^\ell(a_*) := \sum_{k=1}^t \binom{\ell+1}{k} P_t^{(k)}(a_*)$  and  $P_t^{(k)}(a_*) := \sum_{\substack{j_1, \dots, j_k > 0 \\ j_1 + \dots + j_k = t}} a_{j_1} \cdots a_{j_k}$  (the symmetric monic polynomial of weight  $m$  and degree  $k$  in the indeterminates  $a_j$ 's) for all  $m, k, \ell \in \mathbb{N}_+$ , and the formula for  $S(a_n)$  gives the antipode by recursion. From now on, to simplify notation we shall write  $\mathcal{G} := \mathcal{G}^{\text{dif}}$  and  $\mathcal{G}_\infty := \mathcal{G} = \mathcal{G}^{\text{dif}}$ . Note also that the tangent Lie algebra of  $\mathcal{G}^{\text{dif}}$  is just the Lie subalgebra  $W_1^{\geq 1} = \text{Span}(\{d_n \mid n \in \mathbb{N}_+\})$  of the one-sided Witt algebra  $W_1 := \text{Der}(\mathbb{k}[t]) = \text{Span}(\{d_n := t^{n+1} \frac{d}{dt} \mid n \in \mathbb{N} \cup \{-1\}\})$ .

In addition, for all  $\nu \in \mathbb{N}_+$  the subset  $\mathcal{G}^\nu := \{f \in \mathcal{G} \mid a_n(f) = 0, \forall n \leq \nu\}$  is a normal subgroup of  $\mathcal{G}$ ; the corresponding quotient group  $\mathcal{G}_\nu := \mathcal{G}/\mathcal{G}^\nu$  is unipotent, with dimension  $\nu$  and function algebra  $F[\mathcal{G}_\nu]$  (isomorphic to) the Hopf subalgebra of  $F[\mathcal{G}]$  generated by  $a_1, \dots, a_\nu$ . In fact, the  $\mathcal{G}^\nu$ 's form exactly the lower central series of  $\mathcal{G}$  (cf. [Je]). Moreover,  $\mathcal{G}$  is (isomorphic to) the inverse (or projective) limit of these quotient groups  $\mathcal{G}_\nu$  ( $\nu \in \mathbb{N}_+$ ), hence  $\mathcal{G}$  is pro-unipotent; conversely,  $F[\mathcal{G}]$  is the direct (or inductive) limit of the direct system of its graded Hopf subalgebras  $F[\mathcal{G}_\nu]$  ( $\nu \in \mathbb{N}_+$ ). Finally, the set  $\mathcal{G}^{\text{odd}} := \{f \in \mathcal{G}^{\text{dif}} \mid a_{2n-1}(f) = 0 \forall n \in \mathbb{N}_+\}$  is another normal subgroup of  $\mathcal{G}^{\text{dif}}$  (the group of *odd* formal diffeomorphisms<sup>2</sup> after [CK3]), whose function algebra  $F[\mathcal{G}^{\text{odd}}]$  is (isomorphic to) the quotient Hopf algebra  $F[\mathcal{G}^{\text{dif}}] / \left( \{a_{2n-1}\}_{n \in \mathbb{N}_+} \right)$ . The latter has the following description: denoting again the cosets of the  $a_{2n}$ 's with the like symbol, we have  $F[\mathcal{G}^{\text{odd}}] = \mathbb{k}[a_2, a_4, \dots, a_{2n}, \dots]$  with Hopf algebra structure

$$\begin{aligned} \Delta(a_{2n}) &= a_{2n} \otimes 1 + 1 \otimes a_{2n} + \sum_{m=1}^{n-1} a_{2m} \otimes \bar{Q}_{n-m}^m(a_{2*}), & \epsilon(a_{2n}) &= 0 \\ S(a_{2n}) &= -a_{2n} - \sum_{m=1}^{n-1} a_{2m} S(\bar{Q}_{n-m}^m(a_{2*})) = -a_{2n} - \sum_{m=1}^{n-1} S(a_{2m}) \bar{Q}_{n-m}^m(a_{2*}) \end{aligned}$$

where  $\bar{Q}_t^\ell(a_{2*}) := \sum_{k=1}^t \binom{2\ell+1}{k} \bar{P}_t^{(k)}(a_{2*})$  and  $\bar{P}_t^{(k)}(a_{2*}) := \sum_{\substack{j_1, \dots, j_k > 0 \\ j_1 + \dots + j_k = t}} a_{2j_1} \cdots a_{2j_k}$  for all  $m, k, \ell \in \mathbb{N}_+$ . For each  $\nu \in \mathbb{N}_+$  we can consider also the normal subgroup  $\mathcal{G}^\nu \cap \mathcal{G}^{\text{odd}}$  and the corresponding quotient  $\mathcal{G}_\nu^{\text{odd}} := \mathcal{G}^{\text{odd}} / (\mathcal{G}^\nu \cap \mathcal{G}^{\text{odd}})$ : then  $F[\mathcal{G}_\nu^{\text{odd}}]$  is (isomorphic to) the quotient Hopf algebra  $F[\mathcal{G}^{\text{odd}}] / \left( \{a_{2n-1}\}_{(2n-1) \in \mathbb{N}_\nu} \right)$ , in particular it is the Hopf subalgebra of  $F[\mathcal{G}^{\text{odd}}]$  generated by  $a_2, \dots, a_{2[\nu/2]}$ . All the  $F[\mathcal{G}_\nu^{\text{odd}}]$ 's are graded Hopf (sub)algebras forming a direct system with direct limit  $F[\mathcal{G}^{\text{odd}}]$ ; conversely, the  $\mathcal{G}_\nu^{\text{odd}}$ 's form an inverse system with inverse limit  $\mathcal{G}^{\text{odd}}$ . In the sequel we write  $\mathcal{G}^+ := \mathcal{G}^{\text{odd}}$  and  $\mathcal{G}_\nu^+ := \mathcal{G}_\nu^{\text{odd}}$ .

<sup>1</sup>We say *weight* instead of *degree* because we save the latter term for the degree of polynomials.

<sup>2</sup>The fixed-point set of the group homomorphism  $\Phi : \mathcal{G} \rightarrow \mathcal{G}, f \mapsto \Phi(f)$  ( $x \mapsto (\Phi(f))(x) := -f(-x)$ )

For each  $\nu \in \mathbb{N}_+$ , set  $\mathbb{N}_\nu := \{1, \dots, \nu\}$ ; set also  $\mathbb{N}_\infty := \mathbb{N}_+$ . For each  $\nu \in \mathbb{N}_+ \cup \{\infty\}$ , let  $\mathcal{L}_\nu = \mathcal{L}(\mathbb{N}_\nu)$  be the free Lie algebra over  $\mathbb{k}$  generated by  $\{x_n\}_{n \in \mathbb{N}_\nu}$  and let  $U_\nu = U(\mathcal{L}_\nu)$  be its universal enveloping algebra; let also  $V_\nu = V(\mathbb{N}_\nu)$  be the  $\mathbb{k}$ -vector space with basis  $\{x_n\}_{n \in \mathbb{N}_\nu}$ , and let  $T_\nu = T(V_\nu)$  be its associated tensor algebra. Then there are canonical identifications  $U(\mathcal{L}_\nu) = T(V_\nu) = \mathbb{k}\langle\{x_n \mid n \in \mathbb{N}_\nu\}\rangle$ , the latter being the unital  $\mathbb{k}$ -algebra of non-commutative polynomials in the set of indeterminates  $\{x_n\}_{n \in \mathbb{N}_\nu}$ , and  $\mathcal{L}_\nu$  is just the Lie subalgebra of  $U_\nu = T_\nu$  generated by  $\{x_n\}_{n \in \mathbb{N}_\nu}$ . Moreover,  $\mathcal{L}_\nu$  has a basis  $B_\nu$  made of Lie monomials in the  $x_n$ 's ( $n \in \mathbb{N}_\nu$ ), like  $[x_{n_1}, x_{n_2}]$ ,  $[[x_{n_1}, x_{n_2}], x_{n_3}]$ ,  $[[[x_{n_1}, x_{n_2}], x_{n_3}], x_{n_4}]$ , etc.: details can be found e.g. in [Re], Ch. 4–5. In the sequel I shall use these identifications with no further mention. We consider on  $U(\mathcal{L}_\nu)$  the standard Hopf algebra structure given by  $\Delta(x) = x \otimes 1 + 1 \otimes x$ ,  $\epsilon(x) = 0$ ,  $S(x) = -x$  for all  $x \in \mathcal{L}_\nu$ , which is also determined by the same formulas for  $x \in \{x_n\}_{n \in \mathbb{N}_\nu}$  alone. By construction  $\nu \leq \mu$  implies  $\mathcal{L}_\nu \subseteq \mathcal{L}_\mu$ , whence the  $\mathcal{L}_\nu$ 's form a direct system (of Lie algebras) whose direct limit is exactly  $\mathcal{L}_\infty$ ; similarly,  $U(\mathcal{L}_\infty)$  is the direct limit of all the  $U(\mathcal{L}_\nu)$ 's. Finally, with  $\mathbb{B}_\nu$  we shall mean the obvious PBW-like basis of  $U(\mathcal{L}_\nu)$  w.r.t. some fixed total order  $\preceq$  of  $B_\nu$ , namely  $\mathbb{B}_\nu := \{x_{\underline{b}} \mid \underline{b} = b_1 \cdots b_k; b_1, \dots, b_k \in B_\nu; b_1 \preceq \cdots \preceq b_k\}$ . The same construction applies to make out “odd” objects, based on  $\{x_n\}_{n \in \mathbb{N}_\nu^+}$ , with  $\mathbb{N}_\nu^+ := \mathbb{N}_\nu \cap 2\mathbb{N}$  ( $\nu \in \mathbb{N} \cup \{\infty\}$ ), instead of  $\{x_n\}_{n \in \mathbb{N}_\nu}$ ,  $\mathcal{L}_\nu^+ = \mathcal{L}(\mathbb{N}_\nu^+)$ ,  $U_\nu^+ = U(\mathcal{L}_\nu^+)$ ,  $V_\nu^+ = V(\mathbb{N}_\nu^+)$ ,  $T_\nu^+ = T(V_\nu^+)$ , with the obvious canonical identifications  $U(\mathcal{L}_\nu^+) = T(V_\nu^+) = \mathbb{k}\langle\{x_n \mid n \in \mathbb{N}_\nu^+\}\rangle$ ; moreover,  $\mathcal{L}_\nu^+$  has a basis  $B_\nu^+$  made of Lie monomials in the  $x_n$ 's ( $n \in \mathbb{N}_\nu^+$ ), etc. The  $\mathcal{L}_\nu^+$ 's form a direct system whose direct limit is  $\mathcal{L}_\infty^+$ , and  $U(\mathcal{L}_\infty^+)$  is the direct limit of all the  $U(\mathcal{L}_\nu^+)$ 's.

*Warning:* in the sequel, we shall often deal with subsets  $\{\mathbf{y}_b\}_{b \in B_\nu}$  (of some algebra) in bijection with  $B_\nu$ , the fixed basis of  $\mathcal{L}_\nu$ . Then we shall write things like  $\mathbf{y}_\lambda$  with  $\lambda \in \mathcal{L}_\nu$ : this means we extend the bijection  $\{\mathbf{y}_b\}_{b \in B_\nu} \cong B_\nu$  to  $\text{Span}(\{\mathbf{y}_b\}_{b \in B_\nu}) \cong \mathcal{L}_\nu$  by linearity, so that  $\mathbf{y}_\lambda \cong \sum_{b \in B_\nu} c_b b$  iff  $\lambda = \sum_{b \in B_\nu} c_b b$  ( $c_b \in \mathbb{k}$ ). The same kind of convention will be applied with  $B_\nu^+$  instead of  $B_\nu$  and  $\mathcal{L}_\nu^+$  instead of  $\mathcal{L}_\nu$ .

**1.2 The noncommutative Hopf algebra of formal diffeomorphisms.** For all  $\nu \in \mathbb{N}_+ \cup \{\infty\}$ , let  $\mathcal{H}_\nu$  be the Hopf  $\mathbb{k}$ -algebra given as follows: as a  $\mathbb{k}$ -algebra it is simply  $\mathcal{H}_\nu := \mathbb{k}\langle\{\mathbf{a}_n \mid n \in \mathbb{N}_\nu\}\rangle$  (the  $\mathbb{k}$ -algebra of non-commutative polynomials in the set of indeterminates  $\{\mathbf{a}_n\}_{n \in \mathbb{N}_\nu}$ ), and its Hopf algebra structure is given by (for all  $n \in \mathbb{N}_\nu$ )

$$\begin{aligned} \Delta(\mathbf{a}_n) &= \mathbf{a}_n \otimes 1 + 1 \otimes \mathbf{a}_n + \sum_{m=1}^{n-1} \mathbf{a}_m \otimes Q_{n-m}^m(\mathbf{a}_*), & \epsilon(\mathbf{a}_n) &= 0 \\ S(\mathbf{a}_n) &= -\mathbf{a}_n - \sum_{m=1}^{n-1} \mathbf{a}_m S(Q_{n-m}^m(\mathbf{a}_*)) = -\mathbf{a}_n - \sum_{m=1}^{n-1} S(\mathbf{a}_m) Q_{n-m}^m(\mathbf{a}_*) \end{aligned} \quad (1.1)$$

(notation like in §1.1) where the latter formula yields the antipode by recursion. Moreover,  $\mathcal{H}_\nu$  is in fact an  $\mathbb{N}$ -graded Hopf algebra, once generators have been given degree — in the sequel called *weight* — by the rule  $\partial(\mathbf{a}_n) := n$  (for all  $n \in \mathbb{N}_\nu$ ). By construction the various  $\mathcal{H}_\nu$ 's (for all  $\nu \in \mathbb{N}_+$ ) form a direct system, whose direct limit is  $\mathcal{H}_\infty$ : the latter



was originally introduced<sup>3</sup> in [BF], §5.1 (with  $\mathbb{k} = \mathbb{C}$ ), under the name  $\mathcal{H}^{\text{dif}}$ .

Similarly, for all  $\nu \in \mathbb{N}_+ \cup \{\infty\}$  we set  $\mathcal{K}_\nu := \mathbb{k}\langle\{\mathbf{a}_n \mid n \in \mathbb{N}_\nu^+\}\rangle$  (where  $\mathbb{N}_\nu^+ := \mathbb{N}_\nu \cap (2\mathbb{N})$ ): this bears a Hopf algebra structure given by (for all  $2n, 2m+1 \in \mathbb{N}_\nu^+$ )

$$\begin{aligned} \Delta(\mathbf{a}_{2n}) &= \mathbf{a}_{2n} \otimes 1 + 1 \otimes \mathbf{a}_{2n} + \sum_{m=1}^{n-1} \mathbf{a}_{2m} \otimes \bar{Q}_{n-m}^m(\mathbf{a}_{2*}), & \epsilon(\mathbf{a}_{2n}) &= 0 \\ S(\mathbf{a}_{2n}) &= -\mathbf{a}_{2n} - \sum_{m=1}^{n-1} \mathbf{a}_{2m} S(\bar{Q}_{n-m}^m(\mathbf{a}_{2*})) = -\mathbf{a}_{2n} - \sum_{m=1}^{n-1} S(\mathbf{a}_{2m}) Q_{n-m}^m(\mathbf{a}_{2*}) \end{aligned}$$

(notation of §1.1). Indeed, this is an  $\mathbb{N}$ -graded Hopf algebra where generators have degree — called *weight* — given by  $\partial(\mathbf{a}_n) := n$  (for all  $n \in \mathbb{N}_\nu^+$ ). All the  $\mathcal{K}_\nu$ 's form a direct system with direct limit  $\mathcal{K}_\infty$ . Finally, for each  $\nu \in \mathbb{N}_\nu^+$  there is a graded Hopf algebra epimorphism  $\mathcal{H}_\nu \twoheadrightarrow \mathcal{K}_\nu$  given by  $\mathbf{a}_{2n} \mapsto \mathbf{a}_{2n}$ ,  $\mathbf{a}_{2m+1} \mapsto 0$  for all  $2n, 2m+1 \in \mathbb{N}_\nu$ .

Definitions and §1.1 imply that

$$(\mathcal{H}_\nu)_{ab} := \mathcal{H}_\nu / ([\mathcal{H}_\nu, \mathcal{H}_\nu]) \cong F[\mathcal{G}_\nu], \quad \text{via} \quad \mathbf{a}_n \mapsto a_n \quad \forall n \in \mathbb{N}_\nu$$

as  $\mathbb{N}$ -graded Hopf algebras: in other words, the abelianization of  $\mathcal{H}_\nu$  is nothing but  $F[\mathcal{G}_\nu]$ . Thus in a sense one can think at  $\mathcal{H}_\nu$  as a *non-commutative version* (indeed, the “coarsest” one) of  $F[\mathcal{G}_\nu]$ , hence as a “quantization” of  $\mathcal{G}_\nu$  itself: however, this is *not* a quantization in the usual sense, because  $F[\mathcal{G}_\nu]$  is attained through abelianization, *not* via specialization of some deformation parameter. Similarly we have also

$$(\mathcal{K}_\nu)_{ab} := \mathcal{K}_\nu / ([\mathcal{K}_\nu, \mathcal{K}_\nu]) \cong F[\mathcal{G}_\nu^+], \quad \text{via} \quad \mathbf{a}_{2n} \mapsto a_{2n} \quad \forall 2n \in \mathbb{N}_\nu^+$$

as  $\mathbb{N}$ -graded Hopf algebras: in other words, the abelianization of  $\mathcal{K}_\nu$  is just  $F[\mathcal{G}_\nu^+]$ .

In the following I make the analysis explicit for  $\mathcal{H}_\nu$ , the case  $\mathcal{K}_\nu$  being the like (details are left to the reader); I drop the subscript  $\nu$ , which stands fixed, and write  $\mathcal{H} := \mathcal{H}_\nu$ .

**1.3 Deformations.** Let  $\hbar$  be an indeterminate. In this paper we shall consider several Hopf algebras over  $\mathbb{k}[\hbar]$ , which can also be seen as 1-parameter depending families of Hopf algebras over  $\mathbb{k}$ , the parameter being  $\mathbb{k}$ ; each  $\mathbb{k}$ -algebra in such a family can then be thought of as a 1-parameter deformation of any other object in the same family. As a matter of notation, if  $H$  is such a Hopf  $\mathbb{k}[\hbar]$ -algebra I call *fibre* of  $H$  (though of as a deformation) any Hopf  $\mathbb{k}$ -algebra of type  $H/p(\hbar)H$  for some irreducible  $p(\hbar) \in \mathbb{k}[\hbar]$ ; in particular  $H|_{\hbar=c} := H/(\hbar-c)H$ , for any  $c \in \mathbb{k}$ , is called *specialization* of  $H$  at  $\hbar=c$ .

We start from  $\mathcal{H}_\hbar := \mathcal{H}[\hbar] \equiv \mathbb{k}[\hbar] \otimes_{\mathbb{k}} \mathcal{H}$ : this is indeed a Hopf  $\mathbb{k}[\hbar]$ -algebra, namely  $\mathcal{H}_\hbar = \mathbb{k}[\hbar]\langle\{\mathbf{a}_n \mid n \in \mathbb{N}_\nu\}\rangle$  with Hopf structure given by (1.1) again. Set also  $\mathcal{H}(\hbar) := \mathbb{k}(\hbar) \otimes_{\mathbb{k}[\hbar]} \mathcal{H}_\hbar = \mathbb{k}(\hbar) \otimes_{\mathbb{k}} \mathcal{H} = \mathbb{k}(\hbar)\langle\{\mathbf{a}_n \mid n \in \mathbb{N}_\nu\}\rangle$ , a Hopf  $\mathbb{k}(\hbar)$ -algebra ruled by (1.1) too.

<sup>3</sup>However, the formulas in [BF] give the *opposite* coproduct, hence change the antipode accordingly; we made the present choice to make these formulas “fit well” with those for  $F[\mathcal{G}^{\text{dif}}]$  (see below).

## § 2 The Rees deformation $\mathcal{H}_{\hbar}^{\vee}$ .

**2.1 The goal.** The crystal duality principle (cf. [Ga2], §5, or [Ga4]) yields a recipe to produce a 1-parameter deformation  $\mathcal{H}_{\hbar}^{\vee}$  of  $\mathcal{H}$  which is a *quantized universal enveloping algebra* (QUEA in the sequel): namely,  $\mathcal{H}_{\hbar}^{\vee}$  is a Hopf  $\mathbb{k}[\hbar]$ -algebra such that  $\mathcal{H}_{\hbar}^{\vee}\big|_{\hbar=1} = \mathcal{H}$  and  $\mathcal{H}_{\hbar}^{\vee}\big|_{\hbar=0} = U(\mathfrak{g}_-)$ , the universal enveloping algebra of a graded Lie bialgebra  $\mathfrak{g}_-$ . Thus  $\mathcal{H}_{\hbar}^{\vee}$  is a *quantization* of  $U(\mathfrak{g}_-)$ , and the quantum symmetry  $\mathcal{H}$  is a deformation of the classical Poisson symmetry  $U(\mathfrak{g}_-)$ . By definition  $\mathcal{H}_{\hbar}^{\vee}$  is the *Rees algebra* associated to a distinguished *decreasing* Hopf algebra filtration of  $\mathcal{H}$ , so that  $U(\mathfrak{g}_-)$  is just the graded Hopf algebra associated to this filtration. The purpose of this section is to describe explicitly  $\mathcal{H}_{\hbar}^{\vee}$  and its semiclassical limit  $U(\mathfrak{g}_-)$ , hence also  $\mathfrak{g}_-$  itself. This will also provide a direct, independent proof of all the above mentioned results about  $\mathcal{H}_{\hbar}^{\vee}$  and  $U(\mathfrak{g}_-)$  themselves.

**2.2 The Rees algebra  $\mathcal{H}_{\hbar}^{\vee}$ .** Let  $J := \text{Ker}(\epsilon_{\mathcal{H}} : \mathcal{H} \rightarrow \mathbb{k})$  be the augmentation ideal of  $\mathcal{H}$ , and let  $\underline{J} := \{J^n\}_{n \in \mathbb{N}}$  be the  $J$ -adic filtration in  $\mathcal{H}$ . It is easy to show (see [Ga4]) that  $\underline{J}$  is a Hopf algebra filtration of  $\mathcal{H}$ ; since  $\mathcal{H}$  is graded connected we have  $J = \mathcal{H}_+ := \bigoplus_{n \in \mathbb{N}} \mathcal{H}_{(n)}$  (where  $\mathcal{H}_{(n)}$  is the  $n$ -th homogeneous component of  $\mathcal{H}$ ), whence  $\bigcap_{n \in \mathbb{N}} J^n = \{0\}$  and  $\mathcal{H}^{\vee} := \mathcal{H} / \bigcap_{n \in \mathbb{N}} J^n \cong \mathcal{H}$ . We let the *Rees algebra* associated to  $\underline{J}$  be

$$\mathcal{H}_{\hbar}^{\vee} := \mathbb{k}[\hbar] \cdot \sum_{n \geq 0} \hbar^{-n} J^n = \sum_{n \geq 0} \mathbb{k}[\hbar] \hbar^{-n} \cdot J^n = \sum_{n \geq 0} \mathbb{k}[\hbar] (\hbar^{-1} \cdot J)^n \quad (\subseteq \mathcal{H}(\hbar)). \quad (2.1)$$

Letting  $J_{\hbar} := \text{Ker}(\epsilon_{\mathcal{H}_{\hbar}} : \mathcal{H}_{\hbar} \rightarrow \mathbb{k}[\hbar]) = \mathbb{k}[\hbar] \cdot J$  (the augmentation ideal of  $\mathcal{H}_{\hbar}$ ) one has

$$\mathcal{H}_{\hbar}^{\vee} = \sum_{n \geq 0} \hbar^{-n} J_{\hbar}^n = \sum_{n \geq 0} (\hbar^{-1} J_{\hbar})^n \quad (\subseteq \mathcal{H}(\hbar)).$$

For all  $n \in \mathbb{N}_{\nu}$ , set  $\mathbf{x}_n := \hbar^{-1} \mathbf{a}_n$ ; clearly  $\mathcal{H}_{\hbar}^{\vee}$  is the  $\mathbb{k}[\hbar]$ -subalgebra of  $\mathcal{H}(\hbar)$  generated by  $J^{\vee} := \hbar^{-1} J$ , hence by  $\{\mathbf{x}_n\}_{n \in \mathbb{N}_{\nu}}$ , so  $\mathcal{H}_{\hbar}^{\vee} = \mathbb{k}[\hbar] \langle \{\mathbf{x}_n \mid n \in \mathbb{N}_{\nu}\} \rangle$ . Moreover,

$$\begin{aligned} \Delta(\mathbf{x}_n) &= \mathbf{x}_n \otimes 1 + 1 \otimes \mathbf{x}_n + \sum_{m=1}^{n-1} \sum_{k=1}^m \hbar^k \binom{n-m+1}{k} \mathbf{x}_{n-m} \otimes P_m^{(k)}(\mathbf{x}_*), \quad \epsilon(\mathbf{x}_n) = 0 \\ S(\mathbf{x}_n) &= -\mathbf{x}_n - \sum_{m=1}^{n-1} \sum_{k=1}^m \hbar^k \binom{n-m+1}{k} \mathbf{x}_{n-m} S(P_m^{(k)}(\mathbf{x}_*)) = \\ &= -\mathbf{x}_n - \sum_{m=1}^{n-1} \sum_{k=1}^m \hbar^k \binom{n-m+1}{k} S(\mathbf{x}_{n-m}) P_m^{(k)}(\mathbf{x}_*) \end{aligned} \quad (2.2)$$

for all  $n \in \mathbb{N}_{\nu}$ , due to (1.1). From this one sees by hands that the following holds:

**Proposition 2.1.** *Formulas (2.2) make  $\mathcal{H}_{\hbar}^{\vee} = \mathbb{k}[\hbar] \langle \{\mathbf{x}_n, \mid n \in \mathbb{N}_{\nu}\} \rangle$  into a graded Hopf  $\mathbb{k}[\hbar]$ -algebra, embedded into  $\mathcal{H}(\hbar) := \mathbb{k}(\hbar) \otimes_{\mathbb{k}} \mathcal{H}$  as a graded Hopf subalgebra. Moreover,  $\mathcal{H}_{\hbar}^{\vee}$  is a deformation of  $\mathcal{H}$ , for its specialization at  $\hbar = 1$  is isomorphic to  $\mathcal{H}$ , i.e.*

$$\mathcal{H}_{\hbar}^{\vee} \Big|_{\hbar=1} := \mathcal{H}_{\hbar}^{\vee} / (\hbar-1) \mathcal{H}_{\hbar}^{\vee} \cong \mathcal{H} \quad \text{via} \quad \mathbf{x}_n \bmod (\hbar-1) \mathcal{H}_{\hbar}^{\vee} \mapsto \mathbf{a}_n \quad (\forall n \in \mathbb{N}_{\nu})$$

as graded Hopf algebras over  $\mathbb{k}$ .  $\square$

*Remark:* the previous result shows that  $\mathcal{H}_\hbar$  is a deformation of  $\mathcal{H}$ , which is recovered as specialization (of  $\mathcal{H}_\hbar$ ) at  $\hbar = 1$ . Next result instead shows that  $\mathcal{H}_\hbar$  is also a deformation of  $U(\mathcal{L}_\nu)$ , recovered as specialization at  $\hbar = 0$ . Altogether, this gives the top-left horizontal arrow in the frame  $(\boxtimes)$  in the Introduction for  $H = \mathcal{H} := \mathcal{H}_\nu$ , with  $\mathfrak{g}_- = \mathcal{L}_\nu$ .

**Theorem 2.1.**  $\mathcal{H}_\hbar^\vee$  is a QUEA at  $\hbar = 0$ . Namely, the specialization limit of  $\mathcal{H}_\hbar^\vee$  at  $\hbar = 0$  is  $\mathcal{H}_\hbar^\vee \Big|_{\hbar=0} := \mathcal{H}_\hbar^\vee / \hbar \mathcal{H}_\hbar^\vee \cong U(\mathcal{L}_\nu)$  via  $\mathbf{x}_n \bmod \hbar \mathcal{H}_\hbar^\vee \mapsto x_n$  for all  $n \in \mathbb{N}_\nu$ , thus inducing on  $U(\mathcal{L}_\nu)$  the structure of co-Poisson Hopf algebra uniquely provided by the Lie bialgebra structure on  $\mathcal{L}_\nu$  given by  $\delta(x_n) = \sum_{\ell=1}^{n-1} (\ell+1) x_\ell \wedge x_{n-\ell}$  (for all  $n \in \mathbb{N}_\nu$ )<sup>4</sup>. In particular in the diagram  $(\boxtimes)$  for  $H = \mathcal{H} (= \mathcal{H}_\nu)$  we have  $\mathfrak{g}_- = \mathcal{L}_\nu$ .

Finally, the grading  $d$  given by  $d(x_n) := 1$  ( $n \in \mathbb{N}_+$ ) makes  $\mathcal{H}_\hbar^\vee \Big|_{\hbar=0} \cong U(\mathcal{L}_\nu)$  into a graded co-Poisson Hopf algebra; similarly, the grading  $\partial$  given by  $\partial(x_n) := n$  ( $n \in \mathbb{N}_+$ ) makes  $\mathcal{H}_\hbar^\vee \Big|_{\hbar=0} \cong U(\mathcal{L}_\nu)$  into a graded Hopf algebra and  $\mathcal{L}_\nu$  into a graded Lie bialgebra.

*Proof.* First observe that since  $\mathcal{H}_\hbar^\vee = \mathbb{k}[\hbar] \langle \{ \mathbf{x}_n \mid n \in \mathbb{N}_\nu \} \rangle$  and  $U(\mathcal{L}_\nu) = T(V_\nu) = \mathbb{k} \langle \{ x_n \mid n \in \mathbb{N}_\nu \} \rangle$  mapping  $\mathbf{x}_n \bmod \hbar \mathcal{H}_\hbar^\vee \mapsto x_n$  ( $\forall n \in \mathbb{N}_\nu$ ) does really define an isomorphism of algebras  $\Phi: \mathcal{H}_\hbar^\vee / \hbar \mathcal{H}_\hbar^\vee \cong U(\mathcal{L}_\nu)$ . Second, formulas (2.2) give

$$\begin{aligned} \Delta(\mathbf{x}_n) &\equiv \mathbf{x}_n \otimes 1 + 1 \otimes \mathbf{x}_n \bmod \hbar (\mathcal{H}_\hbar^\vee \otimes \mathcal{H}_\hbar^\vee) \\ \epsilon(\mathbf{x}_n) &\equiv 0 \bmod \hbar \mathbb{k}[\hbar], \quad S(\mathbf{x}_n) \equiv -\mathbf{x}_n \bmod \hbar \mathcal{H}_\hbar^\vee \end{aligned}$$

for all  $n \in \mathbb{N}_\nu$ ; comparing with the standard Hopf structure of  $U(\mathcal{L}_\nu)$  this shows that  $\Phi$  is an isomorphism of Hopf algebras too. Finally, as  $\mathcal{H}_\hbar^\vee \Big|_{\hbar=0}$  is cocommutative, a Poisson co-bracket is defined on it by the standard recipe used in quantum group theory, namely

$$\begin{aligned} \delta(x_n) &:= (\hbar^{-1} (\Delta(\mathbf{x}_n) - \Delta^{\text{op}}(\mathbf{x}_n))) \bmod \hbar (\mathcal{H}_\hbar^\vee \otimes \mathcal{H}_\hbar^\vee) = \\ &= \sum_{m=1}^{n-1} \binom{n-m+1}{1} x_{n-m} \wedge P_m^{(1)}(x_*) = \sum_{\ell=1}^{n-1} (\ell+1) x_\ell \wedge x_{n-\ell} \quad \forall n \in \mathbb{N}_\nu. \quad \square \end{aligned}$$

### § 3 The Drinfeld's deformation $(\mathcal{H}_\hbar^\vee)'$ .

**3.1 The goal.** The second step in the crystal duality principle is to build a second deformation basing upon the Rees deformation  $\mathcal{H}_\hbar^\vee$ . This will be a new Hopf  $\mathbb{k}[\hbar]$ -algebra  $(\mathcal{H}_\hbar^\vee)'$ , contained in  $\mathcal{H}_\hbar^\vee$ , which for  $\hbar = 1$  specializes to  $\mathcal{H}$  and for  $\hbar = 0$  specializes to  $F[K_+]$ , the function algebra of some connected Poisson group  $K_+$ ; in other words,  $(\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=1} = \mathcal{H}$  and  $(\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=0} = F[K_+]$ , the latter meaning that  $(\mathcal{H}_\hbar^\vee)'$  is a *quantized*

<sup>4</sup>Hereafter, I use notation  $a \wedge b := a \otimes b - b \otimes a$ .

function algebra (QFA in the sequel). Therefore  $(\mathcal{H}_{\hbar}^{\vee})'$  is a *quantization* of  $F[K_+]$ , and the quantum symmetry  $\mathcal{H}$  is a deformation of the classical Poisson symmetry  $F[K_+]$ .

In addition, the general theory also describes the relationship between  $K_+$  and the Lie bialgebra  $\mathfrak{g}_- = \mathcal{L}_\nu$  in §2.1, which is  $\mathcal{L}_\nu = \text{coLie}(K_+)$ , so that we can write  $K_+ = G_{\mathcal{L}_\nu}^*$ . Comparing with §2.1, one eventually concludes that the quantum symmetry encoded by  $\mathcal{H}$  is intermediate between the two classical, Poisson symmetries ruled by  $G_{\mathcal{L}_\nu}^*$  and  $\mathcal{L}_\nu$ .

In this section I describe explicitly  $(\mathcal{H}_{\hbar}^{\vee})'$  and its semiclassical limit  $F[G_-]$ , hence  $G_-$  itself too. This yields a direct proof of all above mentioned results about  $(\mathcal{H}_{\hbar}^{\vee})'$  and  $G_-$ .

**3.2 Drinfeld's  $\delta_\bullet$ -maps.** Let  $H$  be any Hopf algebra (over a ring  $R$ ). For every  $n \in \mathbb{N}$ , define the iterated coproduct  $\Delta^n: H \rightarrow H^{\otimes n}$  by  $\Delta^0 := \epsilon$ ,  $\Delta^1 := \text{id}_C$ , and finally  $\Delta^n := (\Delta \otimes \text{id}_C^{\otimes(n-2)}) \circ \Delta^{n-1}$  if  $n > 2$ . For any ordered subset  $\Phi = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$  with  $i_1 < \dots < i_k$ , define the linear map  $j_\Phi: H^{\otimes k} \rightarrow H^{\otimes n}$  by  $j_\Phi(a_1 \otimes \dots \otimes a_k) := b_1 \otimes \dots \otimes b_n$  with  $b_i := \mathbb{1}$  if  $i \notin \Phi$  and  $b_{i_m} := a_m$  for  $1 \leq m \leq k$ ; then set  $\Delta_\Phi := j_\Phi \circ \Delta^k$ ,  $\Delta_\emptyset := \Delta^0$ , and  $\delta_\Phi := \sum_{\Psi \subseteq \Phi} (-1)^{n-|\Psi|} \Delta_\Psi$ ,  $\delta_\emptyset := \epsilon$ . The inverse formula  $\Delta_\Phi = \sum_{\Psi \subseteq \Phi} \delta_\Psi$  also holds. We shall also use the shorthand notation  $\delta_0 := \delta_\emptyset$ ,  $\delta_n := \delta_{\{1,2,\dots,n\}}$  for  $n \in \mathbb{N}_+$ . The following properties of the maps  $\delta_\Phi$  will be used:

- (a)  $\delta_n = (\text{id}_C - u \circ \epsilon)^{\otimes n} \circ \Delta^n$  for all  $n \in \mathbb{N}_+$ , where  $u: R \rightarrow H$  is the unit map;
- (b) the maps  $\delta_n$  are coassociative, that is  $(\text{id}_C^{\otimes s} \otimes \delta_\ell \otimes \text{id}_C^{\otimes(n-1-s)}) \circ \delta_n = \delta_{n+\ell-1}$  for all  $n, \ell, s \in \mathbb{N}$ ,  $0 \leq s \leq n-1$ , and similarly in general for the maps  $\delta_\Phi$ ;
- (c)  $\delta_\Phi(ab) = \sum_{\Lambda \cup Y = \Phi} \delta_\Lambda(a) \delta_Y(b)$  for all finite subset  $\Phi \subseteq \mathbb{N}$  and all  $a, b \in H$ ;
- (d)  $\delta_\Phi(ab - ba) = \sum_{\substack{\Lambda \cup Y = \Phi \\ \Lambda \cap Y = \emptyset}} (\delta_\Lambda(a) \delta_Y(b) - \delta_Y(b) \delta_\Lambda(a))$  for all  $\Phi \neq \emptyset$  and  $a, b \in H$ .

**3.3 Drinfeld's algebra  $(\mathcal{H}_{\hbar}^{\vee})'$ .** Using Drinfeld's  $\delta_\bullet$ -maps of §3.2, we define

$$(\mathcal{H}_{\hbar}^{\vee})' := \left\{ \eta \in \mathcal{H}_{\hbar}^{\vee} \mid \delta_n(\eta) \in \hbar^n (\mathcal{H}_{\hbar}^{\vee})^{\otimes n} \forall n \in \mathbb{N} \right\} \quad (\subseteq \mathcal{H}_{\hbar}^{\vee}). \quad (3.1)$$

Now I describe  $(\mathcal{H}_{\hbar}^{\vee})'$  and its specializations at  $\hbar = 1$  and  $\hbar = 0$ , in several steps.

*Step I:* A direct check shows that  $\tilde{\mathbf{x}}_n := \hbar \mathbf{x}_n = \mathbf{a}_n \in (\mathcal{H}_{\hbar}^{\vee})'$ , for all  $n \in \mathbb{N}_\nu$ . Indeed, we have of course  $\delta_0(\tilde{\mathbf{x}}_n) = \epsilon(\tilde{\mathbf{x}}_n) \in \hbar^0 \mathcal{H}_{\hbar}^{\vee}$  and  $\delta_1(\tilde{\mathbf{x}}_n) = \tilde{\mathbf{x}}_n - \epsilon(\tilde{\mathbf{x}}_n) \in \hbar^1 \mathcal{H}_{\hbar}^{\vee}$ . Moreover,  $\delta_2(\tilde{\mathbf{x}}_n) = \sum_{m=1}^{n-1} \tilde{\mathbf{x}}_{n-m} \otimes Q_m^{n-m}(\tilde{\mathbf{x}}_*) = \sum_{m=1}^{n-1} \sum_{k=1}^m \hbar^{k+1} \binom{n-m+1}{k} \mathbf{x}_{n-m} \otimes P_m^{(k)}(\mathbf{x}_*) \in \hbar^2 (\mathcal{H}_{\hbar}^{\vee} \otimes \mathcal{H}_{\hbar}^{\vee})$ . Since in general  $\delta_\ell = (\delta_{\ell-1} \otimes \text{id}) \circ \delta_2$  for all  $\ell \in \mathbb{N}_+$ , we have

$$\delta_\ell(\tilde{\mathbf{x}}_n) = (\delta_{\ell-1} \otimes \text{id})(\delta_2(\tilde{\mathbf{x}}_n)) = \sum_{m=1}^{n-1} \sum_{k=1}^m \hbar^k \binom{n-m+1}{k} \delta_{\ell-1}(\mathbf{x}_{n-m}) \otimes P_m^{(k)}(\mathbf{x}_*)$$

whence induction gives  $\delta_\ell(\tilde{\mathbf{x}}_n) \in \hbar^\ell (\mathcal{H}_{\hbar}^{\vee})^{\otimes \ell}$  for all  $\ell \in \mathbb{N}$ , thus  $\tilde{\mathbf{x}}_n \in (\mathcal{H}_{\hbar}^{\vee})'$ , q.e.d.

*Step II:* Using property (c) in §3.2 one easily checks that  $(\mathcal{H}_{\hbar}^{\vee})'$  is a  $\mathbb{k}[\hbar]$ -subalgebra of  $\mathcal{H}_{\hbar}^{\vee}$  (see [Ga2–3], Proposition 3.5 for details). In particular, by *Step I* and the very definitions this implies that  $(\mathcal{H}_{\hbar}^{\vee})'$  contains  $\mathcal{H}_{\hbar}$ .

*Step III:* Using property (d) in §3.2 one easily sees that  $(\mathcal{H}_{\hbar}^\vee)' \Big|_{\hbar=0}$  is commutative (cf. [Ga2–3], Theorem 3.8 for details): this means  $[a, b] \equiv 0 \pmod{\hbar} (\mathcal{H}_{\hbar}^\vee)'$ , that is  $[a, b] \in \hbar (\mathcal{H}_{\hbar}^\vee)'$  hence also  $\hbar^{-1}[a, b] \in (\mathcal{H}_{\hbar}^\vee)'$ , for all  $a, b \in (\mathcal{H}_{\hbar}^\vee)'$ . In particular, we get  $[\mathbf{x}_n, \mathbf{x}_m] := \hbar [\mathbf{x}_n, \mathbf{x}_m] = \hbar^{-1}[\tilde{\mathbf{x}}_n, \tilde{\mathbf{x}}_m] \in (\mathcal{H}_{\hbar}^\vee)'$  for all  $n, m \in \mathbb{N}_\nu$ , whence iterating (and recalling  $\mathcal{L}_\nu$  is generated by the  $\mathbf{x}_n$ 's) we get  $\tilde{\mathbf{x}} := \hbar \mathbf{x} \in (\mathcal{H}_{\hbar}^\vee)'$  for every  $\mathbf{x} \in \mathcal{L}_\nu$ . Hereafter we identify the free Lie algebra  $\mathcal{L}_\nu$  with its image via the natural embedding  $\mathcal{L}_\nu \hookrightarrow U(\mathcal{L}_\nu) = \mathbb{k}\langle \{x_n\}_{n \in \mathbb{N}_\nu} \rangle \hookrightarrow \mathbb{k}[\hbar]\langle \{\mathbf{x}_n\}_{n \in \mathbb{N}_\nu} \rangle = \mathcal{H}_{\hbar}^\vee$  given by  $x_n \mapsto \mathbf{x}_n$  ( $n \in \mathbb{N}_\nu$ ).

*Step IV:* The previous step showed that, if we embed  $\mathcal{L}_\nu \hookrightarrow U(\mathcal{L}_\nu) \hookrightarrow \mathcal{H}_{\hbar}^\vee$  via  $x_n \mapsto \mathbf{x}_n$  ( $n \in \mathbb{N}_\nu$ ) we find  $\widetilde{\mathcal{L}}_\nu := \hbar \mathcal{L}_\nu \subseteq (\mathcal{H}_{\hbar}^\vee)'$ . Let  $\langle \widetilde{\mathcal{L}}_\nu \rangle$  be the  $\mathbb{k}[\hbar]$ -subalgebra of  $(\mathcal{H}_{\hbar}^\vee)'$  generated by  $\widetilde{\mathcal{L}}_\nu$ : then  $\langle \widetilde{\mathcal{L}}_\nu \rangle \subseteq (\mathcal{H}_{\hbar}^\vee)'$ , because  $(\mathcal{H}_{\hbar}^\vee)'$  is a subalgebra. In particular, if  $\mathbf{b}_b \in \mathcal{H}_{\hbar}^\vee$  is the image of any  $b \in B_\nu$  (cf. §1.1) we have  $\tilde{\mathbf{b}}_b := \hbar \mathbf{b}_b \in (\mathcal{H}_{\hbar}^\vee)'$ .

*Step V:* Conversely to Step IV, we have  $\langle \widetilde{\mathcal{L}}_\nu \rangle \supseteq (\mathcal{H}_{\hbar}^\vee)'$ . In fact, let  $\eta \in (\mathcal{H}_{\hbar}^\vee)'$ ; then there are unique  $d \in \mathbb{N}$ ,  $\eta_+ \in \mathcal{H}_{\hbar}^\vee \setminus \hbar \mathcal{H}_{\hbar}^\vee$  such that  $\eta = \hbar^d \eta_+$ ; set also  $\bar{y} := y \pmod{\hbar H_{\hbar}^\vee} \in H_{\hbar}^\vee / \hbar H_{\hbar}^\vee$  for all  $y \in H_{\hbar}^\vee$ . As  $\mathcal{H}_{\hbar}^\vee = \mathbb{k}[\hbar]\langle \{\mathbf{x}_n \mid n \in \mathbb{N}_\nu\} \rangle$  there is a unique  $\hbar$ -adic expansion of  $\eta_+$ , namely  $\eta_+ = \eta_0 + \hbar \eta_1 + \cdots + \hbar^s \eta_s = \sum_{k=0}^s \hbar^k \eta_k$  with all  $\eta_k \in \mathbb{k}\langle \{\mathbf{x}_n \mid n \in \mathbb{N}_\nu\} \rangle$  and  $\eta_0 \neq 0$ . Then  $\bar{\eta}_+ = \bar{\eta}_0 := \eta_0 \pmod{\hbar \mathcal{H}_{\hbar}^\vee}$ , with  $\bar{\eta}_+ = \bar{\eta}_0 \in \mathcal{H}_{\hbar}^\vee \Big|_{\hbar=0} = U(\mathcal{L}_\nu)$  by Theorem 2.1. On the other hand,  $\eta \in (\mathcal{H}_{\hbar}^\vee)'$  implies  $\delta_{d+1}(\eta) \in \hbar^{d+1} (\mathcal{H}_{\hbar}^\vee)^{\otimes(d+1)}$ , whence  $\delta_{d+1}(\eta_+) = \hbar^{-d} \delta_{d+1}(\eta) \in \hbar (\mathcal{H}_{\hbar}^\vee)^{\otimes(d+1)}$  so that  $\delta_{d+1}(\bar{\eta}_0) = 0$ ; the latter implies that the degree  $\partial(\bar{\eta}_0)$  of  $\bar{\eta}_0$  for the standard filtration of  $U(\mathcal{L}_\nu)$  is at most  $d$  (cf. [Ga2–3], Lemma 4.2(d) for a proof). By the PBW theorem,  $\partial(\bar{\eta}_0)$  is also the degree of  $\bar{\eta}_0$  as a polynomial in the  $\bar{x}_b$ 's, hence also of  $\eta_0$  as a polynomial in the  $x_b$ 's ( $b \in B_\nu$ ): then  $\hbar^d \eta_0 \in \langle \widetilde{\mathcal{L}}_\nu \rangle \subseteq (\mathcal{H}_{\hbar}^\vee)'$  (using *Step III*), hence we find

$$\eta_{(1)} := \hbar^{d+1} (\eta_1 + \hbar \eta_2 + \cdots + \hbar^{s-1} \eta_s) = \eta - \hbar^d \eta_0 \in (\mathcal{H}_{\hbar}^\vee)'.$$

Thus we can apply our argument again, with  $\eta_{(1)}$  instead of  $\eta$ . Iterating we find  $\partial(\bar{\eta}_k) \leq d+k$ , whence  $\hbar^{d+k} \eta_k \in \langle \widetilde{\mathcal{L}}_\nu \rangle$  ( $\subseteq (\mathcal{H}_{\hbar}^\vee)'$ ) for all  $k$ , thus  $\eta = \sum_{k=0}^s \hbar^{d+k} \eta_k \in \langle \widetilde{\mathcal{L}}_\nu \rangle$ , q.e.d.

An entirely similar analysis clearly works with  $\mathcal{K}_{\hbar}$  taking the role of  $\mathcal{H}_{\hbar}$ , with similar results (*mutatis mutandis*). On the upshot, we get the following description:

**Theorem 3.1.** (a) *With notation of Step III in §3.3 (and  $[a, c] := ac - ca$ ), we have*

$$(\mathcal{H}_{\hbar}^\vee)' = \langle \widetilde{\mathcal{L}}_\nu \rangle = \mathbb{k}[\hbar] \langle \{ \tilde{\mathbf{b}}_b \}_{b \in B_\nu} \rangle \Big/ \left( \left\{ [\tilde{\mathbf{b}}_{b_1}, \tilde{\mathbf{b}}_{b_2}] - \hbar [\widetilde{\mathbf{b}_{b_1}}, \mathbf{b}_{b_2}] \mid \forall b_1, b_2 \in B_\nu \right\} \right).$$

(b)  $(\mathcal{H}_{\hbar}^\vee)'$  is a graded Hopf  $\mathbb{k}[\hbar]$ -subalgebra of  $\mathcal{H}_{\hbar}^\vee$ , and  $\mathcal{H}$  is naturally embedded into  $(\mathcal{H}_{\hbar}^\vee)'$  as a graded Hopf subalgebra via  $\mathcal{H} \hookrightarrow (\mathcal{H}_{\hbar}^\vee)'$ ,  $\mathbf{a}_n \mapsto \tilde{\mathbf{x}}_n$  (for all  $n \in \mathbb{N}_\nu$ ).

(c)  $(\mathcal{H}_{\hbar}^\vee)' \Big|_{\hbar=0} := (\mathcal{H}_{\hbar}^\vee)' / \hbar (\mathcal{H}_{\hbar}^\vee)' = F[G_{\mathcal{L}_\nu}^*]$ , where  $G_{\mathcal{L}_\nu}^*$  is an infinite dimensional

connected Poisson algebraic group with cotangent Lie bialgebra isomorphic to  $\mathcal{L}_\nu$  (with the graded Lie bialgebra structure of Theorem 2.1). Indeed,  $(\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=0}$  is the free Poisson (commutative) algebra over  $\mathbb{N}_\nu$ , generated by all the  $\tilde{\mathbf{x}}_n \Big|_{\hbar=0}$  ( $n \in \mathbb{N}_\nu$ ) with Hopf structure given by (1.1) with  $\tilde{\mathbf{x}}_*$  instead of  $\mathbf{a}_*$ . Thus  $(\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=0}$  is the polynomial algebra  $\mathbb{k}[\{\beta_b\}_{b \in B_\nu}]$  generated by a set of indeterminates  $\{\beta_b\}_{b \in B_\nu}$  in bijection with the basis  $B_\nu$  of  $\mathcal{L}_\nu$ , so  $G_{\mathcal{L}_\nu}^* \cong \mathbb{A}_{\mathbb{k}}^{B_\nu}$  (a (pro)affine  $\mathbb{k}$ -space) as algebraic varieties. Finally,  $F[G_{\mathcal{L}_\nu}^*] = (\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=0} \cong \mathbb{k}[\{\beta_b\}_{b \in B_\nu}]$  bears the natural algebra grading  $d$  of polynomial algebras and the Hopf algebra grading inherited from  $(\mathcal{H}_\hbar^\vee)'$ , respectively given by  $d(\tilde{\mathbf{b}}_b) = 1$  and  $\partial(\tilde{\mathbf{b}}_b) = \sum_{i=1}^k n_i$  for all  $b = [[\dots [x_{n_1}, x_{n_2}], x_{n_3}], \dots], x_{n_k}] \in B_\nu$ .

(d)  $F[\mathcal{G}_\nu]$  is naturally embedded into  $(\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=0} = F[G_{\mathcal{L}_\nu}^*]$  as a graded Hopf subalgebra via  $\mu: F[\mathcal{G}_\nu] \hookrightarrow (\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=0} = F[G_{\mathcal{L}_\nu}^*]$ ,  $a_n \mapsto (\tilde{\mathbf{x}}_n \bmod \hbar (\mathcal{H}_\hbar^\vee)')$  (for all  $n \in \mathbb{N}_\nu$ ); moreover,  $F[\mathcal{G}_\nu]$  freely generates  $F[G_{\mathcal{L}_\nu}^*]$  as a Poisson algebra. Thus there is an algebraic group epimorphism  $\mu_*: G_{\mathcal{L}_\nu}^* \twoheadrightarrow \mathcal{G}_\nu$ , that is  $G_{\mathcal{L}_\nu}^*$  is an extension of  $\mathcal{G}_\nu$ .

(e) Mapping  $(\tilde{\mathbf{x}}_n \bmod \hbar (\mathcal{H}_\hbar^\vee)') \mapsto a_n$  (for all  $n \in \mathbb{N}_\nu$ ) gives a well-defined graded Hopf algebra epimorphism  $\pi: F[G_{\mathcal{L}_\nu}^*] \twoheadrightarrow F[\mathcal{G}_\nu]$ . Thus there is an algebraic group monomorphism  $\pi_*: \mathcal{G}_\nu \hookrightarrow G_{\mathcal{L}_\nu}^*$ , that is  $\mathcal{G}_\nu$  is an algebraic subgroup of  $G_{\mathcal{L}_\nu}^*$ .

(f) The map  $\mu$  is a section of  $\pi$ , hence  $\pi_*$  is a section of  $\mu_*$ . Thus  $G_{\mathcal{L}_\nu}^*$  is a semidirect product of algebraic groups, namely  $G_{\mathcal{L}_\nu}^* = \mathcal{G}_\nu \ltimes \mathcal{N}_\nu$  where  $\mathcal{N}_\nu := \text{Ker}(\mu_*) \trianglelefteq G_{\mathcal{L}_\nu}^*$ .

(g) The analogues of statements (a)–(f) hold with  $\mathcal{K}$  instead of  $\mathcal{H}$ , with  $X^+$  instead of  $X$  for all  $X = \mathcal{L}_\nu, B_\nu, \mathbb{N}_\nu, \mu, \pi, \mathcal{N}_\nu$ , and with  $G_{\mathcal{L}_\nu}^{*+}$  instead of  $G_{\mathcal{L}_\nu}^*$ .

*Proof.* (a) This part follows directly from Step IV and Step V in §3.3.

(b) To show that  $(\mathcal{H}_\hbar^\vee)'$  is a graded Hopf subalgebra we use its presentation in (a). But first recall that, by Step II,  $\mathcal{H}$  embeds into  $(\mathcal{H}_\hbar^\vee)'$  via an embedding which is compatible with the Hopf operations (it is a restriction of the identity on  $\mathcal{H}(\hbar)$ ): then this will be a Hopf algebra monomorphism, up to proving that  $(\mathcal{H}_\hbar^\vee)'$  is a Hopf subalgebra (of  $\mathcal{H}_\hbar^\vee$ ).

Now,  $\epsilon_{\mathcal{H}_\hbar^\vee}$  obviously restricts to give a counit for  $(\mathcal{H}_\hbar^\vee)'$ . Second, we show that  $\Delta((\mathcal{H}_\hbar^\vee)') \subseteq (\mathcal{H}_\hbar^\vee)' \otimes (\mathcal{H}_\hbar^\vee)'$ , so  $\Delta$  restricts to a coproduct for  $(\mathcal{H}_\hbar^\vee)'$ . Indeed, each  $b \in B_\nu$  is a Lie monomial, say  $b = [[[\dots [x_{n_1}, x_{n_2}], x_{n_3}], \dots], x_{n_k}]$  for some  $k, n_1, \dots, n_k \in \mathbb{N}_\nu$ , where  $k$  is its Lie degree: by induction on  $k$  we'll prove  $\Delta(\tilde{\mathbf{b}}_b) \in (\mathcal{H}_\hbar^\vee)' \otimes (\mathcal{H}_\hbar^\vee)'$  (with  $\tilde{\mathbf{b}}_b := \hbar \mathbf{b}_b = \hbar [[[\dots [\mathbf{x}_{n_1}, \mathbf{x}_{n_2}], \mathbf{x}_{n_3}], \dots], \mathbf{x}_{n_k}]$ ).

If  $k = 1$  then  $b = x_n$  for some  $n \in \mathbb{N}_\nu$ . Then  $\tilde{\mathbf{b}}_b = \hbar \mathbf{x}_n = \mathbf{a}_n$  and

$$\Delta(\tilde{\mathbf{b}}_b) = \Delta(\mathbf{a}_n) = \mathbf{a}_n \otimes 1 + 1 \otimes \mathbf{a}_n + \sum_{m=1}^{n-1} \mathbf{a}_{n-m} \otimes Q_m^{n-m}(\mathbf{a}_*) \in \mathcal{H}^{\text{dif}} \otimes \mathcal{H}^{\text{dif}} \subseteq (\mathcal{H}_\hbar^\vee)' \otimes (\mathcal{H}_\hbar^\vee)'.$$

If  $k > 1$  then  $b = [b^-, x_n]$  for some  $n \in \mathbb{N}_\nu$  and some  $b^- \in B_\nu$  expressed by a Lie

monomial of degree  $k - 1$ . Then  $\tilde{\mathbf{b}}_b = \hbar[\mathbf{b}^-, \mathbf{x}_n] = [\tilde{\mathbf{b}}^-, \mathbf{x}_n]$  and

$$\begin{aligned} \Delta(\tilde{\mathbf{b}}_b) &= \Delta([\tilde{\mathbf{b}}^-, \mathbf{x}_n]) = [\Delta(\tilde{\mathbf{b}}^-), \Delta(\mathbf{x}_n)] = \hbar^{-1} [\Delta(\tilde{\mathbf{b}}^-), \Delta(\mathbf{a}_n)] = \\ &= \hbar^{-1} \left[ \sum_{(\tilde{\mathbf{b}}^-)} \tilde{\mathbf{b}}_{(1)}^- \otimes \tilde{\mathbf{b}}_{(2)}^-, \mathbf{a}_n \otimes 1 + 1 \otimes \mathbf{a}_n + \sum_{m=1}^{n-1} \mathbf{a}_{n-m} \otimes Q_m^{n-m}(\mathbf{a}_*) \right] = \\ &+ \sum_{(\tilde{\mathbf{b}}^-)} \sum_{m=1}^{n-1} \left( \hbar^{-1} [\tilde{\mathbf{b}}_{(1)}^-, \mathbf{a}_{n-m}] \otimes \tilde{\mathbf{b}}_{(2)}^- Q_m^{n-m}(\mathbf{a}_*) + \tilde{\mathbf{b}}_{(1)}^- \mathbf{a}_{n-m} \otimes \hbar^{-1} [\tilde{\mathbf{b}}_{(2)}^-, Q_m^{n-m}(\mathbf{a}_*)] \right) \end{aligned}$$

where we used the standard  $\Sigma$ -notation for  $\Delta(\tilde{\mathbf{b}}^-) = \sum_{(\tilde{\mathbf{b}}^-)} \tilde{\mathbf{b}}_{(1)}^- \otimes \tilde{\mathbf{b}}_{(2)}^-$ . By inductive hypothesis we have  $\tilde{\mathbf{b}}_{(1)}^-, \tilde{\mathbf{b}}_{(2)}^- \in (\mathcal{H}_\hbar^\vee)'$ ; then since also  $\mathbf{a}_\ell \in (\mathcal{H}_\hbar^\vee)'$  for all  $\ell$  and since  $(\mathcal{H}_\hbar^\vee)'$  is commutative modulo  $\hbar$  we have

$$\hbar^{-1} [\tilde{\mathbf{b}}_{(1)}^-, \mathbf{a}_n], \hbar^{-1} [\tilde{\mathbf{b}}_{(2)}^-, \mathbf{a}_n], \hbar^{-1} [\tilde{\mathbf{b}}_{(1)}^-, \mathbf{a}_{n-m}], \hbar^{-1} [\tilde{\mathbf{b}}_{(2)}^-, Q_m^{n-m}(\mathbf{a}_*)] \in (\mathcal{H}_\hbar^\vee)'$$

for all  $n$  and  $(n-m)$  above: so the previous formula gives  $\Delta(\tilde{\mathbf{b}}_b) \in (\mathcal{H}_\hbar^\vee)' \otimes (\mathcal{H}_\hbar^\vee)'$ , q.e.d.

Finally, the antipode. Take the Lie monomial  $b = [[[\dots [x_{n_1}, x_{n_2}], x_{n_3}], \dots], x_{n_k}] \in B_\nu$ , so  $\tilde{\mathbf{b}}_b = \hbar \mathbf{b}_b = \hbar [[[\dots [\mathbf{x}_{n_1}, \mathbf{x}_{n_2}], \mathbf{x}_{n_3}], \dots], \mathbf{x}_{n_k}]$ . We prove that  $S(\tilde{\mathbf{b}}_b) \in (\mathcal{H}_\hbar^\vee)'$  by induction on the degree  $k$ . If  $k = 1$  then  $b = x_n$  for some  $n$ , so  $\tilde{\mathbf{b}}_b = \hbar \mathbf{x}_n = \mathbf{a}_n$  and

$$S(\tilde{\mathbf{b}}_b) = S(\mathbf{a}_n) = -\mathbf{a}_n - \sum_{m=1}^{n-1} \mathbf{a}_{n-m} S(Q_m^{n-m}(\mathbf{a}_*)) \in \mathcal{H}^{\text{dif}} \subseteq (\mathcal{H}_\hbar^\vee)', \quad \text{q.e.d.}$$

If  $k > 1$  then  $b = [b^-, x_n]$  for some  $n \in \mathbb{N}_\nu$  and some  $b^- \in B_\nu$  which is a Lie monomial of degree  $k - 1$ . Then  $\tilde{\mathbf{b}}_b = \hbar[\mathbf{b}^-, \mathbf{x}_n] = [\tilde{\mathbf{b}}^-, \mathbf{x}_n] = \hbar^{-1} [\tilde{\mathbf{b}}^-, \mathbf{a}_n]$  and so

$$S(\tilde{\mathbf{b}}_b) = S([\tilde{\mathbf{b}}^-, \mathbf{x}_n]) = \hbar^{-1} [S(\mathbf{a}_n), S(\tilde{\mathbf{b}}^-)] \in \hbar^{-1} [(\mathcal{H}_\hbar^\vee)', (\mathcal{H}_\hbar^\vee)'] \subseteq (\mathcal{H}_\hbar^\vee)'$$

using the fact  $S(\mathbf{a}_n) = S(\tilde{\mathbf{x}}_n) = S(\tilde{\mathbf{b}}_{x_n}) \in (\mathcal{H}_\hbar^\vee)'$  (by the case  $k = 1$ ) along with the inductive assumption  $S(\tilde{\mathbf{b}}^-) \in (\mathcal{H}_\hbar^\vee)'$  and the commutativity of  $(\mathcal{H}_\hbar^\vee)'$  modulo  $\hbar$ .

(c) As a consequence of (a), the  $\mathbb{k}$ -algebra  $(\mathcal{H}_\hbar^\vee)'|_{\hbar=0}$  is a polynomial algebra, namely  $(\mathcal{H}_\hbar^\vee)'|_{\hbar=0} = \mathbb{k}[\{\beta_b\}_{b \in B}]$  with  $\beta_b := \tilde{\mathbf{b}}_b \bmod \hbar (\mathcal{H}_\hbar^\vee)'$  for all  $b \in B_\nu$ . So  $(\mathcal{H}_\hbar^\vee)'|_{\hbar=0}$  is the algebra of regular functions  $F[\Gamma]$  of some (affine) algebraic variety  $\Gamma$ ; as  $(\mathcal{H}_\hbar^\vee)'$  is a Hopf algebra the same is true for  $(\mathcal{H}_\hbar^\vee)'|_{\hbar=0} = F[\Gamma]$ , so  $\Gamma$  is an algebraic group; and since  $F[\Gamma] = (\mathcal{H}_\hbar^\vee)'|_{\hbar=0}$  is a specialization limit of  $(\mathcal{H}_\hbar^\vee)'$ , it is endowed with the Poisson bracket  $\{a|_{\hbar=0}, b|_{\hbar=0}\} := (\hbar^{-1}[a, b])|_{\hbar=0}$  which makes  $\Gamma$  into a Poisson group too.

We compute the cotangent Lie bialgebra of  $\Gamma$ . First,  $\mathfrak{m}_e := \text{Ker}(\epsilon_{F[\Gamma]}) = (\{\beta_b\}_{b \in B_\nu})$  (the ideal generated by the  $\beta_b$ 's) by construction, so  $\mathfrak{m}_e^2 = (\{\beta_{b_1} \beta_{b_2}\}_{b_1, b_2 \in B_\nu})$ . Therefore

the cotangent Lie bialgebra  $Q(F[\Gamma]) := \mathfrak{m}_e / \mathfrak{m}_e^2$  as a  $\mathbb{k}$ -vector space has basis  $\{\bar{\beta}_b\}_{b \in B_\nu}$  where  $\bar{\beta}_b := \beta_b \bmod \mathfrak{m}_e^2$  for all  $b \in B_\nu$ . For its Lie bracket we have (cf. Remark 1.5)

$$\begin{aligned} [\bar{\beta}_{b_1}, \bar{\beta}_{b_2}] &:= \{\beta_{b_1}, \beta_{b_2}\} \bmod \mathfrak{m}_e^2 = \left( \hbar^{-1} [\tilde{\mathbf{b}}_{b_1}, \tilde{\mathbf{b}}_{b_2}] \bmod \hbar (\mathcal{H}_\hbar^\vee)' \right) \bmod \mathfrak{m}_e^2 = \\ &= \left( \hbar^{-1} \hbar^2 [\mathbf{b}_{b_1}, \mathbf{b}_{b_2}] \bmod \hbar (\mathcal{H}_\hbar^\vee)' \right) \bmod \mathfrak{m}_e^2 = \left( \hbar \mathbf{b}_{[b_1, b_2]} \bmod \hbar (\mathcal{H}_\hbar^\vee)' \right) \bmod \mathfrak{m}_e^2 = \\ &= \left( \tilde{\mathbf{b}}_{[b_1, b_2]} \bmod \hbar (\mathcal{H}_\hbar^\vee)' \right) \bmod \mathfrak{m}_e^2 = \beta_{[b_1, b_2]} \bmod \mathfrak{m}_e^2 = \bar{\beta}_{[b_1, b_2]}, \end{aligned}$$

thus the  $\mathbb{k}$ -linear map  $\Psi: \mathcal{L}_\nu \longrightarrow \mathfrak{m}_e / \mathfrak{m}_e^2$  defined by  $b \mapsto \bar{\beta}_b$  for all  $b \in B_\nu$  is a *Lie algebra isomorphism*. As for the Lie cobracket, using the general identity  $\delta = \Delta - \Delta^{\text{op}} \bmod (\mathfrak{m}_e^2 \otimes F[\Gamma] + F[\Gamma] \otimes \mathfrak{m}_e^2)$  (written  $\bmod \widehat{\mathfrak{m}}_e^2$  for short) we get, for all  $n \in \mathbb{N}_\nu$ ,

$$\begin{aligned} \delta(\bar{\beta}_{x_n}) &= (\Delta - \Delta^{\text{op}})(\beta_{x_n}) \bmod \widehat{\mathfrak{m}}_e^2 = \left( (\Delta - \Delta^{\text{op}})(\tilde{\mathbf{x}}_n) \bmod \hbar \left( (\mathcal{H}_\hbar^\vee)' \otimes (\mathcal{H}_\hbar^\vee)' \right) \right) \bmod \widehat{\mathfrak{m}}_e^2 = \\ &= \left( (\mathbf{a}_n \wedge 1 + 1 \wedge \mathbf{a}_n + \sum_{m=1}^{n-1} \mathbf{a}_{n-m} \wedge Q_m^{n-m}(\mathbf{a}_*)) \bmod \hbar (\mathcal{H}_\hbar' \otimes \mathcal{H}_\hbar') \right) \bmod \widehat{\mathfrak{m}}_e^2 = \\ &= \sum_{m=1}^{n-1} \beta_{x_{n-m}} \wedge Q_m^{n-m}(\beta_{x_*}) \bmod \widehat{\mathfrak{m}}_e^2 = \sum_{m=1}^{n-1} \sum_{k=1}^m \binom{n-m+1}{k} \beta_{x_{n-m}} \wedge P_m^{(k)}(\beta_{x_*}) \bmod \widehat{\mathfrak{m}}_e^2 = \\ &= \left( \sum_{m=1}^{n-1} \binom{n-m+1}{1} \beta_{x_{n-m}} \wedge P_m^{(1)}(\beta_{x_*}) \right) \bmod \widehat{\mathfrak{m}}_e^2 = \sum_{\ell=1}^{n-1} (\ell+1) \bar{\beta}_{x_\ell} \wedge \bar{\beta}_{x_{n-\ell}} \end{aligned}$$

because — among other things — one has  $P_m^{(k)}(\beta_{x_*}) \in \mathfrak{m}_e^2$  for all  $k > 1$ : therefore

$$\delta(\bar{\beta}_{x_n}) = \sum_{\ell=1}^{n-1} (\ell+1) \bar{\beta}_{x_\ell} \wedge \bar{\beta}_{x_{n-\ell}} \quad \forall n \in \mathbb{N}_\nu. \quad (3.2)$$

Since  $\mathcal{L}_\nu$  is generated (as a Lie algebra) by the  $x_n$ 's, the last formula shows that the map  $\Psi: \mathcal{L}_\nu \longrightarrow \mathfrak{m}_e / \mathfrak{m}_e^2$  given above is also an isomorphism of *Lie bialgebras*, q.e.d.

Finally, the statements about gradings of  $(\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=0}$  should be trivially clear.

(d) The part about Hopf algebras is a direct consequence of (a) and (b), noting that the  $\tilde{\mathbf{x}}_n$ 's commute modulo  $\hbar (\mathcal{H}_\hbar^\vee)'$ , since  $(\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=0}$  is commutative. Taking spectra (i.e. sets of characters of each Hopf algebras) we get an algebraic group morphism  $\mu_*: G_{\mathcal{L}_\nu}^* \longrightarrow \mathcal{G}_\nu$ , which in fact is *onto* because, as these algebras are polynomial, each character of  $F[\mathcal{G}_\nu]$  does extend to a character of  $F[G_{\mathcal{L}_\nu}^*]$ , so the former arises from restriction of the latter.

(e) Due to the explicit description of  $F[G_{\mathcal{L}_\nu}^*]$  coming from (a) and (b), mapping  $(\tilde{\mathbf{x}}_n \bmod \hbar (\mathcal{H}_\hbar^\vee)') \mapsto a_n$  (for all  $n \in \mathbb{N}_\nu$ ) clearly yields a Hopf algebra epimorphism  $\pi: F[G_{\mathcal{L}_\nu}^*] \longrightarrow F[\mathcal{G}_\nu]$ . Taking spectra gives an algebraic group monomorphism  $\pi_*: \mathcal{G}_\nu \hookrightarrow G_{\mathcal{L}_\nu}^*$  as required.

(f) The map  $\mu$  is a section of  $\pi$  by construction. Then clearly  $\pi_*$  is a section of  $\mu_*$ , which implies  $G_{\mathcal{L}_\nu}^* = \mathcal{G}_\nu \rtimes \mathcal{N}_\nu$  (with  $\mathcal{N}_\nu := \text{Ker}(\mu_*) \trianglelefteq G_{\mathcal{L}_\nu}^*$ ) by general theory.

(g) This ought to be clear from the whole discussion, for all arguments apply again — *mutatis mutandis* — when starting with  $\mathcal{K}$  instead of  $\mathcal{H}$ ; details are left to the reader.  $\square$



*Remark:* Roughly speaking, we can say that the extension  $F[\mathcal{G}_\nu] \hookrightarrow F[G_{\mathcal{L}_\nu}^*]$  is performed simply by adding to  $F[\mathcal{G}_\nu]$  a free Poisson structure, which happens to be compatible with the Hopf structure. Then the Poisson bracket starting from the “elementary” coordinates  $a_n$  (for  $n \in \mathbb{N}_\nu$ ) freely generates new coordinates  $\{a_{n_1}, a_{n_2}\}, \{\{a_{n_1}, a_{n_2}\}, a_{n_3}\}$ , etc., thus enlarging  $F[\mathcal{G}_\nu]$  and generating  $F[G_{\mathcal{L}_\nu}^*]$ . At the group level, this means that  $\mathcal{G}_\nu$  freely Poisson-generates the Poisson group  $G_{\mathcal{L}_\nu}^*$ : new 1-parameter subgroups, build up in a Poisson-free manner from those attached to the  $a_n$ ’s, are freely “pasted” to  $\mathcal{G}_\nu$ , expanding it and building up  $G_{\mathcal{L}_\nu}^*$ . Then the epimorphism  $G_{\mathcal{L}_\nu}^* \xrightarrow{\mu^*} \mathcal{G}_\nu$  is just a forgetful map: it kills the new 1-parameter subgroups and is injective (hence an isomorphism) on the subgroup generated by the old ones. On the other hand, definitions imply that  $F[G_{\mathcal{L}_\nu}^*] / \left( \{F[G_{\mathcal{L}_\nu}^*], F[G_{\mathcal{L}_\nu}^*]\} \right) \cong F[\mathcal{G}_\nu]$ , and with this identification  $F[G_{\mathcal{L}_\nu}^*] \xrightarrow{\pi} F[\mathcal{G}_\nu]$  is just the canonical map, which mods out all Poisson brackets  $\{f_1, f_2\}$ , for  $f_1, f_2 \in F[G_{\mathcal{L}_\nu}^*]$ .

**3.4 Specialization limits.** So far, we have already pointed out (by Proposition 2.1, Theorem 2.1, Theorem 3.1(c)) the following specialization limits of  $\mathcal{H}_\hbar^\vee$  and  $(\mathcal{H}_\hbar^\vee)'$ :

$$\mathcal{H}_\hbar^\vee \xrightarrow{\hbar \rightarrow 1} \mathcal{H}, \quad \mathcal{H}_\hbar^\vee \xrightarrow{\hbar \rightarrow 0} U(\mathcal{L}_\nu), \quad (\mathcal{H}_\hbar^\vee)' \xrightarrow{\hbar \rightarrow 0} F[G_{\mathcal{L}_\nu}^*]$$

as graded Hopf  $\mathbb{k}$ -algebras, with some (co-)Poisson structures in the last two cases. As for the specialization limit of  $(\mathcal{H}_\hbar^\vee)'$  at  $\hbar = 1$ , Theorem 3.1 implies that it is  $\mathcal{H}$ . Indeed, by Theorem 3.1(b)  $\mathcal{H}$  embeds into  $(\mathcal{H}_\hbar^\vee)'$  via  $\mathbf{a}_n \mapsto \tilde{\mathbf{x}}_n$  (for all  $n \in \mathbb{N}_\nu$ ): then

$$[\mathbf{a}_n, \mathbf{a}_m] = [\tilde{\mathbf{x}}_n, \tilde{\mathbf{x}}_m] = \hbar [\widetilde{\mathbf{x}_n, \mathbf{x}_m}] \equiv [\widetilde{\mathbf{x}_n, \mathbf{x}_m}] \pmod{(\hbar-1)} (\mathcal{H}_\hbar^\vee)' \quad (\forall n, m \in \mathbb{N}_\nu)$$

whence, due to the presentation of  $(\mathcal{H}_\hbar^\vee)'$  by generators and relations in Theorem 3.1(a),

$$(\mathcal{H}_\hbar^\vee)' \Big|_{\hbar=1} := (\mathcal{H}_\hbar^\vee)' / (\hbar-1) (\mathcal{H}_\hbar^\vee)' = \mathbb{k} \langle \bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \dots, \bar{\mathbf{x}}_n, \dots \rangle = \mathbb{k} \langle \bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \dots, \bar{\mathbf{a}}_n, \dots \rangle$$

(where  $\bar{\mathbf{c}} := \mathbf{c} \pmod{(\hbar-1)} (\mathcal{H}_\hbar^\vee)'$ ) as  $\mathbb{k}$ -algebras, and the Hopf structure is exactly the one of  $\mathcal{H}$  because it is given by the like formulas on generators. In a nutshell, we have  $(\mathcal{H}_\hbar^\vee)' \xrightarrow{\hbar \rightarrow 1} \mathcal{H}$  as Hopf  $\mathbb{k}$ -algebras. This completes the top part of the diagram (✠) in the Introduction, for  $H = \mathcal{H} (= \mathcal{H}_\nu)$ , because  $\mathcal{H}^\vee := \mathcal{H} / \cap_{n \in \mathbb{N}} J^n = \mathcal{H}$  by §2.2: namely,

$$U(\mathcal{L}_\nu) \xleftarrow[\mathcal{H}_\hbar^\vee]{0 \leftarrow \hbar \rightarrow 1} \mathcal{H} \xleftarrow[(\mathcal{H}_\hbar^\vee)']{1 \leftarrow \hbar \rightarrow 0} F[G_{\mathcal{L}_\nu}^*] \quad .$$

## § 4 The Rees deformation $\mathcal{H}_\hbar'$ .

**4.1 The goal.** The crystal duality principle (cf. [Ga2], [Ga4]) yields also a recipe to produce a 1-parameter deformation  $\mathcal{H}_\hbar'$  of  $\mathcal{H}$  which is a *quantized function algebra* (QFA in the sequel): namely,  $\mathcal{H}_\hbar'$  is a Hopf  $\mathbb{k}[\hbar]$ -algebra such that  $\mathcal{H}_\hbar' \Big|_{\hbar=1} = \mathcal{H}$  and  $\mathcal{H}_\hbar' \Big|_{\hbar=0} = F[G_+]$ , the function algebra of a connected algebraic Poisson group  $G_+$ . Thus

$\mathcal{H}_{\hbar}'$  is a *quantization* of  $F[G_+]$ , and the quantum symmetry  $\mathcal{H}$  is a deformation of the classical Poisson symmetry  $F[G_+]$ . By definition  $\mathcal{H}_{\hbar}'$  is the *Rees algebra* associated to a distinguished *increasing* Hopf algebra filtration of  $\mathcal{H}$ , and  $F[G_+]$  is simply the graded Hopf algebra associated to this filtration. The purpose of this section is to describe explicitly  $\mathcal{H}_{\hbar}'$  and its semiclassical limit  $F[G_+]$ , hence also  $G_+$  itself. This will also provide a direct, independent proof of all the above mentioned results about  $\mathcal{H}_{\hbar}'$  and  $F[G_+]$  themselves.

**4.2 The Rees algebra  $\mathcal{H}_{\hbar}'$ .** Let's consider Drinfeld's  $\delta_{\bullet}$ -maps, as in §3.2, for the Hopf algebra  $\mathcal{H}$ . Using them, we define the  $\delta_{\bullet}$ -filtration  $\underline{D} := \{D_n\}_{n \in \mathbb{N}}$  of  $\mathcal{H}$  by  $D_n := \text{Ker}(\delta_{n+1})$ , for all  $n \in \mathbb{N}$ . It is easy to show (cf. [Ga4]) that  $\underline{D}$  is a Hopf algebra filtration of  $\mathcal{H}$ ; moreover, since  $\mathcal{H}$  is graded connected, we have  $\mathcal{H} = \bigcup_{n \in \mathbb{N}} D_n =: \mathcal{H}'$ . We define the *Rees algebra* associated to  $\underline{D}$  as

$$\mathcal{H}_{\hbar}' := \mathbb{k}[\hbar] \cdot \sum_{n \geq 0} \hbar^{+n} D_n = \sum_{n \geq 0} \mathbb{k}[\hbar] \hbar^{+n} \cdot D_n \quad (\subseteq \mathcal{H}_{\hbar} := \mathcal{H}[\hbar]). \quad (4.1)$$

A trivial check shows that the following intrinsic characterization (inside  $\mathcal{H}_{\hbar}$ ) also holds:

$$\mathcal{H}_{\hbar}' = \{ \eta \in \mathcal{H}_{\hbar} \mid \delta_n(\eta) \in \hbar^n \mathcal{H}_{\hbar}^{\otimes n}, \forall n \in \mathbb{N} \} \quad (\subseteq \mathcal{H}_{\hbar}).$$

We shall describe  $\mathcal{H}_{\hbar}'$  explicitly, and we'll compute its specialization at  $\hbar = 0$  and at  $\hbar = 1$ : in particular we'll show that it is really a QFA and a deformation of  $\mathcal{H}$ , as claimed.

By (4.1), all we need is to compute the filtration  $\underline{D} = \{D_n\}_{n \in \mathbb{N}}$ ; the idea is to describe it in combinatorial terms, based on the non-commutative polynomial nature of  $\mathcal{H}$ .

**4.3 Gradings and filtrations:** Let  $\partial_-$  be the unique Lie algebra grading of  $\mathcal{L}_{\nu}$  given by  $\partial_-(x_n) := n - 1 + \delta_{n,1}$  (for all  $n \in \mathbb{N}_{\nu}$ ). Let also  $d$  be the standard Lie algebra grading associated with the central lower series of  $\mathcal{L}_{\nu}$ , i.e. the one defined by  $d([\dots [x_{s_1}, x_{s_2}], \dots x_{s_k}]) = k - 1$  on any Lie monomial of  $\mathcal{L}_{\nu}$ . As both  $\partial_-$  and  $d$  are Lie algebra gradings,  $(\partial_- - d)$  is a Lie algebra grading too. Let  $\{F_n\}_{n \in \mathbb{N}}$  be the Lie algebra filtration associated with  $(\partial_- - d)$ ; then the down-shifted filtration  $\underline{T} := \{T_n := F_{n-1}\}_{n \in \mathbb{N}}$  is again a Lie algebra filtration of  $\mathcal{L}_{\nu}$ . There is a unique algebra filtration on  $U(\mathcal{L}_{\nu})$  extending  $\underline{T}$ : we denote it  $\underline{\Theta} = \{\Theta_n\}_{n \in \mathbb{N}}$ , and set also  $\Theta_{-1} := \{0\}$ . Finally, for each  $y \in U(\mathcal{L}_{\nu}) \setminus \{0\}$  there is a unique  $\tau(y) \in \mathbb{N}$  with  $y \in \Theta_{\tau(y)} \setminus \Theta_{\tau(y)-1}$ ; in particular  $\tau(b) = \partial_-(b) - d(b)$ ,  $\tau(bb') = \tau(b) + \tau(b')$  and  $\tau([b, b']) = \tau(b) + \tau(b') - 1$  for  $b, b' \in B_{\nu}$ .

We can explicitly describe  $\underline{\Theta}$ . Indeed, let us fix any total order  $\preceq$  on the basis  $B_{\nu}$  of §1.1: then  $\mathcal{X} := \left\{ \underline{b} := b_1 \cdots b_k \mid k \in \mathbb{N}, b_1, \dots, b_k \in B_{\nu}, b_1 \preceq \cdots \preceq b_k \right\}$  is a  $\mathbb{k}$ -basis of  $U(\mathcal{L}_{\nu})$ , by the PBW theorem. It follows that  $\underline{\Theta}$  induces a set-theoretic filtration  $\underline{\mathcal{X}} = \{\mathcal{X}_n\}_{n \in \mathbb{N}}$  of  $\mathcal{X}$  with  $\mathcal{X}_n := \mathcal{X} \cap \Theta_n = \left\{ \underline{b} := b_1 \cdots b_k \mid k \in \mathbb{N}, b_1, \dots, b_k \in B_{\nu}, b_1 \preceq \cdots \preceq b_k, \tau(\underline{b}) = \tau(b_1) + \cdots + \tau(b_k) \leq n \right\}$ , and also that  $\Theta_n = \text{Span}(\mathcal{X}_n)$  for all  $n \in \mathbb{N}$ .

Let us define  $\alpha_1 := \mathbf{a}_1$  and  $\alpha_n := \mathbf{a}_n - \mathbf{a}_1^n$  for all  $n \in \mathbb{N}_\nu \setminus \{1\}$ . This “change of variables” — which switch from the  $\mathbf{a}_n$ ’s to their *differentials*, in a sense — is the key to achieve a complete description of  $\underline{D}$ , via a close comparison between  $\mathcal{H}$  and  $U(\mathcal{L}_\nu)$ .

By definition  $\mathcal{H} = \mathcal{H}_\nu$  is the free associative algebra over  $\{\mathbf{a}_n\}_{n \in \mathbb{N}_\nu}$ , hence (by definition of the  $\alpha$ ’s) also over  $\{\alpha_n\}_{n \in \mathbb{N}_\nu}$ ; so we have an algebra isomorphism  $\Phi: \mathcal{H} \xrightarrow{\cong} U(\mathcal{L}_\nu)$  given by  $\alpha_n \mapsto x_n$  ( $\forall n \in \mathbb{N}_\nu$ ). Via  $\Phi$  we pull back all data and results about gradings, filtrations, PBW bases and so on mentioned above for  $U(\mathcal{L}_\nu)$ ; in particular we set  $\alpha_{\underline{b}} := \Phi(x_{\underline{b}}) = \alpha_{b_1} \cdots \alpha_{b_k}$  ( $b_1, \dots, b_k \in B_\nu$ ),  $\mathcal{A}_n := \Phi(\mathcal{X}_n)$  ( $n \in \mathbb{N}$ ),  $\mathcal{A} := \Phi(\mathcal{X}) = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ . For gradings on  $\mathcal{H}$  we stick to the like notation, i.e.  $\partial_-$ ,  $d$  and  $\tau$ , and similarly for  $\underline{\Theta}$ .

Finally, for all  $a \in \mathcal{H} \setminus \{0\}$  we set  $\kappa(a) := k$  iff  $a \in D_k \setminus D_{k-1}$  (with  $D_{-1} := \{0\}$ ).

Our goal is to prove an identity of filtrations, namely  $\underline{D} = \underline{\Theta}$ , or equivalently  $\kappa = \tau$ . In fact, this would give to the Hopf filtration  $\underline{D}$ , which is defined intrinsically in Hopf algebraic terms, an explicit *combinatorial* description, namely the one of  $\Theta$  explained above.

**Lemma 4.1.**  $Q_t^\ell(\mathbf{a}_*) \in \Theta_t \setminus \Theta_{t-1}$ ,  $Z_t^\ell(\alpha_*) := \left( Q_t^\ell(\mathbf{a}_*) - \binom{\ell+t}{t} \mathbf{a}_1^t \right) \in \Theta_{t-1}$  ( $\ell, t \in \mathbb{N}, t \geq 1$ ).

*Proof.* When  $t = 1$  definitions give  $Q_1^\ell(\mathbf{a}_*) = (\ell+1) \mathbf{a}_1 \in \Theta_1$  and so  $Z_1^\ell(\alpha_*) = (\ell+1) \mathbf{a}_1 - \binom{\ell+1}{1} \mathbf{a}_1 = 0 \in \Theta_0$ , for all  $\ell \in \mathbb{N}$ . Similarly, when  $\ell = 0$  we have  $Q_t^0(\mathbf{a}_*) = \mathbf{a}_t \in \Theta_t$  and so  $Z_t^0(\alpha_*) = \mathbf{a}_t - \binom{1}{1} \mathbf{a}_1^t = \alpha_t \in \Theta_{t-1}$  (by definition), for all  $t \in \mathbb{N}_+$ .

When  $\ell > 0$  and  $t > 1$ , we can prove the claim using two independent methods.

First method: The very definitions imply that the following recurrence formula holds:

$$Q_t^\ell(\mathbf{a}_*) = Q_t^{\ell-1}(\mathbf{a}_*) + \sum_{s=1}^{t-1} Q_{t-s}^{\ell-1}(\mathbf{a}_*) \cdot \mathbf{a}_s + \mathbf{a}_t \quad \forall \ell \geq 1, t \geq 2.$$

From this formula and from the identities  $\mathbf{a}_1 = \alpha_1$ ,  $\mathbf{a}_s = \alpha_s + \alpha_1^s$  ( $s \in \mathbb{N}_+$ ), we argue

$$\begin{aligned} Z_t^\ell(\alpha_*) &:= Q_t^\ell(\mathbf{a}_*) - \binom{\ell+t}{t} \mathbf{a}_1^t = Q_t^{\ell-1}(\mathbf{a}_*) + \sum_{s=1}^{t-1} Q_{t-s}^{\ell-1}(\mathbf{a}_*) \mathbf{a}_s + \mathbf{a}_t - \binom{\ell+t}{t} \mathbf{a}_1^t = \\ &= Z_t^{\ell-1}(\mathbf{a}_*) + \binom{\ell-1+t}{t} \mathbf{a}_1^t + \sum_{s=1}^{t-1} \left( Z_{t-s}^{\ell-1}(\mathbf{a}_*) + \binom{\ell-1+t-s}{t-s} \mathbf{a}_1^{t-s} \right) \mathbf{a}_s + \mathbf{a}_t - \binom{\ell+t}{t} \mathbf{a}_1^t = \\ &= Z_t^{\ell-1}(\mathbf{a}_*) + \sum_{s=1}^{t-1} Z_{t-s}^{\ell-1}(\mathbf{a}_*) (\alpha_s + \alpha_1^s) + \sum_{s=1}^{t-1} \binom{\ell-1+t-s}{t-s} \alpha_1^{t-s} \alpha_s + \alpha_t + \\ &\quad + \sum_{s=1}^{t-1} \binom{\ell-1+t-s}{t-s} \alpha_1^{t-s} \alpha_1^s + \alpha_1^t + \binom{\ell-1+t}{t} \alpha_1^t - \binom{\ell+t}{t} \alpha_1^t = \\ &= Z_t^{\ell-1}(\mathbf{a}_*) + \sum_{s=1}^{t-1} Z_{t-s}^{\ell-1}(\mathbf{a}_*) (\alpha_s + \alpha_1^s) + \sum_{s=1}^{t-1} \binom{\ell-1+t-s}{t-s} \alpha_1^{t-s} \alpha_s + \left( \sum_{r=0}^t \binom{\ell-1+r}{\ell-1} - \right. \\ &\quad \left. - \binom{\ell+t}{t} \right) \alpha_1^t + \alpha_t = Z_t^{\ell-1}(\mathbf{a}_*) + \sum_{s=1}^{t-1} Z_{t-s}^{\ell-1}(\mathbf{a}_*) (\alpha_s + \alpha_1^s) + \sum_{s=1}^{t-1} \binom{\ell-1+t-s}{t-s} \alpha_1^{t-s} \alpha_s + \alpha_t \end{aligned}$$

because of the classical identity  $\binom{\ell+t}{\ell} = \sum_{r=0}^t \binom{\ell-1+r}{\ell-1}$ . Then induction upon  $\ell$  and the very definitions allow to argue that all summands in the final sum belong to  $\Theta_{t-1}$ , hence  $Z_t^\ell(\alpha_*) \in \Theta_{t-1}$  as well. Finally, this implies  $Q_t^\ell(\mathbf{a}_*) = Z_t^\ell(\alpha_*) + \binom{\ell+t}{t} \alpha_1^t \in \Theta_t \setminus \Theta_{t-1}$ .

Second method:  $Q_t^\ell(\mathbf{a}_*) := \sum_{s=1}^t \binom{\ell+1}{s} P_t^{(s)}(\mathbf{a}_*) = \sum_{s=1}^t \binom{\ell+1}{s} \sum_{\substack{j_1, \dots, j_s > 0 \\ j_1 + \dots + j_s = t}} \mathbf{a}_{j_1} \cdots \mathbf{a}_{j_s}$ ,

by definition; then expanding the  $\mathbf{a}_j$ ’s (for  $j > 1$ ) as above we find that  $Q_t^\ell(\mathbf{a}_*) =$

$Q_t^\ell(\alpha_* + \alpha_1^*)$  is a linear combination of monomials  $\alpha_{(j_1)} \cdots \alpha_{(j_s)}$  with  $j_1, \dots, j_s > 0$ ,  $j_1 + \cdots + j_s = t$ ,  $\alpha_{(j_r)} \in \{\alpha_{j_r}, \alpha_1^{j_r}\}$  for all  $r$ . Let  $Q_-$  be the linear combination of those monomials such that  $(\alpha_{(j_1)}, \alpha_{(j_2)}, \dots, \alpha_{(j_s)}) \neq (\alpha_1^{j_1}, \alpha_1^{j_2}, \dots, \alpha_1^{j_s})$ ; the remaining monomials enjoy  $\alpha_{j_1} \cdots \alpha_{j_s} = \alpha_1^{j_1 + \cdots + j_s} = \alpha_1^t$ , so their linear combination giving  $Q_+ := Q_t^\ell(\mathbf{a}_*) - Q_-$  is a multiple of  $\alpha_1^t$ , say  $Q_+ = N \alpha_1^t$ . Now we compute this  $N$ .

By construction,  $N$  is nothing but  $N = Q_t^\ell(1_*) = Q_t^\ell(1, 1, \dots, 1, \dots)$  where the latter is the value of  $Q_t^\ell$  when all indeterminates are set equal to 1; thus we compute  $Q_t^\ell(1_*)$ .

Recall that the  $Q_t^\ell$ 's enter in the definition of the coproduct of  $F[\mathcal{G}^{\text{dif}}]$ : the latter is dual to the (composition) product of series in  $\mathcal{G}^{\text{dif}}$ , thus if  $\{a_n\}_{n \in \mathbb{N}_+}$  and  $\{b_n\}_{n \in \mathbb{N}_+}$  are two countable sets of commutative indeterminates then

$$\begin{aligned} & \left( x + \sum_{n=1}^{+\infty} a_n x^{n+1} \right) \circ \left( x + \sum_{m=1}^{+\infty} b_m x^{m+1} \right) := \\ & = \left( \left( x + \sum_{m=1}^{+\infty} b_m x^{m+1} \right) + \sum_{n=1}^{+\infty} a_n \left( x + \sum_{m=1}^{+\infty} b_m x^{m+1} \right)^{n+1} \right) = x + \sum_{k=0}^{+\infty} c_k x^{k+1} \end{aligned}$$

with  $c_k = Q_k^0(b_*) + \sum_{r=1}^k a_r \cdot Q_{k-r}^r(b_*)$  (cf. §1.1). Specializing  $a_\ell = 1$  and  $a_r = 0$  for all  $r \neq \ell$  we get  $c_{t+\ell} = Q_{t+\ell}^0(b_*) + Q_t^\ell(b_*) = b_{t+\ell} + Q_t^\ell(b_*)$ . In particular setting  $b_* = 1_*$  we have that  $1 + Q_t^\ell(1_*)$  is the coefficient  $c_{\ell+t}$  of  $x^{\ell+t+1}$  in the series

$$\begin{aligned} & (x + x^{\ell+1}) \circ \left( x + \sum_{m=1}^{+\infty} x^{m+1} \right) = (x + x^{\ell+1}) \circ (x \cdot (1-x)^{-1}) = \\ & = x \cdot (1-x)^{-1} + (x \cdot (1-x)^{-1})^{\ell+1} = \sum_{m=0}^{+\infty} x^{m+1} + x^{\ell+1} \left( \sum_{m=0}^{+\infty} x^m \right)^{\ell+1} = \\ & = \sum_{m=0}^{+\infty} x^{m+1} + x^{\ell+1} \sum_{n=0}^{+\infty} \binom{\ell+n}{\ell} x^n = \sum_{s=0}^{\ell-1} x^{s+1} + \sum_{s=\ell}^{+\infty} (1 + \binom{s}{\ell}) x^{s+1} \quad ; \end{aligned}$$

therefore  $1 + Q_t^\ell(1_*) = c_{\ell+t} = 1 + \binom{\ell+t}{\ell}$ , whence  $Q_t^\ell(1_*) = \binom{\ell+t}{\ell}$ . As an alternative approach, one can prove that  $Q_t^\ell(1_*) = \binom{\ell+t}{\ell}$  by induction using the recurrence formula  $Q_t^\ell(\mathbf{x}_*) = Q_t^{\ell-1}(\mathbf{x}_*) + \sum_{s=1}^{t-1} Q_{t-s}^{\ell-1}(\mathbf{x}_*) \mathbf{x}_s + \mathbf{x}_t$  and the identity  $\binom{\ell+t}{\ell} = \sum_{s=0}^t \binom{\ell+t-1}{\ell-1}$ .

The outcome is  $N = Q_t^\ell(1_*) = \binom{\ell+t}{\ell}$  (for all  $t, \ell$ ), thus  $Q_t^\ell(\mathbf{a}_*) - \binom{\ell+t}{\ell} \mathbf{a}_t = Q_- + Q_+ - \binom{\ell+t}{\ell} \mathbf{a}_t = Q_- + N \mathbf{a}_t - \binom{\ell+t}{\ell} \mathbf{a}_t = Q_-$ . Now, by definition  $\tau(\alpha_{j_r}) = j_r - 1$  and  $\tau(\alpha_1^{j_r}) = j_r$ . Therefore if  $\alpha_{(j_r)} \in \{\alpha_{j_r}, \alpha_1^{j_r}\}$  (for all  $r = 1, \dots, s$ ) and  $(\alpha_{(j_1)}, \alpha_{(j_2)}, \dots, \alpha_{(j_s)}) \neq (\alpha_1^{j_1}, \alpha_1^{j_2}, \dots, \alpha_1^{j_s})$ , then  $\tau(\alpha_{(j_1)} \cdots \alpha_{(j_s)}) \leq j_1 + \cdots + j_s - 1 = t - 1$ . Then by construction  $\tau(Q_-) \leq t - 1$ , whence, since  $Z_t^\ell(\alpha_*) := Q_t^\ell(\mathbf{a}_*) - \binom{\ell+t}{\ell} \mathbf{a}_t = Q_-$ , we get also  $\tau(Z_t^\ell(\alpha_*)) \leq t - 1$ , i.e.  $Z_t^\ell(\alpha_*) \in \Theta_{t-1}$ , so  $Q_t^\ell(\mathbf{a}_*) = Z_t^\ell(\alpha_*) + \binom{\ell+t}{\ell} \alpha_1^t \in \Theta_t \setminus \Theta_{t-1}$ .  $\square$

**Proposition 4.1.**  $\underline{\Theta}$  is a Hopf algebra filtration of  $\mathcal{H}$ .

*Proof.* By construction (cf. §4.3)  $\underline{\Theta}$  is an algebra filtration; so to check it is Hopf too we are left only to show that  $(\star) \Delta(\Theta_n) \subseteq \sum_{r+s=n} \Theta_r \otimes \Theta_s$  (for all  $n \in \mathbb{N}$ ), for then  $S(\Theta_n) \subseteq \Theta_n$  (for all  $n$ ) will follow from that by recurrence (and Hopf algebra axioms).

By definition  $\Theta_0 = \mathbb{k} \cdot 1_{\mathcal{H}}$ ; then  $\Delta(1_{\mathcal{H}}) = 1_{\mathcal{H}} \otimes 1_{\mathcal{H}}$  proves  $(\star)$  for  $n = 0$ . For  $n = 1$ , by definition  $\Theta_1$  is the direct sum of  $\Theta_0$  with the (free) Lie (sub)algebra (of  $\mathcal{H}$ ) generated by  $\{\alpha_1, \alpha_2\}$ . Since  $\Delta(\alpha_1) = \alpha_1 \otimes 1 + 1 \otimes \alpha_1$  and  $\Delta(\alpha_2) = \alpha_2 \otimes 1 + 1 \otimes \alpha_2$  and

$$\Delta([x, y]) = [\Delta(x), \Delta(y)] = \sum_{(x), (y)} ([x_{(1)}, y_{(1)}] \otimes x_{(2)}y_{(2)} + x_{(1)}y_{(1)} \otimes [x_{(2)}, y_{(2)}])$$

(for all  $x, y \in \mathcal{H}$ ) we argue  $(\star)$  for  $n = 1$  too. Moreover, for every  $n > 1$  (setting  $Q_0^n(\mathbf{a}_*) = 1 = \mathbf{a}_0$  for short) we have  $\Delta(\alpha_n) = \Delta(\mathbf{a}_n) - \Delta(\mathbf{a}_1^n) = \sum_{k=0}^n \mathbf{a}_k \otimes Q_{n-k}^k(\mathbf{a}_*) - \sum_{k=0}^n \binom{n}{k} \mathbf{a}_1^k \otimes \mathbf{a}_1^{n-k} = \sum_{k=2}^n \alpha_k \otimes Q_{n-k}^k(\mathbf{a}_*) + \sum_{k=0}^{n-1} \alpha_1^k \otimes Z_{n-k}^k(\alpha_*)$ , and therefore  $\Delta(\alpha_n) \in \sum_{r+s=n-1} \Theta_r \otimes \Theta_s$  thanks to Lemma 4.1 (and to  $\alpha_m \in \Theta_{m-1}$  for  $m > 1$ ).

Finally, as  $\Delta([x, y]) = [\Delta(x), \Delta(y)] = \sum_{(x), (y)} ([x_{(1)}, y_{(1)}] \otimes x_{(2)}y_{(2)} + x_{(1)}y_{(1)} \otimes [x_{(2)}, y_{(2)}])$  and similarly  $\Delta(xy) = \Delta(x)\Delta(y) = \sum_{(x), (y)} x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}$  (for  $x, y \in \mathcal{H}$ ), we have that  $\Delta$  does not increase  $(\partial_- - d)$ : as  $\Theta$  is exactly the (algebra) filtration induced by  $(\partial_- - d)$ , it is a Hopf algebra filtration as well.  $\square$

**Lemma 4.2.** (notation of §4.3)

- (a)  $\kappa(a) \leq \partial(a)$  for every  $a \in \mathcal{H} \setminus \{0\}$  which is  $\partial(a)$ -homogeneous.
- (b)  $\kappa(aa') \leq \kappa(a) + \kappa(a')$  and  $\kappa([a, a']) < \kappa(a) + \kappa(a')$  for all  $a, a' \in \mathcal{H} \setminus \{0\}$ .
- (c)  $\kappa(\alpha_n) = \partial_-(\alpha_n) = \tau(\alpha_n)$  for all  $n \in \mathbb{N}_\nu$ .
- (d)  $\kappa([\alpha_r, \alpha_s]) = \partial_-(\alpha_r) + \partial_-(\alpha_s) - 1 = \tau([\alpha_r, \alpha_s])$  for all  $r, s \in \mathbb{N}_\nu$  with  $r \neq s$ .
- (e)  $\kappa(\alpha_b) = \partial_-(\alpha_b) - d(\alpha_b) + 1 = \tau(\alpha_b)$  for every  $b \in B_\nu$ .
- (f)  $\kappa(\alpha_{b_1}\alpha_{b_2}\cdots\alpha_{b_\ell}) = \tau(\alpha_{b_1}\alpha_{b_2}\cdots\alpha_{b_\ell})$  for all  $b_1, b_2, \dots, b_\ell \in B_\nu$ .
- (g)  $\kappa([\alpha_{b_1}, \alpha_{b_2}]) = \kappa(\alpha_{b_1}) + \kappa(\alpha_{b_2}) - 1 = \tau([\alpha_{b_1}, \alpha_{b_2}])$ , for all  $b_1, b_2 \in B_\nu$ .

*Proof.* (a) Let  $a \in \mathcal{H} \setminus \{0\}$  be  $\partial(a)$ -homogeneous. Since  $\mathcal{H}$  is graded, we have  $\partial(\delta_\ell(a)) = \partial(a)$  for all  $\ell$ ; moreover,  $\delta_\ell(a) \in J^{\otimes \ell}$  (with  $J := \text{Ker}(\epsilon_{\mathcal{H}})$ ) by definition, and  $\partial(y) > 0$  for each  $\partial$ -homogeneous  $y \in J \setminus \{0\}$ . Then  $\delta_\ell(a) = 0$  for all  $\ell > \partial(a)$ , whence the claim.

(b) Let  $a \in D_m, b \in D_n$ : then  $ab \in D_{m+n}$  by property (c) in §3.2. Similarly, we have  $[a, b] \in D_{m+n-1} \leq m+n-1$  because of property (d) in §3.2. The claim follows.

(c) By part (a) we have  $\kappa(\mathbf{a}_n) \leq \partial(\mathbf{a}_n) = n$ . Moreover, by definition  $\delta_2(\mathbf{a}_n) = \sum_{k=1}^{n-1} \mathbf{a}_k \otimes Q_{n-k}^k(\mathbf{a}_*)$ , thus  $\delta_n(\mathbf{a}_n) = (\delta_{n-1} \otimes \delta_1)(\delta_2(\mathbf{a}_n)) = \sum_{k=1}^{n-1} \delta_{n-1}(\mathbf{a}_k) \otimes \delta_1(Q_{n-k}^k(\mathbf{a}_*))$  by coassociativity. Since  $\delta_\ell(\mathbf{a}_m) = 0$  for  $\ell > m$ ,  $Q_1^{n-1}(\mathbf{a}_*) = n\mathbf{a}_1$  and  $\delta_1(\mathbf{a}_1) = \mathbf{a}_1$ , we have  $\delta_n(\mathbf{a}_n) = \delta_{n-1}(\mathbf{a}_{n-1}) \otimes (n\mathbf{a}_1)$ , thus by induction  $\delta_n(\mathbf{a}_n) = n! \mathbf{a}_1^{\otimes n}$  ( $\neq 0$ ), whence  $\kappa(\mathbf{a}_n) = n$ . But also  $\delta_n(\mathbf{a}_1^n) = n! \mathbf{a}_1^{\otimes n}$ . Thus  $\delta_n(\alpha_n) = \delta_n(\mathbf{a}_n) - \delta_n(\mathbf{a}_1^n) = 0$  for  $n > 1$ .

Clearly  $\kappa(\alpha_1) = 1$ . For the general case, for all  $\ell \geq 2$  we have

$$\delta_{\ell-1}(\mathbf{a}_\ell) = (\delta_{\ell-2} \otimes \delta_1)(\delta_2(\mathbf{a}_\ell)) = \sum_{k=1}^{\ell-1} \delta_{\ell-2}(\mathbf{a}_k) \otimes \delta_1(Q_{\ell-1-k}^k(\mathbf{a}_*))$$

which by the previous analysis gives  $\delta_{\ell-1}(\mathbf{a}_\ell) = \delta_{\ell-2}(\mathbf{a}_{\ell-2}) \otimes ((\ell-1)\mathbf{a}_2 + \binom{\ell-1}{2}\mathbf{a}_1^2) + \delta_{\ell-2}(\mathbf{a}_{\ell-1}) \otimes \ell\mathbf{a}_1 = (\ell-1)! \cdot \mathbf{a}_1^{\otimes(\ell-2)} \otimes (\mathbf{a}_2 + \frac{\ell-1}{2} \cdot \mathbf{a}_1^2) + \ell \cdot \delta_{\ell-2}(\mathbf{a}_{\ell-1}) \otimes \mathbf{a}_1$ . Iterating we get, for all  $\ell \geq 2$  (with  $\binom{-1}{1} := 0$ , and changing indices)

$$\delta_{\ell-1}(\mathbf{a}_\ell) = \sum_{m=1}^{\ell-1} \frac{\ell!}{m+1} \cdot \mathbf{a}_1^{\otimes(m-1)} \otimes (\mathbf{a}_2 + \frac{m-1}{2} \cdot \mathbf{a}_1^2) \otimes \mathbf{a}_1^{\otimes(\ell-1-m)}.$$

On the other hand, we have also  $\delta_{\ell-1}(\mathbf{a}_1^\ell) = \sum_{m=1}^{\ell-1} \frac{\ell!}{2} \cdot \mathbf{a}_1^{\otimes(m-1)} \otimes \mathbf{a}_1^2 \otimes \mathbf{a}_1^{\otimes(\ell-1-m)}$ . Therefore, for  $\delta_{n-1}(\boldsymbol{\alpha}_n) = \delta_{n-1}(\mathbf{a}_n) - \delta_{n-1}(\mathbf{a}_1^n)$  (for all  $n \in \mathbb{N}_\nu$ ,  $n \geq 2$ ) the outcome is

$$\begin{aligned} \delta_{n-1}(\boldsymbol{\alpha}_n) &= \sum_{m=1}^{n-1} \frac{n!}{m+1} \cdot \mathbf{a}_1^{\otimes(m-1)} \otimes (\mathbf{a}_2 - \mathbf{a}_1^2) \otimes \mathbf{a}_1^{\otimes(n-1-m)} = \\ &= \sum_{m=1}^{n-1} \frac{n!}{m+1} \cdot \boldsymbol{\alpha}_1^{\otimes(m-1)} \otimes \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1^{\otimes(n-1-m)} \quad ; \end{aligned} \quad (4.2)$$

in particular  $\delta_{n-1}(\boldsymbol{\alpha}_n) \neq 0$ , whence  $\boldsymbol{\alpha}_n \notin D_{n-2}$  and so  $\kappa(\boldsymbol{\alpha}_n) = n - 1$ , q.e.d.

(d) Let  $r \neq 1 \neq s$ . From (b)–(c) we get  $\kappa([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]) < \kappa(\boldsymbol{\alpha}_r) + \kappa(\boldsymbol{\alpha}_s) = r + s - 2$ . In addition, we prove that  $\delta_{r+s-3}([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]) \neq 0$ , yielding (d). Property (d) in §3.2 gives

$$\delta_{r+s-3}([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]) = \sum_{\substack{\Lambda \cup Y = \{1, \dots, r+s-3\} \\ \Lambda \cap Y \neq \emptyset}} [\delta_\Lambda(\boldsymbol{\alpha}_r), \delta_Y(\boldsymbol{\alpha}_s)] = \sum_{\substack{\Lambda \cup Y = \{1, \dots, r+s-3\} \\ \Lambda \cap Y \neq \emptyset, |\Lambda|=r-1, |Y|=s-1}} [j_\Lambda(\delta_{r-1}(\boldsymbol{\alpha}_r)), j_Y(\delta_{s-1}(\boldsymbol{\alpha}_s))].$$

Using (4.2) in the form  $\delta_{\ell-1}(\mathbf{a}_\ell) = \sum_{m=1}^{\ell-1} \frac{\ell!}{2} \cdot \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1^{\otimes(\ell-2)} + \boldsymbol{\alpha}_1 \otimes \eta_\ell$  (for some  $\eta_\ell \in \mathcal{H}$ ), and counting how many  $\Lambda$ 's and  $Y$ 's exist with  $1 \in \Lambda$  and  $\{1, 2\} \subseteq Y$ , and — conversely — how many of them exist with  $\{1, 2\} \subseteq \Lambda$  and  $1 \in Y$ , we argue

$$\delta_{r+s-3}([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]) = c_{r,s} \cdot [\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1^{\otimes(r+s-5)} + \boldsymbol{\alpha}_1 \otimes \varphi_1 + \boldsymbol{\alpha}_2 \otimes \varphi_2 + [\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes \boldsymbol{\alpha}_1 \otimes \psi$$

for some  $\varphi_1, \varphi_2 \in \mathcal{H}^{\otimes(r+s-4)}$ ,  $\psi \in \mathcal{H}^{\otimes(r+s-5)}$ , and with

$$c_{r,s} = \frac{r!}{2} \cdot \frac{s!}{3} \cdot \binom{r+s-5}{r-2} - \frac{s!}{2} \cdot \frac{r!}{3} \cdot \binom{s+r-5}{s-2} = \frac{2}{3} \binom{r}{2} \binom{s}{2} (s-r)(r+s-5)! \neq 0.$$

In particular  $\delta_{r+s-3}([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]) = c_{r,s} \cdot [\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1^{\otimes(r+s-5)} + \text{i.t.}$ , where “i.t.” stands for some further *terms* which are *linearly independent* of  $[\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1^{\otimes(r+s-5)}$  and  $c_{r,s} \neq 0$ . Then  $\delta_{r+s-3}([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]) \neq 0$ , q.e.d.

Finally, if  $r > 1 = s$  (and similarly if  $r = 1 < s$ ) things are simpler. Indeed, again (b) and (c) together give  $\kappa([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_1]) < \kappa(\boldsymbol{\alpha}_r) + \kappa(\boldsymbol{\alpha}_1) = (r-1) + 1 = r$ , and we prove that  $\delta_{r-1}([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_1]) \neq 0$ . Like before, property (d) in §3.2 gives (since  $\delta_1(\boldsymbol{\alpha}_1) = \boldsymbol{\alpha}_1$ )

$$\begin{aligned} \delta_{r-1}([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_1]) &= \sum_{\substack{\Lambda \cup Y = \{1, 2, \dots, r-1\} \\ \Lambda \cap Y \neq \emptyset, |\Lambda|=r-1, |Y|=1}} [\delta_\Lambda(\boldsymbol{\alpha}_r), \delta_Y(\boldsymbol{\alpha}_1)] = \sum_{k=1}^{r-1} \left[ \delta_{r-1}(\boldsymbol{\alpha}_r), 1^{\otimes(k-1)} \otimes \boldsymbol{\alpha}_1 \otimes 1^{\otimes(r-1-k)} \right] = \\ &= \sum_{m=1}^{r-1} \frac{r!}{m+1} \cdot \boldsymbol{\alpha}_1^{\otimes(m-1)} \otimes [\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes \boldsymbol{\alpha}_1^{\otimes(n-1-m)} \neq 0 \end{aligned}$$

(e) We perform induction upon  $d(b)$ : the case  $d(b) < 2$  is dealt with in parts (c) and (d), thus we assume  $d(b) \geq 2$ , so  $b = [b', x_\ell]$  for some  $\ell \in \mathbb{N}_\nu$  and some  $b' \in B_\nu$  with  $d(b') = d(b) - 1$ ; then  $\tau(\boldsymbol{\alpha}_b) = \tau([\boldsymbol{\alpha}_{b'}, \boldsymbol{\alpha}_\ell]) = \tau(\boldsymbol{\alpha}_{b'}) + \tau(\boldsymbol{\alpha}_\ell) - 1$ , directly from definitions. Moreover  $\tau(\boldsymbol{\alpha}_\ell) = \kappa(\boldsymbol{\alpha}_\ell)$  by part (c), and  $\tau(\boldsymbol{\alpha}_{b'}) = \kappa(\boldsymbol{\alpha}_{b'})$  by inductive assumption.

From (b) we have  $\kappa(\alpha_b) = \kappa([\alpha_{b'}, \alpha_\ell]) \leq \kappa(\alpha_{b'}) + \kappa(\alpha_\ell) - 1 = \tau(\alpha_{b'}) + \tau(\alpha_\ell) - 1 = \tau(\alpha_b)$ , i. e.  $\kappa(\alpha_b) \leq \tau(\alpha_b)$ ; we must prove the converse, for which it is enough to show

$$\delta_{\tau(\alpha_b)}(\alpha_b) = c_b \cdot [\cdots [\underbrace{[\alpha_1, \alpha_2], \alpha_2}, \dots, \alpha_2] \otimes \alpha_2 \otimes \alpha_1^{\otimes(\tau(\alpha_b)-2)} + \text{i.i.t.}] \quad (4.3)$$

for some  $c_b \in \mathbb{k} \setminus \{0\}$ , where “i.i.t.” means the same as before.

Since  $\tau(\alpha_b) = \tau([\alpha_{b'}, \alpha_\ell]) = \tau(\alpha_{b'}) + \ell - 2$ , using property (d) in §3.2 we get

$$\begin{aligned} \delta_{\tau(\alpha_b)}(\alpha_b) &= \delta_{\tau(\alpha_b)}([\alpha_{b'}, \alpha_\ell]) = \sum_{\substack{\Lambda \cup Y = \{1, \dots, \tau(\alpha_b)\} \\ \Lambda \cap Y \neq \emptyset}} [\delta_\Lambda(\alpha_{b'}), \delta_Y(\alpha_\ell)] = \\ &= \sum_{\substack{\Lambda \cup Y = \{1, \dots, \tau(\alpha_b)\}, \Lambda \cap Y \neq \emptyset \\ |\Lambda| = \tau(\alpha_{b'}), |Y| = \ell - 1}} [j_\Lambda(\delta_{\tau(\alpha_{b'})}(\alpha_{b'})), j_Y(\delta_{\ell-1}(\alpha_\ell))] = \\ &= \sum_{\substack{\Lambda \cup Y = \{1, \dots, \tau(\alpha_b)\}, \Lambda \cap Y \neq \emptyset \\ |\Lambda| = \tau(\alpha_{b'}), |Y| = \ell - 1}} \left[ j_\Lambda(c_{b'} [\cdots [\alpha_1, \alpha_2], \dots, \alpha_2] \otimes \alpha_2 \otimes \alpha_1^{\otimes(\tau(\alpha_{b'})-2)}), j_Y\left(\frac{\ell!}{2} \alpha_2 \otimes \alpha_1^{\otimes(\ell-2)}\right) \right] + \text{i.i.t.} = \\ &= c_{b'} \cdot \frac{\ell!}{2} \cdot \binom{\tau(\alpha_b)-2}{\ell-2} \cdot [\cdots [\underbrace{[\alpha_2, \alpha_1], \alpha_2}, \dots, \alpha_2], \alpha_2] \otimes \alpha_2 \otimes \alpha_1^{\otimes(\tau(\alpha_b)-2)} + \text{i.i.t.} \end{aligned}$$

(using induction about  $\alpha_{b'}$ ); this proves (4.3) with  $c_b = c_{b'} \cdot \frac{\ell!}{2} \cdot \binom{\tau(\alpha_b)-2}{\ell-2} \neq 0$ .

Thus (4.3) holds, yielding  $\delta_{\tau(\alpha_b)}(\alpha_b) \neq 0$ , hence  $\kappa(\alpha_b) \geq \tau(\alpha_b)$ , q.e.d.

(f) The case  $\ell = 1$  is proved by part (e), so we can assume  $\ell > 1$ . By part (b) and the case  $\ell = 1$  we have  $\kappa(\alpha_{b_1} \alpha_{b_2} \cdots \alpha_{b_\ell}) \leq \sum_{i=1}^{\ell} \kappa(\alpha_{b_i}) = \sum_{i=1}^{\ell} \tau(\alpha_{b_i}) = \tau(\alpha_{b_1} \alpha_{b_2} \cdots \alpha_{b_\ell})$ ; so we must only prove the converse inequality. We begin with  $\ell = 2$  and  $d(b_1) = d(b_2) = 0$ , so  $\alpha_{b_1} = \alpha_r$ ,  $\alpha_{b_2} = \alpha_s$ , for some  $r, s \in \mathbb{N}_\nu$ .

If  $r = s = 1$  then  $\kappa(\alpha_r) = \kappa(\alpha_s) = \kappa(\alpha_1) = 1$ , by part (c). Then

$$\delta_2(\alpha_1 \alpha_1) = \delta_2(\mathbf{a}_1 \mathbf{a}_1) = (\text{id} - \epsilon)^{\otimes 2} \Delta(\mathbf{a}_1^2) = 2 \cdot \mathbf{a}_1 \otimes \mathbf{a}_1 = 2 \cdot \alpha_1 \otimes \alpha_1 \neq 0$$

so that  $\kappa(\alpha_1 \alpha_1) \geq 2 = \kappa(\alpha_1) + \kappa(\alpha_1)$ , hence  $\kappa(\alpha_1 \alpha_1) = \kappa(\alpha_1) + \kappa(\alpha_1)$ , q.e.d.

If  $r > 1 = s$  (and similarly if  $r = 1 < s$ ) then  $\kappa(\alpha_r) = r - 1$ ,  $\kappa(\alpha_s) = \kappa(\alpha_1) = 1$ , by part (c). Then property (d) in §3.2 gives

$$\begin{aligned} \delta_r(\alpha_r \alpha_1) &= \sum_{\substack{\Lambda \cup Y = \{1, \dots, r\} \\ |\Lambda| = r-1, |Y| = 1}} \delta_\Lambda(\alpha_r) \delta_Y(\alpha_1) = \sum_{m=1}^r \sum_{k < m} \frac{r!}{m+1} \times \\ &\quad \times (\alpha_1^{\otimes(k-1)} \otimes 1 \otimes \alpha_1^{\otimes(m-1-k)} \otimes \alpha_2 \otimes \alpha_1^{\otimes(r-1-m)}) \times (1^{\otimes(k-1)} \otimes \alpha_1 \otimes 1^{\otimes(r-k)}) + \\ &+ \sum_{m=1}^r \sum_{k > m} \frac{r!}{m+1} (\alpha_1^{\otimes(m-1)} \otimes \alpha_2 \otimes \alpha_1^{\otimes(k-1-m)} \otimes 1 \otimes \alpha_1^{\otimes(r-1-k)}) \times (1^{\otimes(k-1)} \otimes \alpha_1 \otimes 1^{\otimes(r-k)}) = \\ &= \sum_{m=1}^r \frac{r!}{m+1} \cdot \alpha_1^{\otimes(m-1)} \otimes \alpha_2 \otimes \alpha_1^{\otimes(r-1-m)} \neq 0 \end{aligned}$$

so that  $\kappa(\alpha_r \alpha_1) \geq r = \kappa(\alpha_r) + \kappa(\alpha_1)$ , hence  $\kappa(\alpha_r \alpha_1) = \kappa(\alpha_r) + \kappa(\alpha_1)$ , q.e.d.

Finally let  $r, s > 1$  (and  $r \neq s$ ). Then  $\kappa(\alpha_r) = r - 1$ ,  $\kappa(\alpha_s) = s - 1$ , by part (c); then property (d) in §3.2 gives

$$\delta_{r+s-2}(\alpha_r \alpha_s) = \sum_{\substack{\Lambda \cup Y = \{1, \dots, r+s-2\} \\ |\Lambda|=r-1, |Y|=s-1}} \delta_\Lambda(\alpha_r) \cdot \delta_Y(\alpha_s) = \sum_{\substack{\Lambda \cup Y = \{1, \dots, r+s-2\} \\ |\Lambda|=r-1, |Y|=s-1}} j_\Lambda(\delta_{r-1}(\alpha_r)) \cdot j_Y(\delta_{s-1}(\alpha_s)).$$

Using (4.2) in the form  $\delta_{t-1}(\mathbf{a}_t) = \sum_{m=1}^{t-1} \frac{t!}{2} \cdot \alpha_2 \otimes \alpha_1^{\otimes(t-2)} + \alpha_1 \otimes \eta_t$  (for some  $\eta_t \in \mathcal{H}$  and  $t \in \{r, s\}$ ) and counting how many  $\Lambda$ 's and  $Y$ 's exist with  $1 \in \Lambda$  and  $2 \in Y$  and viceversa — actually, it is a matter of counting  $(r-2, s-2)$ -shuffles — we argue

$$\delta_{r+s-2}(\alpha_r \alpha_s) = e_{r,s} \cdot \alpha_2 \otimes \alpha_2 \otimes \alpha_1^{\otimes(r+s-4)} + \alpha_1 \otimes \varphi$$

for some  $\varphi \in \mathcal{H}^{\otimes(r+s-3)}$  with  $e_{r,s} = \frac{r!}{2} \cdot \frac{s!}{2} \cdot \left( \binom{r+s-4}{r-2} + \binom{r+s-4}{s-2} \right) = \frac{r!s!}{2} \cdot \binom{r+s-4}{r-2} \neq 0$ .

In particular  $\delta_{r+s-2}(\alpha_r \alpha_s) = e_{r,s} \cdot \alpha_2 \otimes \alpha_2 \otimes \alpha_1^{\otimes(r+s-4)} + \text{l.i.t.}$ , where “l.i.t.” stands again for some further terms which are linearly independent of  $\alpha_2 \otimes \alpha_2 \otimes \alpha_1^{\otimes(r+s-4)}$  and  $e_{r,s} \neq 0$ . Then  $\delta_{r+s-2}(\alpha_r \alpha_s) \neq 0$ , so  $\kappa(\alpha_r \alpha_1) \geq r + s - 2 = \kappa(\alpha_r) + \kappa(\alpha_1)$ , q.e.d.

Now let again  $\ell = 2$  but  $d(b_1), d(b_2) > 0$ . Set  $\kappa_i := \kappa(\alpha_{b_i})$  for  $i = 1, 2$ . Applying (4.3) to  $b = b_1$  and  $b = b_2$  (and reminding  $\tau \equiv \kappa$ ) gives

$$\begin{aligned} \delta_{\kappa_1 + \kappa_2}(\alpha_{b_1} \alpha_{b_2}) &= \sum_{\Lambda \cup Y = \{1, \dots, \kappa_1 + \kappa_2\}} \delta_\Lambda(\alpha_{b_1}) \delta_Y(\alpha_{b_2}) = \sum_{\substack{\Lambda \cup Y = \{1, \dots, \kappa_1 + \kappa_2\} \\ |\Lambda| = \kappa_1, |Y| = \kappa_2}} j_\Lambda(\delta_{\kappa_1}(\alpha_{b_1})) j_Y(\delta_{\kappa_2}(\alpha_{b_2})) = \\ &= \sum_{\substack{\Lambda \cup Y = \{1, \dots, \kappa_1 + \kappa_2\} \\ |\Lambda| = \kappa_1, |Y| = \kappa_2}} j_\Lambda(c_{b_1} \cdot [\dots [\underbrace{[\alpha_1, \alpha_2], \alpha_2], \dots, \alpha_2}_{d(b_1)+1}] \otimes \alpha_2 \otimes \alpha_1^{\otimes(\kappa_1-2)} + \text{l.i.t.}) \times \\ &\quad \times j_Y(c_{b_2} \cdot [\dots [\underbrace{[\alpha_1, \alpha_2], \alpha_2], \dots, \alpha_2}_{d(b_2)+1}] \otimes \alpha_2 \otimes \alpha_1^{\otimes(\kappa_2-2)} + \text{l.i.t.}) = \\ &= 2 c_{b_1} c_{b_2} \binom{\kappa_1 + \kappa_2 - 4}{\kappa_1 - 2} \cdot [\dots [\underbrace{[\alpha_1, \alpha_2], \dots, \alpha_2}_{d(b_1)+1}] \otimes [\dots [\underbrace{[\alpha_1, \alpha_2], \dots, \alpha_2}_{d(b_2)+1}] \otimes \alpha_2^{\otimes 2} \otimes \alpha_1^{\otimes(\kappa_1 + \kappa_2 - 4)} + \text{l.i.t.} \end{aligned}$$

which proves the claim for  $\ell = 2$ . In addition, we can take this last result as the basis of induction (on  $\ell$ ) to prove the following: for all  $\underline{b} := (b_1, \dots, b_\ell) \in B_\nu^\ell$ , one has

$$\delta_{|\underline{\kappa}|} \left( \prod_{i=1}^\ell \alpha_{b_i} \right) = c_{\underline{b}} \cdot \left( \bigotimes_{i=1}^\ell [\dots [\underbrace{[\alpha_1, \alpha_2], \alpha_2], \dots, \alpha_2}_{d(b_i)+1}] \right) \otimes \alpha_2^{\otimes \ell} \otimes \alpha_1^{\otimes(|\underline{\kappa}|-2\ell)} + \text{l.i.t.} \quad (4.4)$$

for some  $c_{\underline{b}} \in \mathbb{k} \setminus \{0\}$ , with  $|\underline{\kappa}| := \sum_{i=1}^\ell \kappa_i$  and  $\kappa_i := \kappa(\alpha_{b_i})$  ( $i = 1, \dots, \ell$ ). The induction step, from  $\ell$  to  $(\ell + 1)$ , amounts to compute (with  $\kappa_{\ell+1} := \kappa(\alpha_{b_{\ell+1}})$ )

$$\begin{aligned} \delta_{|\underline{\kappa}| + \kappa_{\ell+1}}(\alpha_{b_1} \cdots \alpha_{b_\ell} \cdot \alpha_{b_{\ell+1}}) &= \sum_{\Lambda \cup Y = \{1, \dots, |\underline{\kappa}| + \kappa_{\ell+1}\}} \delta_\Lambda(\alpha_{b_1} \cdots \alpha_{b_\ell}) \delta_Y(\alpha_{b_{\ell+1}}) = \\ &= \sum_{\substack{\Lambda \cup Y = \{1, \dots, |\underline{\kappa}| + \kappa_{\ell+1}\} \\ |\Lambda| = |\underline{\kappa}|, |Y| = \kappa_{\ell+1}}} j_\Lambda(\delta_{|\underline{\kappa}|}(\alpha_{b_1} \cdots \alpha_{b_\ell})) \cdot j_Y(\delta_{\kappa_{\ell+1}}(\alpha_{b_{\ell+1}})) = \end{aligned}$$



$$\begin{aligned}
 &= \sum_{\substack{\Lambda \cup Y = \{1, \dots, |\underline{\kappa}| + \kappa_{\ell+1}\} \\ |\Lambda| = |\underline{\kappa}|, |Y| = \kappa_{\ell+1}}} j_{\Lambda} \left( c_{\underline{b}} \cdot \left( \bigotimes_{i=1}^{\ell} [\dots [\underbrace{[\alpha_1, \alpha_2], \alpha_2], \dots, \alpha_2}_{d(b_i)+1}] \right) \otimes \alpha_2^{\otimes \ell} \otimes \alpha_1^{\otimes (|\underline{\kappa}| - 2\ell)} + l.i.t. \right) \times \\
 &\quad \times j_Y \left( c_{b_{\ell+1}} \cdot [\dots [\underbrace{[\alpha_1, \alpha_2], \alpha_2], \dots, \alpha_2}_{d(b_{\ell+1})+1}] \otimes \alpha_2 \otimes \alpha_1^{\otimes (\kappa_{\ell+1} - 2)} + l.i.t. \right) = \\
 &= c_{\underline{b}} c_{b_{\ell+1}} \cdot (\ell + 1) \binom{|\underline{\kappa}| + \kappa_{\ell+1} - 2(\ell+1)}{|\underline{\kappa}| - 2\ell} \cdot \left( \bigotimes_{i=1}^{\ell} [\dots [\underbrace{[\alpha_1, \alpha_2], \alpha_2], \dots, \alpha_2}_{d(b_i)+1}] \right) \otimes \\
 &\quad \otimes [\dots [\underbrace{[\alpha_1, \alpha_2], \alpha_2], \dots, \alpha_2}_{d(b_{\ell+1})+1}] \otimes \alpha_2^{\otimes (\ell+1)} \otimes \alpha_1^{\otimes (|\underline{\kappa}| + \kappa_{\ell+1} - 2(\ell+1))} + l.i.t.
 \end{aligned}$$

which proves (4.4) for  $(\underline{b}, b_{\ell+1})$  with  $c_{(\underline{b}, b_{\ell+1})} = c_{\underline{b}} c_{b_{\ell+1}} \cdot (\ell + 1) \binom{|\underline{\kappa}| + \kappa_{\ell+1} - 2(\ell+1)}{|\underline{\kappa}| - 2\ell} \neq 0$ . Finally, (4.4) yields  $\delta_{|\underline{\kappa}|}(\alpha_{b_1} \cdots \alpha_{b_{\ell}}) \neq 0$ , so  $\kappa(\alpha_{b_1} \cdots \alpha_{b_{\ell}}) \geq \kappa(\alpha_{b_1}) + \cdots + \kappa(\alpha_{b_{\ell}})$ , q.e.d.

(g) Part (d) proves the claim for  $d(b_1) = d(b_2) = 0$ , that is  $b_1, b_2 \in \{x_n\}_{n \in \mathbb{N}}$ . Moreover, when  $b_2 = x_n \in \{x_m\}_{m \in \mathbb{N}_\nu}$  we can replicate the proof of part (d) to show that  $\kappa([\alpha_{b_1}, \alpha_{b_2}]) = \kappa([\alpha_{b_1}, \alpha_n]) = \partial_-([\alpha_{b_1}, \alpha_n]) - d([\alpha_{b_1}, \alpha_n])$ : but the latter is exactly  $\tau([\alpha_{b_1}, \alpha_{b_2}])$ , q.e.d. Everything is similar if  $b_1 = x_n \in \{x_m\}_{m \in \mathbb{N}_\nu}$ .

Now let  $b_1, b_2 \in B_\nu \setminus \{x_n\}_{n \in \mathbb{N}_\nu}$ . Then (b) gives  $\kappa([\alpha_{b_1}, \alpha_{b_2}]) \leq \kappa(\alpha_{b_1}) + \kappa(\alpha_{b_2}) - 1 = \tau([\alpha_{b_1}, \alpha_{b_2}])$ . Applying (4.3) to  $b = b_1$  and  $b = b_2$  we get, for  $\kappa_i := \kappa(\alpha_{b_i})$  ( $i = 1, 2$ )

$$\begin{aligned}
 \delta_{\kappa_1 + \kappa_2 - 1}([\alpha_{b_1}, \alpha_{b_2}]) &= \sum_{\substack{\Lambda \cup Y = \{1, \dots, \kappa_1 + \kappa_2 - 1\} \\ \Lambda \cap Y \neq \emptyset}} [\delta_{\Lambda}(\alpha_{b_1}), \delta_Y(\alpha_{b_2})] = \\
 &= \sum_{\substack{\Lambda \cup Y = \{1, \dots, \kappa_1 + \kappa_2\} \\ |\Lambda| = \kappa_1, |Y| = \kappa_2}} [j_{\Lambda}(c_{b_1} \cdot [\dots [\underbrace{[\alpha_2, \alpha_1], \alpha_2], \dots, \alpha_2}_{d(b_1)+1}] \otimes \alpha_2 \otimes \alpha_1^{\otimes (\kappa_1 - 2)} + l.i.t.) \times \\
 &\quad \times j_Y(c_{b_2} \cdot [\dots [\underbrace{[\alpha_2, \alpha_1], \alpha_2], \dots, \alpha_2}_{d(b_2)+1}] \otimes \alpha_2 \otimes \alpha_1^{\otimes (\kappa_2 - 2)} + l.i.t.)] = \\
 &= 2 c_{b_1} c_{b_2} \binom{\kappa_1 + \kappa_2 - 4}{\kappa_1 - 2} \left[ [\dots [\underbrace{[\alpha_2, \alpha_1], \dots, \alpha_2}_{d(b_1)+1}], [\dots [\underbrace{[\alpha_2, \alpha_1], \dots, \alpha_2}_{d(b_2)+1}]] \right] \otimes \alpha_2^{\otimes 2} \otimes \alpha_1^{\otimes (\kappa_1 + \kappa_2 - 4)} + l.i.t.
 \end{aligned}$$

(note that  $d(b_i) \geq 1$  because  $b_i \notin \{x_n \mid n \in \mathbb{N}_\nu\}$  for  $i = 1, 2$ ). In particular this means  $\delta_{\kappa_1 + \kappa_2 - 1}([\alpha_{b_1}, \alpha_{b_2}]) \neq 0$ , thus  $\kappa([\alpha_{b_1}, \alpha_{b_2}]) \geq \kappa(\alpha_{b_1}) + \kappa(\alpha_{b_2}) - 1 = \tau([\alpha_{b_1}, \alpha_{b_2}])$ .  $\square$

**Lemma 4.3.** *Let  $V$  be a  $\mathbb{k}$ -vector space, and  $\psi \in \text{Hom}_{\mathbb{k}}(V, V \wedge V)$ . Let  $\mathcal{L}(V)$  be the free Lie algebra over  $V$ , and  $\psi_{d\mathcal{L}} \in \text{Hom}_{\mathbb{k}}(\mathcal{L}(V), \mathcal{L}(V) \wedge \mathcal{L}(V))$  the unique extension of  $\psi$  from  $V$  to  $\mathcal{L}(V)$  by derivations, i.e. such that  $\psi_{d\mathcal{L}}|_V = \psi$  and  $\psi_{d\mathcal{L}}([x, y]) = [x \otimes 1 + 1 \otimes x, \psi_{d\mathcal{L}}(y)] + [\psi_{d\mathcal{L}}(x), y \otimes 1 + 1 \otimes y] = x \cdot \psi_{d\mathcal{L}}(y) - y \cdot \psi_{d\mathcal{L}}(x)$  in the  $\mathcal{L}(V)$ -module  $\mathcal{L}(V) \wedge \mathcal{L}(V)$ ,  $\forall x, y \in \mathcal{L}(V)$ . Let  $K := \text{Ker}(\psi)$ : then  $\text{Ker}(\psi_{d\mathcal{L}}) = \mathcal{L}(K)$ , the free Lie algebra over  $K$ .*

*Proof.* Standard, by universal arguments (for a direct proof see [Ga2], Lemma 10.15).  $\square$

**Lemma 4.4.** *The Lie cobracket  $\delta$  of  $U(\mathcal{L}_\nu)$  preserves  $\tau$ . That is, for each  $\vartheta \in U(\mathcal{L}_\nu)$  in the expansion  $\delta_2(\vartheta) = \sum_{\underline{b}_1, \underline{b}_2 \in \mathbb{B}} c_{\underline{b}_1, \underline{b}_2} \alpha_{\underline{b}_1} \otimes \alpha_{\underline{b}_2}$  (w.r.t. the basis  $\mathbb{B} \otimes \mathbb{B}$ , where  $\mathbb{B}$  is a PBW basis as in §1.1 w.r.t. some total order of  $B_\nu$ ) we have  $\tau(\hat{\underline{b}}_1) + \tau(\hat{\underline{b}}_2) = \tau(\vartheta)$  for some  $\hat{\underline{b}}_1, \hat{\underline{b}}_2$  with  $c_{\hat{\underline{b}}_1, \hat{\underline{b}}_2} \neq 0$ , so  $\tau(\delta(\vartheta)) := \max \{ \tau(\underline{b}_1) + \tau(\underline{b}_2) \mid c_{\underline{b}_1, \underline{b}_2} \neq 0 \} = \tau(\vartheta)$  if  $\delta(\vartheta) \neq 0$ .*

*Proof.* It follows from Proposition 4.1 that  $\tau(\delta(\vartheta)) \leq \tau(\vartheta)$ ; so  $\delta: U(\mathcal{L}_\nu) \rightarrow U(\mathcal{L}_\nu)^{\otimes 2}$  is a morphism of filtered algebras, hence it naturally induces a morphism of graded algebras  $\bar{\delta}: G_{\underline{\Theta}}(U(\mathcal{L}_\nu)) \rightarrow G_{\underline{\Theta}}(U(\mathcal{L}_\nu))^{\otimes 2}$ . Thus proving the claim is equivalent to showing that  $\text{Ker}(\bar{\delta}) = G_{\underline{\Theta} \cap \text{Ker}(\delta)}(\text{Ker}(\delta)) =: \overline{\text{Ker}(\delta)}$ , the latter being embedded into  $G_{\underline{\Theta}}(U(\mathcal{L}_\nu))$ .

By construction,  $\tau(xy - yx) = \tau([x, y]) < \tau(x) + \tau(y)$  for  $x, y \in U(\mathcal{L}_\nu)$ , so  $G_{\underline{\Theta}}(U(\mathcal{L}_\nu))$  is commutative: indeed, it is clearly isomorphic — as an algebra — to  $S(V_\nu)$ , the symmetric algebra over  $V_\nu$ . Moreover,  $\delta$  acts as a derivation, that is  $\delta(xy) = \delta(x) \Delta(y) + \Delta(x) \delta(y)$  (for all  $x, y \in U(\mathcal{L}_\nu)$ ), thus the same holds for  $\bar{\delta}$  too. Like in Lemma 4.3, since  $G_{\underline{\Theta}}(U(\mathcal{L}_\nu))$  is generated by  $G_{\underline{\Theta} \cap \mathcal{L}_\nu}(\mathcal{L}_\nu) =: \overline{\mathcal{L}_\nu}$  it follows that  $\text{Ker}(\bar{\delta})$  is the free (associative sub)algebra over  $\text{Ker}(\bar{\delta}|_{\overline{\mathcal{L}_\nu}})$ , in short  $\text{Ker}(\bar{\delta}) = \langle \text{Ker}(\bar{\delta}|_{\overline{\mathcal{L}_\nu}}) \rangle$ . Now, by definition  $\delta(x_n) = \sum_{\ell=1}^{n-1} (\ell+1) x_\ell \wedge x_{n-\ell}$  (cf. Theorem 2.1) is  $\tau$ -homogeneous, of  $\tau$ -degree equal to  $\tau(x_n) = n-1$ . As  $\delta$  also enjoys  $\delta([x, y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] + [\delta(x), y \otimes 1 + 1 \otimes y]$  (for  $x, y \in \mathcal{L}_\nu$ ) we have that  $\delta|_{\mathcal{L}_\nu}$  is even  $\tau$ -homogeneous, i.e. such that  $\tau(\delta(z)) = \tau(z)$ , for any  $\tau$ -homogeneous  $z \in \mathcal{L}_\nu$  such that  $\delta(z) \neq 0$ ; this implies that the induced map  $\bar{\delta}|_{\overline{\mathcal{L}_\nu}}$  enjoys  $\bar{\delta}|_{\overline{\mathcal{L}_\nu}}(\vartheta) = \bar{0} \iff \delta(\vartheta) = 0$  for any  $\vartheta \in \mathcal{L}_\nu$ , whence  $\text{Ker}(\bar{\delta}|_{\overline{\mathcal{L}_\nu}}) = \overline{\text{Ker}(\delta|_{\mathcal{L}_\nu})}$ . On the upshot we get  $\text{Ker}(\bar{\delta}) = \langle \text{Ker}(\bar{\delta}|_{\overline{\mathcal{L}_\nu}}) \rangle = \langle \overline{\text{Ker}(\delta|_{\mathcal{L}_\nu})} \rangle = \overline{\text{Ker}(\delta)}$ , q.e.d.  $\square$

**Proposition 4.2.**  *$\underline{D} = \underline{\Theta}$ , that is  $D_n = \Theta_n$  for all  $n \in \mathbb{N}$ , or  $\kappa = \tau$ . Therefore, given any total order  $\preceq$  in  $B_\nu$ , the set  $\mathcal{A}_{\leq n} = \mathcal{A} \cap \Theta_n = \mathcal{A} \cap D_n$  of ordered monomials*

$$\mathcal{A}_{\leq n} = \left\{ \alpha_{\underline{b}} = \alpha_{b_1} \cdots \alpha_{b_k} \mid k \in \mathbb{N}, b_1, \dots, b_k \in B_\nu, b_1 \preceq \cdots \preceq b_k, \tau(\underline{b}) \leq n \right\}$$

*is a  $\mathbb{k}$ -basis of  $D_n$ , and  $\mathcal{A}_n := (\mathcal{A}_{\leq n} \bmod D_{n-1})$  is a  $\mathbb{k}$ -basis of  $D_n/D_{n-1}$  ( $\forall n \in \mathbb{N}$ ).*

*Proof.* Both claims about the  $\mathcal{A}_{\leq n}$ 's and  $\mathcal{A}_n$ 's are equivalent to  $\underline{D} = \underline{\Theta}$ . Also,  $\mathcal{A}_n := (\mathcal{A}_{\leq n} \bmod D_{n-1}) = (\mathcal{A}_{\leq n} \setminus \mathcal{A}_{\leq n-1} \bmod D_{n-1})$ , with  $\mathcal{A}_{\leq n} \setminus \mathcal{A}_{\leq n-1} = \{ \alpha_{\underline{b}} \in \mathcal{A} \mid \tau(\underline{b}) = n \}$ .

By Lemma 4.2(f) we have  $\mathcal{A}_{\leq n} = \mathcal{A} \cap \Theta_n \subseteq \mathcal{A} \cap D_n \subseteq D_n$ ; since  $\mathcal{A}$  is a basis,  $\mathcal{A}_{\leq n}$  is linearly independent and is a  $\mathbb{k}$ -basis of  $\Theta_n$  (by definition): so  $\Theta_n \subseteq D_n$  for all  $n \in \mathbb{N}$ .

$n = 0$ : By definition  $D_0 := \text{Ker}(\delta_1) = \mathbb{k} \cdot 1_\nu =: \Theta_0$ , spanned by  $\mathcal{A}_{\leq 0} = \{1_\nu\}$ , q.e.d.

$n = 1$ : Let  $\eta' \in D_1 := \text{Ker}(\delta_2)$ . Let  $\mathbb{B}$  be a PBW-basis of  $\mathcal{H}_\hbar^\vee = U(\mathcal{L}_\nu)$  as in Lemma 4.4; expanding  $\eta'$  w.r.t.  $\mathcal{A}$  we have  $\eta' = \sum_{\alpha_{\underline{b}} \in \mathcal{A}} c_{\underline{b}} \alpha_{\underline{b}} = \sum_{\underline{b} \in \mathbb{B}} c_{\underline{b}} \alpha_{\underline{b}}$ . Then  $\eta := \eta' - \sum_{\tau(\underline{b}) \leq 1} c_{\underline{b}} \alpha_{\underline{b}} = \sum_{\tau(\underline{b}) > 1} c_{\underline{b}} \alpha_{\underline{b}} \in D_1$ , since  $\alpha_{\underline{b}} \in \mathcal{A}_1 \subseteq \Theta_1 \subseteq D_1$  for  $\tau(\underline{b}) \leq 1$ .

Now,  $\alpha_1 := \mathbf{a}_1$  and  $\alpha_s := \mathbf{a}_s - \mathbf{a}_1^s = \hbar(\mathbf{x}_s + \hbar^{s-1} \mathbf{x}_1^s)$  for all  $s \in \mathbb{N}_\nu \setminus \{1\}$  yield

$$\eta = \sum_{\substack{\underline{b} \in \mathbb{B} \\ \tau(\underline{b}) > 1}} c_{\underline{b}} \alpha_{\underline{b}} = \sum_{\substack{\underline{b} \in \mathbb{B} \\ \tau(\underline{b}) > 1}} \hbar^{g(\underline{b})} c_{\underline{b}} (\mathbf{x}_{\underline{b}} + \hbar \chi_{\underline{b}}) \in \mathcal{H}_\hbar^\vee$$

for some  $\chi_{\underline{b}} \in \mathcal{H}_{\hbar}^{\vee}$ : hereafter we set  $g(\underline{b}) := k$  for each  $\underline{b} = b_1 \cdots b_k \in \mathbb{B}$  (i.e.  $g(\underline{b})$  is the degree of  $\underline{b}$  as a monomial in the  $b_i$ 's). If  $\eta \neq 0$ , let  $g_0 := \min \{ g(\underline{b}) \mid \tau(\underline{b}) > 1, c_{\underline{b}} \neq 0 \}$ ; then  $g_0 > 0$ ,  $\eta_+ := \hbar^{-g_0} \eta \in \mathcal{H}_{\hbar}^{\vee} \setminus \hbar \mathcal{H}_{\hbar}^{\vee}$  and

$$0 \neq \overline{\eta_+} = \sum_{g(\underline{b})=g_0} c_{\underline{b}} \overline{\alpha_{\underline{b}}} = \sum_{g(\underline{b})=g_0} c_{\underline{b}} x_{\underline{b}} \in \mathcal{H}_{\hbar}^{\vee} / \hbar \mathcal{H}_{\hbar}^{\vee} = U(\mathcal{L}_{\nu}).$$

Now  $\delta_2(\eta) = 0$  yields  $\delta_2(\overline{\eta_+}) = 0$ , thus  $\sum_{g(\underline{b})=g_0} c_{\underline{b}} x_{\underline{b}} = \overline{\eta_+} \in P(U(\mathcal{L}_{\nu})) = \mathcal{L}_{\nu}$ ; therefore all PBW monomials occurring in the last sum do belong to  $B_{\nu}$  (and  $g_0 = 1$ ). In addition,  $\delta_2(\eta) = 0$  also implies  $\delta_2(\eta_+) = 0$  which yields also  $\delta(\overline{\eta_+}) = 0$  for the Lie cobracket  $\delta$  of  $\mathcal{L}_{\nu}$  arising as semiclassical limit of  $\Delta_{\mathcal{H}_{\hbar}^{\vee}}$  (see Theorem 2.1); therefore  $\overline{\eta_+} = \sum_{b \in B_{\nu}} c_b x_b$  is an element of  $\mathcal{L}_{\nu}$  killed by the Lie cobracket  $\delta$ , i.e.  $\overline{\eta_+} \in \text{Ker}(\delta)$ .

Now we apply Lemma 4.3 to  $V = V_{\nu}$ ,  $\mathcal{L}(V) = \mathcal{L}(V_{\nu}) =: \mathcal{L}_{\nu}$  and  $\psi = \delta|_{V_{\nu}}$ , so that  $\psi_{d\mathcal{L}} = \delta$ . By the formulas for  $\delta$  in Theorem 2.1 we get  $K := \text{Ker}(\psi) = \text{Ker}(\delta|_{V_{\nu}}) = \text{Span}(\{x_1, x_2\})$ , hence  $\mathcal{L}(K) = \mathcal{L}(\text{Span}(\{x_1, x_2\})) = \text{Span}(\{x_b \mid b \in B_{\nu}; \tau(b) = 1\})$ , thus eventually (via Theorem 2.1)  $\text{Ker}(\delta) = \mathcal{L}(K) = \text{Span}(\{x_b \mid b \in B_{\nu}; \tau(b) = 1\})$ .

As  $\overline{\eta_+} \in \text{Ker}(\delta) = \text{Span}(\{x_b \mid b \in B_{\nu}; \tau(b) = 1\})$ , we have  $\overline{\eta_+} = \sum_{b \in B_{\nu}, \tau(b)=1} c_b x_b$ ; but  $c_b = 0$  whenever  $\tau(b) \leq 1$ , by construction of  $\eta$ : thus  $\overline{\eta_+} = 0$ , a contradiction. The outcome is  $\eta = 0$ , whence finally  $\eta' \in \Theta_1$ , q.e.d.

$n > 1$ : We must show that  $D_n = \Theta_n$ , while assuming by induction that  $D_m = \Theta_m$  for all  $m < n$ . Let  $\eta = \sum_{\underline{b} \in \mathbb{B}} c_{\underline{b}} \alpha_{\underline{b}} \in D_n$ ; then  $\tau(\eta) = \max \{ \tau(\underline{b}) \mid c_{\underline{b}} \neq 0 \}$ . If  $\delta_2(\eta) = 0$  then  $\eta \in D_1 = \Theta_1$  by the previous analysis, and we're done. Otherwise,  $\delta_2(\eta) \neq 0$  and  $\tau(\delta_2(\eta)) = \tau(\eta)$  by Lemma 4.4. On the other hand, since  $\underline{D}$  is a Hopf algebra filtration we have  $\delta_2(\eta) \in \sum_{\substack{r+s=n \\ r,s>0}} D_r \otimes D_s = \sum_{\substack{r+s=n \\ r,s>0}} \Theta_r \otimes \Theta_s$ , thanks to the induction; but then  $\tau(\delta_2(\eta)) \leq n$ , by definition of  $\tau$ . Thus  $\tau(\eta) = \tau(\delta_2(\eta)) \leq n$ , which means  $\eta \in \Theta_n$ .  $\square$

**Theorem 4.1.** *For any  $b \in B_{\nu}$  set  $\widehat{\alpha}_b := \hbar^{\kappa(\alpha_b)} \alpha_b = \hbar^{\tau(b)} \alpha_b$ .*

(a) *The set of ordered monomials*

$$\widehat{\mathcal{A}}_{\leq n} := \left\{ \widehat{\alpha}_{\underline{b}} := \widehat{\alpha}_{b_1} \cdots \widehat{\alpha}_{b_k} \mid k \in \mathbb{N}, b_1, \dots, b_k \in B, b_1 \leq \dots \leq b_k, \kappa(\alpha_{\underline{b}}) = \tau(\underline{b}) \leq n \right\}$$

is a  $\mathbb{k}[\hbar]$ -basis of  $D'_n = D_n(\mathcal{H}'_{\hbar}) = \hbar^n D_n$ . So  $\widehat{\mathcal{A}} := \bigcup_{n \in \mathbb{N}} \widehat{\mathcal{A}}_{\leq n}$  is a  $\mathbb{k}[\hbar]$ -basis of  $\mathcal{H}'_{\hbar}$ .

$$(b) \mathcal{H}'_{\hbar} = \mathbb{k}[\hbar] \left\langle \left\{ \widehat{\alpha}_b \right\}_{b \in B_{\nu}} \right\rangle / \left( \left\{ [\widehat{\alpha}_{b_1}, \widehat{\alpha}_{b_2}] - \hbar \widehat{\alpha}_{[b_1, b_2]} \mid \forall b_1, b_2 \in B_{\nu} \right\} \right).$$

(c)  $\mathcal{H}'_{\hbar}$  is a graded Hopf  $\mathbb{k}[\hbar]$ -subalgebra of  $\mathcal{H}_{\hbar}$ .

(d)  $\mathcal{H}'_{\hbar} \Big|_{\hbar=0} := \mathcal{H}'_{\hbar} / \hbar \mathcal{H}'_{\hbar} = \widetilde{\mathcal{H}} = F[\Gamma_{\mathcal{L}_{\nu}}^{\star}]$ , where  $\Gamma_{\mathcal{L}_{\nu}}^{\star}$  is a connected Poisson algebraic group with cotangent Lie bialgebra isomorphic to  $\mathcal{L}_{\nu}$  (as a Lie algebra) with the graded Lie bialgebra structure given by  $\delta(x_n) = (n-2)x_{n-1} \wedge x_1$  (for all  $n \in \mathbb{N}_{\nu}$ ). Indeed,  $\mathcal{H}'_{\hbar} \Big|_{\hbar=0}$

is the free Poisson (commutative) algebra over  $\mathbb{N}_\nu$ , generated by all the  $\bar{\alpha}_n := \widehat{\alpha}_n|_{\hbar=0}$  ( $n \in \mathbb{N}_\nu$ ) with Hopf structure given (for all  $n \in \mathbb{N}_\nu$ ) by

$$\begin{aligned} \Delta(\bar{\alpha}_n) &= \bar{\alpha}_n \otimes 1 + 1 \otimes \bar{\alpha}_n + \sum_{k=2}^{n-1} \binom{n}{k} \bar{\alpha}_k \otimes \bar{\alpha}_1^{n-k} + \sum_{k=1}^{n-1} (k+1) \bar{\alpha}_1^k \otimes \bar{\alpha}_{n-k} \\ S(\bar{\alpha}_n) &= -\bar{\alpha}_n - \sum_{k=2}^{n-1} \binom{n}{k} S(\bar{\alpha}_k) \bar{\alpha}_1^{n-k} - \sum_{k=1}^{n-1} (k+1) S(\bar{\alpha}_1)^k \bar{\alpha}_{n-k}, \quad \epsilon(\bar{\alpha}_n) = 0. \end{aligned}$$

Thus  $\mathcal{H}_\hbar'|_{\hbar=0}$  is the polynomial algebra  $\mathbb{k}[\{\eta_b\}_{b \in B_\nu}]$  generated by a set of indeterminates  $\{\eta_b\}_{b \in B_\nu}$  in bijection with  $B_\nu$ , so  $\Gamma_{\mathcal{L}_\nu}^* \cong \mathbb{A}_{\mathbb{k}}^{B_\nu}$  as algebraic varieties.

Finally,  $\mathcal{H}_\hbar'|_{\hbar=0} = F[\Gamma_{\mathcal{L}_\nu}^*] = \mathbb{k}[\{\eta_b\}_{b \in B_\nu}]$  is a graded Poisson Hopf algebra w.r.t. the grading  $\partial(\bar{\alpha}_n) = n$  (inherited from  $\mathcal{H}_\hbar'$ ) and w.r.t. the grading induced from  $\kappa = \tau$  (on  $\mathcal{H}$ ), and a graded algebra w.r.t. the (polynomial) grading  $d(\bar{\alpha}_n) = 1$  (for all  $n \in \mathbb{N}_+$ ).

(e) The analogues of statements (a)–(d) hold with  $\mathcal{K}$  instead of  $\mathcal{H}$ , with  $X^+$  instead of  $X$  for all  $X = \mathcal{L}_\nu, B_\nu, \mathbb{N}_\nu$ , and with  $\Gamma_{\mathcal{L}_\nu^+}^*$  instead of  $\Gamma_{\mathcal{L}_\nu}^*$ .

*Proof.* (a) This follows from Proposition 4.2 and the definition of  $\mathcal{H}_\hbar'$  in §4.2.

(b) This is a direct consequence of claim (a) and Lemma 4.2(g).

(c) Thanks to claims (a) and (b), we can look at  $\mathcal{H}_\hbar'$  as a Poisson algebra, whose Poisson bracket is given by  $\{x, y\} := \hbar^{-1}[x, y] = \hbar^{-1}(xy - yx)$  (for all  $x, y \in \mathcal{H}_\hbar'$ ); then  $\mathcal{H}_\hbar'$  itself is the free associative Poisson algebra generated by  $\{\widehat{\alpha}_n \mid n \in \mathbb{N}\}$ . Clearly  $\Delta$  is a Poisson map, therefore it is enough to prove that  $\Delta(\widehat{\alpha}_n) \in \mathcal{H}_\hbar' \otimes \mathcal{H}_\hbar'$  for all  $n \in \mathbb{N}_+$ . This is clear for  $\alpha_1$  and  $\alpha_2$  which are primitive; as for  $n > 2$ , we have, like in Proposition 4.1,

$$\begin{aligned} \Delta(\widehat{\alpha}_n) &= \sum_{k=2}^n \hbar^{k-1} \alpha_k \otimes \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) + \sum_{k=0}^{n-1} \hbar^k \alpha_1^k \otimes \hbar^{n-k-1} Z_{n-k}^k(\alpha_*) = \\ &= \sum_{k=2}^n \widehat{\alpha}_k \otimes \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) + \sum_{k=0}^{n-1} \widehat{\alpha}_1^k \otimes \hbar^{n-k-1} Z_{n-k}^k(\alpha_*) \in \mathcal{H}_\hbar' \otimes \mathcal{H}_\hbar' \end{aligned} \quad (4.5)$$

thanks to Lemma 4.1 (with notations used therein). In addition,  $S(\mathcal{H}_\hbar') \subseteq \mathcal{H}_\hbar'$  also follows by induction from (4.5) because Hopf algebra axioms along with (4.5) give

$$S(\widehat{\alpha}_n) = -\widehat{\alpha}_n - \sum_{k=2}^{n-1} S(\widehat{\alpha}_k) \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) - \sum_{k=1}^{n-1} S(\widehat{\alpha}_1^k) \hbar^{n-k-1} Z_{n-k}^k(\alpha_*) \in \mathcal{H}_\hbar'$$

for all  $n \in \mathbb{N}_\nu$  (using induction). The claim follows.

(d) Thanks to (a) and (b),  $\mathcal{H}_\hbar'|_{\hbar=0}$  is a polynomial  $\mathbb{k}$ -algebra as claimed, over the set of indeterminates  $\{\bar{\alpha}_b := \widehat{\alpha}_b|_{\hbar=0} (\in \mathcal{H}_\hbar'|_{\hbar=0})\}_{b \in B_\nu}$ . Furthermore, in the proof of (c) we noticed that  $\mathcal{H}_\hbar'$  is also the free Poisson algebra generated by  $\{\widehat{\alpha}_n \mid n \in \mathbb{N}\}$ ; therefore  $\mathcal{H}_\hbar'|_{\hbar=0}$  is the free commutative Poisson algebra generated by  $\{\bar{\alpha}_n := \widehat{\alpha}_n|_{\hbar=0}\}_{n \in \mathbb{N}}$ . Then formula (4.5) — for all  $n \in \mathbb{N}_\nu$  — describes uniquely the Hopf structure of  $\mathcal{H}_\hbar'$ , hence the formula it yields at  $\hbar = 0$  will describe the Hopf structure of  $\mathcal{H}_\hbar'|_{\hbar=0}$ .

Expanding  $\hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*)$  in (4.5) w.r.t. the basis  $\widehat{\mathcal{A}}$  in (a) we find a sum of terms of  $\tau$ -degree less or equal than  $(n-k)$ , and the sole one achieving equality is  $\widehat{\alpha}_1^{n-k}$ , which

occurs with coefficient  $\binom{n}{k}$ : similarly, when expanding  $\hbar^{n-k-1} Z_{n-k}^k(\alpha_*)$  in (4.5) w.r.t.  $\widehat{\mathcal{A}}$  all summands have  $\tau$ -degree less or equal than  $(n-k-1)$ , and equality holds only for  $\widehat{\alpha}_{n-k}$ , whose coefficient is  $(k+1)$ . Therefore for some  $\eta \in \mathcal{H}_{\hbar}'|_{\hbar=0}$  we have

$$\Delta(\widehat{\alpha}_n) = \sum_{k=2}^n \widehat{\alpha}_k \otimes \binom{n}{k} \widehat{\alpha}_1^{n-k} + \sum_{k=0}^{n-1} (k+1) \widehat{\alpha}_1^k \otimes \widehat{\alpha}_{n-k} + \hbar \eta;$$

this yields the formula for  $\Delta$ , from which the formula for  $S$  follows too as usual.

Finally, let  $\Gamma := \text{Spec}(\mathcal{H}_{\hbar}'|_{\hbar=0})$  be the algebraic Poisson group such that  $F[\Gamma] = \mathcal{H}_{\hbar}'|_{\hbar=0}$ , and let  $\gamma_{\nu} := \text{coLie}(\Gamma)$  be its cotangent Lie bialgebra. Since  $\mathcal{H}_{\hbar}'|_{\hbar=0}$  is Poisson free over  $\{\bar{\alpha}_n\}_{n \in \mathbb{N}_{\nu}}$ , as a Lie algebra  $\gamma_{\nu}$  is free over  $\{d_n := \bar{\alpha}_n \bmod \mathfrak{m}^2\}_{n \in \mathbb{N}_{\nu}}$  (where  $\mathfrak{m} := J_{\mathcal{H}_{\hbar}'|_{\hbar=0}}$ ), so  $\gamma_{\nu} \cong \mathcal{L}_{\nu}$ , via  $d_n \mapsto x_n (n \in \mathbb{N}_+)$  as a Lie algebra. The Lie cobracket is

$$\begin{aligned} \delta_{\gamma_{\nu}}(d_n) &= (\Delta - \Delta^{\text{op}})(\bar{\alpha}_n) \bmod \mathfrak{m}_{\otimes} = \sum_{k=2}^{n-1} \binom{n}{k} \bar{\alpha}_k \wedge \bar{\alpha}_1^{n-k} + \\ &+ \sum_{k=1}^{n-1} (k+1) \bar{\alpha}_1^k \wedge \bar{\alpha}_{n-k} \bmod \mathfrak{m}_{\otimes} = \binom{n}{n-1} \bar{\alpha}_{n-1} \wedge \bar{\alpha}_1 + 2 \bar{\alpha}_1 \wedge \bar{\alpha}_{n-1} \bmod \mathfrak{m}_{\otimes} = \\ &= (n-2) \bar{\alpha}_{n-1} \wedge \bar{\alpha}_1 \bmod \mathfrak{m}_{\otimes} = (n-2) d_{n-1} \wedge d_1 \in \gamma \otimes \gamma \end{aligned}$$

where  $\mathfrak{m}_{\otimes} := (\mathfrak{m}^2 \otimes \mathcal{H}_{\hbar}'|_{\hbar=0} + \mathfrak{m} \otimes \mathfrak{m} + \mathcal{H}_{\hbar}'|_{\hbar=0} \otimes \mathfrak{m}^2)$ , whence  $\Gamma = I_{\mathcal{L}_{\nu}}^*$  as claimed in (d).

Finally, the statements about gradings of  $\mathcal{H}_{\hbar}'|_{\hbar=0} = F[I_{\mathcal{L}_{\nu}}^*]$  hold by construction.

(e) This should be clear from the whole discussion, since all arguments apply again — *mutatis mutandis* — when starting with  $\mathcal{K}$  instead of  $\mathcal{H}$ ; we leave details to the reader.  $\square$

## § 5 Drinfeld's deformation $(\mathcal{H}_{\hbar}')^{\vee}$ .

**5.1 The goal.** Like in §3.1, there is a second step in the crystal duality principle which builds another deformation basing upon the Rees deformation  $\mathcal{H}_{\hbar}'$ . This will be again a Hopf  $\mathbb{k}[\hbar]$ -algebra, namely  $(\mathcal{H}_{\hbar}')^{\vee}$ , which specializes to  $\mathcal{H}$  for  $\hbar = 1$  and for  $\hbar = 0$  instead specializes to  $U(\mathfrak{k}_-)$ , for some Lie bialgebra  $\mathfrak{k}_-$ . In other words,  $(\mathcal{H}_{\hbar}')^{\vee}|_{\hbar=1} = \mathcal{H}$  and  $(\mathcal{H}_{\hbar}')^{\vee}|_{\hbar=0} = U(\mathfrak{k}_-)$ , the latter meaning that  $(\mathcal{H}_{\hbar}')^{\vee}$  is a *quantized universal enveloping algebra* (QUEA in the sequel). Thus  $(\mathcal{H}_{\hbar}')^{\vee}$  is a *quantization* of  $U(\mathfrak{k}_-)$ , and the quantum symmetry  $\mathcal{H}$  is a deformation of the classical Poisson symmetry  $U(\mathfrak{k}_-)$ .

The general theory describes explicitly the relationship between  $\mathfrak{k}_-$  and  $I_{\mathcal{L}_{\nu}}^*$  in §4, which is  $\mathfrak{k}_- = \gamma_{\nu} := \text{coLie}(I_{\mathcal{L}_{\nu}}^*) \cong \mathcal{L}_{\nu}$  (with the structure in Theorem 4.1(d)), the cotangent Lie bialgebra of  $I_{\mathcal{L}_{\nu}}^*$ . Thus, from this and §4 we see that the quantum symmetry encoded by  $\mathcal{H}$  is (also) intermediate between the two classical, Poisson symmetries ruled by  $I_{\mathcal{L}_{\nu}}^*$  and  $\gamma_{\nu}$ .

In this section I describe explicitly  $(\mathcal{H}_{\hbar}')^{\vee}$  and its semiclassical limit  $U(\mathfrak{k}_-)$ , hence  $\mathfrak{k}_-$  itself too. This provides a direct proof of the above mentioned results on  $(\mathcal{H}_{\hbar}')^{\vee}$  and  $\mathfrak{k}_-$ .

**5.2 Drinfeld's algebra**  $(\mathcal{H}_\hbar')^\vee$ . Let  $J' := J_{\mathcal{H}_\hbar'}$ , and define

$$(\mathcal{H}_\hbar')^\vee := \sum_{n \in \mathbb{N}} \hbar^{-n} J'^n = \sum_{n \in \mathbb{N}} (\hbar^{-1} J')^n \quad (\subseteq \mathcal{H}(\hbar)). \quad (5.1)$$

Now I describe  $(\mathcal{H}_\hbar')^\vee$  and its specializations at  $\hbar = 1$  and  $\hbar = 0$ . The main step is

**Theorem 5.1.** *For any  $b \in B_\nu$  set  $\check{\alpha}_b := \hbar^{\kappa(\alpha_b)-1} \alpha_b = \hbar^{\tau(b)-1} \alpha_b = \hbar^{-1} \hat{\alpha}_b$ .*

$$(a) \quad (\mathcal{H}_\hbar')^\vee = \mathbb{k}[\hbar] \left\langle \{ \check{\alpha}_b \}_{b \in B_\nu} \right\rangle / \left( \left\{ [\check{\alpha}_{b_1}, \check{\alpha}_{b_2}] - \check{\alpha}_{[b_1, b_2]} \mid \forall b_1, b_2 \in B_\nu \right\} \right).$$

(b)  $(\mathcal{H}_\hbar')^\vee$  is a graded Hopf  $\mathbb{k}[\hbar]$ -subalgebra of  $\mathcal{H}_\hbar$ .

(c)  $(\mathcal{H}_\hbar')^\vee \Big|_{\hbar=0} := (\mathcal{H}_\hbar')^\vee / \hbar (\mathcal{H}_\hbar')^\vee \cong U(\mathcal{L}_\nu)$  as co-Poisson Hopf algebra, where  $\mathcal{L}_\nu$  bears the Lie bialgebra structure given by  $\delta(x_n) = (n-2)x_{n-1} \wedge x_1$  (for all  $n \in \mathbb{N}_\nu$ ).

Finally, the grading  $d$  given by  $d(x_n) := 1$  ( $n \in \mathbb{N}_+$ ) makes  $(\mathcal{H}_\hbar')^\vee \Big|_{\hbar=0} = U(\mathcal{L}_\nu)$  into a graded co-Poisson Hopf algebra, and the grading  $\partial$  given by  $\partial(x_n) := n$  ( $n \in \mathbb{N}_+$ ) makes  $(\mathcal{H}_\hbar')^\vee \Big|_{\hbar=0} = U(\mathcal{L}_\nu)$  into a graded Hopf algebra and  $\mathcal{L}_\nu$  into a graded Lie bialgebra.

(d) The analogues of statements (a)–(c) hold with  $\mathcal{K}$ ,  $\mathcal{L}_\nu^+$ ,  $B_\nu^+$  and  $\mathbb{N}_\nu^+$  respectively instead of  $\mathcal{H}$ ,  $\mathcal{L}_\nu^+$ ,  $B_\nu$  and  $\mathbb{N}_\nu^+$ .

*Proof.* (a) This follows from Theorem 4.1(b) and the very definition of  $(\mathcal{H}_\hbar')^\vee$  in §5.2.

(b) This is a direct consequence of claim (a) and Theorem 4.1(c).

(c) It follows from claim (a) that mapping  $\check{\alpha}_b \Big|_{\hbar=0} \mapsto b$  ( $\forall b \in B_\nu$ ) yields a well-defined algebra isomorphism  $\Phi: (\mathcal{H}_\hbar')^\vee \Big|_{\hbar=0} \xrightarrow{\cong} U(\mathcal{L}_\nu)$ . In addition, when expanding  $\hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*)$  in (4.5) w.r.t. the basis  $\mathcal{A}$  (see Proposition 4.2) we find a sum of terms of  $\tau$ -degree less or equal than  $(n-k)$ , and equality is achieved only for  $\alpha_1^{n-k}$ , which occurs with coefficient  $\binom{n}{k}$ : similarly, the expansion of  $\hbar^{n-k-1} Z_{n-k}^k(\alpha_*)$  in (4.5) yields a sum of terms whose  $\tau$ -degree is less or equal than  $(n-k-1)$ , with equality only for  $\alpha_{n-k}$ , whose coefficient is  $(k+1)$ . Thus using the relation  $\hat{\alpha}_s = \hbar \check{\alpha}_s$  ( $s \in \mathbb{N}_+$ ) we get

$$\begin{aligned} \Delta(\check{\alpha}_n) &= \check{\alpha}_n \otimes 1 + 1 \otimes \check{\alpha}_n + \sum_{k=2}^{n-1} \check{\alpha}_k \otimes \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) + \sum_{k=1}^{n-1} \check{\alpha}_1^k \otimes \hbar^{n-1} Z_{n-k}^k(\alpha_*) = \\ &= \check{\alpha}_n \otimes 1 + 1 \otimes \check{\alpha}_n + \sum_{k=2}^{n-1} \hbar^{n-k} \check{\alpha}_k \otimes \binom{n}{k} \check{\alpha}_1^{n-k} + \sum_{k=1}^{n-1} \hbar^k (k+1) \check{\alpha}_1^k \otimes \check{\alpha}_{n-k} + \hbar^2 \boldsymbol{\eta} = \\ &= \check{\alpha}_n \otimes 1 + 1 \otimes \check{\alpha}_n + \hbar (n \check{\alpha}_{n-1} \otimes \check{\alpha}_1 + 2 \check{\alpha}_1 \otimes \check{\alpha}_{n-1}) + \hbar^2 \boldsymbol{\chi} \end{aligned}$$

for some  $\boldsymbol{\eta}, \boldsymbol{\chi} \in (\mathcal{H}_\hbar')^\vee \otimes (\mathcal{H}_\hbar')^\vee$ . It follows that  $\Delta(\check{\alpha}_n \Big|_{\hbar=0}) = \check{\alpha}_n \Big|_{\hbar=0} \otimes 1 + 1 \otimes \check{\alpha}_n \Big|_{\hbar=0}$  for all  $n \in \mathbb{N}_\nu$ . Similarly we have  $S(\check{\alpha}_n \Big|_{\hbar=0}) = -\check{\alpha}_n \Big|_{\hbar=0}$  and  $\epsilon(\check{\alpha}_n \Big|_{\hbar=0}) = 0$  for all  $n \in \mathbb{N}_\nu$ , thus  $\Phi$  is an isomorphism of Hopf algebras too. In addition, the Poisson cobracket of  $(\mathcal{H}_\hbar')^\vee \Big|_{\hbar=0}$  inherited from  $(\mathcal{H}_\hbar')^\vee$  is given by

$$\begin{aligned} \delta(\check{\alpha}_n \Big|_{\hbar=0}) &= \left( \hbar^{-1} (\Delta - \Delta^{\text{op}})(\check{\alpha}_n) \right) \text{ mod } \hbar (\mathcal{H}_\hbar')^\vee \otimes (\mathcal{H}_\hbar')^\vee = \\ &= (n \check{\alpha}_{n-1} \wedge \check{\alpha}_1 + 2 \check{\alpha}_1 \wedge \check{\alpha}_{n-1}) \text{ mod } \hbar (\mathcal{H}_\hbar')^\vee \otimes (\mathcal{H}_\hbar')^\vee = (n-2) \check{\alpha}_{n-1} \Big|_{\hbar=0} \wedge \check{\alpha}_1 \Big|_{\hbar=0} \end{aligned}$$

hence  $\Phi$  is also an isomorphism of co-Poisson Hopf algebras, as claimed.

The statements on gradings of  $(\mathcal{H}_\hbar')^\vee \Big|_{\hbar=0} = U(\mathcal{L}_\nu)$  should be clear by construction.

(d) This should be clear from the whole discussion, as all arguments apply again — *mutatis mutandis* — when starting with  $\mathcal{K}$  instead of  $\mathcal{H}$ ; details are left to the reader.  $\square$

**5.3 Specialization limits.** So far, Theorem 4.1(d) and Theorem 5.1(c) prove the following specialization results for  $\mathcal{H}_{\hbar}'$  and  $(\mathcal{H}_{\hbar}')^\vee$  respectively:

$$\mathcal{H}_{\hbar}' \xrightarrow{\hbar \rightarrow 0} F[\Gamma_{\mathcal{L}_\nu}^*] \quad , \quad (\mathcal{H}_{\hbar}')^\vee \xrightarrow{\hbar \rightarrow 0} U(\mathcal{L}_\nu)$$

as graded Poisson or co-Poisson Hopf  $\mathbb{k}$ -algebras. In addition, Theorem 4.1(b) implies that  $\mathcal{H}_{\hbar}' \xrightarrow{\hbar \rightarrow 1} \mathcal{H}' = \mathcal{H}$  as graded Hopf  $\mathbb{k}$ -algebras. Indeed, by Theorem 4.1(b)  $\mathcal{H}$  (or even  $\mathcal{H}_{\hbar}$ ) embeds as an algebra into  $\mathcal{H}_{\hbar}'$ , via  $\alpha_n \mapsto \hat{\alpha}_n$  (for all  $n \in \mathbb{N}_\nu$ ): then

$$[\alpha_n, \alpha_m] \mapsto [\hat{\alpha}_n, \hat{\alpha}_m] = \hbar \hat{\alpha}_{[x_n, x_m]} \equiv \hat{\alpha}_{[x_n, x_m]} \pmod{(\hbar-1)\mathcal{H}_{\hbar}'} \quad (\forall n, m \in \mathbb{N}_\nu)$$

thus, thanks to the presentation of  $\mathcal{H}_{\hbar}'$  in Theorem 4.1(b),  $\mathcal{H}$  is isomorphic to  $\mathcal{H}_{\hbar}'|_{\hbar=1} := \mathcal{H}_{\hbar}' / (\hbar-1)\mathcal{H}_{\hbar}' = \mathbb{k}\langle \hat{\alpha}_1|_{\hbar=1}, \hat{\alpha}_2|_{\hbar=1}, \dots, \hat{\alpha}_n|_{\hbar=1}, \dots \rangle$ , as a  $\mathbb{k}$ -algebra, via  $\alpha_n \mapsto \hat{\alpha}_n|_{\hbar=1}$ . Moreover, the Hopf structure of  $\mathcal{H}_{\hbar}'|_{\hbar=1}$  is given by

$$\Delta(\hat{\alpha}_n|_{\hbar=1}) = \sum_{k=2}^n \hat{\alpha}_k \otimes \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) + \sum_{k=0}^{n-1} \hat{\alpha}_1^k \otimes \hbar^{n-1} Z_{n-k}^k(\alpha_*) \pmod{(\hbar-1)\mathcal{H}_{\hbar}' \otimes \mathcal{H}_{\hbar}'}$$

Now,  $Q_{n-k}^k(\mathbf{a}_*) = Q_{n-k}^k(\alpha_* + \alpha_1^*) = \mathcal{Q}_{n-k}^k(\alpha_*)$  for some polynomial  $\mathcal{Q}_{n-k}^k(\alpha_*)$  in the  $\alpha_i$ 's; let  $\mathcal{Q}_{n-k}^k(\alpha_*) = \sum_s \mathcal{T}_{n-k}^{s,k}(\alpha_*)$  be the splitting of  $\mathcal{Q}_{n-k}^k$  into  $\tau$ -homogeneous summands (i.e., each  $\mathcal{T}_{n-k}^{s,k}(\alpha_*)$  is a homogeneous polynomial of  $\tau$ -degree  $s$ ): then

$$\hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) = \hbar^{n-k} \mathcal{Q}_{n-k}^k(\alpha_*) = \hbar^{n-k} \sum_s \mathcal{T}_{n-k}^{s,k}(\alpha_*) = \sum_s \hbar^{n-k-s} \mathcal{T}_{n-k}^{s,k}(\hat{\alpha}_*)$$

with  $n-k-s > 0$  for all  $s$  (by construction). Since clearly  $\hbar^{n-k-s} \mathcal{T}_{n-k}^{s,k}(\hat{\alpha}_*) \equiv \mathcal{T}_{n-k}^{s,k}(\hat{\alpha}_*) \pmod{(\hbar-1)\mathcal{H}_{\hbar}'}$ , we find  $\hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) = \hbar^{n-k} \mathcal{Q}_{n-k}^k(\alpha_*) = \sum_s \hbar^{n-k-s} \mathcal{T}_{n-k}^{s,k}(\hat{\alpha}_*) \equiv \sum_s \mathcal{T}_{n-k}^{s,k}(\hat{\alpha}_*) \pmod{(\hbar-1)\mathcal{H}_{\hbar}'} = \mathcal{Q}_{n-k}^k(\hat{\alpha}_*)$ , for all  $k$  and  $n$ . Similarly we argue that  $\hbar^{n-1} Z_{n-k}^k(\alpha_*) \equiv Z_{n-k}^k(\hat{\alpha}_*) \pmod{(\hbar-1)\mathcal{H}_{\hbar}'}$ , for all  $k$  and  $n$ . The outcome is that

$$\begin{aligned} \Delta(\hat{\alpha}_n|_{\hbar=1}) &= \sum_{k=2}^n \hat{\alpha}_k \otimes \hbar^{n-k} \mathcal{Q}_{n-k}^k(\alpha_*) + \sum_{k=0}^n \hat{\alpha}_1^k \otimes \hbar^{n-1} Z_{n-k}^k(\alpha_*) \pmod{(\hbar-1)\mathcal{H}_{\hbar}' \otimes \mathcal{H}_{\hbar}'} \\ &= \sum_{k=2}^{n-1} \hat{\alpha}_k \otimes \mathcal{Q}_{n-k}^k(\hat{\alpha}_*) + \sum_{k=0}^{n-1} \hat{\alpha}_1^k \otimes Z_{n-k}^k(\hat{\alpha}_*) \pmod{(\hbar-1)\mathcal{H}_{\hbar}' \otimes \mathcal{H}_{\hbar}'} \end{aligned}$$

On the other hand, we have  $\Delta(\alpha_n) = \sum_{k=2}^n \alpha_k \otimes \mathcal{Q}_{n-k}^k(\alpha_*) + \sum_{k=0}^{n-1} \alpha_1^k \otimes Z_{n-k}^k(\alpha_*)$  in  $\mathcal{H}$ . Thus the *graded algebra* isomorphism  $\Psi: \mathcal{H} \xrightarrow{\cong} \mathcal{H}_{\hbar}'|_{\hbar=1}$  given by  $\alpha_n \mapsto \hat{\alpha}_n|_{\hbar=1}$  preserves the coproduct too. Similarly,  $\Psi$  respects the antipode and the counit, hence it is a graded Hopf algebra isomorphism. In a nutshell, we have (as graded Hopf  $\mathbb{k}$ -algebras)  $\mathcal{H}_{\hbar}' \xrightarrow{\hbar \rightarrow 1} \mathcal{H}' = \mathcal{H}$ . Similarly, Theorem 5.1 implies that  $(\mathcal{H}_{\hbar}')^\vee \xrightarrow{\hbar \rightarrow 1} \mathcal{H}$  as graded

Hopf  $\mathbb{k}$ -algebras. Indeed, Theorem 5.1(a) shows that  $(\mathcal{H}_{\hbar}')^\vee \cong \mathbb{k}[\hbar] \otimes_{\mathbb{k}} U(\mathcal{L}_\nu)$  as graded associative algebras, via  $\check{\alpha}_n \mapsto x_n$  ( $n \in \mathbb{N}_\nu$ ), in particular  $(\mathcal{H}_{\hbar}')^\vee$  is the free associative  $\mathbb{k}[\hbar]$ -algebra over  $\{\check{\alpha}_n\}_{n \in \mathbb{N}_\nu}$ ; then specialization yields a graded algebra isomorphism

$$\Omega: (\mathcal{H}_{\hbar}')^\vee \Big|_{\hbar=1} := (\mathcal{H}_{\hbar}')^\vee / (\hbar-1) (\mathcal{H}_{\hbar}')^\vee \xrightarrow{\cong} \mathcal{H}, \quad \check{\alpha}_n \Big|_{\hbar=1} \mapsto \alpha_n.$$

As for the Hopf structure, in  $(\mathcal{H}_{\hbar}')^\vee \Big|_{\hbar=1}$  it is given by

$$\Delta(\check{\alpha}_n \Big|_{\hbar=1}) = \sum_{k=2}^n \check{\alpha}_k \Big|_{\hbar=1} \otimes \hbar^{n-k} \mathcal{Q}_{n-k}^k(\alpha_*) \Big|_{\hbar=1} + \sum_{k=0}^{n-1} \check{\alpha}_1^k \Big|_{\hbar=1} \otimes \hbar^{n-2} Z_{n-k}^k(\alpha_*) \Big|_{\hbar=1}.$$

As before, split  $\mathcal{Q}_{n-k}^k(\alpha_*)$  as  $\mathcal{Q}_{n-k}^k(\alpha_*) = \sum_s \mathcal{T}_{n-k}^{s,k}(\alpha_*)$ , and split each  $\mathcal{T}_{n-k}^{s,k}(\hat{\alpha}_*)$  into homogeneous components w.r.t. the total degree in the  $\hat{\alpha}_i$ 's, say  $\mathcal{T}_{n-k}^{s,k}(\hat{\alpha}_*) = \sum_r \mathcal{Y}_{r,n}^{s,k}(\hat{\alpha}_*)$ : then  $\hbar^{n-k-s} \mathcal{T}_{n-k}^{s,k}(\hat{\alpha}_*) = \hbar^{n-k-s} \sum_r \mathcal{Y}_{r,n}^{s,k}(\hat{\alpha}_*) = \sum_r \hbar^{n-k-s+r} \mathcal{Y}_{r,n}^{s,k}(\check{\alpha}_*)$ , because  $\hat{\alpha}_* = \hbar \check{\alpha}_*$ . As  $\hbar^{n-k-s+r} \mathcal{Y}_{r,n}^{s,k}(\check{\alpha}_*) \equiv \mathcal{Y}_{r,n}^{s,k}(\check{\alpha}_*) \pmod{(\hbar-1) (\mathcal{H}_{\hbar}')^\vee}$ , we eventually get

$$\hbar^{n-k} \mathcal{Q}_{n-k}^k(\alpha_*) = \sum_{s,r} \hbar^{n-k-s+r} \mathcal{Y}_{r,n}^{s,k}(\check{\alpha}_*) \equiv \sum_{s,r} \mathcal{Y}_{r,n}^{s,k}(\check{\alpha}_*) \pmod{(\hbar-1) (\mathcal{H}_{\hbar}')^\vee} = \mathcal{Q}_{n-k}^k(\mathbf{a}_*)$$

for all  $k$  and  $n$ . Similarly  $\hbar^{n-1} Z_{n-k}^k(\alpha_*) \equiv Z_{n-k}^k(\alpha_*) \pmod{(\hbar-1) (\mathcal{H}_{\hbar}')^\vee}$  ( $\forall k, n$ ). Thus

$$\begin{aligned} \Delta(\check{\alpha}_n \Big|_{\hbar=1}) &= \sum_{k=2}^n \check{\alpha}_k \Big|_{\hbar=1} \otimes \hbar^{n-k} \mathcal{Q}_{n-k}^k(\alpha_*) \Big|_{\hbar=1} + \sum_{k=0}^{n-1} \check{\alpha}_1^k \Big|_{\hbar=1} \otimes \hbar^{n-2} Z_{n-k}^k(\alpha_*) \Big|_{\hbar=1} = \\ &= \sum_{k=2}^n \check{\alpha}_k \Big|_{\hbar=1} \otimes \mathcal{Q}_{n-k}^k(\alpha_*) \Big|_{\hbar=1} + \sum_{k=0}^{n-1} \check{\alpha}_1^k \Big|_{\hbar=1} \otimes Z_{n-k}^k(\alpha_*) \Big|_{\hbar=1}. \end{aligned}$$

On the other hand, one has  $\Delta(\alpha_n) = \sum_{k=2}^n \alpha_k \otimes \mathcal{Q}_{n-k}^k(\alpha_*) + \sum_{k=0}^{n-1} \alpha_1^k \otimes Z_{n-k}^k(\alpha_*)$  in  $\mathcal{H}$ , thus the algebra isomorphism  $\Omega: (\mathcal{H}_{\hbar}')^\vee \Big|_{\hbar=1} \xrightarrow{\cong} \mathcal{H}$  given by  $\hat{\alpha}_n \Big|_{\hbar=1} \mapsto \alpha_n$  also preserves the coproduct; similarly, it also respects the antipode and the counit, hence it is a graded Hopf algebra isomorphism. In a nutshell, we have (as graded Hopf  $\mathbb{k}$ -algebras)  $(\mathcal{H}_{\hbar}')^\vee \xrightarrow{\hbar \rightarrow 1} \mathcal{H}$ . Therefore we have filled in the bottom part of the diagram  $(\boxtimes)$  in the Introduction, for  $H = \mathcal{H}$  ( $:= \mathcal{H}_\nu$ ), because  $\mathcal{H}' := \cup_{n \in \mathbb{N}} D_n = \mathcal{H}$  by §4.2: namely,

$$F[\Gamma_{\mathcal{L}_\nu}^*] \xleftarrow[\mathcal{H}_{\hbar}']{0 \leftarrow \hbar \rightarrow 1} \mathcal{H} \xleftarrow[(\mathcal{H}_{\hbar}')^\vee]{1 \leftarrow \hbar \rightarrow 0} U(\mathcal{L}_\nu)$$

where now in right-hand side  $\mathcal{L}_\nu$  is given the Lie bialgebra structure of Theorems 4.1 and 5.1, and  $\Gamma_{\mathcal{L}_\nu}^*$  is the corresponding dual Poisson group mentioned in Theorem 4.1.

## § 6 Summary and generalizations.

**6.1 Summary.** The analysis in §§2–5 yields a complete description of the *non-trivial* deformations of  $\mathcal{H}$  — namely the Rees deformations  $\mathcal{H}_{\hbar}^\vee$  and  $\mathcal{H}_{\hbar}'$  and the Drinfeld's deformations  $(\mathcal{H}_{\hbar}^\vee)'$  and  $(\mathcal{H}_{\hbar}')^\vee$  — built out of the *trivial* deformation  $\mathcal{H}_{\hbar}$ . In particular

$$\mathfrak{g}_-^\times = (\mathcal{L}_\nu, \delta_\bullet), \quad G_- = G_{\mathcal{L}_\nu}^*, \quad G_+ = \Gamma_{\mathcal{L}_\nu}^*, \quad \mathfrak{g}_+^\times = (\mathcal{L}_\nu, \delta_*) \quad (6.1)$$



(with notation of  $(\mathfrak{X})$ ) where  $\delta_\bullet$  and  $\delta_*$  denote the Lie cobracket on  $\mathcal{L}_\nu$  defined respectively in Theorem 2.1 and in Theorems 4.1 and 5.1. Next result shows that the four objects in (6.1) are really different, though they share some common features:

**Theorem 6.1.**

- (a)  $(\mathcal{H}_\hbar^\vee)' \cong \mathcal{H}_\hbar'$  as Poisson  $\mathbb{k}[\hbar]$ -algebras, but  $(\mathcal{H}_\hbar^\vee)' \not\cong \mathcal{H}_\hbar'$  as Hopf  $\mathbb{k}[\hbar]$ -algebras.
- (b)  $(\mathcal{L}_\nu, \delta_\bullet) \cong (\mathcal{L}_\nu, \delta_*)$  as Lie algebras, but  $(\mathcal{L}_\nu, \delta_\bullet) \not\cong (\mathcal{L}_\nu, \delta_*)$  as Lie bialgebras.
- (c)  $G_{\mathcal{L}_\nu}^* \cong \Gamma_{\mathcal{L}_\nu}^*$  as (algebraic) Poisson varieties, but  $G_{\mathcal{L}_\nu}^* \not\cong \Gamma_{\mathcal{L}_\nu}^*$  as (algebraic) groups.
- (d) The analogues of statements (a)–(c) hold with  $\mathcal{K}$  and  $\mathcal{L}_\nu^+$  instead of  $\mathcal{H}$  and  $\mathcal{L}_\nu$ .

*Proof.* It follows from Theorem 3.1(a) that  $(\mathcal{H}_\hbar^\vee)'$  can be seen as a Poisson Hopf algebra, with Poisson bracket given by  $\{x, y\} := \hbar^{-1}[x, y] = \hbar^{-1}(xy - yx)$  (for all  $x, y \in (\mathcal{H}_\hbar^\vee)'$ ); then  $(\mathcal{H}_\hbar^\vee)'$  is the free Poisson algebra generated by  $\left\{ \tilde{\mathbf{b}}_{x_n} = \tilde{\mathbf{x}}_n = \mathbf{a}_n \mid n \in \mathbb{N} \right\}$ ; since  $\mathbf{a}_n = \boldsymbol{\alpha}_n + (1 - \delta_{1,n})\boldsymbol{\alpha}_1^n$  and  $\boldsymbol{\alpha}_n = \mathbf{a}_n - (1 - \delta_{1,n})\mathbf{a}_1^n$  ( $n \in \mathbb{N}_+$ ) it is also (freely) Poisson-generated by  $\{\boldsymbol{\alpha}_n \mid n \in \mathbb{N}\}$ . We also saw that  $\mathcal{H}_\hbar'$  is the free Poisson algebra over  $\{\hat{\boldsymbol{\alpha}}_n \mid n \in \mathbb{N}\}$ ; thus mapping  $\boldsymbol{\alpha}_n \mapsto \hat{\boldsymbol{\alpha}}_n$  ( $\forall n \in \mathbb{N}$ ) does define a unique Poisson algebra isomorphism  $\Phi: (\mathcal{H}_\hbar^\vee)' \xrightarrow{\cong} \mathcal{H}_\hbar'$ , given by  $\tilde{\boldsymbol{\alpha}}_b := \hbar^{-d(b)}\boldsymbol{\alpha}_b \mapsto \hat{\boldsymbol{\alpha}}_b$ , for all  $b \in B_\nu$ . This proves the first half of (a), and then also (taking semiclassical limits and spectra) of (c).

The group structure of either  $G_{\mathcal{L}_\nu}^*$  or  $\Gamma_{\mathcal{L}_\nu}^*$  yields a Lie cobracket onto the cotangent space at the unit point of the above, isomorphic Poisson varieties: this cotangent space identifies with  $\mathcal{L}_\nu$ , and the two cobrackets are given respectively by  $\delta_\bullet(x_n) = \sum_{\ell=1}^{n-1} (\ell+1) x_\ell \wedge x_{n-\ell}$  for  $G_{\mathcal{L}_\nu}^*$  (by Theorem 3.1) and by  $\delta_*(x_n) = (n-2)x_{n-1} \wedge x_1$  for  $\Gamma_{\mathcal{L}_\nu}^*$  (by Theorem 4.1), for all  $n \in \mathbb{N}_\nu$ . It follows that  $\text{Ker}(\delta_\bullet) = \{0\} \neq \text{Ker}(\delta_*)$ , which implies that the two Lie coalgebra structures on  $\mathcal{L}_\nu$  are not isomorphic. This proves (b), and also means that  $G_{\mathcal{L}_\nu}^* \not\cong \Gamma_{\mathcal{L}_\nu}^*$  as (algebraic) groups, hence  $F[G_{\mathcal{L}_\nu}^*] \not\cong F[\Gamma_{\mathcal{L}_\nu}^*]$  as Hopf  $\mathbb{k}$ -algebras, and so  $(\mathcal{H}_\hbar^\vee)' \not\cong \mathcal{H}_\hbar'$  as Hopf  $\mathbb{k}[\hbar]$ -algebras, which ends the proof of (c) and (a) too.

Finally, claim (d) should be clear: one applies the like arguments *mutatis mutandis*, and everything follows as before.  $\square$

**6.2 Generalizations.** Plenty of features of  $\mathcal{H} = \mathcal{H}^{\text{dif}}$  are shared by a whole bunch of graded Hopf algebras, which usually arose in connection with some physical problem or some (co)homological topic and all bear a nice combinatorial content; essentially, most of them can be described as “formal series” over indexing sets — replacing  $\mathbb{N}$  — of various (combinatorial) nature: planar trees (with or without labels), forests, graphs, Feynman diagrams, etc. Besides the ice-breaking examples in physics provided by Connes and Kreimer (cf. [CK1–3]), which are all commutative or cocommutative Hopf algebras, other non-commutative non-cocommutative examples (like the one of  $\mathcal{H}^{\text{dif}}$ ) are introduced in [BF], roughly through a “disabelianization process” applied to the commutative Hopf algebras of Connes and Kreimer. A very general analysis and wealth of examples in this context is due to Foissy (see [Fo1–3]), who also makes an interesting study of  $\delta_\bullet$ -maps and

of the functor  $H \mapsto H'$  ( $H$  a Hopf  $\mathbb{k}$ -algebra). Other examples, issued out of topological motivations, can be found in the works of Loday et al.: see e.g. [LR], and references therein.

When performing the like analysis, as we did for  $\mathcal{H}$ , for a graded Hopf algebra  $H$  of the afore mentioned type, *the arguments used for  $\mathcal{H}$  apply essentially the same, up to minor changes, and give much the same results.* To give an example, the Hopf algebras considered by Foissy are non-commutative polynomial, say  $H = \mathbb{k}\langle\{x_i\}_{i \in \mathcal{I}}\rangle$  for some index set  $\mathcal{I}$ : then one finds  $\mathcal{H}_\hbar^\vee|_{\hbar=0} = U(\mathfrak{g}_-) = U(\mathcal{L}_{\mathcal{I}})$  where  $\mathcal{L}_{\mathcal{I}}$  is the free Lie algebra over  $\mathcal{I}$ .

This opens the way to apply the methods presented in this paper to all these graded Hopf algebras, of great interest for their applications in mathematical physics or in topology (or whatever); the simplest case of  $\mathcal{H}^{\text{dif}}$  plays the role of a toy model which realizes a clear and faithful pattern for many common features of all Hopf algebras of this kind.

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FABIO GAVARINI

UNIVERSITÀ DEGLI STUDI DI ROMA “TOR VERGATA” — DIPARTIMENTO DI MATEMATICA  
 VIA DELLA RICERCA SCIENTIFICA 1, I-00133 ROMA, ITALY — E-MAIL: GAVARINI@MAT.UNIROMA2.IT