

ON THE CONSTRUCTION OF CHEVALLEY SUPERGROUPS

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Abstract

We give a description of the construction of Chevalley supergroups, providing some explanatory examples. We avoid the discussion of the $A(1, 1)$, $P(3)$ and $Q(n)$ cases, for which our construction holds, but the exposition becomes more complicated. We shall not in general provide complete proofs for our statements, instead we will make an effort to convey the key ideas underlying our construction. A fully detailed account of our work is scheduled to appear in [9].

1 Introduction

The notion of Chevalley group, introduced by Chevalley in 1955, provided a unified combinatorial construction of all simple algebraic groups over a generic field k . The consequences of Chevalley’s work were many and have had tremendous impact in the following decades. His construction was motivated by issues linked to the problem of the classification of semisimple algebraic groups: he provided an existence theorem for such groups, essentially exhibiting an example of simple group for each of the predicted possibility. In the course of this discussion, he discovered new examples of finite simple groups, which had escaped to the group theorists up to then. Later on, in the framework of a modern treatment of algebraic geometry, his work was instrumental to show that all simple algebraic groups are algebraic schemes over \mathbb{Z} and to study arithmetic questions over arbitrary fields.

We may say that we have similar motivations: we want a unified approach to describe all algebraic supergroups, which have Lie superalgebras of classical type and we also want to give new examples of supergroups, over arbitrary fields. For instance, our discussion enables us to provide an explicit construction of algebraic supergroups associated with the exceptional and the strange Lie superalgebras. To our knowledge these supergroups have not been examined before, though an approach in the differential setting can be very well carried through via the language of *super Harish-Chandra pairs*. In such approach a supergroup is understood as a pair (G_0, \mathfrak{g}) , consisting of an ordinary group G_0 and a super Lie algebra \mathfrak{g} , with even part $\mathfrak{g}_0 = \text{Lie}(G_0)$, together with some natural compatibility conditions involving the adjoint action of the group G_0 on \mathfrak{g} . It is clear that in positive characteristic this method shows severe limitations.

In the present work we outline the construction of the Chevalley supergroups associated with Lie superalgebras of classical type. We shall not present complete proofs for our statements, they will appear in [9], however we shall concentrate on the key ideas and examples that will help to understand our construction.

In our statements, we shall leave out the strange Lie superalgebra $Q(n)$ and some low dimensional cases, which can be treated very well with the same method, with minor modifications, but present extra difficulties that make our construction and notation opaque.

Essentially, we are going to follow Chevalley's recipe and push it as far as we can, before resorting to more sophisticated algebraic geometry techniques, when the supergeometric nature of our objects forces us to do so.

We start with a complex Lie superalgebra of classical type \mathfrak{g} , together with a fixed Cartan subalgebra \mathfrak{h} , and we define the *Chevalley basis* of \mathfrak{g} . This is an homogeneous basis of \mathfrak{g} , as super vector space, whose elements have the brackets expressed as a linear combination of the basis elements with just *integral* coefficients. Consequently they give us an integral form of \mathfrak{g} , that we call $\mathfrak{g}_{\mathbb{Z}}$ the *Chevalley Lie superalgebra* associated with \mathfrak{g} and \mathfrak{h} . Such integral form gives raise to the Kostant integral form $K_{\mathbb{Z}}(\mathfrak{g})$ of the universal enveloping superalgebra $U(\mathfrak{g})$ of \mathfrak{g} . $K_{\mathbb{Z}}(\mathfrak{g})$ is free over \mathbb{Z} with basis given by the ordered monomials in the divided powers of the root vectors and the binomial coefficients in the generators of \mathfrak{h} in the Chevalley basis:

$$X^m/m!, \binom{H_i}{n}, \alpha \in \Delta \text{ (root system) and } m, n \in \mathbb{N}.$$

Next, we look at a faithful rational representation of \mathfrak{g} in a finite dimensional complex vector space V . Inside V we can find an *integral lattice* M which is invariant under the action of $K_{\mathbb{Z}}(\mathfrak{g})$ and its stabilizer \mathfrak{g}_V in \mathfrak{g} defines an integral form of \mathfrak{g} . In complete analogy with Chevalley, for an arbitrary field k , we can give the following key definitions:

$$V_k := k \otimes_{\mathbb{Z}} M, \quad \mathfrak{g}_k := k \otimes_{\mathbb{Z}} \mathfrak{g}_V, \quad U_k := k \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{g}).$$

We could even take k to be a commutative ring, however for the scope of the present work and to stress the analogy with Chevalley's construction, we prefer the restrictive hypothesis of k to be a field.

This is the point where our construction departs dramatically from Chevalley's one. In fact, starting from the faithful representation V_k of \mathfrak{g}_k , Chevalley defines the Chevalley group G_V as generated by the exponentials $\exp(tX_{\alpha}) := 1 + tX_{\alpha} + (t^2/2)X_{\alpha}^2 + \dots$, for $t \in k$ and X_{α} the root vector corresponding to the root α in the Chevalley basis. Such an expression makes sense since the X_{α} 's act as nilpotent elements. If we were to repeat without changes this construction in the super setting, we shall find only ordinary groups over k associated with the Lie algebra \mathfrak{g}_0 , the even part of \mathfrak{g} . This is because over a field, we cannot see any supergeometric behaviour; the only thing we can recapture is the underlying classical object. For this reason, we need to go beyond Chevalley's construction and build our supergroups as *functors*.

We define \mathbf{G} the *Chevalley supergroup* associated with \mathfrak{g} and the faithful representation V , as the functor $\mathbf{G} : (\text{salg}) \rightarrow (\text{sets})$, with $\mathbf{G}(A)$ the subgroup of $\text{GL}(A \otimes V_k)$ generated by $\mathbf{G}_0(A)$ and the elements $1 + \theta_{\beta}X_{\beta}$, for $\beta \in \Delta_1$. In other words we have:

$$\mathbf{G}_V(A) = \langle \mathbf{G}_0(A), 1 + \theta_{\beta}X_{\beta} \rangle \subset \text{GL}(A \otimes V_k), \quad A \in (\text{salg}), \quad \theta_{\beta} \in A_1$$

where (salg) and (sets) are the categories of commutative superalgebras and sets respectively and (as always) we use X_{β} to denote also the image of the root vector X_{β} in the chosen faithful representation V_k . \mathbf{G}_0 is the functor of points of the (reductive) algebraic supergroup associated to \mathfrak{g}_0 and the representation V_k .

This is a somehow natural generalization of what Chevalley does in his original construction: he provides the k -points of the algebraic group scheme constructed starting from a complex semisimple Lie algebra and a faithful

representation, for all the fields k , while we give the A -points of the supergroup scheme for any commutative k -superalgebra A .

Once this definition is properly established, we need to show that \mathbf{G} is the functor of points of an algebraic supergroup, in other words, that it is representable. This is the price to pay when we employ the language of the functor of points: it is much easier to define geometric objects, however we need to prove representability in order to speak properly of supergroup schemes. As customary, we use the same letter to denote both the superscheme and its functor of points.

We shall obtain the representability of \mathbf{G} by showing that

$$\mathbf{G} \cong \mathbf{G}_0 \times \mathbf{A}^{0|N}$$

where $\mathbf{A}^{0|N}$ is the functor of points of an affine superspace of dimension $0|N$. Once this isomorphism is established the representability follows at once, since both \mathbf{G}_0 and $\mathbf{A}^{0|N}$ are representable, i.e. they are the functors of points of superschemes, hence their product is.

The next question we examine is how much our construction depends on the chosen representation. In complete analogy to Chevalley approach, we show that if we have two representations V and V' , with weight lattices $L_V \subset L_{V'}$, then there is a surjective morphism $\mathbf{G}_{V'} \rightarrow \mathbf{G}_V$, with kernel in the center of $\mathbf{G}_{V'}$. This implies right away that our construction depends only on the weight lattice of the chosen representation V and in particular it shows that it is independent from the choice of the lattice M inside V .

This paper is organized as follows.

In section 2 we review quickly some facts of algebraic supergeometry and the theory of Lie superalgebras.

In sections 3 and 4 we go to the heart of the construction of Chevalley's supergroups going through all the steps detailed above.

Finally in section 5 we provide some insight into our construction with some examples and observations.

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2 Supergeometry in the algebraic setting

Let k be the ground field.

A *super vector space* V is a vector space with \mathbb{Z}_2 grading: $V = V_0 \oplus V_1$, the elements in V_0 are called *even* and the elements in V_1 are called *odd*. Hence we have a function p called the *parity* defined only on homogeneous elements. A *superalgebra* A is a super vector space with multiplication preserving parity; A is *commutative* if $xy = (-1)^{p(x)p(y)}yx$ for all x, y homogeneous elements in A . All superalgebras are assumed to be commutative unless otherwise specified and their category is denoted with (salg) .

Definition 2.1. A *superspace* $S = (|S|, \mathcal{O}_S)$ is a topological space $|S|$ endowed with a sheaf of commutative superalgebras \mathcal{O}_S such that the stalk $\mathcal{O}_{S,x}$ is a local superalgebra for all $x \in |S|$.

A *morphism* $\phi : S \rightarrow T$ of superspaces consists of a pair $\phi = (|\phi|, \phi^*)$, where $|\phi| : |S| \rightarrow |T|$ is a morphism of topological spaces and $\phi^* : \mathcal{O}_T \rightarrow \phi_*\mathcal{O}_S$ is a sheaf morphism such that $\phi_x^*(\mathfrak{m}_{|\phi|(x)}) = \mathfrak{m}_x$ where $\mathfrak{m}_{|\phi|(x)}$ and \mathfrak{m}_x are the maximal ideals in the stalks $\mathcal{O}_{T,|\phi|(x)}$ and $\mathcal{O}_{S,x}$ respectively and ϕ_x^* is the morphism induced by ϕ^* on the stalks and $\phi_*\mathcal{O}_S$ is the sheaf on $|T|$ defined as $\phi_*\mathcal{O}_S(V) := \mathcal{O}_S(\phi^{-1}(V))$.

The next example of superspace turns out to be extremely important, as $\underline{\text{Spec}} A$, for a commutative superalgebra A , is the local model for superschemes, very much in the same way as $\underline{\text{Spec}} A_0$ is the local model for ordinary schemes for A_0 a commutative algebra.

Example 2.2. Let $A \in (\text{salg})$ and let \mathcal{O}_{A_0} be the structural sheaf of the ordinary scheme $\underline{\text{Spec}}(A_0) = (\text{Spec}(A_0), \mathcal{O}_{A_0})$, where $\text{Spec}(A_0)$ denotes the prime spectrum of the commutative ring A_0 . Now A is a module over A_0 , so we have a sheaf \mathcal{O}_A of \mathcal{O}_{A_0} -modules over $\text{Spec}(A_0)$ with stalk A_p , the p -localization of the A_0 -module A , at the prime $p \in \text{Spec}(A_0)$.

$\underline{\text{Spec}}(A) := (\text{Spec}(A_0), \mathcal{O}_A)$ is a superspace, as one can readily check.

Given $f : A \rightarrow B$ a superalgebra morphism, one can define $\underline{\text{Spec}} f : \underline{\text{Spec}} B \rightarrow \underline{\text{Spec}} A$ in a natural way, very similarly to the ordinary setting, thus making $\underline{\text{Spec}}$ a functor $\underline{\text{Spec}} : (\text{salg}) \rightarrow (\text{sets})$, where (salg) is the category of superalgebras and (sets) the category of sets (see [5] ch. 5 or [8] ch. 1, for more details).

Definition 2.3. Given a superspace X , we say it is an *affine superscheme* if it is isomorphic to $\underline{\text{Spec}}(A)$ for some commutative superalgebra A . We say that X is a *superscheme* if it is locally isomorphic to an affine superscheme.

Example 2.4. The *affine superspace* $\mathbb{A}_{\mathbb{k}}^{p|q}$, also denoted $\mathbb{k}^{p|q}$, is defined as

$$\mathbb{A}_{\mathbb{k}}^{p|q} := \mathbb{k}[x_1 \dots x_p] \otimes \wedge(\xi_1 \dots \xi_q)$$

where $\wedge(\xi_1 \dots \xi_q)$ is the exterior algebra generated by the indeterminates ξ_1, \dots, ξ_q .

The formalism of the functor of points that we borrow from algebraic geometry allows us to handle supergeometric objects that would be otherwise very difficult to treat using just the superschemes language.

Definition 2.5. Let X be a superscheme. Its *functor of points* is the functor defined on the objects as

$$h_X : (\text{salg}) \longrightarrow (\text{sets}) , \quad h_X(A) := \text{Hom}(\underline{\text{Spec}}(A), X)$$

and on the arrows as $h_X(f)(\phi) := \phi \circ \underline{\text{Spec}}(f)$.

Since the category of affine superschemes is equivalent to the category of commutative superalgebras (ref. [5, 8]) we have that, when X is affine, its functor of points is equivalently defined as follows:

$$h_X(A) = \text{Hom}(\mathcal{O}(X), A), \quad h_X(f)(\phi) = f \circ \phi$$

where $\mathcal{O}(X)$ is the superalgebra of global sections of the structure sheaf on X .

If h_X is group valued, i. e. it is valued in the category (groups) of groups, we say that X is a *supergroup*. When X is affine, this is equivalent to the fact that $\mathcal{O}(X)$ is a (commutative) *Hopf superalgebra*. More in general, we call *supergroup functor* any functor $G : (\text{salg}) \longrightarrow (\text{groups})$.

Any representable supergroup functor is the same as an affine supergroup. Following a customary abuse of notation, we shall then use the same letter to denote both the superscheme X and its functor of points h_X .

As always, Yoneda's lemma plays a crucial role, allowing us to use natural transformations between the functors of points of superschemes and the morphisms of the superschemes themselves interchangeably.

Proposition 2.6. (*Yoneda's Lemma*) Let \mathcal{C} be a category, and let R, S be two objects in \mathcal{C} . Consider the two functors $h_R, h_S : \mathcal{C} \rightarrow (\text{sets})$ defined on the objects by $h_R(A) := \text{Hom}(R, A)$, $h_S(A) := \text{Hom}(S, A)$ and on the arrows by $h_R(f)(\phi) := f \circ \phi$, $h_S(f)(\psi) := f \circ \psi$.

Then there exists a one-to-one correspondence between the natural transformations and the morphisms

$$\{h_R \rightarrow h_S\} \longleftrightarrow \text{Hom}(R, S).$$

This has an immediate corollary.

Corollary 2.7. *Two affine superschemes are isomorphic if and only if their functors of points are isomorphic.*

The next examples turn out to be very important in the sequel.

Examples 2.8.

(1) *Super vector spaces as superschemes.* Let V be a super vector space. For any superalgebra A we define $V(A) := (A \otimes V)_0 = A_0 \otimes V_0 \oplus A_1 \otimes V_1$. This is a representable functor in the category of superalgebras, whose representing object is $\text{Pol}(V)$, the algebra of polynomial functions on V . Hence any super vector space can be equivalently viewed as an affine superscheme. If $V = k^{m|n}$, that is $V_0 \cong k^p$ and $V_1 \cong k^q$, V is the functor of points of the affine superspace described in 2.4.

(2) *$\text{GL}(V)$ as an algebraic supergroup.* Let V be a finite dimensional super vector space of dimension $p|q$. For any superalgebra A , let $\text{GL}(V)(A) := \text{GL}(V(A))$ be the set of isomorphisms $V(A) \rightarrow V(A)$. If we fix a homogeneous basis for V , we see that $V \cong k^{p|q}$. In this case, we also denote $\text{GL}(V)$ with $\text{GL}(p|q)$. Now, $\text{GL}(p|q)(A)$ is the group of invertible matrices of size $(p+q)$ with diagonal block entries in A_0 and off-diagonal block entries in A_1 . It is well known that the functor $\text{GL}(V)$ is representable; see (e.g.), [25], Ch. 3, for further details.

We end our minireview of supergeometry by introducing the concept of *Lie superalgebra* and stating the Kac's classification theorem for Lie superalgebras of classical type.

We assume now $\text{char}(k) \neq 2, 3$.

Definition 2.9. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a super vector space. We say that \mathfrak{g} is a Lie superalgebra, if we have a bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which satisfies the following properties (as usual for all $x, y \in \mathfrak{g}$ homogeneous):

- *Anti-symmetry:*

$$[x, y] + (-1)^{p(x)p(y)}[y, x] = 0$$

- *Jacobi identity:*

$$(-1)^{p(x)p(z)}[x, [y, z]] + (-1)^{p(y)p(x)}[y, [z, x]] + (-1)^{p(z)p(y)}[z, [x, y]] = 0.$$

The standard example is $\text{End}(V)$ the endomorphisms of the super vector space V , with $\text{End}(V)_0$ the endomorphisms preserving parity and $\text{End}(V)_1$ the endomorphisms reversing parity. The bracket is defined as:

$$[X, Y] := XY - (-1)^{|X||Y|}YX.$$

If $V := k^{p|q} = k^p \oplus k^q$, with $V_0 := k^p$ and $V_1 := k^q$, we write $\text{End}(k^{p|q}) := \text{End}(V)$ or $\mathfrak{gl}(p|q) := \text{End}(V)$. In this case $\text{End}(V)_0$ consists of diagonal block matrices, while $\text{End}(V)_1$ consists of off diagonal block matrices all with entries in k .

In $\text{End}(k^{p|q})$ we can define the *supertrace* as follows:

$$\text{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \text{tr}(A) - \text{tr}(D) \quad .$$

There is an important class of Lie superalgebras, namely the *simple Lie superalgebras* that have been classified by Kac (see [16]).

Definition 2.10. A non abelian Lie superalgebra \mathfrak{g} is *simple* if it has no nontrivial homogeneous ideals. A Lie superalgebra \mathfrak{g} is called of *classical type* if it is simple and \mathfrak{g}_1 is completely reducible as a \mathfrak{g}_0 -module. Furthermore, \mathfrak{g} is said to be *basic* if, in addition, it admits a non-degenerate, invariant bilinear form.

We now give a list of Lie superalgebras of classical type, sending the reader to [16], [20] for the details.

Examples 2.11. (1) — $\mathfrak{sl}(m|n)$. Define $\mathfrak{sl}(m|n)$ as the subset of $\mathfrak{gl}(m|n)$ consisting of all matrices with supertrace zero. This is a Lie subalgebra of $\mathfrak{gl}(m|n)$, with the following \mathbb{Z}_2 -grading:

$$\mathfrak{sl}(m|n)_0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathfrak{gl}(1) , \quad \mathfrak{sl}(m|n)_1 = f_m \otimes f'_n \oplus f'_m \otimes f_n$$

where f_r is the defining representation of $\mathfrak{sl}(r)$ and f'_r is its dual (for any r). When $m \neq n$ this is a Lie superalgebra of *classical type*.

(2) — $\mathfrak{osp}(p|q)$. Let ϕ denote a nondegenerate consistent supersymmetric bilinear form in $V := k^{p|q}$. This means that V_0 and V_1 are mutually orthogonal and the restriction of ϕ to V_0 is a symmetric and to V_1 a skewsymmetric form (in particular, $q = 2n$ is even). We define in $\mathfrak{gl}(p|q)$ the subalgebra $\mathfrak{osp}(p|q) := \mathfrak{osp}(p, |q)_0 \oplus \mathfrak{osp}(p|q)_1$ by setting, for all $s \in \{0, 1\}$,

$$\mathfrak{osp}(p|q)_s := \left\{ \ell \in \mathfrak{gl}(p|q) \mid \phi(\ell(x), y) = -(-1)^{s|x|} \phi(x, \ell(y)) \quad \forall x, y \in k^{p|q} \right\}$$

and we call $\mathfrak{osp}(p|q)$ the *orthosymplectic* Lie superalgebra. Note that $\mathfrak{osp}(0|q)$ is the symplectic Lie algebra $\mathfrak{sp}(q)$, while $\mathfrak{osp}(p|0)$ is the orthogonal Lie algebra $\mathfrak{so}(p)$.

Again, all the $\mathfrak{osp}(p|q)$'s are Lie superalgebras of *classical type*. Moreover, if $m, n \geq 2$, we have:

$$\begin{aligned} \mathfrak{osp}(2m+1|2n)_0 &= \mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n) , & \mathfrak{osp}(2m|2n)_0 &= \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n) \\ \mathfrak{osp}(p|2n)_1 &= f_p \otimes f_{2n} \quad \forall p > 2 , & \mathfrak{osp}(2|2n)_1 &= f_{2n}^{\oplus 2} \end{aligned}$$

We now introduce some terminology in order to be able to state the classification theorem.

Definition 2.12. Define the following Lie superalgebras:

- (1) $A(m, n) := \mathfrak{sl}(m+1|n+1)$, $A(n, n) := \mathfrak{sl}(n+1|n+1) / kI_{2n}$, $\forall m \neq n$;
- (2) $B(m, n) := \mathfrak{osp}(2m+1|2n)$, $\forall m \geq 0, n \geq 1$;
- (3) $C(n) := \mathfrak{osp}(2|2n-2)$, for all $n \geq 2$;
- (4) $D(m, n) := \mathfrak{osp}(2m|2n)$, for all $m \geq 2, n \geq 1$;
- (5) $P(n) := \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{gl}(n+1|n+1) \mid \begin{array}{l} A \in \mathfrak{sl}(n+1) \\ B^t = B, C^t = -C \end{array} \right\}$
- (6) $Q(n) := \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{gl}(n+1|n+1) \mid B \in \mathfrak{sl}(n+1) \right\} / kI_{2(n+1)}$.

Theorem 2.13. *Let k be an algebraically closed field of characteristic zero. Then the Lie superalgebras of classical type are either isomorphic to a simple Lie algebra or to one of the following Lie superalgebras:*

$$A(m, n), \quad m \geq n \geq 0, \quad m + n > 0; \quad B(m, n), \quad m \geq 0, \quad n \geq 1; \quad C(n), \quad n \geq 3$$

$$D(m, n), \quad m \geq 2, \quad n \geq 1; \quad P(n), \quad n \geq 2; \quad Q(n), \quad n \geq 2$$

$$F(4); \quad G(3); \quad D(2, 1; a), \quad a \in k \setminus \{0, -1\}.$$

For the definition of $F(4)$, $G(3)$, $D(2, 1; a)$, and for the proof, we refer to [16].

3 Chevalley basis and Kostant integral form

The main ingredient to construct a Chevalley supergroup starting from a complex Lie superalgebra \mathfrak{g} of classical type is the *Chevalley basis*. This is an homogeneous basis for \mathfrak{g} , consisting of elements that have brackets expressed as *integral* combinations of the basis elements. Consequently a Chevalley basis determines what is called the *Chevalley Lie algebra* $\mathfrak{g}_{\mathbb{Z}}$ of \mathfrak{g} , which is an integral form of \mathfrak{g} .

Assume \mathfrak{g} to be a Lie superalgebra of classical type different from $A(1, 1)$, $P(3)$, $Q(n)$. We want to leave out these pathological cases for which our construction holds, but with a more complicated set of statements and proofs. We invite the reader to go to [9] for a complete and unified treatment of all of these cases. We also consider $D(2, 1; a)$ for only integral values for the coefficient a .

Let us fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} , that is a maximal solvable Lie subalgebra of \mathfrak{g} . The adjoint action of \mathfrak{h} on \mathfrak{g} gives the usual *root space* decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}$$

where

$$\mathfrak{g}_{\alpha} := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$$

and $\Delta = \Delta_0 \cup \Delta_1$ with

$$\Delta_0 := \{ \alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq \{0\} \} = \{ \text{even roots of } \mathfrak{g} \}.$$

$$\Delta_1 := \{ \alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq \{0\} \} = \{ \text{odd roots of } \mathfrak{g} \}.$$

As in the ordinary setting we shall call Δ *root system* and the \mathfrak{g}_α 's the *root spaces*. If we fix a simple system (see [16] for its definition) the root system splits into positive and negative roots, exactly as in the ordinary setting:

$$\Delta = \Delta^+ \amalg \Delta^-, \quad \Delta_0 = \Delta_0^+ \amalg \Delta_0^-, \quad \Delta_1 = \Delta_1^+ \amalg \Delta_1^-.$$

- Observation 3.1.**
1. Notice that the definition allows $\Delta_0 \cap \Delta_1 \neq \emptyset$, as in fact happens for $\mathfrak{g} = Q(n)$, where the roots are simultaneously all even and odd and the root spaces have all dimension 1|1.
 2. Δ_0 is the root system of the reductive Lie algebra \mathfrak{g}_0 , while Δ_1 is the set of weights of the representation of \mathfrak{g}_0 in \mathfrak{g}_1 .

If \mathfrak{g} is not of type $P(n)$ or $Q(n)$, there is an even non-degenerate, invariant bilinear form on \mathfrak{g} , whose restriction to \mathfrak{h} is in turn an invariant bilinear form on \mathfrak{h} . On the other hand, if \mathfrak{g} is of type $P(n)$ or $Q(n)$, then such a form on \mathfrak{h} exists because \mathfrak{g}_0 is simple (of type A_n), though it does not come by restricting an invariant form on the whole \mathfrak{g} .

If (x, y) denotes such form, we can identify \mathfrak{h}^* with \mathfrak{h} , via $H'_\alpha \mapsto (H'_\alpha)$. We can then transfer $(,)$ to \mathfrak{h}^* in the natural way: $(\alpha, \beta) = (H'_\alpha, H'_\beta)$. Define $H_\alpha := 2 \frac{H'_\alpha}{(H'_\alpha, H'_\alpha)}$ when the denominator is non zero. When $(H'_\alpha, H'_\alpha) = 0$ such renormalization can be found in detail in [14]. We call H_α the *coroot* associated with α .

We summarize in the next proposition all the relevant properties of the root system, sending the reader to [16, 20, 21] for the complete story.

Proposition 3.2. *Let \mathfrak{g} be a Lie superalgebra of classical type, as above, that is $\mathfrak{g} \neq A(1, 1), P(3), Q(n)$, and let $n \in \mathbb{N}$.*

$$(a) \quad \Delta_0 \cap \Delta_1 = \emptyset,$$

(b) $-\Delta_0 = \Delta_0$, $-\Delta_1 \subseteq \Delta_1$. If $\mathfrak{g} \neq P(n)$, then $-\Delta_1 = \Delta_1$.

(c) Let $\mathfrak{g} \neq P(2)$, and $\alpha, \beta \in \Delta$, $\alpha = c\beta$, with $c \in \mathbb{K} \setminus \{0\}$. Then $\alpha, \beta \in \Delta_r$ ($r = 0, 1$) $\Rightarrow c = \pm 1$, $\alpha \in \Delta_r, \beta \in \Delta_s, r \neq s \Rightarrow c = \pm 2$.

(d) $\dim_{\mathbb{K}}(\mathfrak{g}_\alpha) = 1$ for each $\alpha \in \Delta$.

We are finally ready to give the definition of *Chevalley basis*.

Definition 3.3. We define a *Chevalley basis* of a Lie superalgebra \mathfrak{g} as above any homogeneous basis

$$B = \{H_1 \dots H_\ell, X_\alpha, \alpha \in \Delta\}$$

of \mathfrak{g} as complex vector space, with the following requirements:

(a) $\{H_1, \dots, H_\ell\}$ is a basis of the complex vector space \mathfrak{h} . Moreover

$$\mathfrak{h}_{\mathbb{Z}} := \text{Span}_{\mathbb{Z}}\{H_1, \dots, H_\ell\} = \text{Span}_{\mathbb{Z}}\{H_\alpha \mid \alpha \in \Delta\}.$$

(b) $[H_i, H_j] = 0$, $[H_i, X_\alpha] = \alpha(H_i) X_\alpha$, $\forall i, j \in \{1, \dots, \ell\}, \alpha \in \Delta$;

(c) $[X_\alpha, X_{-\alpha}] = \sigma_\alpha H_\alpha \quad \forall \alpha \in \Delta \cap (-\Delta)$

with H_α as after 3.2, and $\sigma_\alpha := -1$ if $\alpha \in \Delta_1^-$, $\sigma_\alpha := 1$ otherwise;

(d) $[X_\alpha, X_\beta] = c_{\alpha, \beta} X_{\alpha+\beta} \quad \forall \alpha, \beta \in \Delta : \alpha \neq -\beta$, with $c_{\alpha, \beta} \in \mathbb{Z}$.

More precisely,

- If $(\alpha, \alpha) \neq 0$, or $(\beta, \beta) \neq 0$, $c_{\alpha, \beta} = \pm(r+1)$ or (only if $\mathfrak{g} = P(n)$), $c_{\alpha, \beta} = \pm(r+2)$, where r is the length of the α -string through β .
- If $(\alpha, \alpha) \neq 0 = (\beta, \beta) = 0$, $c_{\alpha, \beta} = \beta(\alpha)$.

Notice that this definition clearly extends to direct sums of finitely many of the \mathfrak{g} 's under the above hypotheses.

Definition 3.4. If B is a Chevalley basis of a Lie superalgebra \mathfrak{g} as above, we set

$$\mathfrak{g}_{\mathbb{Z}} := \text{span}_{\mathbb{Z}}\{B\} \subset \mathfrak{g}$$

and we call it the *Chevalley superalgebra* of \mathfrak{g} .

Observe that $\mathfrak{g}_{\mathbb{Z}}$ is a Lie superalgebra over \mathbb{Z} inside \mathfrak{g} . Since a Chevalley basis B of \mathfrak{g} is unique up to a choice of a sign for each root vector and the choice of the H_i 's we have that $\mathfrak{g}_{\mathbb{Z}}$ is independent of the choice of B (but of course depends on the choice of \mathfrak{h} the Cartan subalgebra).

The existence of a Chevalley basis for the families A, B, C, D is a known result; for example an almost explicit Chevalley basis for types B, C and D is in [23], while for A is a straightforward calculation. More in general, an abstract existence result, with a uniform proof, is given in [14] for all *basic* types. In [9] we provide an existence theorem for *all* cases giving both a case by case analysis, comprehending all Lie superalgebras of classical type and a uniform proof, that however leaves out the $P(n)$ case.

We now turn to another important ingredient for our construction: the *Kostant \mathbb{Z} -form*.

Definition 3.5. Let \mathfrak{g} be a complex Lie superalgebra of classical type over \mathbb{C} and let $B = \{H_1 \dots H_\ell, X_\alpha, \alpha \in \Delta\}$ be a Chevalley basis. We define the *Kostant superalgebra*, $K_{\mathbb{Z}}(\mathfrak{g})$, the \mathbb{Z} -superalgebra inside $U(\mathfrak{g})$, generated by

$$X_\alpha^{(n)}, \quad X_\gamma, \quad \binom{H_i}{n} \quad \forall \alpha \in \Delta_0, n \in \mathbb{N}, \gamma \in \Delta_1, i = 1, \dots, \ell,$$

where

$$X_\alpha^{(n)} := X_\alpha^n / n! \quad \binom{H}{n} := \frac{H(H-1) \cdots (H-n+1)}{n!} \in U(\mathfrak{g})$$

for all H in \mathfrak{h} . These are called respectively *divided powers* and *binomial coefficients*.

Notice that we can remove all the binomial coefficients corresponding to coroots H_i 's relative to even roots and still generate the superalgebra $K_{\mathbb{Z}}(\mathfrak{g})$. In fact a classical result (see [22] pg 9) tells us that the even divided powers generate all such binomial coefficients. Unfortunately we cannot obtain the odd coroot binomial coefficients and this is because the X_γ , for $\gamma \in \Delta_1$ appear only in degree one.

As in the ordinary setting (see [22] pg 7) we have a PBW type of result for $K_{\mathbb{Z}}(\mathfrak{g})$ providing us with a \mathbb{Z} -basis for the Kostant superalgebra. The proof is very similar to the ordinary setting and we send the reader to [9] for more details.

Theorem 3.6. *The Kostant superalgebra $K_{\mathbb{Z}}(\mathfrak{g})$ is a free \mathbb{Z} -module. For any given total order \preceq of the set $\Delta \cup \{1, \dots, \ell\}$, a \mathbb{Z} -basis of $K_{\mathbb{Z}}(\mathfrak{g})$ is the set of ordered “PBW-like monomials”, i.e. all products without repetitions: of factors of type:*

$$X_{\alpha}^{(n_{\alpha})}, \quad \binom{H_i}{n_i}, \quad X_{\gamma}$$

$\alpha \in \Delta_0$, $i \in \{1, \dots, \ell\}$, $\gamma \in \Delta_1$ and $n_{\alpha}, n_i \in \mathbb{N}$ — taken in the right order with respect to \preceq .

4 Chevalley supergroups

This section is devoted to the construction of *Chevalley supergroups* and to prove they are supergroup schemes.

Let \mathfrak{g} be a complex Lie superalgebra of classical type, $B = \{H_1 \dots H_{\ell}, X_{\alpha}, \alpha \in \Delta\}$ a Chevalley basis of \mathfrak{g} and $K_{\mathbb{Z}}(\mathfrak{g})$ its Kostant superalgebra. We start with a finite dimensional complex representation V of \mathfrak{g} and the notion of *admissible lattice* in V .

Definition 4.1. Let V be a complex finite dimensional representation for \mathfrak{g} . We say that V is *rational* if $\mathfrak{h}_{\mathbb{Z}} := \text{Span}_{\mathbb{Z}}(H_1, \dots, H_{\ell})$ acts diagonally on V with integral eigenvalues.

Notice that this condition is automatic for semisimple Lie algebras, while it is actually restrictive for some Lie superalgebras as the next example shows.

Example 4.2. Let $\mathfrak{g} = \mathfrak{sl}(m|n)$ and \mathfrak{h} the diagonal matrices, so that $\mathfrak{h}_{\mathbb{Z}} = \text{Span}_{\mathbb{Z}}\{E_{m,m} + E_{m+1,m+1}, E_{ii} - E_{i+1,i+1}, i \neq m\}$, where E_{ij} denotes an elementary matrix. Let V be a representation with highest weight $\Lambda = \lambda_1 \epsilon_1 + \dots + \lambda_m \epsilon_m + \mu_1 \delta_1 + \dots + \mu_n \delta_n$, where $\epsilon_i : \mathfrak{h} \rightarrow \mathbb{C}$, $\epsilon_i(E_{jj}) = \delta_{ij}$ and similarly for δ_k . We have that (see [16]) V is finite dimensional if and only if $\lambda_i - \lambda_{i+1}, \mu_j - \mu_{j+1} \in \mathbb{Z}^+$, $i = 1 \dots m - 1$, $j = 1 \dots n - 1$, in other words if and only if $\Lambda(H_i) \in \mathbb{Z}^+$ for $i \neq m$. There are hence no conditions on $\Lambda(H_m) = \lambda_m + \mu_1$. Consequently if we pick any (non integral) complex number for such a sum and we build the induced module, we shall obtain a finite dimensional representation for \mathfrak{g} where H_m acts diagonally, with a complex, non integral eigenvalue.

Let us now fix V a finite dimensional rational complex semisimple representation of \mathfrak{g} .

We say that an integral lattice M in V is *admissible* if it is $K_{\mathbb{Z}}(\mathfrak{g})$ -stable.

As in the ordinary setting any rational complex finite dimensional semisimple representation of \mathfrak{g} admits an admissible lattice M , which is generated by the highest weight vector v and it is the sum of its weight components M_{μ} . In particular if V is simple we have:

$$M = K_{\mathbb{Z}}(\mathfrak{g}) \cdot v, \quad M = \bigoplus M_{\mu}.$$

The next proposition establishes the existence of an integral form of \mathfrak{g} stabilizing the admissible lattice M inside the representation V . We send the reader to [9] §5 for the proof.

Theorem 4.3. *Let \mathfrak{g} , V and M as above. Define:*

$$\mathfrak{g}_V = \{X \in \mathfrak{g} \mid X.M \subseteq M\}.$$

If V is faithful, then

$$\mathfrak{g}_V = \mathfrak{h}_V \oplus \left(\bigoplus_{\alpha \in \Delta} \mathbb{Z} X_{\alpha} \right), \quad \mathfrak{h}_V := \{H \in \mathfrak{h} \mid \mu(H) \in \mathbb{Z}, \forall \mu \in \Lambda\}$$

where Λ is the set of all weights of V . In particular, \mathfrak{g}_V is a lattice in \mathfrak{g} , and it is independent of the choice of the admissible lattice M (but not of course of V).

We end this discussion by saying that $\mathfrak{g}_{\mathbb{Z}}$ corresponds to the adjoint representation of \mathfrak{g} and that in general all the integral forms \mathfrak{g}_V lie between the two integral forms \mathfrak{g}_{roots} and $\mathfrak{g}_{weights}$ corresponding respectively to the root and the fundamental weight representations:

$$\mathfrak{g}_{roots} \subset \mathfrak{g}_V \subset \mathfrak{g}_{weights}.$$

We now start the construction of the Chevalley supergroup associated with the data \mathfrak{g} and V .

Let k be a generic field.

Definition 4.1 and Theorem 4.3 allow us to move from the complex field to a generic field quite easily as the next definition shows.

Definition 4.4. Let \mathfrak{g} be a complex Lie superalgebra of classical type (as usual $\mathfrak{g} \neq A(1, 1), P(2), Q(n)$). Let V be a faithful rational complex representation of \mathfrak{g} , M an admissible lattice in V .

Define:

$$\mathfrak{g}_k := k \otimes_{\mathbb{Z}} \mathfrak{g}_V, \quad V_k := k \otimes_{\mathbb{Z}} M, \quad U_k(\mathfrak{g}) := k \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{g}).$$

We are now ready to define the super equivalent of the one-parameter subgroups in the classical theory. As we shall see, homogeneous one-parameter subgroups appear in the super setting with three different dimensions: $1|0$, $0|1$ and $1|1$. In order to keep the analogy with the ordinary setting, we nevertheless have preferred to keep the terminology *one-parameter subgroup*, though in the supersetting the term “one” can be misleading.

Definition 4.5. Let $X_\alpha, X_\beta, X_\gamma$ be root vectors in the Chevalley basis, $\alpha \in \Delta_0, \beta, \gamma \in \Delta_1$, with $[X_\beta, X_\beta] = 0, [X_\gamma, X_\gamma] \neq 0$.

We define *homogeneous one-parameter subgroups* the following supergroup functors from the categories of superalgebras to the category of sets:

$$\begin{aligned} x_\alpha(A) &:= \{ \exp(t X_\alpha) \mid t \in A_0 \} = \\ &= \left\{ \left(1 + t X_\alpha + t^2 \frac{X_\alpha^2}{2} + \dots \right) \mid t \in A_0 \right\} \subset \text{GL}(V_k)(A), \\ x_\beta(A) &:= \{ \exp(\vartheta X_\beta) \mid \vartheta \in A_1 \} = \left\{ (1 + \vartheta X_\beta) \mid \vartheta \in A_1 \right\} \subset \text{GL}(V_k)(A), \\ x_\gamma(A) &:= \{ \exp(\vartheta X_\gamma + t X_\gamma^2) \mid \vartheta \in A_1, t \in A_0 \} = \\ &= \left\{ (1 + \vartheta X_\gamma) \exp(t X_\gamma^2) \mid \vartheta \in A_1, t \in A_0 \right\} \subset \text{GL}(V_k)(A). \end{aligned}$$

Notice that the infinite sums reduce to finite ones since X_α and X_β act as nilpotent operators on V_k . As usual we identify a generic root vector X_α with its image under the representation of $U_k(\mathfrak{g})$ in V_k (the divided powers come at hand exactly at this point).

One can readily see that the functors $x_\alpha, x_\beta, x_\gamma$ are representable, hence they are algebraic supergroups in the sense of §2 and their representing Hopf superalgebras are respectively $k[x], k[\xi]$ and $k[x, \xi]$, where as usual the roman letters correspond to even elements while the greek letters to odd ones. The

comultiplication is coadditive except for the element x in $k[x, \xi]$: $x \mapsto 1 \otimes x + x \otimes 1 + \xi \otimes \xi$. It is very clear by looking at the Hopf superalgebras representing x_α , x_β and x_γ that the superdimensions of these supergroups are respectively $1|0, 0|1$ and $1|1$. It is not hard to see that these are all of the allowed superdimensions for homogeneous one-parameter subgroups (see [9] for more details).

As an abuse of notation we shall sometimes write for $t \in A_0$ and $\theta \in A_1$:

$$\begin{aligned} x_\alpha(t) &:= \exp(t X_\alpha), \\ x_\beta(\theta) &:= \exp(\vartheta X_\beta) = 1 + \vartheta X_\beta \\ x_\gamma(\mathbf{t}) &:= \exp(\vartheta X_\gamma + t X_\gamma^2), \quad \mathbf{t} = (t, \theta). \end{aligned}$$

We now turn to the definition of the generators of what classically is the *maximal torus*.

Definition 4.6. For any $\alpha \in \Delta \subseteq \mathfrak{h}^*$, let $H_\alpha \in \mathfrak{h}_\mathbb{Z}$ as in 3.2. Let $V = \bigoplus_\mu V_\mu$ be the splitting of V into weight spaces. As V is rational, we have $\mu(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta$. Define:

$$h_\alpha(t).v := t^{\mu(H_\alpha)} v \in V_k(A) \quad \forall v \in (V_k)_\mu, \mu \in \mathfrak{h}^* \quad t \in A^\times, \quad A \in (\text{salg})$$

Notice that this defines an operator $h_\alpha(t) \in \text{GL}(V_k)(A)$. Hence we can define:

$$h_H(t) := \prod h_\alpha^{a_\alpha}(t) \in \text{GL}(V_k)(A), \quad H = \sum a_\alpha H_\alpha.$$

We have immediately that h_H defines a supergroup functor:

$$h_H : (\text{salg}) \longrightarrow (\text{sets}), \quad h_H(A) := \{h_H(t) \mid t \in A^\times\}$$

which is clearly representable, its Hopf superalgebra being given by $k[x, x^{-1}]$ with comultiplication $x \mapsto x \otimes x$.

We now want to define an ordinary algebraic group associated with the ordinary Lie algebra \mathfrak{g}_0 , the even part of \mathfrak{g} . One must exert some care at this point, since a Chevalley basis for \mathfrak{g}_0 is not in general the even part or even a subset of a Chevalley basis for \mathfrak{g} , even if $\mathfrak{h} = \mathfrak{h}_0$, that is the Cartan subalgebras for \mathfrak{g}_0 and \mathfrak{g} coincide. Let us illustrate the first of these phenomenons with simple example. Let us look at $A(2, 1)$. $\mathfrak{h}_\mathbb{Z} = (\mathfrak{h}_\mathbb{Z})_0 = \text{Span}_\mathbb{Z}\{H_1, H_2, H_3, H_4\}$,

with $H_i = E_{ii} - E_{i+1,i+1}$, $i \neq 3$, $H_3 = E_{33} + E_{44}$, where the E_{ij} 's are the elementary matrices with 1 in the (i, j) -th position and zero elsewhere. We have only one odd coroot H_3 . This is an even vector, that we would however miss if we were to consider just the even coroots, that is the coroots corresponding to even root spaces. These are the coroots of the roots in Δ_0 the root system associated with $A_2 \oplus A_1$ the *simple* even part of \mathfrak{g} , which in this case is not the same as $\mathfrak{g}_0 = A_2 \oplus A_1 \oplus \mathbb{C}$, which is *reductive*. To produce an instance of the second phenomenon is more complicated. The point is that we can have that the span of the odd coroots may contain some of the even coroots and consequently we can omit those even coroots, so that the Chevalley basis will not be a subset of the Chevalley basis of the even part. This happens for example in the $D(m, n)$ case.

It is possible to construct a reductive algebraic group \mathbf{G}_0 overcoming these difficulties. \mathbf{G}_0 will encode also the information contained in the extra odd coroot (it is in fact possible always to reduce to the case of just one odd coroot) and such construction is explained in detail in [9]. The group \mathbf{G}_0 is constructed following Chevalley's philosophy, but taking into account the extra odd coroot, which would be otherwise missing. On local superalgebras \mathbf{G}_0 is described as follows. Let \mathbf{G}'_0 be the ordinary algebraic group scheme associated with the semisimple part of \mathfrak{g} (which could be smaller than \mathfrak{g}_0 as in the $A(m, n)$ case) and let $T : (\text{salg}) \rightarrow (\text{sets})$, $T(A) = \langle h_H(A) \mid H \in \mathfrak{h}_{\mathbb{Z}} \rangle$. This is in general larger than the maximal torus T_0 in G_0 , since it contains the extra odd root (though one must be aware of some exceptions as we detail in the observation below). If A is a local superalgebra we define:

$$\mathbf{G}_0(A) = \langle \mathbf{G}'_0(A), T(A) \rangle.$$

It is possible to show that this definition extends to any superalgebra A and that the functor so obtained is representable (see [9] section 5).

Observation 4.7. We want to observe that there are cases in which the missing odd root can be somehow recovered without extra work. Let us look at the example of $\mathfrak{osp}(1|2)$. The roots are $\alpha, 2\alpha$ and the corresponding coroots are $H_\alpha = 2H_{2\alpha}$, $H_{2\alpha}$ (the relation between coroots depends on the chosen normalization). Consequently, we have that by taking just the even coroot $H_{2\alpha}$, we can get both the coroots H_α and $H_{2\alpha}$, so in this case it is not necessary to add anything more, in other words $\mathbf{G}_0 = \mathbf{G}'_0$. Clearly this phenomenon is observed for all the superalgebras $B(m, n)$. Notice that the

even coroot 2α corresponds to the adjoint representation of the even part \mathfrak{sl}_2 of $\mathfrak{osp}(1|2)$. As we shall see in our construction, this will tell us that we cannot obtain from $\mathfrak{osp}(1|2)$ a Chevalley supergroup whose reduced group is SL_2 and in fact we shall see that there is only one algebraic supergroup associated with $\mathfrak{osp}(1|2)$ and its reduced algebraic group is PSL_2 . This fact has consequences on the questions regarding which supergroups can be built using our method and we plan to fully explore this in a forthcoming paper.

We are finally ready for the definition of *Chevalley supergroup functor*.

Definition 4.8. Let \mathfrak{g} be a complex Lie superalgebra of classical type and V a faithful rational complex representation of \mathfrak{g} . We call the *Chevalley supergroup*, associated to \mathfrak{g} and V , the functor $\mathbf{G} : (\mathrm{salg}) \longrightarrow (\mathrm{grps})$ defined as:

$$\begin{aligned} \mathbf{G}(A) &:= \left\langle \mathbf{G}_0(A), x_\beta(A) \mid \beta \in \Delta_1 \right\rangle = \\ &= \left\langle \mathbf{G}_0(A), 1 + \theta_\beta X_\beta \mid \beta \in \Delta_1, \theta_\beta \in A_1 \right\rangle \subset \mathrm{GL}(V_k(A)). \end{aligned}$$

In other words $\mathbf{G}(A)$ is the subgroup of $\mathrm{GL}(V_k(A))$ generated by $\mathbf{G}_0(A)$ described above and the 0|1 one-parameter subgroups $x_\beta(A)$ with $\beta \in \Delta_1$. \mathbf{G} is defined on the arrows in the natural way, since $\mathbf{G}(A)$ is a subgroup of $\mathrm{GL}(V_k(A))$.

From the classical theory (see [10] 5.7) we know that on local algebras, since \mathbf{G}_0 is reductive:

$$\mathbf{G}_0(A) = \left\langle x_\alpha(A), h_i(A) \mid i = 1, \dots, \ell, \alpha \in \Delta_0 \right\rangle.$$

Consequently on local superalgebras we then have:

$$\mathbf{G}(A) = \left\langle x_\alpha(A), h_i(A) \mid i = 1, \dots, \ell, \alpha \in \Delta \right\rangle.$$

We call *Chevalley supergroup functor* the functor $G : (\mathrm{salg}) \longrightarrow (\mathrm{grps})$ defined as:

$$G(A) = \left\langle x_\alpha(A), h_i(A) \mid i = 1, \dots, \ell, \alpha \in \Delta \right\rangle.$$

In [9] we explore more deeply the relation between the two functors \mathbf{G} and G and we show that \mathbf{G} is the *sheafification* of G . This important property sheds light on our construction and it is actually needed in the key proofs, since it provides a more explicit way to handle the Chevalley supergroups. Nevertheless, given the scope of the present work, we shall not give the definition of sheafification of a functor, in order to avoid the technicalities involved, that are not adding any insight into our construction. For all the details we send the reader to the appendix in [9] and, for the ordinary setting to [24], where the sheafification of functors is fully explained and to [10] 5.7.6 for its application to reductive groups.

The fact that we have defined the Chevalley supergroup \mathbf{G} as a functor does not automatically imply that it is *representable*, in other words, that it is the functor of points of an algebraic supergroup scheme. This is a new question specific to the supersetting, in fact in the ordinary setting, the definition of Chevalley group is given only on fields, the group is exhibited as an abstract group and only later one shows it has an algebraic scheme structure. On the other hand in the supergeometric environment looking at superobjects on fields only will not give us much information since the odd coordinates disappear when we look at points over a field, thus leaving us with just the underlying ordinary group. In other words $\mathbf{G}(k) = \mathbf{G}_0(k)$ for all fields k , since the θ_β 's in Definition 4.8 are nilpotent.

In order to prove the representability of \mathbf{G} , we shall give a series of lemmas regarding G , which is more accessible than \mathbf{G} , since we know its generators for all $A \in (\text{salg})$. As in the ordinary setting the key to the theory are the explicit formulas for the commutators. The proof is a straightforward generalization of the corresponding proofs for the ordinary setting (see [22] §3), which we stated as (1) of 4.11.

Before this in order to properly state our results and the intermediate steps to obtain them, we need to define the following auxiliary sets.

Definition 4.9. For any $A \in (\text{salg})$, we define the subsets of $G(A)$

$$G_1(A) := \left\{ \prod_{i=1}^n x_{\gamma_i}(\vartheta_i) \mid n \in \mathbb{N}, \gamma_1, \dots, \gamma_n \in \Delta_1, \vartheta_1, \dots, \vartheta_n \in A_1 \right\}$$

$$G_0^\pm(A) := \left\{ \prod_{i=1}^n x_{\alpha_i}(t_i) \mid n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \Delta_0^\pm, t_1, \dots, t_n \in A_0 \right\}$$

$$G_1^\pm(A) := \left\{ \prod_{i=1}^n x_{\gamma_i}(\vartheta_i) \mid n \in \mathbb{N}, \gamma_1, \dots, \gamma_n \in \Delta_1^\pm, \vartheta_1, \dots, \vartheta_n \in A_1 \right\}$$

$$\begin{aligned} G^\pm(A) &:= \left\{ \prod_{i=1}^n x_{\beta_i}(\mathbf{t}_i) \mid n \in \mathbb{N}, \beta_1, \dots, \beta_n \in \Delta^\pm, \mathbf{t}_1, \dots, \mathbf{t}_n \in A_0 \times A_1 \right\} = \\ &= \langle G_0^\pm(A), G_1^\pm(A) \rangle \end{aligned}$$

Moreover, fixing any total order \preceq on Δ_1^\pm , and letting $N_\pm = |\Delta_1^\pm|$, we set

$$G_1^{\pm, <}(A) := \left\{ \prod_{i=1}^{N_\pm} x_{\gamma_i}(\vartheta_i) \mid \gamma_1 \prec \dots \prec \gamma_{N_\pm} \in \Delta_1^\pm, \vartheta_1, \dots, \vartheta_{N_\pm} \in A_1 \right\}$$

and for any total order \preceq on Δ_1 , and letting $N := |\Delta| = N_+ + N_-$, we set

$$G_1^<(A) := \left\{ \prod_{i=1}^N x_{\gamma_i}(\vartheta_i) \mid \gamma_1 \prec \dots \prec \gamma_N \in \Delta_1, \vartheta_1, \dots, \vartheta_N \in A_1 \right\}$$

Note that for special choices of the order, one has $G_1^<(A) = G_1^{-, <}(A) \cdot G_1^{+, <}(A)$ or $G_1^<(A) = G_1^{+, <}(A) \cdot G_1^{-, <}(A)$.

Remark 4.10. Note that $G_1(A)$, $G_0^\pm(A)$, $G_1^\pm(A)$ and $G^\pm(A)$ are subgroups of $G(A)$, while $G_1^{\pm, <}(A)$ and $G_1^<(A)$ instead are *not*, in general.

Lemma 4.11. 1. Let $\alpha, \beta \in \Delta_0$, $A \in (\text{salg})$ and $t, u \in A_0$. Then there exist $c_{ij} \in \mathbb{Z}$ such that

$$(x_\alpha(t), x_\beta(u)) = \prod x_{i\alpha + j\beta}(c_{ij} t^i u^j) \in G_0(A).$$

2. Let $\alpha \in \Delta_0$, $\gamma \in \Delta_1$, $A \in (\text{salg})$ and $t \in A_0$, $\vartheta \in A_1$. Then there exist $c_s \in \mathbb{Z}$ such that

$$(x_\gamma(\vartheta), x_\alpha(t)) = \prod_{s>0} x_{\gamma + s\alpha}(c_s t^s \vartheta) \in G_1(A),$$

(the product being finite). More precisely, with $\varepsilon_k = \pm 1$ and $r \in \mathbb{Z}$,

$$(1 + \vartheta X_\gamma, x_\alpha(t)) = \prod_{s>0} \left(1 + \prod_{k=1}^s \varepsilon_k \cdot \binom{s+r}{r} \cdot t^s \vartheta X_{\gamma + s\alpha} \right)$$

where the factors in the product are taken in any order (as they do commute).

3. Let $\gamma, \delta \in \Delta_1$, $A \in (\text{salg})$, $\vartheta, \eta \in A_1$. Then (notation of Definition 3.3)

$$(x_\gamma(\vartheta), x_\delta(\eta)) = x_{\gamma+\delta}(-c_{\gamma,\delta} \vartheta \eta) = (1 - c_{\gamma,\delta} \vartheta \eta X_{\gamma+\delta}) \in G_0(A)$$

if $\delta \neq -\gamma$; otherwise, for $\delta = -\gamma$, we have

$$(x_\gamma(\vartheta), x_{-\gamma}(\eta)) = (1 - \vartheta \eta H_\gamma) = h_\gamma(1 - \vartheta \eta) \in G_0(A).$$

4. Let $\alpha, \beta \in \Delta$, $A \in (\text{salg})$, $t \in U(A_0)$, $\mathbf{u} \in A_0 \times A_1 = A$. Then

$$h_\alpha(t) x_\beta(\mathbf{u}) h_\alpha(t)^{-1} = x_\beta(t^{\beta(H_\alpha)} \mathbf{u}) \in G_{p(\beta)}(A)$$

where $p(\beta)$ denotes as usual the parity of a root β , that is $p(\beta) = 0$ if $\beta \in \Delta_0$ and $p(\beta) = 1$ if $\beta \in \Delta_1$.

We are still under the simplifying assumption $\mathfrak{g} \neq Q(n)$ hence $\Delta_0 \cap \Delta_1 = \emptyset$. We stress that our results hold for *all* Lie superalgebras of classical type, but we choose in the present work for clarity of exposition to restrict ourselves to $\mathfrak{g} \neq A(1, 1), P(3), Q(n)$.

As a direct consequence of the commutation relations, we have the following proposition involving the sets we have introduced: G^\pm , etc. The proof is a simple exercise.

Theorem 4.12. *Let $A \in (\text{salg})$. There exist set-theoretic factorizations*

$$G(A) = G_0(A) G_1(A) = G_1(A) G_0(A)$$

$$G^\pm(A) = G_0^\pm(A) G_1^\pm(A) = G_1^\pm(A) G_0^\pm(A).$$

This decomposition has a further refinement that we state down below, whose proof is harder and we send the reader to [9] §5.3 for the details.

Theorem 4.13. *For any $A \in (\text{salg})$ we have*

$$G(A) = G_0(A) G_1^<(A) = G_1^<(A) G_0(A)$$

From the previous results we have that a generic $g \in G(A)$ can be factorized (once we choose a suitable ordering on the roots):

$$g = g_0 g_1^+ g_1^-, \quad g_0 \in G_0(A), \quad g_1^\pm = G_1^{\pm, <}(A).$$

The next theorem gives us the key to the representability of \mathbf{G} , by stating the uniqueness of the above decomposition. Again for the proof see [9], 5.3.

Theorem 4.14. *Let the notation be as above. For any $A \in (\text{salg})$, the group product gives the following bijection:*

$$G_0(A) \times G_1^{-, <}(A) \times G_1^{+, <}(A) \xrightarrow{\sim} G(A)$$

and all the similar bijections obtained by permuting the factors $G_1^{\pm, <}(A)$ and the factor $G_0(A)$.

As one can readily see, the functors $G_1^{\pm, <} : (\text{salg}) \rightarrow (\text{sets})$ are representable and they are the functor of points of an odd dimensional affine super space: $G_1^{\pm, <} \cong \mathbb{A}^{0|N^\pm}$, for $N^\pm = |\Delta_1^\pm|$. Then this, together with the definition of \mathbf{G} gives:

$$\mathbf{G} \cong \mathbf{G}_0 \times G_1^{-, <} \times G_1^{+, <} = \mathbf{G}_0 \times \mathbf{A}^{0|M}$$

for $M = N^+ + N^-$. Consequently \mathbf{G} is representable, since it is the direct product of representable functors. We have sketched the proof of the main result of the paper:

Theorem 4.15. *The Chevalley supergroup $\mathbf{G} : (\text{salg}) \rightarrow (\text{sets})$,*

$$\mathbf{G}(A) := \left\langle \mathbf{G}_0(A), x_\beta(A) \mid \beta \in \Delta_1 \right\rangle$$

is representable.

The next proposition establishes how much the Chevalley supergroup scheme \mathbf{G} we have built depends on the chosen representation. It turns out that two different complex \mathfrak{g} -representations V and V' (as in beginning of Sec. 4), with weight lattices $L_{V'} \subset L_V$ of the same complex Lie superalgebra \mathfrak{g} of classical type give raise to a morphism between the corresponding Chevalley supergroups, with kernel inside the center of \mathbf{G} , as it happens in the ordinary setting. This is actually expected, since the kernel is related with the fundamental group, which is a topological invariant, unchanged by the supergeneralization.

Theorem 4.16. *Let \mathbf{G} and \mathbf{G}' be two Chevalley supergroups constructed using faithful complex representations V and V' of the same complex Lie superalgebra of classical type \mathfrak{g} . Let $L_V, L_{V'}$ be the corresponding lattices of weights. If $L_V \supseteq L_{V'}$, then there exists a unique morphism $\phi : \mathbf{G} \rightarrow \mathbf{G}'$ such that $\phi_A(1 + \vartheta X_\alpha) = 1 + \vartheta X'_\alpha$, and $\text{Ker}(\phi_A) \subseteq Z(\mathbf{G}(A))$, for every local algebra A . Moreover, ϕ is an isomorphism if and only if $L_V = L_{V'}$.*

We observe that this theorem tells us that our construction of \mathbf{G} does not depend on the chosen representation V , but only on the weight lattice of V . In particular \mathbf{G} is independent of the choice of an admissible lattice.

In the end we want to ask the following question: does our construction provide all the algebraic supergroups whose Lie superalgebra is of classical type? The answer to this question is positive and we plan to explore furtherly the topics in a forthcoming paper.

5 Examples and further topics

In this final section we want to discuss some examples and to indicate possible further developments and applications of the theory we have described.

We start by discussing how our construction can be generalized to other Lie superalgebras, provided some conditions are satisfied.

We list down below some requirements a Lie superalgebra must satisfy so that we can try to replicate our construction.

We start from a complex Lie superalgebra $\mathfrak{g} = \langle X_a \mid a \in \mathcal{A} \rangle$ generated (as Lie superalgebra) by the homogeneous elements X_a , where $a \in \mathcal{A}$ a finite set of indices, and a complex finite dimensional representation V .

We assume the following.

1. \mathfrak{g} admits a basis $B \supset \{X_a\}_{a \in \mathcal{A}}$ and an integral form $\mathfrak{g}_{\mathbb{Z}} = \text{span}_{\mathbb{Z}}\{B\}$ in which all the brackets are integral combinations of elements in B ;
2. There exists a suitable \mathbb{Z} -subalgebra of $U(\mathfrak{g})$ denoted by $U_{\mathbb{Z}}(\mathfrak{g}) \subset U(\mathfrak{g})$ admitting a PBW theorem. In other words, $U_{\mathbb{Z}}(\mathfrak{g})$ is a free \mathbb{Z} -module with a basis consisting of suitable monomials, which form also a basis for $U(\mathfrak{g})$.
3. V contains an integral lattice M stable under $U_{\mathbb{Z}}(\mathfrak{g})$;
4. There is well defined algebraic group \mathbf{G}_0 over \mathbb{Z} , whose k -points embed into $\text{GL}(k \otimes M)$ and whose corresponding Lie algebra is \mathfrak{g}_0 . This will allow us to consider its functor of points $\mathbf{G}_0 : (\text{salg}) \longrightarrow (\text{sets})$.

If the requirements listed above are satisfied, then we can certainly give the same definition as in 4.8 and define the *Chevalley-type supergroup functor*.

Notice that the first part, up to section 3 is devoted to prove (1)–(3) for \mathfrak{g} of classical type. (4) for the classical type is discussed in [9] section 4.

Definition 5.1. Let \mathfrak{g} and V be as above. Define as before (compare before 4.5):

$$\mathfrak{g}_k := k \otimes \mathfrak{g}_V, \quad V_k := k \otimes M, \quad U_k(\mathfrak{g}) := k \otimes U_{\mathbb{Z}}(\mathfrak{g}).$$

Define the *Chevalley-type supergroup functor* as the functor $\mathbf{G} : (\text{salg}) \rightarrow (\text{sets})$ given on the objects by:

$$\mathbf{G}(A) = \langle \mathbf{G}_0(A), \quad 1 + \theta X_b, \quad X_b \text{ odd}, \quad \theta \in A_0 \rangle \subset \text{GL}(V_k)(A).$$

In other words $\mathbf{G}(A)$ is the subgroup of $\text{GL}(V_k)$ generated by the A -points of $\mathbf{G}_0(A)$ and the elements $1 + \theta X_b$. Again we identify an element X_b with its image in the representation V_k .

As we have already remarked after Definition 4.8, this definition does not ensure \mathbf{G} to be representable, hence to be rightfully called a supergroup scheme, and in fact a key role in the proof of the representability of this functor in the case of \mathfrak{g} of classical type, is played by the commutation relations between the elements generating the group $G(A)$.

Before we go to the representability issues, let us give an example of Lie superalgebra together with a class of representations, which is *not* of classical type and yet it satisfies the requirements listed above, hence it admits a Chevalley-type supergroup functor.

Example 5.2. Let us consider the Heisenberg Lie superalgebra \mathcal{H} , which is generated by an even generator e and by $2n$ odd generators $a_i, b_i, i = 1 \dots n$, with the only non zero brackets:

$$[a_i, b_j] = \delta_{ij}e, \quad i, j = 1 \dots n.$$

Define the following irreducible faithful complex representation (see [16] §1.1). Let $V = \wedge(\xi_1 \dots \xi_n)$, the complex exterior algebra with generators $\xi_1 \dots \xi_n$. V is a complex representation for \mathcal{H} by setting:

$$e \cdot u = \alpha u, \quad a_i \cdot u = \frac{\partial u}{\partial \xi_i}, \quad b_i \cdot u = \alpha \xi_i u, \quad \alpha \in \mathbb{C}.$$

Assume we take $\alpha \in \mathbb{Z}$.

If we set $\{X_a\}_{a=1,\dots,2n} = \{a_i, b_i, i = 1 \dots n\}$ and $B = \{e, a_i, b_i, i = 1 \dots n\}$ we have immediately satisfied item (1). As for item (2), we have that:

$$\left\{ \binom{e}{n} a_{i_1} \dots a_{i_p} b_{j_1} \dots b_{j_q} \right\}, \quad 1 \leq i_1 < \dots < i_p \leq n, \quad 1 \leq j_1 < \dots < j_q$$

is a \mathbb{Z} -basis of

$$U_{\mathbb{Z}}(\mathcal{H}) := \left\langle \binom{e}{n}, a_i, b_j \right\rangle \subset U(\mathcal{H}).$$

V contains the following integral lattice stable under $U_{\mathbb{Z}}(\mathcal{H})$:

$$M = \text{span}_{\mathbb{Z}} \left\{ \xi_{i_1} \dots \xi_{i_m} \mid 1 \leq i_1 < \dots < i_m \leq n \right\}$$

Finally certainly a_i and b_i act as nilpotent operators and consequently also item (3) is satisfied.

As for item (4), we have immediately that $\mathbf{G}_0 \cong k$ is an algebraic group, the additive group of the affine line. In the representation V_k the elements in $\mathbf{G}_0(k)$ act as follows:

$$h_e(t) \cdot u = t^\alpha u$$

hence $\mathbf{G}_0(k)$ is embedded into $\text{GL}(V_k)$ as the diagonal matrices:

$$\mathbf{G}_0(k) = \left(\begin{array}{cccc} t^\alpha & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & t^\alpha \end{array} \right) \subset \text{GL}(V_k).$$

Its functor of points is hence given simply by taking $t \in A_0$:

$$G_0(A) = \left(\begin{array}{cccc} t^\alpha & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & t^\alpha \end{array} \right) \subset \text{GL}(V_k), \quad t \in A_0.$$

We now go to the problem of representability of the functor \mathbf{G} we have defined. Besides the technical problems involved, our proof of the representability issue for the Chevalley supergroup functor discussed in 4 relies on the following facts:

1. $\mathbf{G}(A) = \mathbf{G}_0(A)G_1^<(A)$, where $G_1^<(A) = \{(1 + \theta_{a_1} X_{a_1}) \dots (1 + \theta_{a_n} X_{a_n})\}$ where $a_i < a_{i+1}$ in an order on \mathcal{A} , the index set of the indices a_i .

2. The above decomposition is unique, that is $\mathbf{G}(A) = \mathbf{G}_0(A) \times G_1^<(A)$ is unique.
3. $G_1^< \cong \mathbf{A}^{0|N}$ for a suitable N .

Clearly this leads immediately to the representability of the functor \mathbf{G} , since it is the direct product of two representable functors.

Coming back to our example of the Heisenberg superalgebra, by a direct calculation very similar to the one in section 4 one sees that the commutator:

$$(1 + \theta X_a, 1 + \eta X_b) = 1 + c\theta\eta e, \quad c \in \mathbb{Z}.$$

Notice that $1 + c\theta\eta e$ acts on $u \in V_k$ as a diagonal matrix with entries $1 + c\theta\eta t^\alpha$. This is an element in $G_0(A)$ since $(1 + c\theta\eta t)^\alpha = 1 + c\theta\eta t^\alpha$. By repeating the reordering arguments as in [9] 5.15 and 5.16 one can show that properties (1)–(3) are satisfied, hence giving us the representability of the Chevalley-type supergroup functor for the Heisenberg Lie superalgebra. Consequently we have define the Heisenberg supergroup associated to the Heisenberg Lie superalgebra in the following way:

$$\mathbf{G}(A) = \langle \mathbf{G}_0(A), 1 + \theta_i a_i, 1 + \eta_i b_i \rangle \subset \mathrm{GL}(V)(A).$$

We now want consider an important question: What are the algebraic supergroups, that we can construct using this method, in other words using the Chevalley supergroup construction? One could be tempted to say that we can obtain always an algebraic supergroup whose reduced group corresponds to a reductive group with Lie algebra the even part of the given Lie superalgebra of classical type. The situation, however, is more complicated and this is not always the case. Let us examine an interesting example.

Using the method of Super Harish-Chandra pairs (SHCP) (see for example [17]) we have that there exists a simply connected¹ Lie supergroup corresponding to the Lie superalgebra $\mathfrak{osp}(p|q)$. Since $\mathfrak{osp}(p|q) = \mathfrak{so}(p) \otimes \mathfrak{sp}(q)$, this supergroup can be rightfully called the *spin supergroup* and it is very important in physics. For example in [25] ch. 5, Varadarajan discusses its construction for arbitrary signatures.

¹A *simply connected* supergroup is a supergroup with simply connected underlying topological space.

However, with our construction, it is not always possible to reach the simply connected supergroup associated with the given Lie superalgebra. Let us clarify this phenomenon with an example.

Example 5.3. Let us consider:

$$\mathfrak{osp}(1|2) := \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ \beta & & B \\ -\alpha & & \end{pmatrix} \mid B \in \mathfrak{sl}_2 \right\} \subset \mathfrak{gl}(1|2).$$

A basis is given by:

$$e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

where $\text{span}_{\mathbb{C}}\{e, f, h\} \cong \mathfrak{sl}_2$ and the other non zero brackets are:

$$[e, y] = -x, \quad [f, x] = -y, \quad [x, x] = 2e, \quad [y, y] = -2f,$$

$$[x, y] = h, \quad [h, x] = x, \quad [h, y] = -y.$$

It is clear that a Cartan subalgebra over \mathbb{C} can be chosen as $\mathfrak{h} = \text{span}_{\mathbb{C}}\{h\}$ and a Chevalley basis for \mathfrak{osp} is

$$\mathcal{B} = \{x, y, e, f, h\}.$$

Now we go to the choice of the module V , that determines the Chevalley supergroup. In order to obtain a supergroup with simply connected underlying topological space, we must choose for $\mathfrak{sl}(2) \subset \mathfrak{osp}(1|2)$ the fundamental representation, that is $\mathfrak{g}_V = \{h/2, e, f\}$. However as one can see right away, such choice is forbidden, since otherwise we would have:

$$[h/2, x] = x/2, \quad [h/2, y] = -y/2.$$

Consequently the only Chevalley supergroup associated with $\mathfrak{osp}(1|2)$ has as underlying topological space the adjoint group for $\text{SL}(2)$, that is $\text{PSL}(2)$.

This shows that our method will not yield the spin supergroup, and we believe that this Lie supergroup is not algebraic. We plan to explore this and further topics in this section in a forthcoming paper.

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