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Strassen's law of the iterated logarithm for diffusion processes for small time[1]

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Abstract

We study the Strassen's law of the iterated logarithm for diffusion processes for small values of the parameter. For the Brownian Motion this result can be obtained by time reversal, a technique which is not easy to reproduce for diffusion processes. A number of examples and applications are discussed. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

Let B be a k-dimensional Brownian Motion. The law of the iterated logarithm (LIL) states that

$$\limsup_{u\to+\infty} \frac{|\, B_u \,|}{\sqrt{u \log\log u}} = \sqrt{2}.$$

This law has been extended by Strassen (1965) in a functional form: for $t \in [0,1]$ and $u > e$ set

$$\tilde{B}_u(t) = \frac{B_{ut}}{\sqrt{u \log\log u}},$$

so that for every u $\tilde{B}_u$ is a r.v. taking values in the space of paths $\mathscr{C}$. Let $\mathscr{K}$ be the set of the absolutely continuous paths f satisfying $f(0) = 0$ and $\int_0^1 |\, f'(t) \,|^2 \,\mathrm{d}t \leqslant 2$. Then $\{\tilde{B}_u\}_{u>e}$ is relatively compact in the uniform topology and $\mathscr{K}$ is the set of its limit points as $u \to +\infty$.

An extension of this law to diffusion processes was proved by Baldi (1986). Consider a diffusion process Y starting from x and let $\{\Gamma_\alpha\}_{\alpha>0}$ be a suitable family of contractions with fixed point x for every $\alpha > 0$. For $t \in [0,1]$ and $u > e$ set

$$\tilde{Z}_u(t) = \Gamma_{\sqrt{u \log\log u}}(Y_{ut}).$$

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Under suitable assumptions on the diffusion coefficients of the SDE driving $Y, \{\tilde{Z}_u\}_{u>e}$ is relatively compact in the space of the continuous paths starting from x equipped with the supnorm and the set of the limit points as $u \to +\infty$ is given by a compact set that can be explicitly described as the level set of the rate function associated with a suitable system of small random perturbations.

One might ask if these LILs hold also when $u \to 0^+$. It is well known for the Brownian motion that

$$\limsup_{u\to 0^+} \frac{|\, B_u \,|}{\sqrt{u \log\log u^{-1}}} = \sqrt{2}. \tag{1.1}$$

A functional version of this law has been studied by Mueller (1981) and recently Gantert (1993) has given a new proof of this Strassen's law for small time: for $t \in [0,1]$ and $u < e^{-1}$, setting

$$B_u(t) = \frac{1}{\sqrt{u \log\log u^{-1}}} B_{ut}, \tag{1.2}$$

then $\{B_u\}_{0<u<\bar{u}}$ is relatively compact and the set of its limit points as $u \to 0^+$ is $\mathscr{K}$ again. It should be remarked that whereas Eq. (1.1) can be easily derived from the classical LIL by time reversal, Eq. (1.2) requires some work.

In this paper we study a functional LIL for diffusion processes (not necessarily Gaussian): if we consider the family $\{Z_u\}_{0<u<\bar{u}}$ defined by

$$Z_u(t) = \Gamma_{\sqrt{u \log\log u^{-1}}}(Y_{ut})$$

for $t \in [0,1]$ and $u < \bar{u}(\leqslant e^{-1})$, then we prove that it is relatively compact for $u \to 0^+$ and we determine its limit set C. Such a result holds in the space of the explosive paths if the explosion time of Y is finite with positive probability. As in the case $u \to +\infty$, C is a level set of the rate function λ associated to a system of small random perturbation $(\tilde{b}_u, \tilde{\sigma}_u)$, where $\tilde{b}_u$ and $\tilde{\sigma}_u$ are strictly connected with the diffusion coefficients of Z_u (see Eq. (2.4)). This is the content of Section 3, where a general theorem is proved making use of results of large deviations, which are summarized in Section 2.

In the following sections we give examples. First, we treat a functional LIL for iterated Ito integrals, from which a related result for the Lévy's stochastic area process follows, thus giving a small-time counterpart of the LIL obtained by Berthuet (1986). Also, we show an LIL for small time for the principal invariant diffusion of the Heisenberg group, as a consequence of a more general result concerning invariant diffusions of simply connected nilpotent Lie groups.

It is worth to point out that in most of the examples above the limit sets turn out to be the same both for large and small time, although, in general, such a property does not hold. It should also be pointed out that, unlike the case $u \to +\infty$, in these results the drift coefficient plays no role.

Finally, it is worth mentioning that the technique used to prove the main theorem is similar to the one which is developed in Baldi (1986). One could even say that our proof consists on the remark that the techniques developed for $u \to +\infty$ also work for $u \to 0^+$. However, this allows us to study some non-trivial examples which have attracted some interest in recent years. See Gantert (1993) in which the case of the functional LIL for $u \to 0^+$ has been studied using time reversal, a completely different technique.

2. Some preliminary results

Let U be an open set of $\mathbb{R}^m$, $\mathscr{C}^U$ be the set of the continuous paths $u : [0,1] \to U$ and $\mathscr{C}_x^U$ the subset of $\mathscr{C}^U$ of the paths starting from x, i.e. $u(0) = x$. We shall write $\mathscr{C}_x^m$ and $\mathscr{C}^m$ when $U = \mathbb{R}^m$. Endowed with the supnorm, $\mathscr{C}_x^U$ is a Banach space. We shall denote by d the induced metric.

Let $\mathscr{H}_m$ be the *Cameron–Martin* space, i.e. the subset of $\mathscr{C}_0^m$ of the absolutely continuous paths whose derivative is square integrable on $[0,1]$. It is a Hilbert space with scalar product $(h,g)_{\mathscr{H}_m} = (h',g')_{L^2([0,1])}$, where h' and g' denote the derivatives of h and g, respectively. We shall denote by $|\cdot|_1$ the norm on $\mathscr{H}_m$ induced by the scalar product: $|h|_1 = \|h'\|_{L^2([0,1])}$. We set $\mathscr{K}_m = \{f \in \mathscr{H}_m;\ \frac{1}{2}\,|f|_1^2 \leqslant 1\}$.

Let $(\Omega, \mathscr{F}, P)$ be a probability space on which a k-dimensional Brownian Motion B is defined and consider the following diffusion process Y on U:

$$\begin{aligned} \mathrm{d}Y_t &= \tilde{b}(Y_t)\,\mathrm{d}t + \tilde{\sigma}(Y_t)\,\mathrm{d}B_t, \\ Y_0 &= x, \end{aligned} \tag{2.1}$$

where $x \in U$, $\tilde{\sigma}$ is a $m \times k$ matrix field on U, $\tilde{b}$ is a vector field on U. Its infinitesimal generator is

$$\tilde{L} = \frac{1}{2} \sum_{i,j} \tilde{a}_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_i \tilde{b}(y) \frac{\partial}{\partial y_i}, \tag{2.2}$$

where $\tilde{a} = \tilde{\sigma}\tilde{\sigma}^t$. We first suppose that

Assumption (G). The explosion time

$$\tau = \inf\{t > 0;\ Y_t \notin U\}$$

is a.s. infinite.

Such hypothesis will be relaxed afterwards.

Now, we want to define a family of diffusion processes $\{Z_u\}_{u\in(0,1)}$, whose elements are suitable transformations of Y which allow to deduce a law of the iterated logarithm for Y. To this purpose we need the following definition, which has been preliminarly introduced in Baldi (1986):

Definition 2.1. For $\alpha > 0$, let $\Gamma_\alpha : U \to U$ be a $\mathscr{C}^2$ bijective transformation having continuous derivatives up to order 2. The family $\{\Gamma_\alpha\}_{\alpha>0}$ is said to be a *system of contractions centered at* x if

(a) $\Gamma_\alpha(x) = x$ for every $\alpha > 0$;

(b) if $\alpha \geqslant \beta$ then $|\Gamma_\alpha(y) - \Gamma_\alpha(z)| \leqslant |\Gamma_\beta(y) - \Gamma_\beta(z)|$ for every y, $z \in U$;

(c) $\Gamma_1 = \mathrm{Id}$ and $\Gamma_{\alpha^{-1}} = \Gamma_\alpha^{-1}$. Moreover, for every compact subset K of U and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\alpha\beta - 1| < \delta$ then

$$|\Gamma_\alpha \circ \Gamma_\beta(y) - y| < \varepsilon$$

for every $y \in K$.

Let $\bar{u} = \mathrm{e}^{-1} \wedge \bar{t}$. For $u \in (0, \bar{u})$, set

$$G(u) = \log\log u^{-1}, \qquad \psi(u) = \sqrt{uG(u)}$$

and for $t \in [0,1]$

$$Z_u(t) = \Gamma_{\psi(u)}(Y_{ut}), \tag{2.3}$$

where Y is defined by Eq. (2.1) and Γ is a system of contraction centered at x, the starting point of Y. By Ito's formula and time change, Z_u is the solution of the SDE

$$\mathrm{d}Z_u(t) = \tilde{b}_u(Z_u(t))\,\mathrm{d}t + \frac{1}{\sqrt{G(u)}}\tilde{\sigma}_u(Z_u(t))\,\mathrm{d}B_t^{(u)},$$

$$Z_u(0) = x, \tag{2.4}$$

where $B_t^{(u)} = 1/\sqrt{u}B_{ut}$ is a k-dimensional Brownian Motion and

$$\tilde{b}_u(y) = u\tilde{L}\Gamma_{\psi(u)}(z)\Big|_{z=\Gamma^{-1}_{\psi(u)}(y)}, \tag{2.5}$$

$$\tilde{\sigma}_u(y) = \psi(u)\mathrm{grad}\,\Gamma_{\psi(u)}(z)\Big|_{z=\Gamma^{-1}_{\psi(u)}(y)} \cdot \tilde{\sigma}(\Gamma^{-1}_{\psi(u)}(y)), \tag{2.6}$$

where $\tilde{L}$ is defined by Eq. (2.2).

From now on we set the following hypothesis:

Assumption (A)**.** (i) There exist a matrix field σ and a vector field b on U such that

$$\lim_{u\to 0^+} \tilde{b}_u(y) = b(y),$$

$$\lim_{u\to 0^+} \tilde{\sigma}_u(y) = \sigma(y)$$

uniformly on compact subsets of U.

(ii) $\tilde{b}_u, \tilde{\sigma}_u, b, \sigma$ are Lipschitz continuous on compact subsets of U.

(iii) For $f \in \mathscr{H}_k$ and $x \in U$ the solution $g = S_x(f)$ of the Cauchy problem

$$g_t' = b(g_t) + \sigma(g_t)f_t',$$

$$g_0 = x \tag{2.7}$$

is defined on the whole interval $[0,1]$.

Assumption (A) not only ensures the uniqueness of the solution Z_u of Eq. (2.4), but also allows us to apply some well-known results of large deviation theory to the system of small random perturbation $(\tilde{b}_u, \tilde{\sigma}_u)$, which are only an improvement of the classical results due to Azencott and Priouret (see e.g. Azencott, 1980; Priouret, 1982) and whose proofs can be found in Baldi and Chaleyat-Maurel (1986) (see, in particular, Theorems 1.1 and 2.1). They can be summarized in the following way.

We recall the Cramér functional λ and the Cramér transform Λ: $\lambda : \mathscr{C}_x^U \to [0,+\infty]$

$$\lambda(g) = \inf\{\tfrac{1}{2} \mid f\mid_1^2;\ g = S_x(f)\} \tag{2.8}$$

and $+\infty$ if the above set is empty;

$$\Lambda : \mathscr{B}(\mathscr{C}_x^U) \to [0,+\infty]$$

$$\Lambda(A) = \inf_{g\in A} \lambda(g), \tag{2.9}$$

where $\mathscr{B}(\mathscr{C}_x^U)$ denotes the Borel σ-field.

Under Assumption (A), λ is lower semicontinuous and for $a \in \mathbb{R}$ the set $\{g;\ \lambda(g)\leqslant a\}$ is a compact subset of U. Moreover, Assumption (A) ensures that the family of probabilities on $\mathscr{B}(\mathscr{C}_x^U)$ induced by $\{Z_u\}_{0<u<\bar{u}}$ satisfies a large deviation principle with the good rate function λ, i.e. for every $A \in \mathscr{B}(\mathscr{C}_x^U)$, if A is open then

$$\liminf_{u\to 0^+} \frac{1}{G(u)} \log P(Z_u \in A) \geqslant -\Lambda(A) \tag{2.10}$$

and if A is closed then

$$\limsup_{u\to 0^+} \frac{1}{G(u)} \log P(Z_u \in A) \leqslant -\Lambda(A). \tag{2.11}$$

Finally, under Assumption (A) also the following result holds:

Theorem 2.2. *Let $a > 0$ be fixed. For every $\rho > 0$, $R>0$, there exists $u_0 > 0, \alpha_0 >0$ such that for every f with $\lambda(f)\leqslant a$ if $g = S_x(f)$ then*

$$P\left(d\left(\frac{1}{\sqrt{G(u)}}B^{(u)}, f\right) < \alpha, \qquad d(Z_u, g) > \rho\right) \leqslant \exp\left\{-\frac{R}{G(u)}\right\}$$

for every $u<u_0$, $\alpha<\alpha_0$.

3. The main theorem

In this section we shall prove the following:

Theorem 3.1. *Under Assumption* (A) *the family $\{Z_u\}_{0<u<\bar{u}}$ is relatively compact and the set*

$$C = \{g;\ \lambda(g)\leqslant 1\}$$

is the limit set of $\{Z_u\}_{0<u<\bar{u}}$ when $u \to 0^+$ a.s.

In order to prove Theorem 3.1 we need the next results.

Proposition 3.2. *Under Assumption* (A),

$$\lim_{u\to 0^+} d(Z_u, C) = 0 \quad \text{a.s.}$$

Proof. If $C'_\varepsilon = \{g;\ d(g,C)\geqslant\varepsilon\}$, then there exists $\delta>0$ such that $\Lambda(C'_\varepsilon)>1+\delta$. Indeed, if by contradiction $\Lambda(C'_\varepsilon)=1$, by Eq. (2.9) one could build a sequence $\{g_n\}_n\subset C'_\varepsilon$ such that $\lim_{n\to\infty}\lambda(g_n)=\Lambda(C'_\varepsilon)=1$. Obviously, for every n large $g_n\in\{f;\ \lambda(f)\leqslant 2\}$ which is compact: there exists a subsequence $\{g_{n_k}\}_k$ converging to $g\in C'_\varepsilon$. By the lower semicontinuity of λ

$$1=\liminf_{k\to\infty}\lambda(g_{n_k})\geqslant\lambda(g)$$

so that $g\in C$. Therefore, $\Lambda(C'_\varepsilon)>1+\delta$ for some $\delta>0$. By Eq. (2.11)

$$\limsup_{u\to 0^+}\frac{1}{G(u)}\log P(Z_u\in C'_\varepsilon)\leqslant -(1+\delta).$$

Thus, for j large and for every $c<1$

$$P(Z_{c^j}\in C'_\varepsilon)\leqslant\exp\{-(1+\delta)G(c^j)\}=\frac{\text{const}}{j^{1+\delta}}$$

so that by Borel–Cantelli lemma $P(\limsup_j\{d(Z_{c^j},C)\geqslant\varepsilon\})=0$, i.e. $\lim_{j\to\infty} d(Z_{c^j},C)=0$ a.s.

Using a procedure similar to that developed in Deuschel and Stroock (1986) (Lemma 1.4.3) or Baldi (1986) (Section 2), it is not hard to prove now that $\lim_{u\to 0^+} d(Z_u,C)=0$ a.s. and we give here a sketch of this proof.

Set $c\in(0,1)$ and j sufficiently large in order that $d(Z_{c^j},C)<\varepsilon$, ε being arbitrarily small. Thus, for $u\in[c^{j+1},c^j]$,

$$\begin{aligned} d(Z_u,C) &\leqslant \|Z_u-Z_{c^j}\|+d(Z_{c^j},C)\\ &<\varepsilon+\|\Gamma_{\psi(u)}\circ\Gamma^{-1}_{\psi(c^j)}(Z_{c^j})-Z_{c^j}\|+\sup_{t\in[0,1]}|\,\Gamma_{\psi(u)}(Y_{ut})-\Gamma_{\psi(u)}(Y_{c^jt})\,|\,. \end{aligned}$$

By the first part of this proof $\{Z_{c^j}\}_j$ is a.s. norm bounded: for every j sufficiently large, $Z_{c^j}\in K\equiv\{z;\ |\,z\,|\leqslant M\}$ and notice that M may be choosen such that $K\subset U$. Moreover, for every $u\in[c^{j+1},c^j]$,

$$1\geqslant\frac{\psi(u)}{\psi(c^j)}\geqslant\sqrt{c},$$

so that if c is sufficiently close to 1, using (b) of Definition 2.1, we obtain

$$\|\Gamma_{\psi(u)}\circ\Gamma^{-1}_{\psi(c^j)}(Z_{c^j})-Z_{c^j}\|\leqslant\varepsilon.$$

Furthermore, recalling that $\psi(u)$ is increasing for any small value of u, by (b) of Definition 2.1

$$|\Gamma_{\psi(u)}(Y_{ut})-\Gamma_{\psi(u)}(Y_{c^jt})|\leqslant|\Gamma_{\psi(c^{j+1})}(Y_{ut})-\Gamma_{\psi(c^{j+1})}(Y_{c^jt})|.$$

We can then write

$$d(Z_u, C) < 2\varepsilon + \sup_{t\in[0,1]} |\Gamma_{\psi(c^{j+1})}(Y_{ut}) - \Gamma_{\psi(c^{j+1})}(Y_{c^j t})|$$

$$\leqslant 2\varepsilon + \sup_{\substack{0\leqslant s\leqslant 1\\ cs\leqslant t\leqslant s}} |\Gamma_{\psi(c^{j+1})}(Y_{ut}) - \Gamma_{\psi(c^{j+1})}(Y_{c^j s})|.$$

Adding and substracting the terms $Z_{c^{j+1}}(t)$ and $Z_{c^{j+1}}(s)$ and using (c) of Definition 2.1, we have

$$d(Z_u, C) \leqslant 4\varepsilon + \sup_{\substack{0\leqslant s\leqslant 1\\ cs\leqslant t\leqslant s}} |Z_{c^{j+1}}(t) - Z_{c^{j+1}}(s)|.$$

Let now $g \in C$ such that $\|Z_{c^{j+1}} - g\| < d(Z_{c^{j+1}}, C) + \varepsilon$. Then, in particular, $\|Z_{c^{j+1}} - g\| < 2\varepsilon$ and the above estimate may be written as

$$d(Z_u, C) \leqslant 8\varepsilon + \sup_{g\in C} \sup_{\substack{0\leqslant s\leqslant 1\\ cs\leqslant t\leqslant s}} |g(t) - g(s)|.$$

Recalling that C is compact, Ascoli–Arzelà's theorem ensures the existence of a c such that the second term of the r.h.s. does not exceed ε, and this concludes the proof. □

Proposition 3.3. *Let $g \in C$ be such that $\lambda(g) < 1$. Then for any $\varepsilon > 0$ there exists $c_\varepsilon < 1$ such that for every $c < c_\varepsilon$*

$$P(d(Z_{c^j}, g) < \varepsilon \text{ i.o.}) = 1.$$

Proof. The statement is well known when $k = m$, $\tilde{b} = 0$, $\tilde{\sigma} = \mathrm{Id}$, i.e. as

$$Z_u(t) \equiv B_u(t) = \frac{B_{ut}}{\sqrt{uG(u)}}$$

and the compact C coincides with the set $\mathscr{K}_k \equiv \{f \in \mathscr{H}_k;\ \frac{1}{2}|f|_1^2 \leqslant 1\}$ (see Gantert, 1993 or Mueller, 1981): for any $f \in \mathscr{K}_k$,

$$P\left(\left\| \frac{1}{\sqrt{G(c^j)}} B^{(c^j)} - f \right\| < \varepsilon \text{ i.o.}\right) = 1.$$

For the general case we shall make use of the last equation and of Theorem 2.1. Let $g \in C$ be such that $\lambda(g) < 1$: there exists $f \in \mathscr{K}_k$ such that $g = S_x(f)$. For $\varepsilon > 0$ and $\alpha > 0$, setting

$$F_j = \left\{ \left\| \frac{1}{\sqrt{G(c^j)}} B^{(c^j)} - f \right\| < \alpha \right\}, \qquad H_j = \{\|Z_{c^j} - g\| < \varepsilon\}.$$

we obtain by Theorem 2.2

$$P(F_j \cap H_j^c) \leqslant \exp\left\{ -\frac{2}{G(c^j)} \right\} = \frac{\text{const}}{j^2}$$

being $R>2$, $c<1$, j large and α sufficiently small. Then by Borel–Cantelli lemma $P(F_j \cap H_j^c \text{ i.o.}) = 0$ so that

$$1 = P(F_j \text{ i.o.}) \leqslant P(F_j \cap H_j \text{ i.o.}) + P(F_j \cap H_j^c \text{ i.o.}) \leqslant P(H_j \text{ i.o.}). \qquad \square$$

Proof of Theorem 3.1. By Proposition 3.2 for any $\varepsilon>0$ there exists a.s. u_0 such that $d(Z_u, C)<\varepsilon/2$ for every $u<u_0$. By compactness of C one can find a finite number of balls of radius ε whose union contains $\{Z_u\}_{0<u<u_0}$. Now, let

$$\eta : [u_0, \bar{u}] \to \mathscr{C}_x^U,$$

$$u \mapsto Z_u.$$

By Definition 2.1 and Eq. (2.3), η is continuous a.s. so that $\{Z_u\}_{u_0 \leqslant u \leqslant \bar{u}}$ is compact a.s.

The second part of the proof of Theorem 3.1 now follows from Proposition 3.3 which ensures that all points in C are actually limit points. $\square$

Corollary 3.4. *Let E be a topological space and let $F : \mathscr{C}_x^U \to E$ be a continuous mapping. Then $\{F(Z_u)\}_{0<u<\bar{u}}$ is relatively compact and its limit set is $F(C)$.*

We analyze now the asymptotic behavior of the family $\{Z_u\}_u$ when the explosion time of Y is finite with positive probability. In such a case, the diffusion Y takes its values in the set $\mathscr{E}_x^U$ of the explosive paths (see [Azencott, 1980, Ch. III] for details): if $U \cup \delta$ denotes the Alexandroff's compactification of the open set U, then $\mathscr{E}_x^U$ is given by the continuous trajectories $g : [0,1] \to U \cup \delta$ such that $g_0 = x$ and such that if $g_{t_0} = \delta$, $0 \leqslant t_0 \leqslant 1$, then $g_t = \delta$ for any $t \in [t_0, 1]$. The explosion time $\tau(g)$ of $g \in \mathscr{E}_x^U$ is defined as

$$\tau(g) = \inf\{t>0 : \; g_t = \delta\}$$

and $\tau(g) = +\infty$ if $g_t \in U$ for every $t \in [0,1]$. The topology on $\mathscr{E}_x^U$ is induced by the following convergence: $\{g_n\}_n \subset \mathscr{E}_x^U$ converges to $g \in \mathscr{E}_x^U$ if $\{g_n\}_n$ converges to g uniformly on the compact subsets of $[0, \tau(g)[$. Notice that the space $\mathscr{C}_x^U$ of the continuous paths from $[0,1]$ to U, starting from x easily becomes the open subset $\{g \in \mathscr{E}_x[0,1], U); \; \tau(g) = +\infty\}$ of $\mathscr{E}_x^U$.

Let now $\{\Gamma_a\}_{\alpha>0}$ be a family of contraction on U, centered at x (as in Definition 2.1). In the case we are going to study, also the process

$$Z_t^u \equiv \Gamma_{\psi(u)}(Y_{ut})$$

as $u \in (0, \bar{u})$, $\bar{u} < e^{-1}$, takes its values in $\mathscr{E}_x^U$. Under an additional assumption on the family $\{\Gamma_\alpha\}_\alpha$, one can generalize Theorem 3.1 to the space $\mathscr{E}_x^U$. We first prove an easy result which analyzes the compactness properties in the space $\mathscr{E}_x^U$. The symbols $\bar{A}^{\mathscr{C}_x^U}$ and $\bar{A}^{\mathscr{E}_x^U}$ denote the closure of the set A in $\mathscr{C}_x^U$ and $\mathscr{E}_x^U$, respectively.

Lemma 3.5. *Let A be a relatively compact subset of $\mathscr{C}_x^U$. Then*

$$\bar{A}^{\mathscr{C}_x^U} = \bar{A}^{\mathscr{E}_x^U}$$

and so A is relatively compact in $\mathscr{E}_x^U$ too.

Proof. Let $g \in \bar{A}^{\mathscr{E}_x^U}$: there exists a sequence $\{g_n\}_n \subset A$ such that $g_n \overset{n\to\infty}{\longrightarrow} g$ in $\mathscr{E}_x([0,1],U)$. Since $\bar{A}^{\mathscr{C}_x^U}$ is a compact subset of $\mathscr{C}_x^U$, one can determine a subsequence $\{g_{n_k}\}_k$ converging in $\mathscr{C}_x^U$ to $\tilde{g} \in \mathscr{C}_x^U$. If $\tau(g) < \infty$ then for every $t < \tau(g)$ one obviously has $g_t = \tilde{g}_t$ and thus $\delta = \lim_{t\uparrow\tau(g)} g_t = \lim_{t\uparrow\tau(g)} \tilde{g}_t \in U$ which gives a contradiction. Thus, $\bar{A}^{\mathscr{C}_x^U} = \bar{A}^{\mathscr{E}_x^U}$. □

Theorem 3.6. *Suppose that*

$$\lim_{\alpha\to\infty} \Gamma_\alpha(y) = x,$$

uniformly on the compact subsets of U. Then, under Assumption (A), the family $\{Z_u\}_{0<u<\bar{u}}$ *is a.s. relatively compact in* $\mathscr{E}_x^U$ *and the set*

$$C = \{g \in \mathscr{C}_x^U;\ \lambda(g) \leqslant 1\}$$

determines its limit points as $u \to 0^+$.

Proof. We shall make use here of standard localization arguments which allow to apply Theorem 3.1.

Let G be an open subset of U containing x such that, for a suitable $\delta > 0$, the closure of its δ-neighborhood G^δ is strictly contained in U. Let us denote φ a $\mathscr{C}^\infty$ function such that $\varphi \equiv 1$ on G and $\varphi \equiv 0$ out of G^δ. Set $\hat{b} = \tilde{b} \cdot \varphi$ and $\hat{\sigma} = \tilde{\sigma} \cdot \varphi$. Let now $\hat{Y}$ denote the (strong) solution of Eq. (2.1) with diffusion coefficients $\hat{b}$ and $\hat{\sigma}$. Since both drift and diffusion coefficient are equal to zero out of G^δ and recalling that $\bar{G}^\delta \subset U$, the explosion time of the diffusion $\hat{Y}$ is equal to infinity, a.s. Set $\hat{b}_u$ and $\hat{\sigma}_u$ the fields defined in Eqs. (2.5) and (2.6), respectively, built using $\hat{b}$ and $\hat{\sigma}$. It follows that

$$\hat{b}_u(z) = \varphi(\Gamma^{-1}_{\psi(u)}(z))\, b_u(z) \quad \text{and} \quad \hat{\sigma}_u(z) = \varphi(\Gamma^{-1}_{\psi(u)}(z))\sigma_u(z)$$

and thus, recalling that $\varphi \equiv 1$ near x, part (i) of Assumption (A) ensures that

$$\hat{b}_u(z) \to b(z) \quad \hat{\sigma}_u(z) \to \sigma(z) \quad u \to 0^+ \text{ a.s.}$$

uniformly on the compact subsets of U. This allows to prove that Assumption (A) is verified by the diffusion coefficients driving $\hat{Y}$. Therefore, setting $\hat{Z}_t^u = \Gamma_{\psi(u)}(\hat{Y}_{ut})$ and using Theorem 3.1, the family $\{\hat{Z}^u\}_{0<u<\bar{u}}$ is a.s. relatively compact in $\mathscr{C}_x^U$ and the set C is the set of its limit points as u tends to 0.

Now, if $\hat{\tau}_G$ denotes the exit time of $\hat{Y}$ from G, we can write

$$\{Z^u\}_{u<\bar{u}} \equiv \{\hat{Z}^u\}_{0<u\leqslant\hat{\tau}_G\wedge\bar{u}} \cup \{Z^u\}_{\hat{\tau}_G\wedge\bar{u}\leqslant u\leqslant\bar{u}}.$$

For any fixed $u_0 > 0$, the map $\eta : [u_0, +\infty) \to \mathscr{E}_x^U, \eta(u)(t) = Z_t^u \equiv \Gamma_{\psi(u)}(Y_{ut})$ is continuous and thus $\{Z^u\}_{\hat{\tau}_G\wedge\bar{u}\leqslant u\leqslant\bar{u}} = \eta([\hat{\tau}_G \wedge \bar{u}, \bar{u}])$ is a.s. compact in $\mathscr{E}_x^U$. Moreover,

$$\{Z^u\}_{0<u<\hat{\tau}_G\wedge\bar{u}} \equiv \{\hat{Z}^u\}_{0<u<\hat{\tau}_G\wedge\bar{u}} \subset \{\hat{Z}^u\}_{u\leqslant\bar{u}}. \tag{3.1}$$

Since this last set is relatively compact in $\mathscr{C}_x^U$ (as previously proved), by Lemma 3.5 it is actually relatively compact in $\mathscr{E}_x^U$. Thus, the first statement is proved. Concerning the second one, it is sufficient to recall Eq. (3.1): the limit set has to coincide with the limit set of $\{\hat{Z}^u\}_{0<u<\bar{u}}$ and therefore it is just C. □

4. Applications

The results contained in Section 3 can be applied to the simplest case.

Proposition 4.1. *Suppose that $\tilde{b}$ and $\tilde{\sigma}$ are Lipschitz continuous on the compact subsets of $\mathbb{R}^m$ and set Y the solution of* Eq. (2.1). *Then the family*

$$Z_u(t) = x + \frac{Y_{ut} - x}{\sqrt{u \log\log u^{-1}}}$$

is a.s. *relatively compact in $\mathscr{E}_x^U$ and the set of its limit points when $u \to 0^+$ is*

$$C = \{g;\ g_t = x + \tilde{\sigma}(x) f_t,\ f \in \mathscr{K}_k\}.$$

Proof. Let us suppose $x = 0$ (otherwise, take $\tilde{Y}_t = Y_t - x$). To apply Theorem 3.6 we only need to check Assumption (A) with

$$\Gamma_\alpha y = \frac{y}{\alpha}.$$

Indeed, in such a case by Eqs. (2.4) and (2.5)

$$\tilde{\sigma}_u(y) = \tilde{\sigma}(\psi(u)y), \qquad \tilde{b}_u(y) = \frac{u}{\psi(u)}\tilde{b}(\psi(u)y).$$

By Lipschitz conditions one easily has

$$\lim_{u\to 0^+} \tilde{\sigma}_u(y) = \tilde{\sigma}(0), \qquad \lim_{u\to 0^+} \tilde{b}_u(y) = 0$$

uniformly on compact sets. Thus (i) and (ii) hold. Moreover, the solution of the Cauchy problem (2.6) is

$$g_t = \tilde{\sigma}(0) f_t$$

which is actually defined on $[0,1]$. □

Obviously, if Y is defined as the solution of Eq. (2.1) up to time 1, the above compactness property hold in $\mathscr{C}_x^U$.

The counterpart of the Strassen's law of the iterated logarithm for small time becomes now an immediate consequence of the above proposition: if B denote an m-dimensional Brownian Motion, setting for $t \in [0,1]$ and $u \in (0, \mathrm{e}^{-1})$

$$B_u(t) = \frac{B_{ut}}{\sqrt{u \log\log u^{-1}}}$$

then the family $\{B_u\}$ is relatively compact and its limit set as $u \to 0^+$ is $\mathscr{K}_m \equiv \{f \in \mathscr{H}_m : \frac{1}{2}|f|_1^2 \leqslant 1\}$. Such property has been proved by Gantert and Mueller (see

Gantert, 1993; Mueller, 1981). It is worth to mention that although we made use of a particular case of their results (see Proposition 3.3) Theorem 3.1 may be proved independently from them so that the Strassen's counterpart of the LIL is actually a consequence of Theorem 3.1. Indeed, in Proposition 3.3 we referred to Gantert (1993) and Mueller (1981) only to deduce that for any $f \in \mathscr{K}_k$,

$$P\left(\left\|\frac{1}{\sqrt{G(c^j)}}B^{(c^j)} - f\right\| < \varepsilon \text{ i.o.}\right) = 1$$

and this equality can be proved, for example, using large deviation estimates and the second Borel–Cantelli theorem.

We observe, moreover, that if $A_x = \tilde{\sigma}^t(x)\tilde{\sigma}(x)$ is a non-degenerate matrix then setting $Q_x = A_x^{-1}\tilde{\sigma}^t(x)$ one has

$$C = \left\{x + g;\ \ g \in \mathscr{H}_m, \tfrac{1}{2}|Q_x g|_1^2 \leqslant 1\right\}.$$

In particular, if $\tilde{\sigma}(x)$ is a unit matrix then

$$C = x + \mathscr{K}_m.$$

We can also give a non-functional LIL (possibly well known) applying Corollary 3.4 to the function $F : \mathscr{C}_x^m \to \mathbb{R}^m$, $F(g) = g(1)$:

$$\left\{\frac{Y_u - x}{\sqrt{u \log\log u^{-1}}}\right\}_{0<u<\bar{u}}$$

is relatively compact on $\mathbb{R}^m$ and the set of the limit points as $u \to 0^+$ is

$$C' = \{y \in \mathbb{R}^m,\ \ \tfrac{1}{2}|Q_x y|^2 \leqslant 1\}.$$

Remark. The result contained in Proposition 4.1 is actually very simple and of intuitive meaning. However, it is worth to be remarked that it shows that the LIL may not hold simultaneously for large and small time and if this should hold then the limit sets would not necessarily coincide. Indeed, for $s > e$ set

$$V_s(t) = \frac{Y_{st} - x}{\sqrt{s \log\log s}} + x$$

and consider the family $\{V_s\}_s$. Then, for every fixed s, V_s is a diffusion process (see Baldi, 1986) with drift $\tilde{b}_s$ and diffusion coefficient $(\varphi(s))^{-1/2}\tilde{\sigma}_s$, being $\varphi(s) = \sqrt{s \log\log s}$ and

$$\tilde{b}_s(y) = \frac{s}{\varphi(s)}\tilde{b}(\varphi(s)y + x),$$

$$\tilde{\sigma}_s(y) = \tilde{\sigma}(\varphi(s)y + x).$$

Now, in order to apply Theorem 2.1 in Baldi (1986), it is necessary that both $\tilde{b}_s$ and $\tilde{\sigma}_s$ do converge as $s \to +\infty$, uniformly on compact sets. For example, if $\tilde{b}$ is constant and different from 0, then the drift explodes and therefore no convergence result can be deduced for $\{V_s\}_s$ (in contrast to the LIL for small time, which always holds, subject to the Lipschitz property of the diffusion coefficients). If on the contrary $\tilde{b}=0$ and there exists $\lim_{|y|\to\infty}\tilde{\sigma}(y)=\sigma$, uniformly on the compact subsets of $\mathbb{R}^m$, one can then apply Theorem 2.1 in Baldi (1986): the family $\{V_s\}_{s>e}$ is relatively compact and its limit set as $s\to+\infty$ is

$$C_\infty=\{g;\ \ g_t=x+\sigma f_t,\ \ f\in\mathscr{K}_k\}$$

which, compared with the set C given by Proposition 4.1, is not (in general) the set of the limit points of the LIL for small time.

In the following, we study situations which have been treated in Baldi (1986) for $u\to+\infty$. Our aim is to compare them with the limits as $u\to 0^+$.

Let $B_t=(B_1(t),\dots,B_m(t))$ a m-dimensional Brownian Motion and for $\ell\leqslant m$ consider the following iterated Ito integral

$$X(t)=\sum_{i_1\dots i_\ell\in\mathscr{A}}a_{i_1\dots i_\ell}\int_0^t \mathrm{d}B_{i_\ell}(t_\ell)\int_0^{t_\ell}\cdots\int_0^{t_2}\mathrm{d}B_{i_1}(t_1). \tag{4.1}$$

$\mathscr{A}$ being a subset of $\{1,\dots,m\}$. We set for $0<u<\mathrm{e}^{-1}$ and $t\in(0,1)$

$$Z_u(t)=\frac{1}{\psi(u)^\ell}X(ut).$$

Proposition 4.2. *$\{Z_u\}_{0<u<\mathrm{e}^{-1}}$ is relatively compact and the limit points as $u\to 0^+$ are the paths of the form*

$$g(t)=\sum_{i_1\dots i_\ell\in\mathscr{A}}a_{i_1\dots i_\ell}\int_0^t f'_{i_\ell}(t_\ell)\,\mathrm{d}t_\ell\int_0^{t_\ell}\cdots\int_0^{t_2}f'_{i_1}(t_1)\,\mathrm{d}t_1,$$

where $f\in\mathscr{K}_m$.

Proof. We now define a suitable diffusion process whose last component is X_t: let Y_t be the solution of the following SDE:

$$\begin{aligned}
&\mathrm{d}Y_1(t)=\mathrm{d}B_1(t),\\
&\vdots\\
&\mathrm{d}Y_m(t)=\mathrm{d}B_m(t),\\
&\mathrm{d}Y_{i_1i_2}(t)=Y_{i_1}\mathrm{d}B_{i_2}(t),\\
&\vdots\\
&\mathrm{d}Y_{i_1i_2\dots i_k}(t)=Y_{i_1\dots i_{k-1}}\mathrm{d}B_{i_k}(t),\\
&\vdots\\
&\mathrm{d}X(t)=\sum_{i_1\dots i_\ell}a_{i_1\dots i_\ell}Y_{i_1\dots i_{\ell-1}}\mathrm{d}B_{i_\ell}(t).
\end{aligned}\tag{4.2}$$

$Y_t=(Y_1(t),\dots,Y_m(t),Y_{i_1i_2}(t),\dots,Y_{i_1i_2\dots i_k}(t),\dots,X(t))$ can be rewritten in the shortest form

$$Y_t=\int_0^t\tilde{\sigma}(Y_s)\,\mathrm{d}B_s,$$

where $\tilde{\sigma}$ is an $N \times m$ matrix defined by Eq. (4.2). Consider now the following family of contractions:

$$\Gamma_\alpha y = \left(\frac{y_1}{\alpha}, \dots, \frac{y_m}{\alpha}, \frac{y_{i_1 i_2}}{\alpha^2}, \dots, \frac{y_{i_1 i_2 \dots i_k}}{\alpha^k}, \dots, \frac{x}{\alpha^\ell}\right).$$

Note that $\mathrm{grad}_z \Gamma_\alpha z = \Gamma_\alpha$ and $\tilde{\sigma}(\Gamma_\alpha^{-1} z) = \Gamma_\alpha^{-1} \tilde{\sigma}(z)$, so that Eqs. (2.5) and (2.6) become

$$\tilde{b}_u = 0, \quad \tilde{\sigma}_u(y) = \tilde{\sigma}(y) \tag{4.3}$$

and Assumption (A) is verified. Therefore, we can apply Theorem 3.1 to the process

$$V_u(t) = \Gamma_{\psi(u)} Y_{ut}$$

and its limit set is given by the trajectories which solve

$$g'(t) = \tilde{\sigma}(g(t)) f'(t),$$

$$g(0) = 0,$$

where $f = (f_1, \dots, f_m) \in \mathscr{K}_m$. Note that the above Cauchy problem is easy to solve recursively. Consider now the projection $\pi : \mathbb{R}^N \to \mathbb{R}$

$$(y_1, \dots, y_m, \dots, y_{i_1 i_2 \dots i_k}, \dots, x) \mapsto x.$$

Obviously, $Z_u = \pi(V_u)$ and the statement holds by Corollary 3.4. □

Let $F : \mathscr{C}_0^1 \to \mathbb{R}$ be defined by

$$g \mapsto F(g) = g(1).$$

By Proposition 4.2 and Corollary 3.4 $\{F(Z_u)\}_{0<u<e^{-1}}$ is relatively compact. By setting for $f \in \mathscr{H}_m$

$$J(f) = \sum_{i_1 \dots i_\ell \in \mathscr{A}} a_{i_1 \dots i_\ell} \int_0^1 f'_{i_\ell}(t_\ell)\, dt_\ell \int_0^{t_\ell} \dots \int_0^{t_2} f'_{i_1}(t_1)\, dt_1,$$

we can write the limit set C as $u \to 0^+$ for $\{F(Z_u)\}_{0<u<e^{-1}}$ as

$$C = \{J(f);\; f \in \mathscr{K}_m\}.$$

Let

$$M_1 = \max_{f \in \mathscr{K}_m} J(f),$$

$$M_2 = \min_{f \in \mathscr{K}_m} J(f).$$

Then, by Corollary 3.4, it immediately follows that

$$\text{(4.1)} \quad \limsup_{u \to 0^+} \frac{X_u}{(\sqrt{u \log\log u^{-1}})^\ell} = M_1,$$

$$\text{(4.2)} \quad \liminf_{u \to 0^+} \frac{X_u}{(\sqrt{u \log\log u^{-1}})^\ell} = M_2.$$

Example. Let $\mathscr{P}_m$ denote the set of all the permutations of $\{1,\dots,m\}$ and $\varepsilon_{(i_1,\dots,i_m)}$ the signature of the generic permutation $(i_1,\dots,i_m)$. Suppose $\ell = m$ and

$$a_{i_1\dots i_m} = \varepsilon_{(i_1,\dots,i_m)}.$$

Thus, Eq. (4.1) becomes

$$L_m(t) = \sum_{(i_1,\dots,i_m)\in\mathscr{P}_m} \varepsilon_{(i_1,\dots,i_m)} \int_0^t \mathrm{d}B_{i_m}(t_m) \int_0^{t_m} \cdots \int_0^{t_2} \mathrm{d}B_{i_1}(t_1)$$

which is called the *m-dimensional Lévy's Stochastic Area*. By Proposition 4.2

$$\left\{ \frac{L_m(ut)}{(\sqrt{u \log\log u^{-1}})^m} \right\}_{0<u<\mathrm{e}^{-1}}$$

is relatively compact and the set of the limit points as $u \to 0^+$ is given by the paths of the form

$$S(f)_t = \sum_{(i_1,\dots,i_m)\in\mathscr{P}_m} \varepsilon_{(i_1,\dots,i_m)} \int_0^t f'_{i_m}(t_m)\,\mathrm{d}t_m \int_0^{t_m} \cdots \int_0^{t_2} f'_{i_1}(t_1)\,\mathrm{d}t_1$$

as f varies in $\mathscr{K}_m$. By Corollary 3.4 we can also deduce a non-functional LIL as $u \to 0^+$ for the Lévy's stochastic area process:

$$\limsup_{u\to 0^+} \frac{L_m(u)}{(\sqrt{u \log\log u^{-1}})^m} = l_m = -\liminf_{u\to 0^+} \frac{L_m(u)}{(\sqrt{u \log\log u^{-1}})^m},$$

where

$$l_m = \max_{f\in K} S(f)_1 = \max_{f\in K} \sum_\sigma \varepsilon(\sigma) \int_0^1 f'_{\sigma(m)}(t_m)\,\mathrm{d}t_m \int_0^{t_m} \cdots \int_0^{t_2} f'_{\sigma(1)}(t_1)\,\mathrm{d}t_1 \qquad (4.4)$$

(indeed, by symmetry, $-M_2 = M_1 \equiv l_m$).

Remark. For iterated Ito integrals, it is important to observe that the limit set C is just equal to that in the LIL for large time obtained in Baldi (1986) (see Proposition 3.1). Obviously, also the limit values M_1 and M_2 in the non-functional LIL turn out to be the same as $u \to +\infty$ (see again Baldi, 1986, Corollary 3.2). This shows that for the Lévy's stochastic area the limit value l_m in Eq. (4.4) is the same appearing in the non-functional iterated logarithm for large time which has been proved by Berthuet (1986, Section 3, Theorem 3), who also computed the exact value of l_m for any m: setting

- $\tilde{\omega}_m \equiv (\omega_1,\dots,\omega_{[m/2]}) \in \mathbb{R}^{[m/2]}$;
- $r_{2m}(t)$, $t \in \mathbb{R}$, the vector of $\mathbb{R}^{2m}$ whose components are $((\sin\omega_j t, \cos\omega_j t),\ 1\leqslant j\leqslant m)$ $r_{2m+1}(t) = (r_{2m}(t), 1) \in \mathbb{R}^{2m+1}$;
- $\Delta_m(t) = \det(r_m(t_1),\dots,r_m(t_m))$, $t \equiv (t_1,\dots,t_m)$;
- $D_m = \{t \equiv (t_1,\dots,t_m);\ t_1 \leqslant \cdots \leqslant t_m \leqslant 1\}$;
- $M_m = \max_{\tilde{\omega}_m} \int_{D_m} \Delta_m(t)\,\mathrm{d}t$

then

$$l_m = \begin{cases} \dfrac{4^{m/2}}{m} M_m, & m \text{ even}, \\ \dfrac{4^{m/2}}{m} \dfrac{M_m}{\sqrt{2}}, & m \text{ odd}. \end{cases}$$

It therefore seems that the same situation of the classical LIL turns out, in which the behavior of the Brownian Motion for small time can be easily deduced by time reversal from that as $u \to +\infty$, an analogy which allows to suggest that some time-reversal principles might hold also for the Lévy's stochastic area process.

5. Invariant diffusions on nilpotent Lie groups

In this section we shall prove a functional LIL for $u \to 0^+$ for particular invariant diffusions of nilpotent Lie groups. Moreover, we shall apply this result to the principal invariant diffusion of the Heisenberg group. Again we shall compare our result to the one treated in Baldi (1986). Let us begin by recalling some useful notions.

Let $\mathfrak{g} = (\Re, [\cdot, \cdot])$ be a nilpotent real Lie algebra whose underlying vector space is $\mathbb{R}^N$ and $G = (\Re, \circ)$ be the nilpotent simply connected Lie group whose product is defined by the Campbell–Hausdorff formula

$$g \circ h = g + h + \tfrac{1}{2}[g,h] + \tfrac{1}{12}[g,[g,h]] + \tfrac{1}{12}[h,[h,g]] + \cdots .$$

We set $\mathfrak{g}_1 = \mathfrak{g}$, $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1], \ldots, \mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k], \ldots, \mathfrak{g}_{\ell+1} = \{0\}$ the central lower series of g. Let

$$e_1, e_2, \ldots, e_{i_1}, e_{i_1+1}, \ldots, e_{i_2}, \ldots, e_\ell = e_N$$

be a basis of $\mathfrak{g}$ adapted to the central lower series, i.e. for every $j = 1, 2, \ldots, N$ $e_j \in \mathfrak{g}_k \setminus \mathfrak{g}_{k+1}$ if $i_k < j \leqslant i_{k+1}$ $(i_0 = 1)$. For $g \in \mathfrak{g}$ we denote by g^k the kth coordinate on such a basis.

Let V_k, $k = 1, 2, \ldots, \ell$, be spanned by $\{e_{i_k}, \ldots, e_{i_{k+1}-1}\}$: for every $g \in g$ we can write

$$g = \sum_{k=1}^{\ell} g_k,$$

where $g_k \in V_k$. For $\alpha > 0$ consider the transformation

$$D_\alpha g = \sum_{k=1}^{\ell} \alpha^k g_k,$$

which is an endomorphism of the Lie algebra $\mathfrak{g}$ such that $D_\alpha^{-1} = D_{\alpha^{-1}}$. By setting $\Gamma_\alpha \equiv D_{\alpha^{-1}}$, i.e.

$$\Gamma_\alpha g = \sum_{k=1}^{\ell} \frac{1}{\alpha^k} g_k \tag{5.1}$$

then $\{\Gamma_\alpha\}_{\alpha > 0}$ is a system of contraction centered at 0.

Let $x_0, x_1, \dots, x_r$ be in $\mathfrak{g}$ and $X_0, X_1, \dots, X_r$ be the corresponding left invariant vector fields (i.e. $X_i f(y) = \mathrm{d}/\mathrm{d}t\ f(y \circ tx_i)|_{t=0}$, $y \in \mathfrak{R}$, $f \in \mathscr{C}^\infty(\mathfrak{R})$). We write

$$X_i(y) = \sum_{j=1}^N X_i^j(y) \frac{\partial}{\partial y_j},$$

where $X_i^j(y)$, $j = 1, \dots N$, are the coordinates of $X_i(y)$ with respect to $\{e_1, \dots, e_N\}$.

Consider now the differential operator on G

$$L = X_0 + \frac{1}{2} \sum_{i=1}^r X_i^2. \tag{5.2}$$

A diffusion process Y on G whose generator is of the form given by Eq. (5.2) is said to be *left invariant.* If B is a k-dimensional Brownian Motion then Y is the solution of the SDE

$$\mathrm{d}Y(t) = \tilde{b}(Y(t))\,\mathrm{d}t + \tilde{\sigma}(Y(t))\,\mathrm{d}B_t,$$

$$Y(0) = 0,$$

where

$$\tilde{b}(y) = X_0(y) + \frac{1}{2} \sum_{i=1}^r \operatorname{grad} X_i(y) \cdot X_i(y)$$

and $\tilde{\sigma}(y)$ is an $N \times k$ matrix such that

$$\tilde{\sigma}_{ji}(y) = X_i^j(y).$$

For $t \in [0,1]$ and $u \in (0, \mathrm{e}^{-1})$ we set

$$Z_u(t) = \Gamma_{\psi(u)}(Y_{ut}),$$

where Γ_α is defined by Eq. (5.1). Let $f \in \mathscr{H}_k$ and $S_0(f) : [0,1] \to G$ be the solution of

$$\begin{aligned} \varphi'(t) &= \sum_{i=1}^r X_i(\varphi(t)) f_i'(t), \\ \varphi(0) &= 0. \end{aligned} \tag{5.3}$$

Remark 5.1. $S_0(f)$ is well defined: by Baldi (1986), Lemma 4.3, there exists a unique solution of Eq. (5.3) up to time 1.

For $i = 1, \dots, r$ let us denote $x_i = \sum_{j=1}^{\ell} x_{ij}$, where $x_{ij} \in V_j$, and let X_{ij} be the left invariant vector field associated to x_{ij}, $j = 1, \dots, \ell$.

Set

$$C = \{\varphi : [0,1] \to G;\ \varphi = S_0(f),\ f \in \mathscr{K}_k\}.$$

Theorem 5.2. *Suppose that*

$$x_0 \in V_1 \oplus V_2 \quad \textit{and} \quad x_i \in V_1,\ i = 1,\dots,k. \tag{5.4}$$

Then $\{Z_u\}_{u\in(0,\bar u)}$ *is relatively compact and* C *is the limit set as* $u \to 0^+$ a.s.

We shall make use of the following lemma, whose proof may be found in Baldi (1986).

Lemma 5.3. *Let* $x \in \mathfrak{g}$ *and* X *be the corresponding left invariant vector field. If* $x \in V_i$ *then for* $\alpha > 0$ *and* $y \in G$

$$\alpha^{-i} D_\alpha X(y) = X(D_\alpha y),$$

$$\operatorname{grad} X(y) = \alpha^i D_\alpha^{-1} \operatorname{grad} X(D_\alpha y) \cdot D_\alpha.$$

Proof of Theorem 5.2. In order to apply Theorem 3.1, we have to prove that Assumption (A) holds. Indeed we show that

$$\tilde\sigma_u(y) \equiv \tilde\sigma(y),$$

$$\lim_{u\to 0^+} \tilde b_u(y) = 0$$

uniformly on compact sets.
By Lemma 5.3

$$\tilde\sigma(\Gamma_{\psi(u)} y) = \psi(u)^{-1} \Gamma_{\psi(u)}^{-1} \tilde\sigma(y)$$

and keeping in mind Eq. (2.5) $\tilde\sigma_u(y) \equiv \tilde\sigma(y)$. Again by Lemma 5.3

$$\begin{aligned}\tilde L \Gamma_{\psi(u)} z &= X_0 \Gamma_{\psi(u)}(z) + \frac{1}{2}\sum_{i=1}^r X_i^2 \Gamma_{\psi(u)}(z)\\ &= \Gamma_{\psi(u)} X_0(z) + \frac{1}{2}\sum_{i=1}^r \Gamma_{\psi(u)} \operatorname{grad} X_i(z) \cdot X_i(z)\end{aligned}$$

from which we have

$$\begin{aligned}\tilde b_u(y) &= u\Gamma_{\psi(u)} X_0(\Gamma_{\psi(u)}^{-1} y) + \frac{u}{2}\sum_{i=1}^r \Gamma_{\psi(u)} \operatorname{grad} X_i(\Gamma_{\psi(u)}^{-1} y) \cdot X_i(\Gamma_{\psi(u)}^{-1} y)\\ &= u\psi(u)^{-1} \Gamma_{\psi(u)} \Gamma_{\psi(u)}^{-1} X_0(y) + \frac{u}{2}\sum_{i=1}^r \Gamma_{\psi(u)} \cdot \psi(u)^{-1}\\ &\quad \cdot \Gamma_{\psi(u)}^{-1} \operatorname{grad} X_i(y) \cdot \Gamma_{\psi(u)} \cdot \psi(u)^{-1} \cdot \Gamma_{\psi(u)}^{-1} X_i(y)\\ &= \frac{u}{\psi(u)}[X_{01}(y) + X_{02}(y)/\psi(u)] + \frac{u}{2\psi^2(u)} \sum_{i=1}^r \operatorname{grad} X_i(y) \cdot X_i(y)\end{aligned}$$

and $\lim_{u\to 0^+} \tilde{b}_u(y)=0$ uniformly on compact sets. Thus, part (i) of Assumption (A) holds. Also, (ii) holds for the smoothness of $\tilde{\sigma}_u$ and $\tilde{b}_u$ and by Remark 5.1(iii) is verified.

Finally, note that C is actually the same set which appears in Theorem 3.1. □

Remark. It is easy to see that if there exists $i\geqslant 1$ such that $x_i \notin V_1$ then the divergence of $\tilde{\sigma}_u$ as $u \to 0^+$ will follow; again, if $x_0 \notin V_1 \oplus V_2$ then $\lim_{u\to 0^+} \tilde{b}_u = \infty$. Therefore, condition (5.4) is actually equivalent to statement (i) of Assumption (A), and thus to the whole Assumption (A). Therefore, Theorem 5.2 holds for any *principal* invariant diffusions.

Remark. As well as in the case of iterated Ito integrals, also for invariant diffusions of simply connected nilpotent Lie groups the limit set C coincides with the limit set of iterated logarithm for large time (see Baldi, 1986, Theorem 4.1). However, we point out that a difference arises in the two situations. Indeed, in the present framework the ILLs both for small and large time might not hold simultaneously: while for our result we must require Eq. (5.4), when $u\to +\infty$ the statement depends only on the vector field X_0 since the constraint is $x_0 \notin V_1$, if $x_0\neq 0$. On the other hand, the next example shows a particular case in which the above hypothesis are all satisfied (because $x_0 = 0$ and $x_i\in V_1$, $i\geqslant 1$).

Example. The principal invariant diffusion of the Heisenberg group.

Let $\mathfrak{g}$ be the Lie algebra generated by $\{e_1,e_2,e_3\}$, with

$$[e_1,e_2] = e_3, \qquad [e_1,e_3] = [e_2,e_3] = 0.$$

Here $\ell=2$, V_1 is spanned by $\{e_1,e_2\}$ and V_2 by $\{e_3\}$. Let X_1 and X_2 be the left invariant vector fields associated to e_1 and e_2, respectively:

$$X_1(y) = \frac{\partial}{\partial y_1} - \frac{1}{2}y_2\frac{\partial}{\partial y_3},$$

$$X_2(y) = \frac{\partial}{\partial y_2} + \frac{1}{2}y_1\frac{\partial}{\partial y_3}.$$

The principal invariant diffusion of the Heisenberg group is the diffusion process Y whose generator is

$$L = \tfrac{1}{2}(X_1^2 + X_2^2),$$

i.e. Y is the solution of the SDE which starts from 0 with drift $\tilde{b} = 0$ and diffusion coefficient

$$\tilde{\sigma}(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{1}{2}y_2 & \frac{1}{2}y_1 \end{pmatrix}.$$

Then

$$Y_t = \left(B_1(t), B_2(t), \tfrac{1}{2}\int_0^t \{B_1(s)\,\mathrm{d}B_2(s) - B_2(s)\,\mathrm{d}B_1(s)\}\right).$$

In such a case Eq. (5.1) works in the following way:

$$\Gamma_\alpha x = \left(\frac{x^1}{\alpha}, \frac{x^2}{\alpha}, \frac{x^3}{\alpha^2} \right),$$

so that for $0 < u < e^{-1}$ and $t \in [0,1]$

$$Z_u(t) = \left(\frac{B_1(ut)}{\psi(u)}, \frac{B_2(ut)}{\psi(u)}, \frac{1}{2\psi(u)^2} \int_0^{ut} \{B_1(s)\,dB_2(s) - B_2(s)\,dB_1(s)\} \right).$$

Theorem 5.2 states that $\{Z_u\}_{0<u<e^{-1}}$ is relatively compact and its limit points as $u \to 0^+$ are the trajectories φ of the form

$$\varphi_t = \left(f_1(t), f_2(t), \tfrac{1}{2} \int_0^t \{f_1(s)f_2'(s) - f_2(s)f_1'(s)\}\,ds \right),$$

where $f \in \mathscr{K}_2$. By Corollary 3.8 also $\{Z_u(1)\}_{0<u<e^{-1}}$ is relatively compact and the limit set is the subset of $\mathbb{R}^3$ of the points (x^1, x^2, x^3) such that

$$x^1 = f_1(1), \quad x^2 = f_2(1), \quad x^3 = \frac{1}{2} \int_0^1 \{f_1'(s)f_2(s) - f_1(s)f_2'(s)\}\,ds$$

as f varies in $\mathscr{K}_2$.

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