Classification of semigroups of linear fractional maps in the unit ball

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Abstract

We give a complete classification up to conjugation of continuous semigroups of linear fractional self-maps of the unit ball.

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Introduction

In a recent paper, Cowen and MacCluer [11] introduced a class of holomorphic self-maps of the unit ball $\mathbb{B}^n$, called *linear fractional self-maps* of $\mathbb{B}^n$, which generalize the automorphisms of $\mathbb{B}^n$ as well as the linear fractional maps in one variable. Linear fractional self-maps in $\mathbb{B}^n$ for $n > 1$ present analogies and differences with respect to their relatives for $n = 1$. They provide a family of holomorphic self-maps of $\mathbb{B}^n$ quite easy to handle which possesses many interesting geometric and analytic properties.

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Due to the importance of one-dimensional linear fractional maps in iteration and composition operator theory, linear fractional self-maps of $\mathbb{B}^n$ have deserved a quite deep consideration, with the belief that they can play an important role also in similar problems in several variables. In [4] Bisi and the first author provide a classification of linear fractional maps up to conjugation with automorphisms of $\mathbb{B}^n$ and study cyclicity properties of their associated composition operators. In [23], Richman provides a simple criterion to say when a linear fractional map has range in the unit ball, while in [12] Cowen, Crosby, Horine, Ortiz Albino, Richman, Yeow and Zerbe discuss another classification of linear fractional maps based on the "characteristic domain" introduced by Cowen in [10] with the purpose of linearizing holomorphic self-maps of the unit disc; in [15] Khatskevich, Reich and Shoikhet deal with linear fractional solutions to functional equations in Hilbert spaces. Linear fractional maps are also basic in [8], where the first named author and Gentili solve the so-called Schröder equation for holomorphic self-maps of $\mathbb{B}^n$ with no fixed points.

On the other hand, instead of considering just one map and its iterates (a "discrete semigroup") one can consider a continuous semigroup of holomorphic self-maps of $\mathbb{B}^n$. In case $n = 1$, Berkson and Porta (see [3] and also [1], [9]) proved that these objects are "holomorphically linearizable" and they can be considered essentially continuous semigroups of linear fractional maps. In several complex variables, similar linearization properties are known only in some special cases (see de Fabritiis [13]). Nonetheless, we believe that complete understanding of semigroups of linear fractional self-maps of $\mathbb{B}^n$ can help in dealing with the general case.

In this paper we deal with continuous semigroups of linear fractional maps of the unit ball. We provide a complete classification of such analytic objects up to conjugation with injective linear fractional maps (not necessarily with range in the unit ball), essentially proving that semigroups of linear fractional self-maps of $\mathbb{B}^n$ are linearizable. The classification is constructed by selecting and normalizing suitable geometric invariants, in the spirit of [5] and [4], but it should be noted that some of these linearization results are new also in the case of a single linear fractional map. In particular we base our classification on the presence or not of (common) fixed points in $\mathbb{B}^n$. If there are common fixed points—the elliptic case—the semigroup is essentially given by a matrix semigroup of the type $Z \mapsto e^{tM}Z$, with M being dissipative and asymptotically stable (see Theorem 3.2). In case the semigroup has no common fixed points in $\mathbb{B}^n$, then all the iterates share a common fixed point on $\partial\mathbb{B}^n$, the Denjoy-Wolff point. In this case, the semigroup is hyperbolic or parabolic according to the value of the "boundary dilatation coefficient" (see Section 1 and the Appendix). For the hyperbolic and parabolic case we provide a general form (Theorem 5.1 and Theorem 6.1) and several simpler forms according to geometrical invariants the semigroup might have (see Sections 5 and 6).

The plan of the paper is the following. In the first section we recall some preliminary geometric results and fix notations. In the second section we deal with fixed and invariant slices for linear fractional maps and relate these geometric objects to algebraic properties of linear fractional maps. In the third section we examine the case of elliptic semigroups and prove the linearization theorem. In section four we provide a basic "model" for a linear fractional map with no fixed points in $\mathbb{B}^n$, which will be the base of subsequent classifications. In section five we give the classification of hyperbolic semigroups of linear fractional maps and discuss their properties according to normal forms that we obtain. In section six we deal with the parabolic case. Finally, in the Appendix we give a short proof of the basic (and partially new) classification in elliptic, hyperbolic and parabolic types in the setting of strongly convex domains.

1 Preliminary results

Let $\langle \cdot, \cdot \rangle$ be the standard Hermitian product in $\mathbb{C}^n$, $\| \cdot \|$ the associated norm and $\mathbb{B}^n := \{Z \in \mathbb{C}^n : \|Z\| < 1\}$ the unit ball. As a matter of notation, we usually write $Z \in \mathbb{C}^n$ as a column vector and use the decomposition $Z = (z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$.

In this section, we recall some general definitions and results about linear fractional maps in the unit ball and semigroups with the aim of fixing notations which will be used throughout the paper.

Following [11], we say that a map $\varphi : \mathbb{B}^n \to \mathbb{C}^n$ is a *linear fractional map* if there exist a complex $n \times n$ matrix $\Gamma \in \mathbb{C}^{n\times n}$, two column vectors B and C in $\mathbb{C}^n$, and a complex number $D \in \mathbb{C}$ satisfying

$$\text{(i) } |D| > \|C\|\,; \qquad \text{(ii) } D\Gamma \neq BC^*,$$

such that

$$\varphi(Z) = \frac{\Gamma Z + B}{\langle Z, C\rangle + D}, \qquad Z \in \mathbb{B}^n. \tag{1}$$

Condition (i) implies that $\langle Z, C\rangle + D \neq 0$ for every $z \in \mathbb{B}^n$ and therefore, φ is actually holomorphic in a neighborhood of the closed ball. In fact, $\varphi \in \mathrm{Hol}(r\mathbb{B}^n; \mathbb{C}^n)$ for some $r > 1$. On the other hand, condition (ii) just says that φ is not constant.

If the image $\varphi(\mathbb{B}^n) \subset \mathbb{B}^n$, then we say that φ is a linear fractional self-map of $\mathbb{B}^n$ and write $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$.

If $\Omega \subset \mathbb{C}^n$ is a domain and $\psi : \Omega \to \Omega$ is holomorphic, we call the couple (Ω, ψ) an *iteration couple*. For instance, if $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ then $(\mathbb{B}^n, \varphi)$ is an iteration couple.

Definition 1.1 *Let (Ω, ψ) and (Ω', ψ') be two iteration couples. We say that the two couples are conjugated if there exists a biholomorphic map $\sigma : \Omega \to \Omega'$ such that $\psi = \sigma^{-1} \circ \psi' \circ \sigma$. The map σ is called an intertwining map.*

From a dynamical point of view two conjugated iteration couples are undistinguishable. In the sequel we will often transfer a dynamical model from the unit ball to the Siegel half-plane.

Recall that the unit ball $\mathbb{B}^n$ is biholomorphic to the Siegel half-plane $\mathbb{H}^n := \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : \mathsf{Re}\, z > \|w\|^2\}$ via the the generalized Cayley transform σ_C defined as

$$\sigma_C(z, w) := \left(\frac{1+z}{1-z}, \frac{w}{1-z}\right), \quad (z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}.$$

Note that σ_C extends (setting "$\sigma_C(e_1) = \infty$") to a bi-continuous map from $\overline{\mathbb{B}^n}$ onto $\mathrm{cl}_\infty(\mathbb{H}^n)$, the one-point compactification of the closure of $\mathbb{H}^n$.

The iteration couple $(\mathbb{B}^n, \varphi)$ for $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ is quite simple, but nonetheless it is often very useful to consider the conjugated iteration couple $(\mathbb{H}^n, \psi)$, where $\mathbb{H}^n := \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : \mathsf{Re}\, z > \|w\|^2\}$ is the Siegel half-plane and $\psi = \sigma_C \circ \varphi \circ \sigma_C^{-1} : \mathbb{H}^n \to \mathbb{H}^n$, with σ_C is the generalized Cayley transform. Those holomorphic maps ψ appearing in this way will be called linear fractional selfmaps of $\mathbb{H}^n$ and the set of all of them will be denoted by $\mathrm{LFM}(\mathbb{H}^n, \mathbb{H}^n)$.

We mention some results about fixed points and linear fractional maps. The first one is quite well-known (see, *e.g.*, [4, Theorem 2.2]).

Theorem 1.2 *Let $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ with no fixed points in $\mathbb{B}^n$. Then, there exists a unique point $\tau \in \partial\mathbb{B}^n$ such that $\varphi(\tau) = \tau$ and $\langle d\varphi_\tau(\tau), \tau\rangle = \alpha(\varphi)$ with $0 < \alpha(\varphi) \leq 1$.*

The point $\tau \in \partial\mathbb{B}^n$ in Theorem 1.2 is called the *Denjoy-Wolff point* of φ and $\alpha(\varphi)$ the *boundary dilatation coefficient* of φ. We list here some basic properties of Denjoy-Wolff points and boundary dilatation coefficients as needed for our aim (see [20] and [5, Theorem 3.6, Proposition 4.2 and Theorem 5.1]):

Proposition 1.3 *Let $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ with no fixed points in $\mathbb{B}^n$, let $\tau \in \partial\mathbb{B}^n$ be its Denjoy-Wolff point and $\alpha(\varphi)$ the boundary dilatation coefficient. Then*

(1) For all $z \in \mathbb{B}^n$ it follows that $\lim_{m\to\infty} \varphi_m(z) = \tau$.
(2) If $v \in \mathbb{C}^n$ then $\langle d\varphi_\tau(v), \tau\rangle = \alpha(\varphi)\langle v, \tau\rangle$.
(3) $\alpha(\varphi)$ is an eigenvalue of $d\varphi_\tau$.

(4) If $v \in \mathbb{C}^n$ is an eigenvector for $d\varphi_\tau$ such that $\langle v, \tau \rangle \neq 0$ then $d\varphi_\tau(v) = \alpha(\varphi)v$.

Now we are ready to give a first definition which divides $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ in three big families.

Definition 1.4 *Let $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$. If φ has some fixed point in $\mathbb{B}^n$ we call it elliptic. If φ has no fixed points in $\mathbb{B}^n$ and $\alpha(\varphi)$ is the boundary dilatation coefficient of φ at its Denjoy-Wolff point, we say that φ is hyperbolic if $\alpha(\varphi) < 1$ while we say it is parabolic if $\alpha(\varphi) = 1$.*

According to [4, Theorems 3.1 and 3.2] a non-elliptic linear fractional map has at most two fixed points on $\partial\mathbb{B}^n$ and a parabolic linear fractional map has only one fixed point on $\partial\mathbb{B}^n$ (its Denjoy-Wolff point). A hyperbolic linear fractional map might have one or two fixed points on $\partial\mathbb{B}^n$.

The main contribution of this paper is to provide a "dynamical classification" of semigroups of linear fractional self-maps of the ball.

To begin with, we recall that for a domain $\Omega \subset \mathbb{C}^n$ a *continuous (one-parameter) semigroup* in $\mathrm{Hol}(\Omega, \Omega)$ is a continuous homomorphism

$$[0, +\infty) \ni t \mapsto \varphi_t \in \mathrm{Hol}(\Omega; \Omega)$$

from the additive semigroup of non-negative real numbers into the composition semigroup of all holomorphic self-maps of Ω (with the compact-open topology). The functions φ_t are sometimes called the *iterates* of the semigroup (φ_t). Such a semigroup extends to a continuous group action of $\mathbb{R}$ on Ω whenever it is possible to extend the semigroup continuously to $\mathbb{R}$.

Three basic properties of a semigroup (φ_t) of holomorphic self-maps of a domain $\Omega \subset \mathbb{C}^n$, which we will tacitely use throughout the paper, are:

(1) for all $t \geq 0$ the map $z \mapsto \varphi_t(z)$ is injective.
(2) for all $z \in \Omega$, the map $(0, 1) \ni t \mapsto \varphi_t(z) \in \Omega$ is analytic.
(3) If $\varphi_{t_0} \in \mathsf{Aut}(\Omega)$ for some $t_0 > 0$ then $\varphi_t \in \mathsf{Aut}(\Omega)$ for all $t \geq 0$.

For a proof of the previous assertions see, *e.g.*, [1, Section 2.5.3].

We say that a continuous semigroup in $\mathrm{Hol}(\mathbb{B}^n; \mathbb{B}^n)$ is a *semigroup of linear fractional maps* if $\varphi_t \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ for all $t \geq 0$. Even in this case we can talk about elliptic, hyperbolic and parabolic semigroups. For this we need to exploit the following theorem.

Theorem 1.5 *Let (φ_t) be a continuous semigroup in $\mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. Then, either all the iterates have a common fixed point in $\mathbb{B}^n$ or no φ_t $(t > 0)$ has a* *fixed point in* $\mathbb{B}^n$ *and they share the same Denjoy-Wolff point* $\tau \in \partial\mathbb{B}^n$. *In this case, there exists* $0 < r \leq 1$ *such that* $\alpha_t = r^t$, *where* $\alpha_t := \alpha(\varphi_t)$ *denotes the boundary dilatation coefficient of* φ_t *(for* $t > 0$*) at* τ.

Such a theorem is due to M. Abate (see [1]) in case of strongly convex domains except for the behavior of the boundary dilatation coefficient, while it is proved for the unit ball by L. Aizenberg and D. Shoikhet in [2]. In the appendix we give a complete *short* proof of such a result in the context of strongly convex domains.

Corollary 1.6 *Let* (φ_t) *be a continuous semigroup in* $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$. *If for some* $t_0 > 0$ *the iterate* φ_{t_0} *is elliptic (respectively hyperbolic; respectively parabolic), then for all* $t > 0$ *the iterates* φ_t *are elliptic (respectively hyperbolic; respectively parabolic).*

In particular we can safely give the following definition.

Definition 1.7 *Let* (φ_t) *be a continuous semigroup in* $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$. *If* φ_1 *has some fixed point in* $\mathbb{B}^n$ *we call* (φ_t) *elliptic. If* φ_1 *has no fixed points in* $\mathbb{B}^n$ *and* $\tau \in \partial\mathbb{B}^n$ *is its Denjoy-Wolff point, we say that* (φ_t) *is hyperbolic (respectively parabolic) if* φ_1 *is hyperbolic (respectively parabolic) and we call* τ *the Denjoy-Wolff point of* (φ_t).

If (φ_t) is a semigroup of $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ we say that a point $z \in \overline{\mathbb{B}^n}$ is a fixed point for the semigroup if $\varphi_t(z) = z$ for all $t \geq 0$.

2 Slices and complex geodesics

A *slice* S of $\mathbb{B}^n$ is a non-empty subset of $\mathbb{B}^n$ of the form $S = \mathbb{B}^n \cap V$, where V is a one-dimensional affine subspace of $\mathbb{C}^n$.

Slices can be nicely described by holomorphic functions. Namely, given a slice S of $\mathbb{B}^n$, there exists an injective proper map $f \in \mathrm{Hol}(\mathbb{D}; \mathbb{C}^n)$ from the unit disc $\mathbb{D}$ to $\mathbb{B}^n$ such that $f(\mathbb{D}) = S$. These maps are called *complex geodesics* (associated to S) because they are isometries between the Poincaré metric on $\mathbb{D}$ and the Bergmann metric on $\mathbb{B}^n$ (see, *e.g.*, [1] for details). Given a slice S and an associated complex geodesic $f : \mathbb{D} \to \mathbb{B}^n$, any other complex geodesic associated to S is given by $f \circ \theta$ with $\theta \in \mathsf{Aut}(\mathbb{D})$.

The prototype of a slice is $S_0 := \mathbb{B}^n \cap \mathbb{C}e_1$, where $e_1 = (1, 0, \ldots, 0)$ and the associated complex geodesic is $f_0(\zeta) = (\zeta, 0, \ldots, 0)$. Since the group of automorphisms $\mathsf{Aut}(\mathbb{B}^n)$ sends slices onto slices and acts transitively on $\mathbb{P}(T\mathbb{B}^n)$ (namely for any couple of points $Z, W \in \mathbb{B}^n$ and any couples of non zero directions $v \in T_Z\mathbb{B}^n$ and $v' \in T_W\mathbb{B}^n$ there exists $\Phi \in \mathsf{Aut}(\mathbb{B}^n)$ such that $\Phi(Z) = W$

and $d\Phi_Z(v) = \lambda v'$ for some $\lambda \in \mathbb{C} \setminus \{0\}$), it follows that for any slice S in $\mathbb{B}^n$ there exists an automorphism $\Phi \in \mathsf{Aut}(\mathbb{B}^n)$ such that $S = \Phi(S_0)$ and a complex geodesic associated to S is given by $\Phi \circ f_0 : \mathbb{D} \to \mathbb{B}^n$.

Transferring, as we will often do, everything to $\mathbb{H}^n = \{(z, w) : \mathsf{Re}\, z > \|w\|^2\}$ via the Cayley transform σ_C, we see that a slice $S \subset \mathbb{B}^n$ such that $e_1 \in \overline{S}$ corresponds to a slice $S' \subset \mathbb{H}^n$ given by $\{(z, w) \in \mathbb{H}^n : w = \text{const}\}$. The "prototype" slice S_0 corresponds now to the slice $S_0' := \{(z, w) \in \mathbb{H}^n : w = 0\}$ in $\mathbb{H}^n$ and the complex geodesic $f_0 : \mathbb{D} \to \mathbb{B}^n$ to the complex geodesic $f_0' : \mathbb{D} \to \mathbb{H}^n$ defined as $f_0'(\zeta) = ((1 + \zeta)(1 - \zeta)^{-1}, 0, \dots, 0)$.

In [11] it is proven that if S is a slice in $\mathbb{B}^n$ and $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ then there exists a slice S' in $\mathbb{B}^n$ such that $\varphi(S) \subseteq S'$. In case $f(S) \subseteq S$, if $f : \mathbb{D} \to \mathbb{B}^n$ is a complex geodesic associated to S, we can define

$$\varphi_f := f|_S^{-1} \circ \varphi \circ f.$$

Such a map $\varphi_f \in \mathrm{LFM}(\mathbb{D}, \mathbb{D})$ depends on f but, since any other complex geodesic f' associated to S is given by $f' = f \circ \theta$ for some $\theta \in \mathsf{Aut}(\mathbb{D})$, it follows that φ_f is conjugated to $\varphi_{f'}$. Therefore φ_f can be used to understand properties of φ invariant by conjugation.

We say that a slice S of $\mathbb{B}^n$ *passes through* some point $Z \in \overline{\mathbb{B}^n}$ if $Z \in \overline{S}$. Likewise, we say that $v \in \mathbb{C}^n \setminus \{0\}$ is a *direction vector* of S if

$$v \in V_S := \mathrm{span}\{s - s' : s, s' \in S\} = T_Z S \quad \text{for any } Z \in S.$$

This one-dimensional vector space V_S is called the *direction subspace of* S. It is clear that $S = (Z + V_S) \cap \mathbb{B}^n$ for any $Z \in \overline{S}$. We say that a slice S passes through Z with direction $v \in \mathbb{C}^n \setminus \{0\}$, if $S = (Z + \mathbb{C}v) \cap \mathbb{B}^n$.

Proposition 2.1 *Let $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ be non-elliptic with Denjoy-Wolff point $\tau \in \partial\mathbb{B}^n$. Let S be a slice in $\mathbb{B}^n$ passing through τ with direction subspace V_S. Then, for every $v \in V_S \setminus \{0\}$ it follows that $\langle d\varphi_\tau(v), \tau\rangle \neq 0$. In other words,*

$$\widehat{S} := (\tau + d\varphi_\tau(V_S)) \cap \mathbb{B}^n$$

is a well-defined slice in $\mathbb{B}^n$. Moreover, $\varphi(S) \subseteq \widehat{S}$.

PROOF. Let $\alpha := \alpha(\tau)$ be the boundary dilatation coefficient of φ at τ. Let $v \in V_S \setminus \{0\}$. Since $\mathbb{C}v \cap \mathbb{B}^n \neq \emptyset$ then $\langle v, \tau\rangle \neq 0$. By Proposition 1.3.(2) we have

$$\langle d\varphi_\tau(v), \tau\rangle = \alpha\langle v, \tau\rangle \neq 0.$$

In particular, since $V_S = \mathbb{C}v$, it follows that $d\varphi_\tau(V_S)$ is a one-dimensional subspace of $\mathbb{C}^n$ and the previous computation implies that $S' := (d\varphi_\tau(V_S) +$

$\tau) \cap \mathbb{B}^n \neq \emptyset$ and thus it is a slice in $\mathbb{B}^n$.

We are left to show that $\varphi(S) \subseteq S'$. We know that there exists a slice $\tilde{S}$ of $\mathbb{B}^n$ such that $\varphi(S) \subseteq \tilde{S}$. Since $\varphi(\tau) = \tau$, it is enough to show that $V_S = V_{\tilde{S}}$. To see this, let $v \in V_{\tilde{S}} \setminus \{0\}$. Up to change v with $e^{i\theta}v$ for some $\theta \in \mathbb{R}$, we can assume that there exists $\varepsilon > 0$ such that $\tau + \lambda v \in \mathbb{B}^n$, whenever $0 < \lambda < \varepsilon$. Then

$$\frac{1}{\lambda}(\varphi(\tau + \lambda v) - \varphi(\tau)) \in V_{\tilde{S}}.$$

Letting λ goes to 0 we deduce that $d\varphi_\tau(v) \in V_{\tilde{S}}$ and then $V_S = V_{\tilde{S}}$ as wanted. $\square$

Remark 2.2 *The proof of the above proposition can be adapted to certain elliptic situations. Namely, if $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ has a fixed point $b \in \mathbb{B}^n$, S is a slice in $\mathbb{B}^n$ passing through b with direction subspace V_S and $d\varphi_b(V_S)$ is one-dimensional, then*

$$\varphi(S) = (b + d\varphi_b(V_S)) \cap \mathbb{B}^n.$$

Proposition 2.3 *Let $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ be non-elliptic with Denjoy-Wolff point $\tau \in \partial\mathbb{B}^n$ and boundary dilatation coefficient $\alpha(\varphi)$. Let S be a slice in $\mathbb{B}^n$ with direction subspace V_S. The followings are equivalent:*

(1) The slice S is invariant (as a set) for φ.
(2) The slice S passes through τ and $d\varphi_\tau(V_S) = V_S$.
(3) The slice S passes through τ and some—and hence any—$v \in V_S$ verifies $d\varphi_\tau(v) = \alpha(\varphi)v$.

PROOF. The equivalence of (1) and (2) follows directly from Proposition 2.1 as soon as we realize that all invariant slices must contain τ in their closure. Indeed, if $S \subset \mathbb{B}^n$ were an invariant slice for φ not passing through τ, then $\lim_{m\to\infty} \varphi_m(Z) \neq \tau$ for all $Z \in S$, contradicting Proposition 1.3.(1). If (2) holds, then any $v \in V_S$ is an eigenvector of $d\varphi_\tau$ and (3) follows from Proposition 1.3.(4). Conversely, if (3) holds then V_S is $d\varphi_\tau$-invariant and then (2) holds. $\square$

A finite collection $\{S_1, ..., S_p\}$ of slices of $\mathbb{B}^n$ is said to be *independent* if the family of the corresponding one-dimensional direction subspaces $\{V_{S_1}, ..., V_{S_p}\}$ spans a p-dimensional subspace of $\mathbb{C}^n$.

If $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ we let $\sharp\mathrm{inv}(\varphi)$ to be the dimension of the space spanned by the direction subspaces V_S of all φ-invariant slices $S \subset \mathbb{B}^n$. By Proposition 2.3 if φ has no fixed points in $\mathbb{B}^n$ the number $\sharp\mathrm{inv}(\varphi)$ coincides with the dimension of the *inner space*

$$\mathbb{A}(\varphi) := \mathrm{span}\{v \in \mathbb{C}^n : d\varphi_\tau(v) = \alpha(\varphi)v, \langle v, \tau\rangle \neq 0\}$$

introduced in [5] (see also [4, Theorem 2.4]).

We examine now invariant slices for semigroups.

Theorem 2.4 *Let (φ_t) be a continuous non-elliptic semigroup in* $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ *with $\tau \in \partial\mathbb{B}^n$ as the common Denjoy-Wolff point. Let α_t denote the boundary dilatation coefficient of φ_t at τ and consider the inner space of φ_t*

$$\mathbb{A}_t := span\{w \in \mathbb{C}^n : d(\varphi_t)_\tau w = \alpha_t w,\ \langle w, \tau\rangle \neq 0,\}.$$

Also let $\mathbb{A} := \cap_{t\geq 0}\mathbb{A}_t$. Then

(1) If $\mathbb{A}_{t_0} = \{0\}$ for some $t_0 > 0$ then $\mathbb{A}_t = \{0\}$ for all $t > 0$ and φ_t has no invariant slices in $\mathbb{B}^n$ for all $t > 0$.
(2) If $\mathbb{A}_{t_0} \neq \{0\}$ for some $t_0 > 0$ then $\mathbb{A} \neq \{0\}$.

Moreover, if $p :=$ $\dim\mathbb{A} > 0$ then (φ_t) has exactly p common independent invariant slices in $\mathbb{B}^n$.

PROOF. If $\mathbb{A}_t = \{0\}$ for all $t > 0$ then, by Propositions 2.1 and 2.3, no φ_t has any invariant slice.

So, assume that $\mathbb{A}_{t_0} \neq \{0\}$ for some $t_0 > 0$ and let $d = \dim\mathbb{A}_{t_0}$. Notice that $d \leq n$. First of all, $\mathbb{A}_{t_0} = \ker(d(\varphi_{t_0})_\tau - \alpha_{t_0}I)$ because clearly $\mathbb{A}_{t_0} \subseteq \ker(d(\varphi_{t_0})_\tau - \alpha_{t_0}I)$ and, conversely, if $v \in \ker(d(\varphi_{t_0})_\tau - \alpha_{t_0}I)$ is such that $\langle v, \tau\rangle = 0$ then for any $w \in \mathbb{A}_{t_0}$ with $\langle w, \tau\rangle \neq 0$ (and there must exist such a w because $\mathbb{A}_{t_0} \neq \{0\}$) it follows that $\langle v-w, \tau\rangle \neq 0$ and since $v = w+(v-w) \in \mathbb{A}_{t_0}$ then $v \in \mathbb{A}_{t_0}$.

Now we claim that $d(\varphi_t)_\tau\mathbb{A}_{t_0} \subseteq \mathbb{A}_{t_0}$ for every $t \geq 0$. To see this, let $w \in \mathbb{A}_{t_0}$ and $t \geq 0$. Since

$$d(\varphi_{t_0})_\tau d(\varphi_t)_\tau w = d(\varphi_{t_0} \circ \varphi_t)_\tau w = d(\varphi_t)_\tau d(\varphi_{t_0})_\tau w = \alpha_{t_0} d(\varphi_t)_\tau w,$$

then $d(\varphi_t)_\tau w \in \ker(d(\varphi_{t_0})_\tau - \alpha_{t_0}I) = \mathbb{A}_{t_0}$.

Now, let $K := \mathbb{B}^n \cap (\mathbb{A}_{t_0} + \tau)$. Since $\mathbb{A}_{t_0}$ is $d(\varphi_t)_\tau$-invariant for all $t \geq 0$ then by Proposition 2.1 it follows that $\varphi_t(K) \subseteq K$ for all $t \geq 0$. The set K is equivalent to a ball of dimension d by means of an affine map (to see this from an algebraic point of view conjugate with rotations in such a way that $\tau = e_1$ and $\mathbb{A}_{t_0}$ is spanned by $\{e_1, \ldots, e_d\}$, cfr [4, Lemma 4.1]). Let $\theta : K \to \mathbb{B}^d$ be the affine transformation mapping K to the ball of dimension d in $\mathbb{C}^d$.

Then we have a well defined semigroup $t \longmapsto \eta_t := \theta \circ \varphi_t|_K \circ \theta^{-1}$ of linear fractional maps of $\mathbb{B}^d$. It is clear that (η_t) is non-elliptic, its Denjoy-Wolff point is $x := \theta(\tau) \in \partial\mathbb{B}^d$ and the boundary dilatation coefficient of η_t at x is still α_t. Moreover, by construction, $d(\eta_{t_0})_\tau = \alpha_{t_0} I$.

The statement (2) of the theorem will follow as soon as we show that there exists $v \in \mathbb{C}^d$ such that $\langle v, x\rangle \neq 0$ and $d(\eta_v)_t v = \alpha_t v$ for all $t \geq 0$, because then $d\theta_x^{-1}(v) \in \mathbb{A}_t$ for all $t \geq 0$.

To this aim, we examine the continuous application $t \longmapsto d(\eta_t)_x$. This is clearly a continuous semigroup of matrices and therefore there exists a matrix $M \in \mathbb{C}^{d\times d}$ such that $d(\eta_t)_x = \exp(tM)$. Write $M = P^{-1}JP$ with J a Jordan blocks matrix. Then $\exp(tM) = P^{-1}\exp(tJ)P$. Since $d(\eta_{t_0})_\tau = \alpha_{t_0} I$ then $\exp(t_0 J) = \alpha_{t_0} I$ which means that J is diagonal, with diagonal entries a_j, $j = 1, \ldots, d$. Since P is invertible, there exists $j \in \{1, \ldots, d\}$ such that the vector $v = P^{-1}e_j$ satisfies $\langle v, x\rangle \neq 0$. Now

$$d(\eta_t)_x(v) = \exp(tM)(P^{-1}e_j) = P^{-1}\exp(tJ)e_j = \exp(ta_j)P^{-1}e_j = \exp(ta_j)v.$$

Therefore by Proposition 1.3.(4) it follows that $\exp(ta_j) = \alpha_t$ and we are done.

The last assertion follows easily from the very definition of $\mathbb{A}$. $\square$

Remark 2.5 *The argument in the proof of Theorem 2.4 shows that if $t_0 > 0$ is such that $\mathbb{A}_{t_0} = \mathbb{A}$, then, for all $t \geq 0$, $\mathbb{A}_{t_0}$ is φ_t-invariant, $d(\varphi_t)_\tau$-invariant and the restriction of $d(\varphi_t)_\tau$ (viewed as a linear map) to $\mathbb{A}_{t_0}$ is diagonalizable.*

3 Classification of elliptic semigroups

In this section we deal with elliptic semigroups of linear fractional self-maps of $\mathbb{B}^n$. Hervé's theorem (see, *e.g.*, [1] or [24]) states that the fixed points set of a holomorphic self-map of $\mathbb{B}^n$ is either empty or it is a slice of $\mathbb{B}^n$ (that is the intersection of $\mathbb{B}^n$ with an affine complex space). Accordingly, if $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ has a non-empty fixed points set in $\mathbb{B}^n$ then such a set is a *p-dimensional slice* of $\mathbb{B}^n$. Namely, it is the non-empty intersection between $\mathbb{B}^n$ and a p-dimensional affine space of $\mathbb{C}^n$ with $p \geq 0$. We classify an elliptic semigroup according to the dimension of its common fixed points set and to the action of the differentials on its tangent space.

As a matter of notation whenever $\varphi \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and $Z_0 \in \mathbb{B}^n$ is a fixed point of φ, we define the *unitary space* of φ at the point Z_0 as

$$L_U(\varphi, Z_0) := \bigoplus_{|\lambda|=1} \ker(d\varphi_{Z_0} - \lambda I)^n.$$

In other words, $L_U(\varphi, Z_0)$ is the (direct) sum of all generalized eigenspaces of $d\varphi_{Z_0}$ associated to the different eigenvalues of modulus 1. The dimension of $L_U(\varphi, Z_0)$ is called the *unitary index* of φ at Z_0 and it is usually denoted by $u(\varphi, Z_0)$.

We begin with showing that the above index can be consistently defined in the context of semigroups.

Lemma 3.1 *Let (φ_t) be an elliptic semigroup of* $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$*. Then there exists a non-negative integer p such that $u(\varphi_t, Z_0) = p$, for every common fixed point $Z_0 \in \mathbb{B}^n$ of the semigroup and for every $t > 0$.*

PROOF. Let us first suppose that (φ_t) has only one common fixed point $Z_0 \in \mathbb{B}^n$. Up to conjugation, we may assume that $Z_0 = O$. Therefore,

$$\varphi_t(Z) = \frac{A_t Z}{\langle Z, C_t \rangle + 1},$$

for some $A_t \in \mathbb{C}^{n\times n}$ and $C_t \in \mathbb{C}^n$. Now $[0, +\infty) \ni t \mapsto d(\varphi_t)_O = A_t \in \mathbb{C}^{n\times n}$ is a continuous matrix semigroup, so there exists $M \in \mathbb{C}^{n\times n}$ such that $A_t = e^{tM}$. Note that by Schwarz's Lemma $\|A_t\| \leq 1$, for all $t \geq 0$. This implies (see for instance [21, p. 428]) that the real part of each eigenvalue of M is non-positive and those eigenvalues of M whose real part is zero have the same algebraic and geometric multiplicity. In particular, we can deduce that

$$M = P \begin{bmatrix} \lambda_1 i & & & & & \\ & \ddots & & & & \\ & & \lambda_p i & & & \\ & & & J_{p+1}(\lambda_{p+1}) & & \\ & & & & \ddots & \\ & & & & & J_{p+q}(\lambda_{p+q}) \end{bmatrix} P^{-1}$$

where $1 \leq p+q \leq n$, P is an invertible matrix of order n, every λ_k $(k = 1, ..., p)$ is a *real* number and $J_{p+k}(\lambda_{p+k})$ denotes a Jordan block associated to $\lambda_{p+k} \in \mathbb{C}$

with $\mathsf{Re}\,(\lambda_{p+k}) < 0$, for every $k = 1, ..., q$. Therefore

$$e^{Mt} = P \begin{bmatrix} e^{\lambda_1 ti} & & & & & \\ & \ddots & & & & \\ & & e^{\lambda_p ti} & & & \\ & & & \exp(J_{p+1}(\lambda_{p+1})t) & & \\ & & & & \ddots & \\ & & & & & \exp(J_{p+q}(\lambda_{p+q})t) \end{bmatrix} P^{-1}.$$

Since all the diagonal entries of the upper triangular matrix $\exp(J_{p+k}(\lambda_{p+k})t)$, for $k = 1, ..., q$, are equal to $e^{\lambda_{p+k}t}$ then these blocks have eigenvalues with modules strictly less than one; hence the dimension of the sum of generalized eigenspaces of e^{Mt} associated to eigenvalues of modulus one is exactly p. In other words, $u(\varphi_t, O) = p$, for all $t \geq 0$.

Now, suppose that the semigroup (φ_t) has at least two common fixed points. By Herve's theorem any slice joining two different fixed points is fixed for all φ_t's and therefore there exists an affine s-dimensional slice of common fixed points for (φ_t) for some $s \geq 1$. A simple argument (see [4, proof of Theorems 3.1, 3.2]) allows us to assume that, up to conjugation, the common fixed points set for (φ_t) is given by $\mathbb{C}\{e_1, \ldots, e_s\} \cap \mathbb{B}^n$. Therefore for each $t \geq 0$,

$$\varphi_t(z_1, \ldots, z_s, z^{(s)}) = \left(z_1, \ldots, z_s, A_t z^{(s)}\right)$$

where A_t is a matrix of order $n-s$ and $z^{(s)} \in \mathbb{C}^{n-s}$. Then $\varphi_t^{(s)} : (t, z^{(s)}) \mapsto A_t z^{(s)}$ is an elliptic semigroup of linear fractional maps in $\mathbb{B}^{(s)}$ and, if p' is the unitary index of $\varphi_t^{(s)}$ at O then clearly the unitary index of φ_t at $(z_1, \ldots, z_s, O)$ is $p' + s$ for all $(z_1, \ldots, z_s) \in \mathbb{B}^s$, concluding the proof. $\square$

We call $u(\varphi_t)$ the *unitary index of the semigroup* (φ_t), which, thanks to Lemma 3.1, can be safely defined as $u(\varphi_t) := u(\varphi_1, Z_0)$ for some $Z_0 \in \mathbb{B}^n$ such that $\varphi_1(Z_0) = Z_0$.

By Theorem 1.5, if $\mathrm{Fix}(\varphi_{t_0}) = \{Z \in \mathbb{B}^n : \varphi_{t_0}(Z) = Z\}$ is non-empty for some $t_0 > 0$ then $\mathrm{Fix}(\varphi_t)$ is a non-empty affine subset of $\mathbb{B}^n$ for all $t \geq 0$ and therefore the set $\mathbb{F} := \cap_{t\geq 0}\mathrm{Fix}(\varphi_t)$ is a non-empty p-dimensional slice of $\mathbb{B}^n$ with $p \geq 0$.

Before stating the next result we need to recall some concepts from matrix theory. A matrix $M \in \mathbb{C}^{n\times n}$ is said to be *dissipative*, whenever $\mathsf{Re}\, w^*Mw \leq 0$ for all $w \in \mathbb{C}^n$; it is said to be *asymptotically stable* if all of its eigenvalues have negative real part. Recall that the so called Phillips-Lumer's theorem (see, e.g., [27, p. 250]) states that $\|e^{tM}\| \leq 1$ for all t if and only if M is dissipative; while M is asymptotically stable if and only if $e^{tM} \to O \in \mathbb{C}^n$ as t goes to $+\infty$ (see, e.g., [21, Theorem 9.57]).

Theorem 3.2 *Let (φ_t) be an elliptic semigroup of linear fractional self-maps of $\mathbb{B}^n$, let $\mathbb{F} := \cap_{t\geq 0}\mathsf{Fix}(\varphi_t)$ be the corresponding p-dimensional slice of common fixed points of (φ_t) in $\mathbb{B}^n$ and let $u(\varphi_t)$ be the unitary index of the semigroup.*

(1) If $p = 0$ and $u(\varphi_t) > 0$ or if $p \geq 1$ then $(\mathbb{B}^n, \varphi_t)$ is conjugated to $(\mathbb{B}^n, \psi_t)$ with

$$\psi_t(z', z'', z''') = (z', e^{it\Theta}z'', e^{tM}z''')$$

where $(z', z'', z''') \in \mathbb{C}^p \times \mathbb{C}^q \times \mathbb{C}^{n-p-q} \cap \mathbb{B}^n$, $p+q = u(\varphi_t)$, Θ is a diagonal matrix of order q with real entries and M is a dissipative asymptotically stable matrix of order $n-p-q$.

(2) If $p = u(\varphi_t) = 0$ then there exist a dissipative and asymptotically stable matrix M and a complex ellipsoid $\Omega \subset \mathbb{C}^n$ such that $(\mathbb{B}^n, \varphi_t)$ is conjugated to (Ω, e^{tM}).

Conversely, any iteration couple as in (1) and (2) can be realized as an elliptic semigroup of linear fractional self-maps of $\mathbb{B}^n$.

PROOF. (1) First of all, we consider the case $p = 0$ and $q := u(\varphi_t) > 0$. Then, $\mathbb{F} = \{Z_0\}$ for some $Z_0 \in \mathbb{B}^n$. Up to conjugations with automorphisms of the unit ball, we can clearly assume that $Z_0 = O$. Therefore,

$$\varphi_t(Z) = \frac{A_t Z}{\langle Z, C_t\rangle + 1}$$

for some $A_t \in \mathbb{C}^{n\times n}$ and $C_t \in \mathbb{C}^n$. By hypothesis, the unitary space $L_U(\varphi_t, O)$ is q-dimensional and therefore (see the proof of Lemma 3.1) there exists a linear independent subset $\Gamma_1 := \{u_1, ..., u_q\}$ of $\mathbb{C}^n$ such that:

(i) For all $t > 0$, $L_U(\varphi_t, O) = L := \mathrm{span}(\Gamma_1)$.
(ii) For all $t > 0$ and for all $k = 1, ..., q$, it follows that $d(\varphi_t)_O(u_k) = \lambda_k u_k$ for some $\lambda_k \in \mathbb{C}$.

Let us consider the q-dimensional slice $S_L := L \cap \mathbb{B}^n$. By Remark 2.2 and (ii) it follows that for all $t > 0$, $\varphi_t(S_L \cap \mathbb{C}u_k) \subseteq S_L \cap \mathbb{C}u_k$ for all $k = 1, \ldots, q$ and by Schwarz's lemma $\varphi_t|_{S_L\cap\mathbb{C}u_k}$ is an automorphism. Therefore φ_t maps S_L bijectively onto S_L. Since S_L is a ball of dimension q this means that $\varphi_t|_{S_L\cap\mathbb{C}u_k}$ is a semigroup of unitary matrices whose differentials at O are simultaneously diagonalizable. Therefore we can find an orthonormal basis $\Gamma_2 := \{w_1, ..., w_q\}$ of L such that $d(\varphi_t)_O(w_k) = e^{itd_k}w_k$ for $k = 1, ..., q$ and $t > 0$. Up to rotations, we can then assume that $w_j = e_j$, $j = 1, \ldots, q$, that $S_L = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_q\} \cap$

$\mathbb{B}^n$ and that

$$\varphi_t(Z) = \frac{(U'_t z' + A'_t z'', A''_t z'')}{\langle z'', c''_t \rangle + 1},$$

where $Z = (z', z'') \in \mathbb{C}^q \times \mathbb{C}^{n-q} \cap \mathbb{B}^n$, U'_t is a diagonal unitary $q \times q$ matrix with entries e^{itd_k}, $d_k \in \mathbb{R}$, $k = 1, \ldots, q$, $A'_t \in \mathbb{C}^{q\times(n-q)}$, $A''_t \in \mathbb{C}^{(n-q)\times(n-q)}$ and $c''_t \in \mathbb{C}^{n-q}$ with $\|c''_t\| < 1$.

We claim that also $A'_t = O$ and $c''_t = O$. To see this, we first notice that for all $x = (x', O) \in \overline{S_L} \cap \partial\mathbb{B}^n$ we have

$$d(\varphi_t)_x = \begin{pmatrix} U'_t & A'_t - U'_t x' \cdot (c''_t)^* \\ O & A''_t \end{pmatrix}.$$

As a consequence of Rudin's version of the Julia-Wolff-Carathéodory theorem (see [24] or [1]) it follows that $\langle d(\varphi_t)_x(v), \varphi_t(x)\rangle = 0$ for all $v \in T^{\mathbb{C}}_x \partial\mathbb{B}^n$. In particular, if we take $x = \pm e_j$ with $j = 1, \ldots, q$ and $v = e_k$ with $k = q+1, \ldots, n$ and since $\varphi_t(\pm e_j) = \pm e^{itd_j} e_j$, it follows that $\langle d(\varphi_t)_{\pm e_j}(e_k), e_j\rangle = 0$, for all $j = 1, \ldots, q$ and $k = q+1, \ldots, n$. In particular, $A'_t - U'_t x' \cdot (c''_t)^* = O$ for $x' = \pm(e_j)'$, $j = 1, \ldots, q$. Thus, $A'_t = O$ and $c''_t = O$ as wanted.

Since $(d(\varphi_t)_O)$ is a continuous matrix semigroup then $A''_t = e^{tM}$ for some matrix M of order $n-q$. To conclude we just note that, by Schwarz's Lemma, $\|e^{tM}\| \leq 1$ for all t, so that, by Phillips-Lumer's theorem, M is dissipative. In particular, every eigenvalue of M has non-negative real part. By construction all unitary eigenvalues of $d(\varphi_t)_O$ are contained in L and hence all eigenvalues of M has strictly negative real part, as wanted.

Suppose now that $p \geq 1$. Up to conjugation with automorphisms, we can assume that $\mathbb{F} = \mathbb{B}^n \cap \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_p\}$. A direct computation (or see [4, proof of Theorem 3.2]) shows then that $\varphi_t(Z) = (z', A_t z'')$ for $(z', z'') \in \mathbb{C}^p \times \mathbb{C}^{n-p}$ and some $(n-p) \times (n-p)$ matrix A_t with $\|A_t\| \leq 1$. Since $(t, z'') \mapsto A_t z''$ is an elliptic semigroup of linear fractional self-maps of $\mathbb{B}^{n-p}$ with only one common fixed point at O, the result follows arguing as before.

(2) First of all, up to conjugation with automorphisms of $\mathbb{B}^n$ we can assume that $Z_0 = O$. Therefore

$$\varphi_t(Z) = \frac{A_t Z}{\langle Z, C_t \rangle + 1}$$

for some $A_t \in \mathbb{C}^{n\times n}$ and $C_t \in \mathbb{C}^n$. Since $\varphi_{t+s} = \varphi_t \circ \varphi_s$ we have

$$\langle Z, C_{t+s}\rangle - \langle e^{sN} Z, C_t\rangle - \langle Z, C_s\rangle \equiv 0 \tag{2}$$

and $A_t = e^{tN}$ for some matrix N of order n. We claim that N is invertible for otherwise there would be a non-zero vector $w \in \mathbb{C}^n$ such that $Nw = O$ and thus, for all $t > 0$, $e^{tN} w = w$, implying that $u(\varphi_t) \geq 1$, against our hypothesis.

Deriving (2) with respect to s and setting $s = 0$, we obtain $\langle Z, \frac{d}{dt}C_t\rangle - \langle NZ, C_t\rangle - \langle Z, V_0\rangle \equiv 0$, where $V_0 = \frac{d}{dt}C_t|_{t=0}$. Taking into account that $C_0 = O$, we have thus the following system of differential equations

$$\begin{cases} \frac{d}{dt}C_t = N^*C_t + V_0, \\ C_0 = O. \end{cases}$$

Since N is invertible, the solution of the above differential system is given by $C_t = (e^{tN^*} - I)V$ where $V \in \mathbb{C}^n$ is such that $N^*V = V_0$. Since $\varphi_t(\mathbb{B}^n) \subseteq \mathbb{B}^n$ it follows that $\|(e^{tN^*} - I)V\| < 1$. Since the unitary index of the semigroup is zero, Cartan-Carathéodory's theorem (see, e.g. [1]) implies that $e^{tN} = d(\varphi_t)_O \to O$ as t goes to $+\infty$. Hence, $e^{tN^*} \to O$ as $t \to \infty$ and then $\delta := \|V\| \leq 1$. Therefore there exists a unitary matrix U such that $U^*V = \delta e_1$. Conjugating φ_t with the automorphism $Z \mapsto UZ$, we obtain the semigroup

$$\widehat{\varphi_t}(Z) = \frac{e^{tM}Z}{\delta\langle Z, (e^{tM^*} - I)e_1\rangle + 1}, \quad Z \in \mathbb{B}^n$$

where $M = U^*NU$. As in part (1), a joint application of Schwarz's Lemma and Phillips-Lumer theorem shows that M is dissipative and asymptotically stable, since $u(\widehat{\varphi_t}) = u(\varphi_t) = 0$.

Let us now define

$$\sigma(Z) := \frac{Z}{-\delta z_1 + 1}, \quad Z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1} \cap \mathbb{B}^n. \tag{3}$$

The linear fractional map σ is clearly holomorphic and injective in $\mathbb{B}^n$, since $\delta \leq 1$. A direct computation shows that

$$\sigma \circ \widehat{\varphi_t}(Z) = e^{tM}\sigma(Z)$$

for all $t \geq 0$ and $Z \in \mathbb{B}^n$. Thus, setting $\Omega := \sigma(\mathbb{B}^n)$ we have the result.

Finally, from the very construction it follows that every iteration couple as in (1) and (2) can be realized as an elliptic semigroup of linear fractional self-maps of $\mathbb{B}^n$. $\square$

From the previous proof we can better specify the shape of the complex ellipsoid in part (2):

Corollary 3.3 *Let (φ_t) be an elliptic semigroup of linear fractional self-maps of $\mathbb{B}^n$. Suppose that $\cap_{t\geq 0}\mathsf{Fix}(\varphi_t) = \{Z_0\}$ and $u(\varphi_t) = 0$.*

- *If (φ_t) extends analytically beyond the unit ball, i.e., if there exists $\rho > 1$ such that all the iterates of the semigroup are well-defined on $\rho\mathbb{B}^n$, then there* *exist $r \geq 1$ and a dissipative and asymptotically stable matrix $M \in \mathbb{C}^{n\times n}$ such that $(\mathbb{B}^n, \varphi_t)$ is conjugated to the iteration couple (Δ_1, e^{tM}), where Δ_1 is the complex ellipsoid given by*

$$\Delta_1 = \left\{(z,w) \in \mathbb{C}\times\mathbb{C}^{n-1} : \frac{1}{r^2}|z-\sqrt{r^2-1}|^2 + \|w\|^2 < r^2\right\}.$$

- *If the semigroup is not analytic beyond the unit ball then there exists a dissipative and asymptotically stable matrix $M \in \mathbb{C}^{n\times n}$ such that $(\mathbb{B}^n, \varphi_t)$ is conjugated to the iteration couple (Δ_2, e^{tM}), where*

$$\Delta_2 = \left\{(z,w) \in \mathbb{C}\times\mathbb{C}^{n-1} : \mathsf{Re}\,(2z) > \|w\|^2 - 1\right\}.$$

PROOF. By Theorem 3.2 the couple $(\mathbb{B}^n, \varphi_t)$ is conjugated to (Ω, e^{tM}), where $\Omega = \sigma(\mathbb{B}^n)$ and σ is defined in (3). Therefore

$$\begin{aligned}\Omega = \sigma(\mathbb{B}^n) &= \{\tfrac{Z}{-\delta z_1+1} : Z = (z_1, z') \in \mathbb{C}\times\mathbb{C}^{n-1}\cap\mathbb{B}^n\}\\ &= \{W = (w_1, w') \in \mathbb{C}\times\mathbb{C}^{n-1} : |1+\delta w_1|^2 > \|W\|^2\}.\end{aligned}$$

If $\delta < 1$, we see that every iterate φ_t is holomorphic either on the ball (centered at the origin) of radius $\frac{1}{\delta} > 1$ if $\delta \neq 0$ or in the whole $\mathbb{C}^n$ if $\delta = 0$. In both cases, the semigroup extends analytical beyond the unit ball. Then, if we set $r = (1-\delta^2)^{-1/2}$, we find that $\Omega = \Delta_1$.

If $\delta = 1$, from the proof of Theorem 3.2 it follows that all the iterates of the semigroup have the same singularity at the boundary. In this case, direct computations show that $\Omega = \Delta_2$. $\square$

Remark 3.4 *In [11] Cowen and MacCluer prove that if $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ fixes O and the spectrum of $d\varphi_O$ does not contain eigenvalues of modulus 1 then there exists an injective linear fractional map $\sigma : \mathbb{B}^n \to \mathbb{C}^n$ such that $\sigma\circ\varphi = d\varphi_O\circ\sigma$. Their argument (which is indeed a simplified version—and the inspiration—of the proof of (2) above) allows only to state that if $d\varphi_O$ contains eigenvalues of modulus 1 then such an intertwining map σ is defined only on a neighborhood of O (not in all $\mathbb{B}^n$). In the proof of Theorem 3.2, which clearly works also for just one elliptic linear fractional map, we showed that actually one can always obtain an intertwining mapping σ defined in all of $\mathbb{B}^n$, regardless the presence of eigenvalues of modulus 1.*

4 Non-elliptic linear fractional maps

Our first result is somewhat technical and it says that, in the non-elliptic case, we can always obtained a simpler iteration couple transferring the corresponding linear fractional map from $\mathbb{B}^n$ to $\mathbb{H}^n$. We also show how this model can be used to detect simply independent invariant slices.

Recall that if $H \in \mathbb{C}^{n\times n}$ is a hermitian matrix, by the spectral theorem, there exists a unitary $n \times n$ matrix U and a diagonal matrix D such that $H = U^*DU$. If D has entries $d_1, \dots, d_n \in \mathbb{R}$ on the principal diagonal, let D^+ be the diagonal matrix whose entry of position (j, j) is 0 if $d_j = 0$ or d_j^{-1} if $d_j \neq 0$. Then the *pseudo-inverse* (or *generalized inverse*) H^+ of H is defined as $H^+ := U^*D^+U$ (see, e.g., [21]).

Lemma 4.1 *Let $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ be non-elliptic with Denjoy-Wolff point $\tau \in \partial\mathbb{B}^n$ and boundary dilatation coefficient $\alpha = \alpha(\varphi)$. Then, the iteration couple $(\mathbb{B}^n, \varphi)$ is conjugated to the iteration couple $(\mathbb{H}^n, \widetilde{\varphi})$ where*

$$\widetilde{\varphi}(z, w) = \frac{1}{\alpha}(z + \langle w, b\rangle + c, Aw + d), \quad (z, w) \in \mathbb{H}^n, \tag{4}$$

with $c \in \mathbb{C}$, $b, d \in \mathbb{C}^{n-1}$, $A \in \mathbb{C}^{(n-1)\times(n-1)}$ satisfying

(i) $Q := \alpha I - A^*A$ *is a hermitian positive semi-definite matrix,*
(ii) $\alpha\mathsf{Re}\,(c) - \|d\|^2 \geq \left\langle Q^+(A^*d - \frac{1}{2}\alpha b), A^*d - \frac{1}{2}\alpha b\right\rangle$ *where Q^+ is the pseudo-inverse of Q,*
(iii) $A^*d - \frac{1}{2}\alpha b$ *belongs to the space spanned by the columns of Q.*

PROOF. Up to conjugation with a rotation we can suppose that $\tau = e_1$. By Proposition 1.3.(2) it follows that

$$d\varphi_{e_1} = \begin{pmatrix} \alpha & 0 \\ d & A \end{pmatrix} \tag{5}$$

for some $d \in \mathbb{C}^{n-1}$ and $A \in \mathbb{C}^{(n-1)\times(n-1)}$. Conjugating φ with the Cayley transform $\sigma : \mathbb{B}^n \to \mathbb{H}^n$ which maps e_1 to O, namely $\sigma(z, w) := (1 - z, w)(1 + z)^{-1}$ for $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$, we obtain $\varphi' := \sigma \circ \varphi \circ \sigma^{-1} \in \mathrm{LFM}(\mathbb{H}^n, \mathbb{H}^n)$ such that O is its Denjoy-Wolff point and α is boundary dilatation coefficient at O. Moreover, since $T^{\mathbb{C}}_O\partial\mathbb{H}^n = T^{\mathbb{C}}_{e_1}\partial\mathbb{B}^n$ and, taking into account the form of φ (see (1)) and (5), a straightforward computation gives us

$$\varphi'(z, w) = \frac{(\alpha z, Aw + zd)}{cz + \langle w, b\rangle + 1}, \quad \mathsf{Re}\, z > \|w\|^2,$$

for some $c \in \mathbb{C}, b \in \mathbb{C}^{(n-1)}$. Now let

$$G : \mathbb{H}^n \longrightarrow \mathbb{H}^n, \quad G(z, w) := \left(\frac{1}{z}, \frac{w}{z}\right).$$

Then $G \in \mathsf{Aut}(\mathbb{H}^n)$ and $G \circ \sigma = \sigma_C$, the generalized Cayley transform.

Let $\widetilde{\varphi} := G \circ \varphi' \circ G^{-1}$. A direct computation shows that

$$\widetilde{\varphi}(z,w) = \frac{1}{\alpha}\left(z + \langle w, b\rangle + c, Aw + d\right), \quad \mathsf{Re}\, z > \|w\|^2.$$

We prove now that conditions (i), (ii) and (iii) hold.

The matrix Q is obviously hermitian. So, in order to prove (i), let $w \in \mathbb{C}^{n-1}$ with $\|w\| = 1$. Since $(r^2, rw) \in \partial\mathbb{H}^n$ for all $r > 0$, then $\widetilde{\varphi}(r^2, rw) \in \overline{\mathbb{H}^n}$ for every $r > 0$. Writing $\widetilde{\varphi}(Z) = (\varphi_z(Z), \varphi_w(Z)) \in \mathbb{C} \times \mathbb{C}^{n-1}$, we have thus $\mathsf{Re}\, \widetilde{\varphi}_z(r^2, rw) \geq \|\widetilde{\varphi}_w(r^2, rw)\|^2$, namely

$$\alpha + \frac{1}{r}\mathsf{Re}\ \langle w, \alpha b\rangle + \frac{1}{r^2}\alpha \mathsf{Re}\, c \geq \left\| Aw + \frac{1}{r}d \right\|^2.$$

Letting r tend to infinite, we see that $\alpha \geq \|Aw\|^2$. Since w was arbitrary, we get $\alpha \geq \|A\|^2$. It follows immediately that all the (necessarily real) eigenvalues of Q are non-negative and therefore Q is semi-definite positive.

As for the other two conditions, since $\widetilde{\varphi}(\mathbb{H}^n) \subseteq \mathbb{H}^n$ and $\widetilde{\varphi}$ is continuous on $\overline{\mathbb{H}^n}$, we have that $\widetilde{\varphi}(\partial\mathbb{H}^n) \subset \overline{\mathbb{H}^n}$. If we parameterize $\partial\mathbb{H}^n$ as

$$\mathbb{R} \times \mathbb{C}^{n-1} \ni (r, w) \mapsto (\|w\|^2 + ir, w) \in \partial\mathbb{H}^n,$$

then for every $w \in \mathbb{C}^{n-1}$ it holds

$$\alpha \|w\|^2 + \alpha \mathsf{Re}\ \langle w, b\rangle + \alpha \mathsf{Re}\, c \geq \|Aw + d\|^2.$$

Denoting $Q := \alpha I - A^*A$, $\gamma_1 := -\frac{1}{2}(\alpha b - 2A^*d)$ and $\gamma_2 = \frac{1}{2}(\alpha \mathsf{Re}\, c - \|d\|^2)$, the above inequality is equivalent to

$$F(w) := \frac{1}{2} w^* Q w - \mathsf{Re}\ \langle w, \gamma_1\rangle + \gamma_2 \geq 0. \tag{6}$$

Notice that $\gamma_2 \in \mathbb{R}$ and, since Q is hermitian, also $w^*Qw \in \mathbb{R}$. Thus, we have a function $F : \mathbb{C}^{n-1} \to \mathbb{R}$ such that $F(w) \geq 0$, for every $w \in \mathbb{C}^{n-1}$. We claim that $F \geq 0$ if and only if

$$F(Q^+\gamma_1) \geq 0 \quad \text{and} \quad \gamma_1 \in \operatorname{span}_{\mathbb{C}}\{Qe_1, \ldots, Qe_{n-1}\}.$$

Since $Q^+QQ^+ = Q^+$, we see that $F(Q^+\gamma_1) = \gamma_2 - \frac{1}{2}\gamma_1^* Q^+ \gamma_1$ and thus (ii) and (iii) follow.

Thus we are left to prove the claim. The matrix Q is a hermitian positive semi-definite matrix with, say, rank $k \leq n-1$.

By the spectral decomposition theorem, there exists an unitary matrix U of order $n-1$ such that $Q = U\Sigma U^*$, where Σ is a diagonal matrix whose entries are

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_{n-1} = 0.$$

Let
$$\widehat{F}(w) := F(Uw), \quad w = (w_1, ..., w_{n-1}) \in \mathbb{C}^{n-1}.$$
Clearly $F \geq 0$ if and only if $\widehat{F} \geq 0$. Let

$$\nu = (\nu_1, \ldots, \nu_{n-1}) := U^* \gamma_1, \quad \delta := \gamma_2 - \frac{1}{2} \sum_{j=1}^{k} \frac{|\nu_j|^2}{\sigma_j}.$$

Thus
$$\widehat{F}(w) = \frac{1}{2} \sum_{j=1}^{k} \left| \sqrt{\sigma_j} w_j - \frac{\nu_j}{\sqrt{\sigma_j}} \right|^2 + \delta - \sum_{j=k+1}^{n-1} \mathsf{Re}\,(w_j \overline{\nu_j}).$$
Clearly, if $\widehat{F} \geq 0$ then $\nu_{k+1} = \ldots = \nu_{n-1} = 0$, namely, $\gamma_1 \in \mathrm{span}_{\mathbb{C}}\{Qe_1, \ldots, Qe_{n-1}\}$. Under this condition, $\widehat{F}$ assumes its minimum value at the point $x = (\frac{\nu_1}{\sigma_1}, \ldots, \frac{\nu_k}{\sigma_k}, 0, \ldots, 0)$. And thus it is non negative if and only if $\widehat{F}(x) \geq 0$, namely $F(Q^+ \gamma_1) \geq 0$. $\square$

Notice that the intertwining map between the two iteration couples $(\mathbb{B}^n, \varphi)$ and $(\mathbb{H}^n, \widetilde{\varphi})$ in Lemma 4.1 is simply given by a rotation followed by the Cayley transform σ_C. Also, notice that setting $w = 0$ in (6) we obtain

$$\alpha \mathsf{Re}\, c - \|d\|^2 \geq 0,$$

which implies $\mathsf{Re}\, c \geq 0$ because $\alpha > 0$.

As a corollary of the proof of Lemma 4.1 we have the following result.

Proposition 4.2 *Let $\varphi \in \mathrm{LFM}(\mathbb{H}^n, \mathbb{H}^n)$ be non-elliptic with Denjoy-Wolff point ∞ and boundary dilatation coefficient $\alpha = \alpha(\varphi)$. Then*

$$\varphi(z, w) = \frac{1}{\alpha}(z + \langle w, b \rangle + c, Aw + d), \quad (z, w) \in \mathbb{H}^n, \tag{7}$$

where $c \in \mathbb{C}$, $b, d \in \mathbb{C}^{n-1}$, $A \in \mathbb{C}^{(n-1)\times(n-1)}$ satisfy (i), (ii) and (iii) in Lemma 4.1.

Conversely, for any α, A, b, c, d as before that satisfy (i), (ii) and (iii) in Lemma 4.1 the linear fractional map φ defined by (7) *is in* $\mathrm{LFM}(\mathbb{H}^n, \mathbb{H}^n)$.

PROOF. One direction follows from the proof of Lemma 4.1. Conversely, note that if α, A, b, c, d satisfy (i), (ii) and (iii) then (6) is satisfied and then the linear fractional map φ defined by (7) is such that $\varphi(\mathbb{H}^n) \subseteq \mathbb{H}^n$. $\square$

The argument in the proof of Lemma 4.1 also allows us to detect automorphisms of $\mathbb{H}^n$ with Denjoy-Wolff point at ∞ among linear fractional self-maps of $\mathbb{H}^n$. In some sense, this extends [11, Theorem 2.90].

Proposition 4.3 *Let $\varphi \in \mathrm{LFM}(\mathbb{H}^n, \mathbb{H}^n)$ be without fixed points in $\mathbb{H}^n$, with Denjoy-Wolff point ∞ and boundary dilatation coefficient $\alpha := \alpha(\varphi)$. Then $\varphi \in \mathsf{Aut}(\mathbb{H}^n)$ if and only if it is the composition of a rotation in the last $(n-1)$-coordinates and a generalized α-Heisenberg translation. That is,*

$$\varphi(z, w) = \phi_\alpha(z, Uw), \quad (z, w) \in \mathbb{H}^n,$$

where $U \in \mathbb{C}^{(n-1)\times(n-1)}$ is a unitary matrix and

$$\phi_\alpha(z, w) := \frac{1}{\alpha}(z + 2\frac{1}{\sqrt{\alpha}}\langle w, d\rangle + c, \sqrt{\alpha}w + d), \quad (z, w) \in \mathbb{H}^n \tag{8}$$

with $\mathsf{Re}\, c = \|d\|^2$.

PROOF. According to Proposition 4.2 the map φ has the form

$$\varphi(z, w) = \frac{1}{\alpha}(z + \langle w, b\rangle + c, Aw + d), \quad \mathsf{Re}\, z > \|w\|^2.$$

By Alexander's theorem (see, *e.g.*, [24], or [4, Theorem 2.3]) $\varphi \in \mathsf{Aut}(\mathbb{H}^n)$ if and only if $\varphi(\partial\mathbb{H}^n) \subseteq \partial\mathbb{H}^n$. From this the statement follows easily. $\square$

As we promised, we apply the above result to estimate $\sharp\mathrm{inv}(\varphi)$, the dimension of the space spanned by the direction subspaces V_S of all φ-invariant slices $S \subset \mathbb{B}^n$ (see Section 2).

Proposition 4.4 *Let $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ be non-elliptic with Denjoy-Wolff point $\tau \in \partial\mathbb{B}^n$ and boundary dilatation coefficient $\alpha = \alpha(\varphi)$. Let $(\mathbb{H}^n, \widetilde{\varphi})$ with*

$$\widetilde{\varphi}(z, w) = \frac{1}{\alpha}(z + \langle w, b\rangle + c, Aw + d), \quad (z, w) \in \mathbb{H}^n$$

be the iteration couple prescribed in Lemma 4.1, conjugated to the iteration couple $(\mathbb{B}^n, \varphi)$. Then:

(1) The boundary dilatation coefficient α is not an eigenvalue of A if and only if $\sharp\mathrm{inv}(\varphi) = 1$.
(2) If α is an eigenvalue of A, then either $\sharp\mathrm{inv}(\varphi) = 0$ or $\sharp\mathrm{inv}(\varphi) = 1 + \dim\ker(\alpha I - A)$.

Moreover, if $\sharp\mathrm{inv}(\varphi) \geq 1$ then we can assume that $d = 0$ and $Ae_j = \alpha e_j$ for $j = 1, \ldots, \sharp\mathrm{inv}(\varphi) - 1$.

PROOF. Since φ and $\tilde{\varphi}$ are conjugated, then $\sharp\mathrm{inv}(\varphi) = \sharp\mathrm{inv}(\widetilde{\varphi})$. Moreover, by Proposition 2.3 all invariant slices $S' \in \mathbb{H}^n$ for $\widetilde{\varphi}$ are of the form $\{(z, w) \in$

$\mathbb{C} \times \mathbb{C}^{n-1} : w = \text{const}\}$. Hence, to determine $\sharp\text{inv}(\widetilde{\varphi})$, we just need to solve the linear system

$$(A - \alpha I)w = d.$$

Now, assertion (1) follows from the fact that this system has a unique solution if and only if the matrix $(A - \alpha I)$ is invertible.

Otherwise, if $(A - \alpha I)$ is not invertible then the system has either no solutions or the set of all solutions contains $1 + \dim \text{Ker}(\alpha I - A)$ independent solutions, which proves (2).

Now, assume $\sharp\text{inv}(\widetilde{\varphi}) \geq 1$ and let $S' = \{(z, w) \in \mathbb{H}^n : w = w_0\}$ be an invariant slice. We can use a parabolic automorphism Φ of the form (8) (with boundary dilatation coefficient 1) to map $w = O$ to w_0. Then $\Phi^{-1} \circ \widetilde{\varphi} \circ \Phi$ has the slice $S'_0 := \{(z, w) \in \mathbb{H}^n : w = 0\}$ as invariant and therefore the w-component of $\Phi^{-1} \circ \widetilde{\varphi} \circ \Phi$ is of the form Aw/α (there is no d term). Up to this conjugation we can then assume that $\widetilde{\varphi}$ is a linear fractional map with $d = 0$. Now, if we conjugate $\widetilde{\varphi}$ with a rotation $(z, w) \mapsto (z, Uw)$ with U unitary, we see that the w-component of $\widetilde{\varphi}$ becomes $U^* AUw/\alpha$. It is clear (cfr. [4, Lemma 4.1]) that we can choose U in such a way that the eigenvectors of A related to the eigenvalue α are $e_1, \ldots, e_l$ with $l = \dim \text{Ker}(\alpha I - A)$, ending the proof. $\square$

5 Classification of hyperbolic semigroups of linear fractional maps

We begin with the following result which completely classifies hyperbolic semigroups of linear fractional maps.

Theorem 5.1 *Let (φ_t) be a hyperbolic continuous semigroup in* $\text{LFM}(\mathbb{B}^n, \mathbb{B}^n)$. *Then, the iteration couple $(\mathbb{B}^n, (\varphi_t))$ is conjugated to the iteration couple $(\mathbb{H}^n, (\widetilde{\varphi_t}))$ where*

$$\widetilde{\varphi_t}(z, w) = e^{\lambda t}(z + (1 - e^{-\lambda t})c, e^{tM}w + e^{-\lambda t}\left(\int_0^t e^{(\lambda I + M)s} ds\right) d), \quad (z, w) \in \mathbb{H}^n$$

with $\lambda > 0$, $c \geq 0$ $d \in \mathbb{C}^{n-1}$, $M \in \mathbb{C}^{(n-1)\times(n-1)}$ and such that

(i) $Q_t := e^{-\lambda t} I - \exp(tM^*) \exp(tM)$ *is a positive semi-definite hermitian matrix for every $t \geq 0$,*

(ii) $e^{-\lambda t}(1 - e^{-\lambda t})c - \|d_t\|^2 \geq \left\langle Q_t^+ \exp(tM^* d_t), \exp(tM^*) d_t \right\rangle$, *for every $t \geq 0$, where Q_t^+ is the pseudo-inverse of Q_t and $d_t = e^{-\lambda t}\left(\int_0^t e^{(\lambda I + M)s} ds\right) d$,*

(iii) $\exp(M^* t) d_t$ *belongs to the space spanned by the columns of Q_t, for every $t \geq 0$.*

Moreover, given λ, c, d, M as above, there exists a hyperbolic semigroup (φ_t) of linear fractional self-maps of $\mathbb{B}^n$ such that $(\mathbb{B}^n, (\varphi_t))$ is conjugated to $(\mathbb{H}^n, (\widetilde{\varphi_t}))$.

PROOF. According to Theorem 1.5, the boundary dilatation coefficient $\alpha(\varphi_t)$ at the common Denjoy-Wolff point of the semigroup is $e^{-\lambda t}$, for some $\lambda > 0$. By conjugating φ_t via the Cayley transform $\sigma_\tau : \mathbb{B}^n \to \mathbb{H}^n$ which maps τ to ∞ (see Lemma 4.1), the semigroup $(\mathbb{B}^n, (\varphi_t))$ is conjugated to the semigroup $(\mathbb{H}^n, (\varphi_t^1))$ with

$$\varphi_t^1(z, w) = e^{\lambda t}(z + \langle w, b_t \rangle + c_t, A_t w + d_t), \quad (z, w) \in \mathbb{H}^n, \tag{9}$$

where $c_t \in \mathbb{C}$, $b_t, d_t \in \mathbb{C}^{n-1}$ and $A_t \in \mathbb{C}^{(n-1)\times(n-1)}$ satisfy (i), (ii) and (iii) in Lemma 4.1. In particular (i) implies $\|A_t\| \leq \sqrt{e^{-\lambda t}} < 1$, for all $t > 0$.

Applying the algebraic semigroup conditions, we come up with the following four equations for the above coefficients:

$$\begin{cases} 1)\ c_{t+s} = e^{-\lambda s} c_t + c_s + \langle d_s, b_t \rangle \\ 2)\ d_{t+s} = A_t d_s + e^{-\lambda s} d_t \\ 3)\ A_{t+s} = A_t A_s \\ 4)\ b_{t+s} = b_s + A_s^* b_t \end{cases} \qquad t, s \geq 0. \tag{10}$$

Moreover, since φ_0^1 is the identity on $\mathbb{H}^n$, we obtain $c_0 = 0$, $b_0 = O$, $d_0 = O$ and $A_0 = I$. In what follows, recall that $t \mapsto \varphi_t^1$ is real analytic and therefore we can freely differentiate c_t, b_t, d_t and A_t with respect to t.

From equation 3) we see that there exists a matrix $M \in \mathbb{C}^{(n-1)\times(n-1)}$ such that $A_t = \exp(tM)$.

Next, we look at equation 4). Differentiating with respect to s and setting $s = 0$, we obtain the following system of linear differential equations

$$\begin{cases} \dfrac{d}{dt} b_t = M^* b_t + v, \\ b_0 = O, \end{cases}$$

for some vector $v \in \mathbb{C}^{n-1}$. Since $\|A_t\| < 1$, then M^* is invertible. Therefore, we have $b_t = (\exp(tM^*) - I)b$ for some vector $b \in \mathbb{C}^{n-1}$ such that $M^* b = -v$.

Now, consider the following Heisenberg translation

$$\eta(z, w) = (z + 2\langle w, k_2 \rangle + k_1, w + k_2), \quad (z, w) \in \mathbb{H}^n$$

with $(k_1, k_2) \in \partial\mathbb{H}^n$, namely $\operatorname{Re} k_1 = \|k_2\|^2$. By Proposition 4.3 it follows that $\eta \in \mathsf{Aut}(\mathbb{H}^n)$. Let

$$\varphi_t^2 = \eta^{-1} \circ \varphi_t^1 \circ \eta.$$

Straightforward computations show that

$$\varphi_t^2(z,w) = e^{\lambda t}(z + \langle w, b_t + 2k_2 - 2A_t^* k_2\rangle + \widetilde{c}_t, A_t w + \widetilde{d}_t), \quad (z,w) \in \mathbb{H}^n$$

for some $\widetilde{c}_t \in \mathbb{C}$ and $\widetilde{d}_t \in \mathbb{C}^{n-1}$ still satisfying the same algebraic semigroup conditions of c_t and d_t.

We focus our attention to the linear system

$$(A_t^* - I)k_2 = \frac{1}{2} b_t. \tag{11}$$

Substituting the expressions of A_t and b_t as found before, we see that such a system is solved for $k_2 = \frac{1}{2}b$. Therefore, if we choose $(k_1, k_2) = (\|\frac{1}{2}b\|^2, \frac{1}{2}b)$, we have

$$\varphi_t^2(z,w) = e^{\lambda t}(z + \widetilde{c}_t, e^{tM}w + \widetilde{d}_t), \quad (z,w) \in \mathbb{H}^n,$$

with

$$\begin{cases} 1)\ \tilde{c}_{t+s} = e^{-\lambda s}\tilde{c}_t + \tilde{c}_s, \qquad \tilde{c}_0 = 0, \\ 2)\ \tilde{d}_{t+s} = e^{tM}\tilde{d}_s + e^{-\lambda s}\tilde{d}_t,\ \tilde{d}_0 = 0. \end{cases}$$

Arguing as before, passing from algebraic equations to differential equations, we obtain

$$\begin{cases} \widetilde{c}_t = (1 - e^{-\lambda t})c, \text{ for some } c \in \mathbb{C} \\ \widetilde{d}_t = e^{-\lambda t}\left(\int_0^t e^{(\lambda I + M)s} ds\right) d, \text{ for some } d \in \mathbb{C}^{n-1}. \end{cases}$$

We prove now that we can conjugate once more in order to take $c \in \mathbb{R}$. Assume that $c = c_1 + ic_2$ with $c_1, c_2 \in \mathbb{R}$. Let $\nu(z,w) = (z - ic_2, w)$ for $(z,w) \in \mathbb{H}^n$. Then, ν is an automorphism of $\mathbb{H}^n$. A straightforward computation shows that

$$\nu^{-1} \circ \varphi_t^2 \circ \nu(z,w) = e^{\lambda t}(z + (1 - e^{-\lambda t})c_1, e^{tM}w + e^{-\lambda t}\left(\int_0^t e^{(\lambda I + M)s} ds\right) d).$$

The remaining assertions follow now applying Lemma 4.1. $\square$

Remark 5.2 *Condition* (i) *in the above Theorem 5.1 means (in the terminology of dynamical systems) that the matrix* M *is* $\frac{\lambda}{2}$*-uniformly dissipative. That is,* (i) *is equivalent to*

$$(i') \ \operatorname{Re} w^* M w \leq -\frac{\lambda}{2}\|w\|^2, \ \textit{for all } w \in \mathbb{C}^{n-1}.$$

Indeed, Q_t *is positive semi-definite for all* $t \geq 0$ *if and only if* $\left\|e^{(\lambda I + M + M^*)t}\right\| \leq 1$, *for all* $t \geq 0$, *which, by Phillips-Lumer's theorem (see, e.g., [27, p. 250]), it* *is equivalent to* $\mathsf{Re} w^*(\lambda I + M + M^*)w \leq 0$, *for all* $w \in \mathbb{C}^{n-1}$, *which in turns is equivalent to* (i').

A quite interesting consequence of this classification is the link between the conjugation of hyperbolic semigroups in $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ and the classical open question of the classification of automorphisms in $\mathbb{C}^n$, provided by the next corollary whose proof is straightforward from Theorem 5.1.

Corollary 5.3 *Let* (φ_t) *a continuous hyperbolic semigroup in* $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$. *Then, there exist a biholomorphic map* σ *from* $\mathbb{B}^n$ *onto* $\mathbb{H}^n$ *and a continuous group* (ϕ_t) *in* $\mathrm{Aut}(\mathbb{C}^n, \mathbb{C}^n)$ *such that, for every* $t \geq 0$, *the restriction of* ϕ_t *to* $\mathbb{H}^n$ *is exactly* $\sigma \circ \varphi_t \circ \sigma^{-1}$.

As one might suspect, the existence of invariant slices also allows to simplify the model given in Theorem 5.1. We recall that by Theorem 2.4 and assuming that (φ_t) is a semigroup of non-elliptic linear fractional self-maps of $\mathbb{B}^n$ (or of $\mathbb{H}^n$), if φ_{t_0} has some invariant slices for some $t_0 > 0$ then (φ_t) has at least one common invariant slice.

Before examining the case of existence of invariant slices, we comment some examples.

Example 5.4 *Let*

$$\varphi_t(z, w) = (e^{\lambda t} z + (e^{\lambda t} - 1)c, w + t),$$

for $(z, w) \in \mathbb{H}^2$, *where* $\lambda > 0$ *and* $c \geq \lambda^2$. *According to Theorem 5.1 the semigroup* (φ_t) *is a hyperbolic semigroup of* $\mathrm{LFM}(\mathbb{H}^2, \mathbb{H}^2)$ *and clearly there are no invariant slices for* $t > 0$. *Moreover, each hyperbolic semigroup of* $\mathrm{LFM}(\mathbb{H}^2, \mathbb{H}^2)$ *with no invariant slice can be conjugated to a semigroup as above for a certain* $c \geq 0$.

Example 5.5 *Let* $\lambda > 0$ *and let*

$$\varphi_t(z, w) = (e^{\lambda t} z, e^{2\pi i k_1 t} w_1, \ldots, e^{2\pi i k_{n-1} t} w_{n-1}),$$

for $(z, w_1, ...w_{n-1}) \in \mathbb{H}^n$, *with* $k_p = 2^{-p}$, $p = 1, ..., n-1$. *Notice that* φ_t *is a hyperbolic semigroup of* $\mathrm{LFM}(\mathbb{H}^n, \mathbb{H}^n)$ *with only one common invariant slice* $\{(z, w) \in \mathbb{B}^n : w = 0\}$. *However, as* t *varies in* $(0, \infty)$ *the dimension of the inner space* $\mathbb{A}_t$ *varies between* 1 *and* n *(all values are attained) and thus there exist iterates* φ_t *which have up to* n *independent invariant slices.*

The two previous examples are somewhat degenerate as the following remark explains.

Remark 5.6 *Let* $(\mathbb{B}^n, (\varphi_t))$ *be a hyperbolic semigroup of* $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ *with common Denjoy-Wolff point* $\tau \in \partial\mathbb{B}^n$. *Let* $(\mathbb{H}^n, (\tilde{\varphi}_t))$ *be the conjugated semi-*

group of LFM($\mathbb{H}^n, \mathbb{H}^n$) *given by Theorem 5.1. Then, to study the number of common independent invariant slices (in* $\mathbb{H}^n$*), we write down the family of equations*

$$(e^{t(\lambda I+M)} - I)w = (\int_0^t e^{(\lambda I+M)s} ds)d, \quad t \geq 0.$$

Let us see that $(\mathbb{B}^n, (\varphi_t))$ *has a unique invariant slice if and only if* $\lambda I + M$ *is invertible. Firstly, the invertibility of this matrix implies that* $\int_0^t e^{(\lambda I+M)s} ds = (e^{t(\lambda I+M)} - I)(\lambda I + M)^{-1}$*. Moreover, since* $e^{t(\lambda I+M)}$ *tends to* I *as* t *goes to* 0*, there must be some* $t > 0$ *such that* $e^{t(\lambda I+M)}$ *is invertible. Therefore, the only possible common solution to the above family of equations is* $w = (\lambda I+M)^{-1}d$*, hence* $(\mathbb{B}^n, (\varphi_t))$ *has a unique invariant slice. On the other hand, if there exists a unique invariant slice and* $(\lambda I+M)$ *were not invertible, taking any non-zero vector* w *in the kernel of* $(\lambda I+M)$ *it would follow that* $(e^{(\lambda I+M)t} - I)w = O$ *for all* $t \geq 0$*, namely* $e^{-\lambda t}$ *would be an eigenvalue of* e^{tM} *for all* t*, contradicting Proposition 4.4.*

The condition that $\lambda M + I$ *is invertible can be easily translated into an algebraic condition for* (φ_t) *as follows. A direct computation shows that the eigenvalues of* $d(\varphi_t)_\tau$ *are exactly* $e^{-t\lambda}$ *and* $e^{-t\lambda_1}, \ldots, e^{-t\lambda_m}$*, the eigenvalues of* e^{tM}*, with* $\mathsf{Re}\,\lambda_j > 0$ *since* $\|e^{tM}\|^2 \leq e^{-\lambda t} < 1$*. Therefore,* $\lambda M + I$ *is invertible if and only if the algebraic multiplicity of the boundary dilatation coefficient* $e^{-t\lambda}$ *as eigenvalue of* $d(\varphi_t)_\tau$ *is* 1 *for some (and hence any)* $t > 0$*.*

In case of existence of a common invariant slice for (φ_t) we can choose a different conjugation.

Proposition 5.7 *Let* (φ_t) *be a hyperbolic continuous semigroup in* LFM($\mathbb{B}^n, \mathbb{B}^n$)*. Assume that* φ_t *has an invariant slice, for some* $t > 0$*. Then, the iteration couple* $(\mathbb{B}^n, (\varphi_t))$ *is conjugated to the iteration couple* $(\mathbb{H}^n, (\widetilde{\varphi_t}))$ *where*

$$\widetilde{\varphi}_t(z, w) = e^{\lambda t}(z + (1 - e^{-\lambda t})c + \langle w, (e^{M^* t} - I)b\rangle, e^{Mt}w), \quad (z, w) \in \mathbb{H}^n$$

with $\lambda > 0$*,* $c \in \mathbb{C}$*,* $b \in \mathbb{C}^{n-1}$ *and* $M \in \mathbb{C}^{(n-1)\times(n-1)}$ *a* $\frac{\lambda}{2}$*-uniformly dissipative matrix.*

PROOF. By Theorem 5.1 the semigroup $(\mathbb{B}^n, (\varphi_t))$ is conjugated to $(\mathbb{H}^n, (\varphi_t^1))$ where

$$\varphi_t^1(z, w) = e^{\lambda t}(z + (1 - e^{-\lambda t})c, e^{Mt}w + d_t)$$

with d_t, c and M satisfying certain restrictions. By Theorem 2.4 there exists a common invariant slice, say $\{(z, w) \in \mathbb{H}^n : w = w_0\}$. Consider the linear fractional map

$$\eta(z, w) = (z - 2\langle w, w_0\rangle + \|w_0\|^2, w - w_0), \quad (z, w) \in \mathbb{H}^n.$$

Then $\eta \in \mathsf{Aut}(\mathbb{H}^n)$ and η sends the slice $\{(z,w) \in \mathbb{H}^n : w = w_0\}$ onto the slice $\{(z,w) \in \mathbb{H}^n : w = O\}$. Thus $\varphi_t^2 := \eta^{-1} \circ \varphi_t^1 \circ \eta$ is of the form

$$\varphi_t^2(z,w) = e^{\lambda t}(z + c_t + \langle w, b_t \rangle, e^{tM} w),$$

with c_t, b_t satisfying (10). Thus $b_t = (e^{tM^*} - I)b$ and $c_t = (1 - e^{-\lambda t})c$. Finally, Lemma 4.1 and Remark 5.2 give the desired estimate. $\square$

Notice that the compatibility condition in Proposition 5.7 is not enough to guarantee that the semigroup (φ_t) maps $\mathbb{H}^n$ into $\mathbb{H}^n$. Indeed, conditions (ii), (iii) in Lemma 4.1 must also be satisfied.

Remark 5.8 *If the semigroup (φ_t) given at the above proposition has $p+1$ $(p \geq 1)$ common independent invariant slices, it is possible to simplify a little more the model, taking into account the final assertions given at Proposition 4.4. Indeed, up to a suitable rotation, the matrix e^{Mt} in $(\widetilde{\varphi}_t)$, can be replaced by the block matrix*

$$\begin{bmatrix} e^{-\lambda t} I_p & B_t \\ O & D_t \end{bmatrix}$$

where $D_t = e^{Nt}$ for some $N \in \mathbb{C}^{q\times q}$ and $B_t = B \int_0^t e^{Ns} ds$ for some $B \in \mathbb{C}^{p\times q}$.

In case $\varphi \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ has exactly two fixed points on $\partial \mathbb{B}^n$, the situation is much simpler.

Theorem 5.9 *Let (φ_t) be a semigroup in $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$. If for some $t_0 > 0$ the iterate φ_{t_0} has exactly two (distinct) fixed points then (φ_t) is a hyperbolic semigroup with two common fixed points.*

Moreover, the iteration couple $(\mathbb{B}^n, (\varphi_t))$ is conjugated to the iteration couple $(\mathbb{H}^n, (\widetilde{\varphi_t}))$ where

$$\widetilde{\varphi_t}(z,w) = e^{\lambda t}(z, e^{Mt} w), \quad (z,w) \in \mathbb{H}^n$$

with $\lambda > 0$ and $M \in \mathbb{C}^{(n-1)\times(n-1)}$ a $\frac{\lambda}{2}$-uniformly dissipative matrix.

Conversely, for all λ, M as above, there exists a hyperbolic semigroup (φ_t) of $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ with two common fixed points such that the iteration couple $(\mathbb{B}^n, (\varphi_t))$ is conjugated to $(\mathbb{H}^n, (\widetilde{\varphi}_t))$.

PROOF. By [4, Theorem 3.2] the map φ_{t_0} is necessarily hyperbolic and both fixed points belong to $\partial \mathbb{B}^n$. Thus, all φ_t must be hyperbolic by Theorem 1.5. According to Theorem 5.1 the iteration couple $(\mathbb{B}^n, \varphi)$ is conjugated to the iteration couple $(\mathbb{H}^n, \widetilde{\varphi})$ with

$$\widetilde{\varphi}(z, w) = e^{\lambda t}(z + c_t, e^{tM}w + d_t), \quad (z, w) \in \mathbb{H}^n$$

satisfying (i), (ii) and (iii).

Since φ_{t_0} has another boundary fixed point different from its Denjoy-Wolff point, so does $\widetilde{\varphi}_{t_0}$. Let $(\widehat{z}, \widehat{w}) \in \partial\mathbb{H}^n$ be such a fixed point. Hence, $\mathsf{Re}\,\widehat{z} = \|\widehat{w}\|^2$ and

$$\widehat{z} + c_{t_0} = e^{-\lambda t_0}\widehat{z}, \qquad e^{t_0 M}\widehat{w} + d_{t_0} = e^{-\lambda t_0}\widehat{w}.$$

Since $c_{t_0} = (1 - e^{-\lambda t_0})c$, the first equation implies $\hat{z} = -c$ but since $c \geq 0$ then $\hat{z} = c = 0$. Imposing the condition $\widetilde{\varphi}_t(O) \in \overline{\mathbb{H}^n}$ we find $d_t \equiv O$ as wanted. Finally, Theorem 5.1 and Remark 5.2 give the remaining assertions. $\square$

As a corollary we have the following characterization of groups of hyperbolic automorphisms of $\mathbb{B}^n$.

Corollary 5.10 *Let (φ_t) be a hyperbolic group in $\mathsf{Aut}(\mathbb{B}^n)$. Then, the iteration couple $(\mathbb{B}^n, (\varphi_t))$ is conjugated to the iteration couple $(\mathbb{H}^n, (\widetilde{\varphi}_t))$ where*

$$\widetilde{\varphi_t}(z, w) = e^{\lambda t}(z, e^{-\frac{\lambda}{2}t}e^{it\Theta}w), \quad (z, w) \in \mathbb{H}^n,$$

with $\lambda > 0$ and Θ a diagonal $(n-1) \times (n-1)$ matrix with real entries.

PROOF. Combining Theorem 5.9 and Proposition 4.3 we see that the iteration couple $(\mathbb{B}^n, (\varphi_t))$ is conjugated to the iteration couple $(\mathbb{H}^n, (\varphi_t^1))$ where

$$\varphi_t^1(z, w) = e^{\lambda t}(z, \sqrt{e^{-\lambda t}}U_t w), \quad (z, w) \in \mathbb{H}^n.$$

for some unitary matrix $U_t = e^{tH} \in \mathbb{C}^{(n-1)\times(n-1)}$. Thus $H + H^* = O$. By the spectral theorem there exists another unitary matrix V of order $n-1$ such that $V^*HV = i\Theta$, with Θ a diagonal real matrix of order $n-1$. Thus, the statement follows as soon as we conjugate φ_t^1 with the map $\eta \in \mathsf{Aut}(\mathbb{H}^n)$ defined as $\eta(z, w) = (z, Vw)$. $\square$

In our last result of this section we provide a simple model in case the differential at the common Wolff point of an iterate of a hyperbolic semigroup of linear fractional maps (φ_t) is normal:

Proposition 5.11 *Let (φ_t) be a hyperbolic semigroup in $\mathrm{LFM}(\mathbb{B}^n; \mathbb{B}^n)$ and let $\tau \in \partial\mathbb{B}^n$ be the common Wolff point. If $d(\varphi_t)_\tau$ acts normally on $T_\tau^{\mathbb{C}}\partial\mathbb{B}^n$ for* *some (and hence any) $t>0$ then, the iteration couple $(\mathbb{B}^n,(\varphi_t))$ is conjugated to the iteration couple $(\mathbb{H}^n,(\widetilde{\varphi_t}))$ with*

$$\widetilde{\varphi_t}(Z)=(e^{\lambda t}z_1+(e^{\lambda t}-1)c, z'+td', e^{t\Delta}z''+(e^{\Delta t}-I)d'', e^{\frac{\lambda}{2}t}e^{i\Theta t}z'''),$$

where $Z=(z_1,z',z'',z''')\in\mathbb{C}\times\mathbb{C}^p\times\mathbb{C}^q\times\mathbb{C}^r\cap\mathbb{H}^n$, $c\geq 0$, $d'\in\mathbb{C}^p$, $d''\in\mathbb{C}^q$, Δ is a diagonal invertible matrix of order q all of whose entries have real part strictly less than $\frac{\lambda}{2}$, Θ is a diagonal matrix of order r with real entries and $p+q+r=n-1$ $(p,q,r\geq 0)$.

PROOF. Up to conjugation we can assume that (φ_t) is given as in Theorem 5.1. Then the action of the differential of (φ_t) on the complex tangent space at the common Wolff point is represented by e^{tM}, and it is normal at $t>0$ if and only if M is normal. The result follows then from an application of the spectral theorem bearing in mind conditions (i) and (iii) appearing in Theorem 5.1. $\square$

6 The parabolic case

In this section we examine parabolic semigroups of linear fractional maps.

Theorem 6.1 *Let (φ_t) be a parabolic semigroup in $\mathrm{LFM}(\mathbb{B}^n,\mathbb{B}^n)$. Then, the iteration couple $(\mathbb{B}^n,(\varphi_t))$ is conjugated to the iteration couple $(\mathbb{H}^n,(\widetilde{\varphi_t}))$ where*

$$\widetilde{\varphi_t}(z,w)=(z+\langle w,b_t\rangle+c_t, e^{Mt}w+d_t),\quad (z,w)\in\mathbb{H}^n$$

and

$$c_t:=ct+\int_0^t\left\langle e^{Ms}d,b\right\rangle(t-s)ds,\ d_t:=\left(\int_0^t e^{Ms}ds\right)d,\ b_t:=\left(\int_0^t e^{M^*s}ds\right)b,$$

with $c\in\mathbb{C}$, $b,d\in\mathbb{C}^{n-1}$, $M\in\mathbb{C}^{(n-1)\times(n-1)}$ and such that

- *(i)* $Q_t:=I-e^{M^*t}e^{Mt}$ *is a positive semidefinite hermitian matrix, for every* $t\geq 0$,
- *(ii)* $\mathsf{Re}(c_t)-\|d_t\|^2\geq\left\langle Q_t^+(e^{M^*t}d_t-\frac{1}{2}b_t),(e^{M^*t}d_t-\frac{1}{2}b_t)\right\rangle$, *for every $t\geq 0$, where Q_t^+ is the pseudo-inverse of Q_t,*
- *(iii)* $e^{M^*t}d_t-\frac{1}{2}b_t$ *belongs to the space spanned by the columns of Q_t, for every* $t\geq 0$.

Moreover, given c,b,d and M as above, there exists a parabolic semigroup (φ_t) of linear fractional self-maps of $\mathbb{B}^n$ such that the iteration couple $(\mathbb{B}^n,(\varphi_t))$ is conjugated to $(\mathbb{H}^n,(\widetilde{\varphi_t}))$.

PROOF. According to Theorem 1.5, the boundary dilatation coefficient $\alpha(\varphi_t)$ at the common Denjoy-Wolff point of the semigroup is exactly 1. Then, by conjugating φ_t via the Cayley transform $\sigma_\tau : \mathbb{B}^n \to \mathbb{H}^n$ which maps τ to ∞ (see Lemma 4.1), the semigroup $(\mathbb{B}^n, (\varphi_t))$ is conjugated to the semigroup $(\mathbb{H}^n, (\varphi_t^1))$ with

$$\varphi_t^1(z,w) = (z + \langle w, b_t\rangle + c_t, A_t w + d_t), \quad (z,w) \in \mathbb{H}^n, \tag{12}$$

where $c_t \in \mathbb{C}$, $b_t, d_t \in \mathbb{C}^{n-1}$ and $A_t \in \mathbb{C}^{(n-1)\times(n-1)}$ satisfy (i), (ii) and (iii) in Lemma 4.1.

Applying the algebraic semigroup conditions, we come up with the following four equations for the above coefficients:

$$\begin{cases} 1)\ c_{t+s} = c_t + c_s + \langle d_s, b_t\rangle \\ 2)\ d_{t+s} = A_t d_s + d_t \\ 3)\ A_{t+s} = A_t A_s \\ 4)\ b_{t+s} = b_s + A_s^* b_t \end{cases} \qquad t, s \geq 0. \tag{13}$$

Moreover, we have that $c_0 = 0$, $b_0 = O$, $d_0 = O$, and $A_0 = I$.

As in the hyperbolic case, from equation 3) we deduce that there exists a matrix $M \in \mathbb{C}^{(n-1)\times(n-1)}$ such that $A_t = \exp(tM)$. We point out that M is not necessarily invertible now.

Next, arguing as in the hyperbolic case, we solve 2) and 4) to get

$$\begin{cases} b_t = \left(\int_0^t e^{M^* s} ds\right) b, \\ d_t = \left(\int_0^t e^{Ms} ds\right) d, \end{cases}$$

for some vectors $b, d \in \mathbb{C}^{n-1}$.

Finally, to compute c_t we differentiate with respect to t and setting $t = 0$, we obtain that

$$\begin{cases} \frac{d}{ds} c_s = v + \langle d_s, \frac{d}{dt} b_t|_{t=0}\rangle \\ c_0 = 0, \end{cases}$$

for some $v \in \mathbb{C}^{n-1}$. Therefore

$$\langle d_s, \frac{d}{dt} b_t|_{t=0}\rangle = \langle \left(\int_0^s e^{M^* t} dt\right) d, b\rangle.$$

Integrating with respect to s and applying Fubini's theorem, we obtain the wanted expression for c_t. The remaining assertions follow by Lemma 4.1. $\square$

Remark 6.2 *Condition* (i) *in the above theorem means exactly that the matrix* M *is dissipative. Indeed,* Q_t *is positive semidefinite for all* $t \geq 0$ *if and only if* $\left\| e^{Mt} \right\| \leq 1$, *for all* $t \geq 0$ *and the claim follows from Phillips-Lumer's theorem.*

In a similar way as in the hyperbolic case, Theorem 6.1 implies that the classification of parabolic semigroups in $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ can be seen as a part of the classification problem of parabolic groups of automorphisms of $\mathbb{C}^n$.

Once more, the existence of common invariant slices simplifies the model.

Example 6.3 *Let*

$$\varphi_t(z, w) = (z + 2tw + ct + t^2, w + t),$$

where $(z, w) \in \mathbb{H}^2$ *and* $\operatorname{Re} c = 0$. *Then,* (φ_t) *is a parabolic semigroup with no invariant slices. In fact, it is possible to show that each parabolic semigroup of* $\mathrm{LFM}(\mathbb{H}^2, \mathbb{H}^2)$ *with no invariant slices can be conjugated to a semigroup as above for some* $c \in \mathbb{C}$ *with* $\operatorname{Re} c \geq 0$. *Moreover, one—and hence any—of the iterates* φ_t $(t > 0)$ *is an automorphism if and only if* $\operatorname{Re} c = 0$.

Remark 6.4 *Let* $(\mathbb{B}^n, (\varphi_t))$ *be a parabolic semigroup of* $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ *with common Denjoy-Wolff point* $\tau \in \partial\mathbb{B}^n$ *and let* $(\mathbb{H}^n, (\tilde{\varphi}_t))$ *be the conjugated semigroup of* $\mathrm{LFM}(\mathbb{H}^n, \mathbb{H}^n)$ *given by Theorem 6.1. Following the lines of Remark 5.6, we find that* $(\mathbb{B}^n, (\varphi_t))$ *has a unique invariant slice if and only if* M *is invertible.*

Arguing as in Theorem 5.7 we obtain:

Theorem 6.5 *Let* (φ_t) *be a parabolic semigroup in* $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ *with, at least, one common invariant slice. Then, the iteration couple* $(\mathbb{B}^n, (\varphi_t))$ *is conjugated to the iteration couple* $(\mathbb{H}^n, (\widetilde{\varphi_t}))$ *where*

$$\widetilde{\varphi_t}(z, w) = (z + \langle \left(\int_0^t e^{Ms} ds \right) w, b\rangle + ct, e^{Mt} w) \quad (z, w) \in \mathbb{H}^n$$

with $c \in \mathbb{C}$, $b \in \mathbb{C}^{n-1}$ *and* M *is a dissipative matrix of order* $(n-1)$.

It is worth pointing out that the matrix M in the above theorem might be non-invertible, so that in general it is not possible to remove the integral symbol. Similar to the hyperbolic case if the semigroup has $p > 1$ common independent invariant slices, it is possible to simplify a little more the model, following the ideas given in Remark 5.8. We leave details for the general case to the interested reader and concentrate on the case of a unique common invariant slice, where the situation resembles the hyperbolic case:

Proposition 6.6 *Let* (φ_t) *be a parabolic semigroup in* $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ *with a* *unique common invariant slice. Then, the iteration couple* $(\mathbb{B}^n, (\varphi_t))$ *is conjugated to the iteration couple* $(\mathbb{H}^n, (\widehat{\varphi}_t))$ *where*

$$\widehat{\varphi}_t(z,w) = (z + ct, e^{Mt}w + (e^{Mt} - I)d), \quad (z,w) \in \mathbb{H}^n$$

with $c \in \mathbb{C}$, $d \in \mathbb{C}^{n-1}$ *and* M *is a dissipative matrix of order* $n-1$.

PROOF. First we apply Theorem 6.1 in order to conjugate the semigroup (φ_t) to the semigroup

$$\widehat{\varphi_t}(z,w) = (z + \langle w, b_t \rangle + c_t, e^{Mt}w + d_t)$$

with b_t d_t, c_t and M satisfying the corresponding restrictions. In particular, we have that

$$b_t := \left(\int_0^t e^{M^* s} ds \right) b,$$

for some $b \in \mathbb{C}^{n-1}$.

Arguing as in the proof of Theorem 5.1 we come up with equations similar to (11), namely

$$(e^{M^* t} - I)k_2 = \frac{1}{2} \left(\int_0^t e^{M^* s} ds \right) b.$$

Since M is invertible by Remark 6.4, we can solve these equations setting $k_2 := \frac{1}{2}(M^*)^{-1}b$.

Therefore, if we consider the Heisenberg translation

$$\eta(z,w) = (z + 2\langle w, k_2 \rangle + k_1, w + k_2), \quad (z,w) \in \mathbb{H}^n$$

with $(k_1, k_2) \in \partial\mathbb{H}^n$ and $k_2 := \frac{1}{2}(M^*)^{-1}b$ and conjugate the semigroup $(\widehat{\varphi_t})$ with η, then the new semigroup is given by

$$\varphi_t^2 = (z + c_t^2, e^{Mt}w + d_t^2), \quad (z,w) \in \mathbb{H}^n,$$

for some c_t^2, d_t^2 satisfying (13). The remaining assertions follow from Theorem 6.1. $\square$

We end up this section with a classification of parabolic groups of automorphisms of $\mathbb{B}^n$ which naturally follows from our procedure (see also [5] and [14]):

Theorem 6.7 *Let* (φ_t) *be a parabolic group in* $\mathsf{Aut}(\mathbb{B}^n)$.

(i) If the group has no invariant slice, then it can be conjugated to an iteration couple $(\mathbb{H}^n, (\widetilde{\varphi_t}))$ *where*

$$\widetilde{\varphi_t}(z,w) = (z + 2t\langle w, d \rangle + ct + t^2, e^{it\Theta}w + td), \quad (z,w) \in \mathbb{H}^n,$$

with $\mathsf{Re}\, c = 0$*,* d *is a vector of* $\mathbb{C}^{n-1}$ *of norm one and* Θ *is a diagonal matrix of order* $n-1$ *with real entries.*

(ii) If the group has $p+1$ *common independent invariant slices* ($p \geq 0$)*, then it can be conjugated to an iteration couple* $(\mathbb{H}^n, (\widetilde{\varphi_t}))$ *where*

$$\widetilde{\varphi_t}(z, w', w'') = (z + ct, w', e^{it\Theta} w''), \quad (z, w', w'') \in \mathbb{C} \times \mathbb{C}^p \times \mathbb{C}^{n-1-p} \cap \mathbb{H}^n,$$

with $\mathsf{Re}\, c = 0$ *and* Θ *is a diagonal matrix of order* $n-1-p$ *with real entries.*

PROOF. According to Theorem 6.1, we see that the iteration couple $(\mathbb{B}^n, (\varphi_t))$ is conjugated to the iteration couple $(\mathbb{H}^n, (\varphi_t^1))$ where

$$\varphi_t^1(z, w) = (z + \langle w, b_t \rangle + c_t, e^{Mt} + d_t) \quad (z, w) \in \mathbb{H}^n,$$

for some c_t, $b_t = (\int_0^t e^{M^* s} ds) b$, $d_t = (\int_0^t e^{Ms} ds) d$ ($b, d \in \mathbb{C}^{n-1}$) and M satisfying the restrictions mentioned in that theorem. Since each iterate of the semigroup is an automorphism then $\varphi_t(\partial \mathbb{B}^n) = \partial \mathbb{B}^n$ and therefore for every $t \geq 0$:

$$\begin{cases} 1) \ e^{Mt} \text{ is unitary;} \\ 2) \ \mathsf{Re}\,(c_t) = \|d_t\|^2; \\ 3) \ b_t = 2 e^{M^* t} d_t. \end{cases}$$

By condition 1) and Stone's theorem, we see that $e^{Mt} = e^{itH}$ for some Hermitian matrix H of order $n-1$. This, together with condition 3), implies that, for all t,

$$e^{iHt} \left(\int_0^t e^{-iHs} ds \right) b = \left(\int_0^t e^{iHs} ds \right) b = 2 \left(\int_0^t e^{iHs} ds \right) d,$$

and, therefore, $b = 2d$. Moreover, by the spectral theorem there exists a unitary matrix V of order $n-1$ such that $V^* H V = \Theta$, with Θ a real diagonal matrix of order $n-1$. Without lose of generality, we may assume that

$$\Theta = \begin{bmatrix} O & O \\ O & \Lambda \end{bmatrix}$$

with Λ a diagonal matrix of order $n-1-q$ with non-zero elements in the diagonal ($0 \leq q \leq n-1$). Now, conjugating $(\mathbb{H}^n, (\varphi_t^1))$ by $(z, w) \mapsto (z, Vw)$ we obtain a conjugated iteration couple $(\mathbb{H}^n, (\varphi_t^2))$ given by

$$\varphi_t^2(z, w) = (z + 2 \langle (\int_0^t e^{i\Theta s} ds) w, d \rangle + c_t, e^{i\Theta t} w + (\int_0^t e^{i\Theta s} ds) d), \quad (z, w) \in \mathbb{H}^n,$$

for some c_t and d (maybe different from above). Note that

$$\int_0^t e^{i\Theta s} ds = \begin{bmatrix} tI_q & O \\ O & (e^{i\Lambda t} - I)(-i\Lambda^{-1}) \end{bmatrix}.$$

Hence

$$\varphi_t^2(z,w) = (z+2t\langle w', d'\rangle + 2\langle (e^{i\Lambda t} - I)w'', -d''\rangle + c_t, w' + td', e^{i\Lambda t}w'' + (e^{i\Lambda t} - I)d'')$$

with $(z,w) = (z,w',w'') \in \mathbb{C} \times \mathbb{C}^q \times \mathbb{C}^{n-1-q} \cap \mathbb{H}^n$ and $(d',d'') \in \mathbb{C}^q \times \mathbb{C}^{n-1-q}$.

Conjugating now with the Heisenberg transformation

$$\eta_2(z,w',w'') = (z + 2\langle w'', k_2\rangle + k_1, w', w'' + k_2)$$

where $(z,w',w'') \in \mathbb{C}\times\mathbb{C}^q\times\mathbb{C}^{n-1-q}\cap\mathbb{H}^n$ with $(k_1, O, k_2) \in \partial\mathbb{H}^n$ and $k_2 := -d''$, $k_1 = \|k_2\|^2$, we obtain a new iteration couple $(\mathbb{H}^n, (\varphi_t^3))$ where

$$\varphi_t^3(z,w',w'') = (z + 2t\langle w', d'\rangle + c_t, w' + td', e^{i\Lambda t}w'')$$

for some c_t (again, maybe different from above) satisfying equations (13). Thus, arguing as in Theorem 6.1 we obtain that

$$c_t = ct + \|d'\|^2 t^2.$$

for some $c \in \mathbb{C}$ with $\mathsf{Re}\, c = 0$. By Proposition 4.4 the semigroup has no common invariant slices if and only if $d' \neq O$. If $d' = 0$, again by Proposition 4.4, it follows that $p \leq q$ and we are done.

If $d' \neq O$, we conjugate once more with

$$\eta_3(z,w) = (\|d'\|^2 z, \|d'\| w) \quad (z,w) \in \mathbb{H}^n.$$

The new iterates are given by

$$\varphi_t^4(z,w',w'') = (z + 2t\langle w', d'\rangle + ct + t^2, w' + td', e^{i\Lambda t}w'')$$

where $c \in \mathbb{C}$ with $\mathsf{Re}\, c = 0$ and d' has norm one, as wanted. $\square$

A Appendix

The aim of this appendix is to give a short proof of Theorem 1.5. Actually we will prove a more general (and partially new) result for semigroups of holomorphic self-maps in strongly convex domains.

Let $D \subset \mathbb{C}^n$ be a strongly convex domain with C^3 boundary. Let $f \in \mathrm{Hol}(D, D)$ be a holomorphic self-map of D and denote

$$\mathsf{Fix}(f) := \{z \in D : f(z) = z\}.$$

Recall (see, *e.g.*, [1]) that either $\mathsf{Fix}(f) \neq \emptyset$ or there exists a unique point $\tau(f) \in \partial D$ such that the sequence of iterates $\{f^k\}$ converges uniformly on compacta to the constant function $D \ni z \mapsto \tau(f)$. Such a point $\tau(f)$ is called the *Denjoy-Wolff point* of f.

Moreover, fix $z_0 \in D$, assume $f \in \mathrm{Hol}(D, D)$ has no fixed points in D and let $\tau(f)$ be its Wolff point. If k_D is the Kobayashi distance in D (see, *e.g.*, [16] for its definition and properties), we define the *boundary dilatation coefficient* $\alpha(f)$ of f at $\tau(f)$ to be

$$\frac{1}{2}\log\alpha(f) := \liminf_{w\to\tau(f)}[k_D(z_0, w) - k_D(z_0, f(w))].$$

Note that if $D = \mathbb{D}$ the unit disc in $\mathbb{C}$, then by the Julia-Wolff-Carathéodory theorem $\alpha(f) = f'(\tau(f))$, the multiplier of f at $\tau(f)$. Likewise, if $D = \mathbb{B}^n$ the number $\alpha(f)$ coincides with the usual boundary dilatation coefficient (for instance, when $f \in \mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$, the number $\alpha(f)$ coincides with the number bearing the same name in Theorem 1.2).

Theorem A.1 *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with C^3 boundary. Let (F_t) be a continuous one-parameter semigroup of holomorphic self-maps of D. Then*

- *either $\bigcap_{t\geq 0}\mathsf{Fix}(F_t) \neq \emptyset$,*
- *or $\mathsf{Fix}(F_t) = \emptyset$ for all $t > 0$, there exists a unique $\tau \in \partial D$ such that τ is the Denjoy-Wolff point of F_t for all $t > 0$ and there exists $0 < r \leq 1$ such that $\alpha(F_t) = r^t$.*

This theorem has been proved for the unit ball by Aizenberg and Shoikhet in [2]. In [1], Abate gave a proof of this result without dealing with the boundary dilatation coefficient $\alpha(F_t)$.

It is worth noticing that the corresponding statement for a (discrete) family of commuting mappings is false (see [6]).

In order to prove Theorem A.1, we need the following lemma:

Lemma A.2 *Let $\mathcal{A}$ be an indices set and $\varphi_{\nu\in\mathcal{A}} \in Hol(D, D)$ a family of commuting holomorphic self-maps of D. If $\bigcap_{\nu\in\mathcal{A}}\mathsf{Fix}(\varphi_\nu) = \emptyset$ then there exist $m \in \mathbb{N}$ and $s_1, \ldots, s_m \in \mathcal{A}$ such that $\bigcap_{j=1}^m \mathsf{Fix}(\varphi_{s_j}) = \emptyset$.*

PROOF. Recall that by [25] the set $\mathsf{Fix}(\varphi_\nu)$ is a holomorphic retract of D and in particular it is an open connected submanifold of D. Let $d_\nu := \dim \mathsf{Fix}(\varphi_\nu)$ (here we agree to set $d_\nu = -1$ if $\mathsf{Fix}(\varphi_\nu) = \emptyset$). We set $d_0 = \min d_\nu$. If $d_0 < 0$ then there exists ν_0 such that $\mathsf{Fix}(\varphi_{\nu_0}) = \emptyset$ and the result is proved. Assume that $d_0 \geq 0$. Actually $d_0 > 0$ because if $d_0 = 0$ then $\mathsf{Fix}(\varphi_{\nu_0})$ is a single point and since it is clearly invariant for all φ_ν (recall the family commutes) then it follows that $\mathsf{Fix}(\varphi_{\nu_0})$ is fixed for all φ_ν against our hypothesis.

Thus $d_0 > 0$. Let φ_{ν_0} be such that $d_{\nu_0} = d_0$. Now consider the sets $A^1_\nu := \mathsf{Fix}(\varphi_\nu) \cap \mathsf{Fix}(\varphi_{\nu_0})$ varying $\nu \in \mathcal{A}$. Every A^1_ν is an open connected submanifold of D since $A^1_\nu := \pi_\nu \circ \pi_{\nu_0}(D)$ where $\pi_j : D \to \mathsf{Fix}(\varphi_j)$ is the holomorphic retraction. Let $d_1 := \min \dim A^1_\nu$. Then $d_1 < d_0$. Indeed if $d_1 = d_0$ then $\mathsf{Fix}(\varphi_{\nu_0})$ would be contained in $\mathsf{Fix}(\varphi_\nu)$ for all $\nu \in \mathcal{A}$, against the hypothesis. If $d_1 < 0$ we are done. Otherwise it is easy to see that $d_1 > 0$. Let $A^1_{\nu_1}$ be such that $d_{\nu_1} = d_1$. This set is invariant for all φ_ν. Define $A^2_\nu := A^1_{\nu_1} \cap \mathsf{Fix}(\varphi_\nu)$. Again A^2_ν is an open connected submanifold of D. Let $d_2 := \min \dim A^2_\nu$. Arguing as before one finds that $d_2 < d_1$. Continuing in this way we can find a strictly decreasing sequence and thus after (at most) $n-1$ steps we are done. $\square$

PROOF OF THEOREM A.1 Assume that $\mathsf{Fix}(F_{t_0}) \neq \emptyset$ for some $t_0 > 0$. Let $\mathcal{C} := \{t \in (0,\infty) : \mathsf{Fix}(F_t) \neq \emptyset\}$. Let

$$\mathcal{D} := \bigcap_{t \in \mathcal{C}} \mathsf{Fix}(F_t).$$

If $\mathcal{D} = \emptyset$, by Lemma A.2 we can find $s_0, \ldots, s_m \in \mathcal{C}$ such that $\bigcap_{j=1}^m \mathsf{Fix}(F_{s_j}) = \emptyset$. Without loss of generality we can suppose that $M := \bigcap_{j=1}^{m-1} \mathsf{Fix}(F_{s_j}) \neq \emptyset$. By [25] there exists a holomorphic retraction $\pi_M : D \to M$ such that $M = \pi_M(D)$ (the holomorphic retraction π_M is the composition of the holomorphic retractions of $\mathsf{Fix}(F_{s_j})$, $j = 1, \ldots, m-1$). Now we can consider $f := F_{s_m} \circ \pi_M$. We have $f(D) = F_{s_m}(\pi_M(D)) = F_{s_m}(M) \subset M$ and $f^k = F^k_{s_m} \circ \pi_M$. But then $\mathsf{Fix}(f) = \emptyset$ and by Abate's theory [1] $f^k(z) \to \partial D$ for $k \to \infty$ and $z \in D$. This contradicts the fact that $\{F^k_{s_m}(z)\}$ stays bounded in D for all $z \in D$ since $\mathsf{Fix}(F_{s_m}) \neq \emptyset$. Therefore $\mathcal{D} \neq \emptyset$ and it is clearly an open connected submanifold of D, since it is actually given as the intersection of finitely many holomorphic retracts of D and therefore a holomorphic retract of D, $\mathcal{D} = \pi_{\mathcal{D}}(D)$.

We want to show that $\mathcal{C} = (0,\infty)$. Assume that this is not the case. It is easy to see that $\mathcal{D}$ is invariant for F_t for all t. Thus we can consider the continuous one parameter semigroup ϕ_t of holomorphic self-maps of $\mathcal{D}$ defined by

$$\phi_t := F_t|_{\mathcal{D}}.$$

Notice that $\phi_t(z) \equiv z$ for all $t \in \mathcal{C}$. Let $t_0 > 0$ be such $t_0 \in \mathcal{C}$. Therefore for all $t \geq 0$

$$\phi_{t+t_0} = \phi_t \circ \phi_{t_0} = \phi_t.$$

In particular $\phi_t^k = \phi_{tk} = \phi_{tk \mod (0,t_0)}$. Assume $t \notin \mathcal{C} \cup \{0\}$. Then $F_t^k(z) = \phi^k(z) \to \partial D$ for $k \to \infty$ and $z \in D$. Let k_ν be a subsequence such that $tk_\nu \to t_1 \mod [0,t_0]$. Then $\phi_t^{k_\nu}(z) \to \phi_{t_1}(z) \notin \partial D$, against $\phi_t^{k_\nu}(z) \to \partial D$. Thus $\mathcal{C} = (0,\infty)$ and we are done.

Assume now that $\mathsf{Fix}(F_{t_0}) = \emptyset$ for some $t_0 > 0$. Let $\tau := \tau(F_{t_0}) \in \partial D$ be the Wolff point of F_{t_0}. Clearly $\tau(F_{nt_0}) = \tau(F_{t_0}^n) = \tau$ for all $n \in \mathbb{N}$ and thus $\tau(F_{qt_0}) = \tau$ for all $q \in \mathbb{Q}^+$. Since we already proved that $\mathsf{Fix}(F_t) = \emptyset$ for all $t > 0$, by Joseph-Kwack Theorem (see [17] and also [7, Theorem 3.10.(2)]) it follows that $\tau(F_t) = \tau$ for all $t > 0$.

Now we are left to show that if $\mathsf{Fix}(F_t) = \emptyset$ for all $t > 0$ then there exists $0 < r \leq 1$ such that $\alpha(F_t) = r^t$. Let $\alpha(t) := \alpha(F_t)$. If we prove that

(1) $\alpha : [0,\infty) \to (0,1]$ is measurable
(2) $\alpha(0) = 1$,
(3) $\alpha(t+s) = \alpha(t)\alpha(s)$ for all $t, s \geq 0$,

then the result will follow from standard arguments. The first property follows from the fact that $\alpha : \mathrm{Hol}(D,D) \to (0,1]$ is lower semicontinuous (where $\mathrm{Hol}(D,D)$ is endowed with the compact-open topology), see [7]. The second property is obvious. As for the third one, one needs to use a Julia-Wolff-Carathéodory-type theorem for strongly convex domains, due to Abate [1]. For the reader's convenience, we recall here how it works.

Let $z_0 \in D$. By Lempert's work (see [18] and [1]) given any point $z \in \overline{D}$ there exists a unique complex geodesic $\varphi : \mathbb{D} \to D$, *i.e.*, a holomorphic isometry between $k_{\mathbb{D}}$ and k_D, such that φ extends smoothly past the boundary, $\varphi(0) = z_0$ and $\varphi(t) = z$, with $t \in (0,1)$ if $z \in D$ and $t = 1$ if $z \in \partial D$. Moreover for any such a complex geodesic there exists a holomorphic retraction $p : D \to \varphi(\mathbb{D})$, *i.e.* p is a holomorphic self-map of D such that $p \circ p = p$ and $p(z) = z$ for any $z \in \varphi(\mathbb{D})$. We call such a p the *Lempert projection* associated to φ. Furthermore we let $\tilde{p} := \varphi^{-1} \circ p$ and call it the *left inverse* of φ, for $\tilde{p} \circ \varphi = Id_{\mathbb{D}}$. The triple $(\varphi, p, \tilde{p})$ is the so-called *Lempert projection device.*

Let $(\varphi, p, \tilde{p})$ be the Lempert projection device associated to the complex geodesic such that $\varphi(1) = \tau$. Consider the following function $T : D \to \mathbb{C}$,

$$T_t(z) := \frac{1 - \tilde{p} \circ F_t(z)}{1 - \tilde{p}(z)}.$$

By Abate's theorem (see *Theorem 2.7.14 in [1]*) it follows that if $\gamma : [0,1) \to D$ is a continuous curve such that $\lim_{u \to 1} \gamma(u) = \tau$, $\lim_{u\to 1} k_D(\gamma(u), p(\gamma(u))) = 0$, and $p(\gamma(u))$ tends to τ non-tangentially (a curve with such properties is said to be τ-*special and restricted*), then

$$\lim_{u\to 1} T_t(\gamma(u)) = \alpha(t).$$

By *Proposition 3.4 in [6]* it follows that $[0,1) \ni u \mapsto F_t(\varphi(u))$ is τ-special and restricted. Then we have

$$T_{t+s}(\varphi(u)) = \frac{1-\tilde{p}\circ F_t(F_s(\varphi(u)))}{1-\tilde{p}(F_s(\varphi(u)))}\cdot\frac{1-\tilde{p}\circ F_s(\varphi(u))}{1-\tilde{p}(\varphi(u))},$$

and taking the limit as $u \to 1$ it follows that $\alpha(t+s) = \alpha(t)\alpha(s)$ concluding the proof. $\square$

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