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BRAIDINGS OF POISSON GROUPS WITH QUASITRIANGULAR DUAL

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Abstract. Let $\mathfrak{g}$ be a quasitriangular Lie bialgebra over a field k of characteristic zero, and let $\mathfrak{g}^*$ be its dual Lie bialgebra. We prove that the formal Poisson group $F[[\mathfrak{g}^*]]$ is a braided Hopf algebra. More generally, we prove that if $\big(U_h, R\big)$ is any quasitriangular QUEA, then $\Big({U_h}', Ad(R)\big|_{{U_h}'\otimes{U_h}'}\Big)$ — where ${U_h}'$ is defined by Drinfeld — is a braided QFSHA. The first result is then just a consequence of the existence of a quasitriangular quantization (U_h, R) of $U(\mathfrak{g})$ and of the fact that ${U_h}'$ is a quantization of $F[[\mathfrak{g}^*]]$.

Introduction

Let $\mathfrak{g}$ be a Lie Lie bialgebra over a field k of characteristic zero; let $\mathfrak{g}^*$ be the dual Lie bialgebra of $\mathfrak{g}$; finally denote $F[[\mathfrak{g}^*]]$ the algebra of functions on the formal Poisson group associated to $\mathfrak{g}^*$. If $\mathfrak{g}$ is quasitriangular, endowed with the r–matrix r, this gives $\mathfrak{g}$ some additional properties. A question then rises: what new structure one obtains on the dual bialgebra $\mathfrak{g}^*$? In this work we shall show that the topological Poisson Hopf algebra $F[[\mathfrak{g}^*]]$ is a braided Poisson algebra (we'll give the definition later on). This was already proved for $\mathfrak{g} = \mathfrak{sl}(2,k)$ by Reshetikhin (cf. [Re]), and generalised to the case where $\mathfrak{g}$ is Kac-Moody of finite (cf. [G1]) or affine (cf. [G2]) type by the first author.

In order to prove the result, we shall use quantization of universal enveloping algebras. After Etingof-Kazhdan (cf. [EK]), each Lie bialgebra admits a quantization $U_h(\mathfrak{g})$, namely a topological Hopf algebra over $k[[h]]$ whose specialisation at $h=0$ is isomorphic to $U(\mathfrak{g})$ as a co-Poisson Hopf algebra; in addition, if $\mathfrak{g}$ is quasitriangular and r is its r–matrix, then such a $U_h(\mathfrak{g})$ exists which is quasitriangular too, as a Hopf algebra, with au R–matrix $R_h \,(\in U_h(\mathfrak{g})\otimes U_h(\mathfrak{g})\,)$ such that $R_h \equiv 1 + r\,h \mod h^2$ (where we have identified, as vector spaces, $U_h(\mathfrak{g}) \cong U(\mathfrak{g})[[h]]\,$).

Now, after Drinfel'd (*cf.* [Dr]), for any quantised universal enveloping algebra U one can define also a certain Hopf subalgebra U' such that, if the semiclassical limit of U is

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$U(\mathfrak{g})$ (with $\mathfrak{g}$ a Lie bialgebra), then the semiclassical limit of U' is $F[[\mathfrak{g}^*]]$. In our case, when considering $U_h(\mathfrak{g})'$ one can observe that the R–matrix does not belong, *a priori*, to $U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'$; nevertheless, we prove that its adjoint action $\mathfrak{R}_h := \mathrm{Ad}(R_h) : U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) \longrightarrow U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$, $x \otimes y \mapsto R_h \cdot (x \otimes y) \cdot R_h^{-1}$, stabilises $U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'$, hence it induces by specialisation an operator $\mathfrak{R}_0$ on $F[[\mathfrak{g}^*]] \otimes F[[\mathfrak{g}^*]]$. Finally, the properties which make R_h an R–matrix imply that $\mathfrak{R}_h$ is a braiding operator, whence the same holds for $\mathfrak{R}_0$: thus the pair $\big(F[[\mathfrak{g}^*]], \mathfrak{R}_0\big)$ is braided Poisson algebra.

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§ 1. Recallings and definitions

1.1 The classical objects. Let k be a fixed field of characteristic zero. In the following k will be the ground field of all the objects — Lie algebras and bialgebras, Hopf algebras, etc. — which we'll introduce.

Following [CP], §1.3, we call Lie bialgebra a pair $(\mathfrak{g}, \delta_\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra and $\delta_\mathfrak{g} \colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is a antisymmetric linear map — called Lie cobracket — such that its dual $\delta_\mathfrak{g}^* \colon \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ be Lie bracket and that δ_g itself be a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. Then it happens that also $\mathfrak{g}^*$, the linear dual of $\mathfrak{g}$, is a Lie bialgebra on its own. Following [CP], §2.1.B, we call quasitriangular Lie bialgebra a pair $(\mathfrak{g}, r)$ such that $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a solution of the classical Yang-Baxter equation (CYBE) $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ and $\mathfrak{g}$ be a Lie bialgebra with respect to the cobracket $\delta = \delta_\mathfrak{g}$ defined by $\delta(x) = [x, r]$; the element r is then called r–matrix of $\mathfrak{g}$.

If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra; if, in addition, $\mathfrak{g}$ is a Lie bialgebra, then $U(\mathfrak{g})$ is in fact a co-Poisson Hopf algebra (cf. [CP], §6.2.A).

Let $\mathfrak{g}$ be any Lie algebra: then we call function algebra on the formal group associated to $\mathfrak{g}$, or simply formal group associated to $\mathfrak{g}$, the space $F[[\mathfrak{g}]] := U(\mathfrak{g})^*$ linear dual of $U(\mathfrak{g})$. As $U(\mathfrak{g})$ is a Hopf algebra, its dual $F[[\mathfrak{g}]]$ is on its own a formal Hopf algebra (following [Di], Ch. 1). Note that, if G is a connected algebraic group whose tangent Lie algebra is $\mathfrak{g}$, letting $F[G]$ be the Hopf algebra of regular functions on G and letting $\mathfrak{m}_e$ be the maximal ideal of $F[G]$ of functions vanishing at the unit point $e \in G$, the formal Hopf algebra $F[[\mathfrak{g}]]$ is nothing but the $\mathfrak{m}_e$–adic completion of $F[G]$ (cf. [On], Ch. I). When, in addition, $\mathfrak{g}$ is a Lie bialgebra, $F[[\mathfrak{g}]]$ is in fact a formal Poisson Hopf algebra (cf. [CP], §6.2.A).

1.2 Braidings and quasitriangularity. Let H be a Hopf algebra in a tensor category $(\mathcal{A}, \otimes)$ (cf. [CP], §5): H is called braided (cf. [Re], Définition 2) if there exists an algebra automorphism $\mathfrak{R}$ of $H \otimes H$, called braiding operator of H, different from the flip $\sigma \colon H^{\otimes 2} \to H^{\otimes 2}$, $a \otimes b \mapsto b \otimes a$, and such that

$$\mathfrak{R} \circ \Delta = \Delta^{\mathrm{op}}$$

$$(\Delta \otimes id) \circ \mathfrak{R} = \mathfrak{R}_{13} \circ \mathfrak{R}_{23} \circ (\Delta \otimes id)\,, \qquad (id \otimes \Delta) \circ \mathfrak{R} = \mathfrak{R}_{13} \circ \mathfrak{R}_{12} \circ (id \otimes \Delta)$$

where Δ^{op} is the opposite comultiplication, i. e. $\Delta^{\mathrm{op}}(a) = \sigma \circ \Delta(a)$, and $\mathfrak{R}_{12}$, $\mathfrak{R}_{13}$, and $\mathfrak{R}_{23}$ are the automorphisms of $H \otimes H \otimes H$ defined by $\mathfrak{R}_{12} = \mathfrak{R} \otimes id$, $\mathfrak{R}_{23} = id \otimes \mathfrak{R}$, $\mathfrak{R}_{13} = (\sigma \otimes id) \circ (id \otimes \mathfrak{R}) \circ (\sigma \otimes id)$.

Finally, when H is, in addition, a Poisson Hopf algebra, we'll say that it is braided — as a Poisson Hopf algebra — if it is braided — as a Hopf algebra — by a braiding which is also an automorphism of Poisson algebra.

If the pair $(H, \mathfrak{R})$ is a braided algebra, it follows from the definition that $\mathfrak{R}$ satisfies the quantum Yang-Baxter equation — QYBE in the sequel — in $End(H^{\otimes 3})$, that is

$$\mathfrak{R}_{12} \circ \mathfrak{R}_{13} \circ \mathfrak{R}_{23} = \mathfrak{R}_{23} \circ \mathfrak{R}_{13} \circ \mathfrak{R}_{12}$$

which implies that, for all $n \in \mathbb{N}$ the braid group $\mathcal{B}_n$ acts on $H^{\otimes n}$, from which one can also obtain some knot invariants, according to the recipe given in [CP], §15.12.

A Hopf algebra H (in a tensor category) is said to be quasitriangular (cf. [Dr], [CP]) if there exists an invertible element $R \in H \otimes H$, called the R–matrix of H, such that

$$R \cdot \Delta(a) \cdot R^{-1} = \mathrm{Ad}(R)(\Delta(a)) = \Delta^{\mathrm{op}}(a)$$
$$(\Delta \otimes id)(R) = R_{13} R_{23} \; , \qquad (id \otimes \Delta)(R) = R_{13} R_{12}$$

where $R_{12}, R_{13}, R_{23} \in H^{\otimes 3}$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (\sigma \otimes id)(R_{23}) = (id \otimes \sigma)(R_{12})$. Then it follows from the identities above that R satisfies the QYBE in $H^{\otimes 3}$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \; .$$

Thus, the tensor products of H–modules are endowed with an action of the braid group. Moreover, it is clear that if (H, R) is quasitriangular, then $\big(H, \mathrm{Ad}(R)\big)$ is braided.

1.3 The quantum objects. Let $\mathcal{A}$ be the category whose objets are the $k[[h]]$–modules which are topologically frees and complete in h–adic sense, and the morphisms are the $k[[h]]$–linear continuous maps. For all V, W in $\mathcal{A}$, we define $V \otimes W$ to be the projective limit of the $k[[h]]/(h^n)$–modules $\big(V/h^n V\big) \otimes_{k[[h]]/(h^n)} \big(W/h^n W\big)$: this makes $\mathcal{A}$ into a tensor category (see [CP] for further details). After Drinfel'd (cf. [Dr]), we call quantised universal enveloping algebra — QUEA in the sequel — any Hopf algebra in the category $\mathcal{A}$ whose semiclassical limit (= specialisation at $h = 0$) is the universal enveloping algebra of a Lie bialgebra. Similarly, we call quantised formal series Hopf algebra — QFSHA in the sequel — any Hopf algebra in the category $\mathcal{A}$ whose semiclassical limit is the function algebra of a formal group.

In the sequel, we shall need the following result:

Theorem 1.4. *(cf. [EK]) Let $\mathfrak{g}$ be a Lie bialgebra. Then there exists a QUEA $U_h(\mathfrak{g})$ whose semiclassical limit is isomorphic to $U(\mathfrak{g})$; furthermore, there exists an isomorphism of $k[[h]]$–modules $U_h(\mathfrak{g}) \cong U(\mathfrak{g})[[h]]$.*

In addition, if $\mathfrak{g}$ is quasitriangular, with r–matrix r, then there exists a QUEA $U_h(\mathfrak{g})$ as above and an element $R_h \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ such that $\big(U_h(\mathfrak{g}), R_h\big)$ be a quasitriangular Hopf algebra and $R_h = 1 + r\,h + O\left(h^2\right)$ (with $O\left(h^2\right) \in h^2 \cdot H \otimes H$). □

1.5 The Drinfeld's functor. Let H be a Hopf algebra over $k[[h]]$. For all $n \in \mathbb{N}$, define $\Delta^n \colon H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := id_H$, and $\Delta^n := \big(\Delta \otimes id_H^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ if $n > 2$. For all ordered subset $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, define the homomorphism $j_\Sigma \colon H^{\otimes k} \longrightarrow H^{\otimes n}$ by $j_\Sigma(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_{i_m} := a_m$ for $1 \leq m \leq k$; then set $\Delta_\Sigma := j_\Sigma \circ \Delta^k$. Finally, define

$\delta_n \colon H \longrightarrow H^{\otimes n}$ by $\delta_n := \sum_{\Sigma \subseteq \{1,\dots,n\}} (-1)^{n-|\Sigma|} \Delta_\Sigma$, for all $n \in \mathbb{N}_+$. More in general, for all $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, with $i_1 < \dots < i_k$, define

$$\delta_\Sigma := \sum_{\Sigma' \subseteq \Sigma} (-1)^{|\Sigma|-|\Sigma'|} \Delta_{\Sigma'} \; ; \tag{1.1}$$

(in particular, $\delta_{\{1,\dots,n\}} = \delta_n$). Thanks to the inclusion-exclusion principle, this is equivalent to

$$\Delta_\Sigma = \sum_{\Sigma' \subseteq \Sigma} \delta_{\Sigma'} \tag{1.2}$$

for all $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \dots < i_k$. Finally, define

$$H' := \big\{\, a \in H \,\big|\, \delta_n(a) \in h^n H^{\otimes n} \,\big\} ,$$

a subspace of H which we consider endowed with the induced topology. Then we have

Theorem 1.6. *(cf. [Dr], §7, ou [G3]) Let H be a Hopf algebra in the category $\mathcal{A}$. Then H' is a QFSHA. Moreover, if $H = U_h(\mathfrak{g})$ is a QUEA, with $U(\mathfrak{g})$ as semiclassical limit, then the semiclassical limit of $U_h(\mathfrak{g})'$ is $F[[\mathfrak{g}^*]]$.* □

§ 2. The main results

From the technical point of view, the main result of this paper concerns the general framework of quasitriangular Hopf algebras:

Theorem 2.1. *Let H be a quasitriangular Hopf algebra in the category $\mathcal{A}$, and let R be its R–matrix. Then, the inner automorphism $\mathrm{Ad}(R) \colon H \otimes H \to H \otimes H$ restricts to an automorphism of $H' \otimes H'$, and the pair $\Big(H', \, \mathrm{Ad}(R)\big|_{H' \otimes H'} \Big)$ is a braided Hopf algebra in the category $\mathcal{A}$.* □

The proof of this theorem will be given in section 3. Nevertheless, we can already get out of it as a consequence the main result announced by the title and in the introduction, which gives us a geometrical interpretation of the classical r–matrix:

Theorem 2.2. *Let $\mathfrak{g}$ be a quasitriangular Lie bialgebra. Then the topological Poisson Hopf algebra $F[[\mathfrak{g}^*]]$ is braided. Moreover, there exists a quantisation of $F[[\mathfrak{g}^*]]$ which is a braided Hopf algebra whose braiding operator specialises into that of $F[[\mathfrak{g}^*]]$.*

Proof. Let r be the r–matrix of $\mathfrak{g}$. By Theorem 1.4, there exists a quasitriangular QUEA $\big(U_h(\mathfrak{g}), R_h\big)$ whose semiclassical limit is exactly $\big(U(\mathfrak{g}), \, r\big)$: that is, $U_h(\mathfrak{g}) / h\, U_h(\mathfrak{g}) \cong U(\mathfrak{g})$ and $(R-1)/h \equiv r \mod h\, U_h(\mathfrak{g})^{\otimes 2}$; and by Theorem 1.6, the semiclassical limit of $U_h(\mathfrak{g})'$ is $F[[\mathfrak{g}^*]]$. Let $\mathfrak{R}_h := \mathrm{Ad}(R_h)$: then Theorem 2.1 ensures that $\Big(U_h(\mathfrak{g})', \, \mathfrak{R}_h\big|_{U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'} \Big)$ is a braided Hopf algebra, hence its semiclassical limit $\bigg(F[[\mathfrak{g}^*]], \, \Big(\mathfrak{R}_h\big|_{U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'} \Big) \Big|_{h=0} \bigg)$

is braided as well. Furthermore, as $\mathfrak{R}_h$ is an algebra automorphism and the Poisson bracket of $F[[\mathfrak{g}^*]]$ is given by $\{a,b\} = \big([\alpha,\beta]/h\big)\big|_{h=0}$ for all $a, b \in F[[\mathfrak{g}^*]]$ and $\alpha, \beta \in {U_h(\mathfrak{g})}'$ such that $\alpha|_{h=0} = a$, $\beta|_{h=0} = b$, we have that $\Big(\mathfrak{R}_h\big|_{U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'}\Big)\Big|_{h=0}$ is also an automorphism of Poisson algebra. □

The theorem above gives a geometrical interpretation of the r–matrix of a quasitriangular Lie bialgebra. This very result had been proved for $\mathfrak{g} = \mathfrak{sl}(2,k)$ by Reshetikhin (cf. [Re]), and generalised to the case when $\mathfrak{g}$ is Kac-Moody of finite type (cf. [G1], where a more precise analysis is carried on) or affine type (cf. [G2]) by the first author.

Theorem 2.2 has also an important consequence. Let $\mathfrak{g}$ and $\mathfrak{g}^*$ be as above, let $\mathfrak{R}$ be the braiding of $F[[\mathfrak{g}^*]]$, and let $\mathfrak{e}$ be the (unique) maximal ideal of $F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]] = F[[\mathfrak{g}^*]] \otimes F[[\mathfrak{g}^*]]$ (topological tensor product, following [Di], Ch. 1). Now, $\mathfrak{R}$ is an algebra automorphism, hence $\mathfrak{R}(\mathfrak{e}) = \mathfrak{e}$, and $\mathfrak{R}$ induces an automorphism of vector space $\overline{\mathfrak{R}}\colon \mathfrak{e}\big/\mathfrak{e}^2 \to \mathfrak{e}\big/\mathfrak{e}^2$; in addition, $\mathfrak{e}\big/\mathfrak{e}^2 \cong \mathfrak{g} \oplus \mathfrak{g}$, and since $\mathfrak{R}$ is also an automorphism of Poisson algebra, one has that $\overline{\mathfrak{R}}$ is a Lie algebra automorphism of $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{e}\big/\mathfrak{e}^2$; the other properties of the braiding $\mathfrak{R}$ make so that $\overline{\mathfrak{R}}$ have other corresponding properties. Finally, the dual $\overline{\mathfrak{R}}^*\colon \mathfrak{g}^* \oplus \mathfrak{g}^* \to \mathfrak{g}^* \oplus \mathfrak{g}^*$ is a Lie coalgebra automorphism of $\mathfrak{g}^* \oplus \mathfrak{g}^*$, enjoying many other properties dual of those of $\overline{\mathfrak{R}}$. In particular, $\mathfrak{R}$, $\overline{\mathfrak{R}}$ and $\overline{\mathfrak{R}}^*$ are solutions of the QYBE, whence there is an action of the braid group $\mathcal{B}_n$ on $F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]]^{\otimes n}$, on $(\mathfrak{g} \oplus \mathfrak{g})^{\otimes n}$, and on $(\mathfrak{g}^* \oplus \mathfrak{g}^*)^{\otimes n}$ ($n \in \mathbb{N}$), and from that one can obtain knot invariants (following [CP], §15.12). Now, such automorphisms of $\mathfrak{g} \oplus \mathfrak{g}$ and of $\mathfrak{g}^* \oplus \mathfrak{g}^*$ have been introduced in [WX], §9, related to the so-called "global R–matrix", which also yields a geometrical interpretation of the classical r–matrix: comparing our results with those of [WX], as well as the functoriality properties of our construction, will be the matter of a forthcoming article.

§ 3. Proof of theorem 2.1

In this section (H,R) will be a quasitriangular Hopf algebra as in the statement of Theorem 2.1. We want to study the adjoint action of R on $H \otimes H$, where the latter is endowed with ite natural structure of Hopf algebra; we denote by $\tilde{\Delta}$ its coproduct, defined by $\tilde{\Delta} := s_{23} \circ (\Delta \otimes id_H \otimes id_H) \circ (id_H \otimes \Delta)$ where s_{23} denotes the flip in the positions 2 and 3. We'll denote also $I := 1 \otimes 1$ the unit in $H \otimes H$. After our definition of tensor product in $\mathcal{A}$, we have $\big(H \otimes H\big)' = H' \otimes H'$. Our goal is to show that, although R do not necessarily belong to $\big(H \otimes H\big)'$, its adjoint action $a \mapsto R \cdot a \cdot R^{-1}$ leaves stable $\big(H \otimes H\big)' = H' \otimes H'$.

First of all set, for $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, always with $i_1 < \cdots < i_k$,

$R_\Sigma := R_{2i_1-1,2i_k} R_{2i_1-1,2i_{k-1}} \cdots R_{2i_1-1,2i_1} R_{2i_2-1,2i_k} \cdots R_{2i_{k-1}-1,2i_k} R_{2i_k-1,2i_1} \cdots R_{2i_k-1,2i_1}$

(product of k^2 terms) where $R_{i,j} := j_{\{i,j\}}(R)$, defining $j_{\{r,s\}}\colon H \otimes H \longrightarrow H^{\otimes 2n}$ as before. We shall always write $|\Sigma|$ for the cardinality of Σ (here $|\Sigma| = k$).

Lemma 3.1. *In $\big(H \otimes H\big)^{\otimes n}$, for all $\Sigma \subseteq \{1, \dots, n\}$, we have:* $\tilde{\Delta}_\Sigma(R) = R_\Sigma$.

Proof. With no loss of generality, we'll prove the result for $\Sigma = \{1, \dots, n\}$, i.e.

$\tilde{\Delta}_{\{1,\dots,n\}}(R) = R_{\{1,\dots,n\}} = R_{1,2n} \cdot R_{1,2n-2} \cdots R_{1,2} \cdot R_{3,2n} \cdots R_{2n-3,2} \cdot R_{2n-1,2n} \cdots R_{2n-1,2}$.

The result is evident at rank $n = 1$. Assume it be true at rank n, and prove it at rank $n+1$; by definition of $\tilde{\Delta}$ and by the properties of the R–matrix we have

$$\tilde{\Delta}_{\{1,\dots,n+1\}}(R) = \Big(\tilde{\Delta} \otimes id_{H \otimes H}{}^{\otimes n-1}\Big)\big(\tilde{\Delta}_{\{1,\dots,n\}}(R)\big) = \Big(\tilde{\Delta} \otimes id_{H \otimes H}{}^{\otimes n-1}\Big)\big(R_{\{1,\dots,n\}}\big)$$
$$= s_{23}\big(\Delta \otimes id_H^{\otimes 2n}\big)\Big(id_H \otimes \Delta \otimes id_H^{\otimes (2n-2)}\Big)\big(R_{1,2n} \cdots R_{1,2} \cdots R_{3,2} \cdots R_{2n-1,2}\big)$$
$$= s_{23}\big(\Delta \otimes id_H^{\otimes 2n}\big)\left(R_{1,2n+1} \cdots R_{1,3} R_{1,2} \cdots R_{4,3} R_{4,2} \cdots R_{2n,3} R_{2n,2}\right) =$$
$$= s_{23}(R_{1,2n+2} R_{2,2n+2} \cdots R_{1,4} R_{2,4} R_{1,3} R_{2,3} \cdots R_{5,4} R_{5,3} \cdots R_{2n+1,4} R_{2n+1,3})$$
$$= R_{1,2n+2} R_{3,2n+2} \cdots R_{1,4} R_{3,4} \cdot R_{1,2} R_{3,2} \cdots R_{5,4} \cdot R_{5,2} \cdots R_{2n+1,4} R_{2n+1,2}$$
$$= R_{1,2n+2} \cdots R_{1,4} R_{1,2} R_{3,2n+2} \cdots R_{3,4} R_{3,2} \cdots R_{5,4} R_{5,2} \cdots R_{2n+1,4} R_{2n+1,2}$$
$$= R_{\{1,\dots,n+1\}}\,, \quad \text{q.e.d.} \quad \square$$

From now on we shall use the notation $C_b^a := \binom{b}{a}$ for all $a,\, b \in \mathbb{N}$.

Lemma 3.2. *For all $a \in \big(H \otimes H\big)'$, and for all set Σ such that $|\Sigma| > i$, we have*

$$\tilde{\Delta}_\Sigma(a) = \sum_{\Sigma' \subseteq \Sigma,\ |\Sigma'| \leq i} (-1)^{i-|\Sigma'|}\, C_{|\Sigma|-1-|\Sigma'|}^{i-|\Sigma'|}\, \tilde{\Delta}_{\Sigma'}(a) + O\big(h^{i+1}\big)\,.$$

Proof. It is enough to prove the claim for $\Sigma = \{1, \dots, n\}$, with $n > i$. Due to (1.2), we have

$$\tilde{\Delta}_{\{1,\dots,n\}}(a) = \sum_{\bar{\Sigma} \subseteq \{1,\dots,n\}} \delta_{\bar{\Sigma}}(a) = \sum_{\bar{\Sigma} \subseteq \{1,\dots,n\},\ |\bar{\Sigma}| \leq i} \delta_{\bar{\Sigma}}(a) + O\big(h^{i+1}\big)$$
$$= \sum_{\bar{\Sigma} \subseteq \{1,\dots,n\},\ |\bar{\Sigma}| \leq i}\ \sum_{\Sigma' \subseteq \bar{\Sigma}} (-1)^{|\bar{\Sigma}|-|\Sigma'|}\, \tilde{\Delta}_{\Sigma'}(a) + O\big(h^{i+1}\big)$$
$$= \sum_{\Sigma' \subseteq \{1,\dots,n\},\ |\Sigma'| \leq i} \tilde{\Delta}_{\Sigma'}(a) \sum_{\Sigma' \subseteq \bar{\Sigma},\ |\bar{\Sigma}| \leq i} (-1)^{|\bar{\Sigma}|-|\Sigma'|} + O\big(h^{i+1}\big)$$
$$= \sum_{\Sigma' \subseteq \{1,\dots,n\},\ |\Sigma'| \leq i} \tilde{\Delta}_{\Sigma'}(a)\, (-1)^{i-|\Sigma'|}\, C_{n-1-|\Sigma'|}^{i-|\Sigma'|} + O\big(h^{i+1}\big)\,, \quad \text{q.e.d.} \quad \square$$

Before going on with the main result, we need still another minor technical fact about the binomial coefficients: one can easily prove it using the formal series expansion of $(1-X)^{-(r+1)}$, namely $(1-X)^{-(r+1)} = \sum_{k=0}^{\infty} C_{k+r}^r X^k$.

Lemma 3.3. *Let $r,\, s,\, t \in \mathbb{N}$ be such that $r < t$. Then we have the following relations (where we set $C_u^v := 0$ if $v > u$):*

$$(a) \quad \sum_{d=0}^{t} (-1)^d\, C_{d-1}^r\, C_t^d = -(-1)^r\,, \qquad (b) \quad \sum_{d=0}^{t} (-1)^d\, C_{d+s}^r\, C_t^d = 0\,. \quad \square$$

Finally, here is the main result of this section:

Proposition 3.4. *For all* $a \in \big(H \otimes H\big)'$ *, we have* $R\, a\, R^{-1} \in \big(H \otimes H\big)'$ *.*

Proof. As we have to show that $R\, a\, R^{-1}$ belongs to $\big(H \otimes H\big)'$, we have to consider the terms $\delta_n\big(R\, a\, R^{-1}\big)$, $n \in \mathbb{N}$. For this we go and re-write $\delta_{\{1,\dots,n\}}\big(R\, a\, R^{-1}\big)$ by using Lemma 3.1 and the fact that $\tilde{\Delta}$ and more in general $\tilde{\Delta}_{\{i_1,\dots,i_k\}}$, for $k \leq n$, are algebra morphisms; then $\delta_{\{1,\dots,n\}}\big(R\, a\, R^{-1}\big) = \sum\limits_{\Sigma \subseteq \{1,\dots,n\}} (-1)^{n-|\Sigma|} R_\Sigma\, \tilde{\Delta}_\Sigma(a)\, R_\Sigma^{-1}$.

We shall prove by induction on i that

$$\delta_{\{1,\dots,n\}}\big(R\, a\, R^{-1}\big) = O\big(h^{i+1}\big) \qquad \text{for all} \quad 0 \leq i \leq n-1 \; . \tag{$\star$}$$

In other words, we'll see that all the terms of the expansion truncated at the order $n-1$ are zero, hence $\delta_n\big(R\, a\, R^{-1}\big) = O(h^n)$, whence our claim.

For $i = 0$, we have, for each Σ: $\tilde{\Delta}_\Sigma(a) = \epsilon(a) I^{\otimes n} + O(h)$ and $R_\Sigma = I^{\otimes n} + O(h)$, and similarly $R_\Sigma^{-1} = I^{\otimes n} + O(h)$, whence $\delta_{\{1,\dots,n\}}\big(R\, a\, R^{-1}\big) = \sum\limits_{k=1}^{n} C_n^k (-1)^{n-k} \epsilon(a)\, I^{\otimes n} +$ $+ O(h) = O(h)$, thus the result $(\star)$ is true for $i = 0$.

Let's assume the result $(\star)$ proved for all $i' < i$. Write the h–adic expansions of R_Σ and R_Σ^{-1} in the form $R_\Sigma = \sum_{\ell=0}^{\infty} R_\Sigma^{(\ell)}\, h^\ell$ and $R_\Sigma^{-1} = \sum_{m=0}^{\infty} R_\Sigma^{(-m)}\, h^m$. By the previous proposition, we have an approximation of $\tilde{\Delta}_\Sigma(a)$ at the order j

$$\tilde{\Delta}_\Sigma(a) \;=\; \sum_{\Sigma' \subseteq \Sigma,\ |\Sigma'| \leq j} (-1)^{j-|\Sigma'|}\, C_{|\Sigma|-1-|\Sigma'|}^{j-|\Sigma'|}\, \tilde{\Delta}_{\Sigma'}(a) + O\big(h^{j+1}\big) \; .$$

Then we have the following approximation of $\delta_{\{1,\dots,n\}}\big(R\, a\, R^{-1}\big)$:

$$\delta_{\{1,\dots,n\}}\big(R\, a\, R^{-1}\big) = \sum_{\Sigma \subseteq \{1,\dots,n\}} \sum_{\ell+m \leq i} (-1)^{n-|\Sigma|}\, R_\Sigma^{(\ell)}\, \tilde{\Delta}_\Sigma(a)\, R_\Sigma^{(-m)}\, h^{\ell+m} + O\big(h^{i+1}\big) =$$

$$= \sum_{j=0}^{i} \sum_{\ell+m=i-j} \Bigg(\sum_{\substack{\Sigma \subseteq \{1,\dots,n\} \\ |\Sigma| > j}} \sum_{\substack{\Sigma' \subseteq \Sigma \\ |\Sigma'| \leq j}} (-1)^{n-|\Sigma|} (-1)^{j-|\Sigma'|}\, C_{|\Sigma|-1-|\Sigma'|}^{j-|\Sigma'|}\, R_\Sigma^{(\ell)}\, \tilde{\Delta}_{\Sigma'}(a)\, R_\Sigma^{(-m)} +$$

$$+ \sum_{\substack{\Sigma \subseteq \{1,\dots,n\} \\ |\Sigma| \leq j}} (-1)^{n-|\Sigma|}\, R_\Sigma^{(\ell)}\, \tilde{\Delta}_\Sigma(a)\, R_\Sigma^{(-m)} \Bigg) h^{\ell+m} + O\big(h^{i+1}\big) =$$

$$= \sum_{j=0}^{i} \sum_{\ell+m+j=i} \sum_{\substack{\Sigma' \subseteq \{1,\dots,n\} \\ |\Sigma'| \leq j}} \Bigg(\sum_{\substack{\Sigma \subseteq \{1,\dots,n\} \\ \Sigma' \subseteq \Sigma,\ |\Sigma| > j}} (-1)^{n-|\Sigma|} (-1)^{j-|\Sigma'|}\, C_{|\Sigma|-1-|\Sigma'|}^{j-|\Sigma'|}\, R_\Sigma^{(\ell)}\, \tilde{\Delta}_{\Sigma'}(a)\, R_\Sigma^{(-m)} +$$

$$+ (-1)^{n-|\Sigma'|}\, R_{\Sigma'}^{(\ell)}\, \tilde{\Delta}_{\Sigma'}(a)\, R_{\Sigma'}^{(-m)} \Bigg) h^{\ell+m} + O\big(h^{i+1}\big) \; .$$

We denote (E) the last expression in brackets, and we'll show that this expression is zero, whence $\delta_n\big(R\, a\, R^{-1}\big) = O\big(h^{i+1}\big)$.

Let's look first at the terms corresponding to $\ell + m = 0$, that is $j = i$. Then we find back $\delta_{\{1,\dots,n\}}(a)$, which is in $O\big(h^{i+1}\big)$ by assumption. Therefore, by now on in the sequel of the computation we assume $\ell + m > 0$.

Consider first how the terms $R_\Sigma^{(\ell)}$ and $R_\Sigma^{(-m)}$ act on $\big(H \otimes H\big)'^{\otimes n}$ (respectively on the left and on the right) for $\ell + m$ fixed (and positive), say $\ell + m = S$.

Taking the truncated expansion of each $R_{i,j}$ which occurs in R_Σ, we see that $R_\Sigma^{(\ell)}$ and $R_\Sigma^{(-m)}$ are sums of products of at most ℓ and m terms respectively, each one acting on at most two tensor of $\big(H \otimes H\big)'^{\otimes n}$. We re-write $\sum_{\ell+m=S} R_\Sigma^{(\ell)}\, \tilde{\Delta}_{\Sigma'}(a)\, R_\Sigma^{(-m)}$ by gathering together the terms of the sum which act on the same factors of $\big(H \otimes H\big)'^{\otimes n}$: we'll denote the set of positions of this factors by Σ''.

Now, if i belongs to Σ'', in the identification $(H \otimes H)^{\otimes n} = H^{\otimes 2n}$ (such as we chose it to define R_Σ) the index i corresponds to the pair $(2i-1, 2i)$; but then R_Σ and R_Σ^{-1}, and then also each $R_\Sigma^{(\ell)}$ and each $R_\Sigma^{(-m)}$, may act non-trivially on the i–th factor of $\tilde{\Delta}_{\Sigma'}(a)$ only if one of $2i-1$ and $2i$ (or even both of them) occurs in the explicit written expression of R_Σ (in $H^{\otimes 2n}$), hence only if $i \in \Sigma$: thus $\Sigma'' \subseteq \Sigma$. Then we set $\sum_{\ell+m=S} R_\Sigma^{(\ell)}\, \tilde{\Delta}_{\Sigma'}(a)\, R_\Sigma^{(-m)} = \sum_{\Sigma''\subseteq\Sigma} A^{(S)}_{\Sigma',\Sigma,\Sigma''}(a)$.

Now consider $\bar{\Sigma} \supseteq \Sigma$. From the very definition we have $R_{\bar{\Sigma}} = R_\Sigma + \mathcal{A}$, where $\mathcal{A}$ is a sum of terms which contain factors $R^{(s)}_{2i-1,2j}$ with $\{i,j\} \not\subseteq \Sigma$: to see this, it is enough to expand every factor $R_{a,b}$ in $R_{\bar{\Sigma}}$ as $R_{a,b} = 1^{\otimes 2n} + O(h)$. Similarly, we have also $R_{\bar{\Sigma}}^{(\ell)} = R_\Sigma^{(\ell)} + \mathcal{A}'$, and similarly $R_{\bar{\Sigma}}^{(-m)} = R_\Sigma^{(-m)} + \mathcal{A}''$. This implies that $A^{(S)}_{\Sigma'',\bar{\Sigma},\Sigma'}(a) = A^{(S)}_{\Sigma'',\Sigma,\Sigma'}(a)$, and so the $A^{(S)}_{\Sigma'',\Sigma,\Sigma'}(a)$ do not depend on Σ; then we write

$$\sum_{\ell+m=S} R_\Sigma^{(\ell)}\, \tilde{\Delta}_{\Sigma'}(a)\, R_\Sigma^{(-m)} = \sum_{\Sigma''\subseteq\Sigma} A^{(S)}_{\Sigma',\Sigma''}(a)\ .$$

In the sequel we re-write (E) using the $A^{(S)}_{\Sigma',\Sigma''}(a)$. In the following we'll denote by $\delta_{\Sigma''\subseteq\Sigma'}$ the function whose value is 1 if $\Sigma'' \subseteq \Sigma'$ and 0 if not.

Then we obtain a new expression for $\delta_{\{1,\dots,n\}}\big(R\, a\, R^{-1}\big)$, namely

$$\delta_{\{1,\dots,n\}}\big(R\, a\, R^{-1}\big) = \sum_{j=0}^{i-1} \sum_{\substack{\Sigma'\subseteq\{1,\dots,n\}\\ |\Sigma'|\leq j}} \Bigg(\sum_{\substack{\Sigma\subseteq\{1,\dots,n\}\\ \Sigma'\subseteq\Sigma,\ |\Sigma|>j}} (-1)^{n-|\Sigma|}\, (-1)^{j-|\Sigma'|}\, C^{j-|\Sigma'|}_{|\Sigma|-1-|\Sigma'|} \times$$

$$\times \sum_{\Sigma''\subseteq\Sigma} A^{(i-j)}_{\Sigma',\Sigma''}(a) + (-1)^{n-|\Sigma'|} \sum_{\Sigma''\subseteq\Sigma'} A^{(i-j)}_{\Sigma',\Sigma''}(a) \Bigg)\, h^{i-j} + O\big(h^{i+1}\big) =$$

$$= \sum_{j=0}^{i-1} \sum_{\substack{\Sigma'\subseteq\{1,\dots,n\}\\ |\Sigma'|\leq j}} h^{i-j} \sum_{\Sigma''\subseteq\{1,\dots,n\}} A^{(i-j)}_{\Sigma',\Sigma''}(a) \times$$

$$\times \Bigg(\sum_{\substack{\Sigma\subseteq\{1,\dots,n\}\\ \Sigma'\subseteq\Sigma,\ \Sigma''\subseteq\Sigma,\ |\Sigma|>j}} (-1)^{n-|\Sigma|}\, (-1)^{j-|\Sigma'|}\, C^{j-|\Sigma'|}_{|\Sigma|-1-|\Sigma'|} + (-1)^{n-|\Sigma'|}\, \delta_{\Sigma''\subseteq\Sigma'} \Bigg) + O\big(h^{i+1}\big)\ .$$

We denote $\big(E'\big)_{\Sigma',\Sigma''}$ the new expression in brackets; in other words, for fixed Σ' and Σ'', with $\big|\Sigma'\big| \leq j$, we set

$$\big(E'\big)_{\Sigma',\Sigma''} := \sum_{\substack{\Sigma\subseteq\{1,\dots,n\} \\ \Sigma'\subseteq\Sigma,\ \Sigma''\subseteq\Sigma,\ |\Sigma|>j}} (-1)^{n-|\Sigma|}\,(-1)^{j-|\Sigma'|}\,C^{j-|\Sigma'|}_{|\Sigma|-1-|\Sigma'|} + (-1)^{n-|\Sigma'|}\,\delta_{\Sigma''\subseteq\Sigma'}$$

(by the way, we remark that this is a purely combinatorial expression); we shall show that this expression is zero when Σ' and Σ'' are such that $\big|\Sigma'\cup\Sigma''\big| \leq j-i+\big|\Sigma'\big|$ and $\big|\Sigma'\big| \leq j$. In force of the following lemma, this will be enough to prove Proposition 3.4.

Lemma 3.5.
(a) We have $j < i$ and $i \leq n-1$, hence $j \leq n-2$.
(b) For all $S > 0$, in the expression $\sum_{\ell+m=S} R_\Sigma^{(\ell)}\,\tilde{\Delta}_{\Sigma'}(a)\,R_\Sigma^{(-m)} = \sum_{\Sigma''\subseteq\Sigma} A^{(S)}_{\Sigma',\Sigma''}(a)$ we have that $A^{(S)}_{\Sigma',\Sigma''}(a) = 0$ for all Σ', Σ'' such that $\big|\Sigma'\cup\Sigma''\big| > S + \big|\Sigma'\big|$.

Proof. The first part of the statement is trivial; to prove the second, we study the adjoint action of R_Σ on $\big(H\otimes H\big)^{\otimes n}$.

First of all, on $k\cdot I^{\otimes n}$ the action of these elements gives a zero term because one gets the term at the order S of the h–adic expansion of $R_\Sigma \cdot R_\Sigma^{-1} = 1$ (for $S > 0$).

Second, let us consider $\Sigma \subseteq \{1,\dots,n\}$, and let us study the action on $\big(H\otimes H\big)_{\Sigma'} := j_{\Sigma'}\Big(\big(H\otimes H\big)^{\otimes|\Sigma|}\Big)$ ($\subseteq \big(H\otimes H\big)^{\otimes n}$). We know that R_Σ is a product of $|\Sigma|^2$ terms of type $R_{a,b}$, with $a,b\in\big\{\,2i-1,2j\,\big|\,i,j\in\Sigma\,\big\}$; so let's analyse what happens when one computes the product $P := R_\Sigma\cdot x\cdot R_\Sigma^{-1}$ if $x\in\big(H\otimes H\big)_\Sigma$.

Consider the rightmost factor $R_{a,b}$: if $a,b\notin\big\{\,2j-1,2j\,\big|\,j\in\Sigma'\,\big\}$, then when computing P one gets $P := R_\Sigma\,x\,R_\Sigma^{-1} = R_\star\,R_{a,b}\,x\,R_{a,b}^{-1}\,R_\star^{-1} = R_\star\,x\,R_\star^{-1}$ (where $R_\star := R_\Sigma\,R_{a,b}^{-1}$). Similarly, moving further on from right to left along R_Σ one can discard all factors $R_{c,d}$ of this type, namely those such that $c,d\notin\big\{\,2j-1,2j\,\big|\,j\in\Sigma'\,\big\}$. Thus the first factor whose adjoint action is non-trivial will be necessarily of type $R_{\bar{a},\bar{b}}$ with one of the two indices belonging to $\big\{\,2j-1,2j\,\big|\,j\in\Sigma'\,\big\}$, say for instance $\bar{a}$. Notice that the new index $\bar{a}$ ($\in\{1,2,\dots,2n-1,2n\}$) — which "marks" a tensor factor in $H^{\otimes 2n}$ — corresponds to a new index $j_{\bar{a}}$ ($\in\{1,\dots,n\}$) — marking a tensor factor of $\big(H\otimes H\big)^{\otimes n}$. So for the following factors — i.e. on the left of $R_{\bar{a},\bar{b}}$ — one has to repeat the same analysis, but with the set $\big\{\,2j-1,2j\,\big|\,j\in\Sigma'\cup\{j_{\bar{a}}\}\,\big\}$ instead of $\big\{\,2j-1,2j\,\big|\,j\in\Sigma'\,\big\}$; therefore, as $R_{\bar{a},\bar{b}}$ might act in non-trivial way on at most $\big|\Sigma'\big|$ factors of $\big(H\otimes H\big)^{\otimes n}$, similarly the factor which is the closest on its left may act in a non-trivial way on at most $\big|\Sigma'\big|+1$ factors. The upset is that the adjoint action of R_Σ is non-trivial on at most $\big|\Sigma'\big|+\big|\Sigma\big|$ factors of $\big(H\otimes H\big)^{\otimes n}$.

Now consider the different terms $R_\Sigma^{(\ell)}$ and $R_\Sigma^{(-m)}$, with $\ell+m=S$, and study the products $R_\Sigma^{(\ell)}\cdot x\cdot R_\Sigma^{(-m)}$, with $x\in\big(H\otimes H\big)_\Sigma$. We already know that $R_\Sigma^{(\ell)}$ and $R_\Sigma^{(-m)}$ are sums of products, denoted P_+ and P_- , of at most ℓ and m terms respectively, of type $R_{i,j}^{(\pm k)}$; the terms $A^{(S)}_{\Sigma',\Sigma''}(a)$ then are nothing but sums of terms of type $P_+\,\tilde{\Delta}_{\Sigma'}(a)\,P_-$, where in addition the products P_+ and P_- have their "positions" in Σ''. Now, since each P_+ and each P_- is a product of at most ℓ and m factors $R_{i,j}^{(\pm k)}$, one can refine the previous argument. Consider only the term at the order S of the h–adic expansion of $P := R_\Sigma \, x \, R_\Sigma^{-1} = R_\star \, R_{a,b} \, x \, R_{a,b}^{-1} \, R_\star^{-1} = R_\star \, x \, R_\star^{-1}$: whenever there are factors of type $R_{a,b}^{(k)}$ or $R_{a,b}^{(t)}$, for fixed a, b — not belonging to $\big\{ 2j-1, 2j \,\big|\, j \in \Sigma' \big\}$ — which appear in $R_\Sigma^{(\ell)}$ or $R_\Sigma^{(-m)}$, for some ℓ or m, the total contribution of all these terms in the sum $\sum_{\ell+m=S} R_\Sigma^{(\ell)} \, x \, R_\Sigma^{(-m)}$ will be zero (this follows from the fact that $R_\star \, R_{a,b} \, x \, R_{a,b}^{-1} \, R_\star^{-1} = R_\star \, x \, R_\star^{-1}$). In addition, since now we are dealing only with S factors in total, we conclude that $A_{\Sigma',\Sigma''}^{(S)}(a) = 0$ if $\big|\Sigma' \cup \Sigma''\big| > S + \big|\Sigma'\big|$. □

Now we shall compute $\big(E'\big)_{\Sigma',\Sigma''}$. Thanks to the previous remark, we can limit ourselves to consider the pairs $\big(\Sigma', \Sigma''\big)$ such that $\big|\Sigma' \cup \Sigma''\big| \leq i-j+m+\big|\Sigma'\big| \leq i-j+j = i \leq n-1$. Then one can always find at least two $\Sigma \subseteq \{1, \dots, n\}$ such that $|\Sigma| > j$ and $\Sigma' \cup \Sigma'' \subseteq \Sigma$, which make us sure that there will always be at least two terms in the calculation which is to follow (such a condition will guarantee the vanishing of the expression $\big(E'\big)_{\Sigma',\Sigma''}$). We distinguish three cases:

(I) If $\Sigma'' \subseteq \Sigma'$, then the expression $\big(E'\big)_{\Sigma',\Sigma''}$ becomes

$$\big(E' : 1\big)_{\Sigma',\Sigma''} = \sum_{\substack{\Sigma \subseteq \{1,\dots,n\} \\ \Sigma' \subseteq \Sigma, \; |\Sigma| > j}} (-1)^{n-|\Sigma|} \, (-1)^{j-|\Sigma'|} \, C_{|\Sigma|-1-|\Sigma'|}^{j-|\Sigma'|} + (-1)^{n-|\Sigma'|} \, .$$

Gathering together the Σ's which share the same cardinality d, a simple computation gives

$$\big(E' : 1\big)_{\Sigma',\Sigma''} = \sum_{d=j+1}^{n} (-1)^{n-d} \, (-1)^{j-|\Sigma'|} \, C_{d-1-|\Sigma'|}^{j-|\Sigma'|} \, C_{n-|\Sigma'|}^{d-|\Sigma'|} + (-1)^{n-|\Sigma'|} \, .$$

Now, this last expression is zero by Lemma 3.3, for it corresponds to a sum of type $\sum_{k=r+1}^{t} (-1)^{t+r-k} \, C_{k-1}^{r} \, C_t^k + (-1)^t = \sum_{k=0}^{t} (-1)^{t+r-k} \, C_{k-1}^{r} \, C_t^k + (-1)^t$ (where $C_u^v := 0$ if $v > u$) with $r, t \in \mathbb{N}_+$ and $r < t$: in our case we set $t = n - \big|\Sigma'\big|$, $r = j - \big|\Sigma'\big|$ and $k = d - \big|\Sigma'\big|$; one verifies that one has just $j - \big|\Sigma'\big| < n - \big|\Sigma'\big|$ because $j < n$.

(II) If $\Sigma'' \not\subseteq \Sigma'$ and $\big|\Sigma' \cup \Sigma''\big| > j$, then the expression $\big(E'\big)_{\Sigma',\Sigma''}$ becomes

$$\big(E' : 2\big)_{\Sigma',\Sigma''} = \sum_{\substack{\Sigma \subseteq \{1,\dots,n\} \\ \Sigma' \cup \Sigma'' \subseteq \Sigma}} (-1)^{n-|\Sigma|} \, (-1)^{j-|\Sigma'|} \, C_{|\Sigma|-1-|\Sigma'|}^{j-|\Sigma'|} \, .$$

Gathering together the Σ's which share the same cardinality d, a simple computation gives

$$\big(E' : 2\big)_{\Sigma',\Sigma''} = \sum_{d=|\Sigma' \cup \Sigma''|}^{n} (-1)^{n-d} \, (-1)^{j-|\Sigma'|} \, C_{d-1-|\Sigma'|}^{j-|\Sigma'|} \, C_{n-|\Sigma' \cup \Sigma''|}^{d-|\Sigma' \cup \Sigma''|} \, .$$

Again, the last expression is zero thanks to Lemma 3.3, for it corresponds to a sum of type $\sum_{k=0}^{t} (-1)^{t+r-k} \, C_{k+s}^{r} \, C_t^k$ with $r, t, s \in \mathbb{N}_+$ and $r < t$: in our case we set

$t = n - \big|\Sigma' \cup \Sigma''\big|$, $r = j - \big|\Sigma'\big|$, $s = \big|\Sigma' \cup \Sigma''\big| - \big|\Sigma'\big| - 1$ and $k = d - \big|\Sigma' \cup \Sigma''\big|$; then one verifies that $j - \big|\Sigma'\big| < n - \big|\Sigma'\big|$ for $j < n$ and $\big|\Sigma' \cup \Sigma''\big| - \big|\Sigma'\big| - 1 \geq 0$ since $\Sigma'' \not\subseteq \Sigma'$.

(III) If $\Sigma'' \not\subseteq \Sigma'$ and $\big|\Sigma' \cup \Sigma''\big| \leq j$, then the expression $\big(E'\big)_{\Sigma',\Sigma''}$ becomes

$$\big(E' : 3\big)_{\Sigma',\Sigma''} = \sum_{\substack{\Sigma \subseteq \{1,\dots,n\} \\ \Sigma' \cup \Sigma'' \subseteq \Sigma,\ |\Sigma| > j}} (-1)^{n-|\Sigma|} (-1)^{j-|\Sigma'|} C^{j-|\Sigma'|}_{|\Sigma|-1-|\Sigma'|} .$$

Gathering together the Σ's which share the same cardinality d, a simple computation gives

$$\big(E' : 3\big)_{\Sigma',\Sigma''} = \sum_{d=j+1}^{n} (-1)^{n-d} (-1)^{j-|\Sigma'|} C^{j-|\Sigma'|}_{d-1-|\Sigma'|} C^{d-|\Sigma' \cup \Sigma''|}_{n-|\Sigma' \cup \Sigma''|} .$$

But again the last expression is zero because of Lemma 3.3, for it corresponds to a sum of type $\sum_{k=j+1-|\Sigma' \cup \Sigma''|}^{t} (-1)^{t+r-k} C^r_{k+s} C^k_t = \sum_{k=0}^{t} (-1)^{t+r-k} C^r_{k+s} C^k_t$ (where $C^v_u := 0$ if $v > u$) with $r, t, s \in \mathbb{N}_+$ and $r < t$: here again we set $t = n - \big|\Sigma' \cup \Sigma''\big|$, $r = j - \big|\Sigma'\big|$, $s = \big|\Sigma' \cup \Sigma''\big| - \big|\Sigma'\big| - 1$ and $k = d - \big|\Sigma' \cup \Sigma''\big|$; one has, always for the same reasons, $j - \big|\Sigma'\big| < n - \big|\Sigma'\big|$ and $\big|\Sigma' \cup \Sigma''\big| - \big|\Sigma'\big| - 1 \geq 0$.

Therefore, one has always $\big(E'\big)_{\Sigma',\Sigma''} = 0$, whence $(E) = 0$, which ends the proof. □

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