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ON QUANTUM MARKOV CHAINS ON CAYLEY TREE I: UNIQUENESS OF THE ASSOCIATED CHAIN WITH XY-MODEL ON THE CAYLEY TREE OF ORDER TWO

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In this paper we study forward quantum Markov chains (QMC) defined on Cayley tree. A construction of such QMC is provided, namely we construct states on finite volumes with boundary conditions, and define QMC as a weak limit of those states which depends on the boundary conditions. Using the provided construction, we investigate QMC associated with XY-model on a Cayley tree of order two. We prove uniqueness of QMC associated with such a model, this means the QMC does not depend on the boundary conditions.

Keywords: Quantum Markov chain; Cayley tree; XY-model; uniqueness.

AMS Subject Classification: 46L53, 60J99, 46L60, 60G50, 82B10, 81Q10, 94A17

1. Introduction

Nowadays, it is known that Markov fields play an important role in classical probability, in physics, in biological and neurological models and in an increasing number of technological problems such as image recognition. Therefore, it is quite natural to forecast that the quantum analogue of these models will also play a relevant role. The quantum analogues of Markov processes were first constructed in Ref. 1, where the notion of quantum Markov chain (QMC) on infinite tensor product algebras was

introduced. Nowadays, QMC have become a standard computational tool in solid state physics, and several natural applications have emerged in quantum statistical mechanics and quantum information theory. The reader is referred to Refs. 19, 23–25, 34 and the references cited therein, for recent developments of the theory and the applications.

A first attempt to construct a quantum analogue of classical Markov fields has been done in Refs. 28, 4, 6 and 9. These papers extend to fields the notion of *quantum Markov state* introduced in Ref. 8 as a subclass of the QMC introduced in Ref. 1. In Ref. 7, a definition of quantum Markov states and chains has been proposed, which extend a proposed one in Ref. 33, and includes all the presently known examples. Note that in the mentioned papers, quantum Markov fields were considered over multidimensional integer lattice $\mathbb{Z}^d$. This lattice has so-called amenability condition. Therefore, it is natural to investigate quantum Markov fields over non-amenable lattices. One of the simplest non-amenable lattice is a Cayley tree. First attempts to investigate QMC over such trees was done in Ref. 12, such studies were related to the investigation of thermodynamic limit of valence-bond-solid models on a Cayley tree.[18] There, it was constructed finitely correlated states as ground states of VBS-model on Cayley tree. The mentioned considerations naturally suggest the study of the following problem: the extension to fields the notion of QMC. In Ref. 10 we have introduced a hierarchy of notions of Markovianity for states on discrete infinite tensor products of C^*-algebras and for each of these notions we constructed some explicit examples. We showed that the construction of Ref. 8 can be generalized to trees. It is worth to note that, in a different context and for quite different purposes, the special role of trees was already emphasized in Ref. 28. Noncommutative extensions of classical Markov fields, associated with Ising and Potts models on Cayley tree, were investigated in Refs. 31 and 32. In the classical case, Markov fields on trees were also considered in Refs. 35–40.

In this paper, we continue our investigations started in Ref. 10. In Ref. 10 we have studied backward QMC defined on the Cayley tree. Note that shift invariant backward QMC chains can also be considered as an extension of C^*-finitely correlated states defined in Ref. 19 to the Cayley trees. But the forward QMC cannot be described by the finitely correlated ones (see Remark 3.4 below). Therefore, in Sec. 3, we provide a construction of forward QMC. Namely we construct states on finite volumes with boundary conditions, and define QMC as a weak limit of those states which depends on the boundary conditions. There, we involve some methods used in the theory of Gibbs measures on trees (see Ref. 22). Such a construction extends those provided in Refs. 2 and 8. In Sec. 4, by means of the provided construction we investigate QMC associated with XY-model on a Cayley tree of order two. For that model, in a QMC scheme, we prove uniqueness of the limiting state, i.e. the state does not depend on the boundary conditions. Note that whether or not the resulting states have a physical interest is a question that cannot be solved on a purely mathematical ground. We have to stress that classical XY-model have been

investigated by many authors on a 1D-lattice,[30, 38] and also on a Cayley tree.[14] In a quantum setting such a model was studied in Refs. 13, 27, 26 and 21.

2. Preliminaries

Recall that a Cayley tree Γ^k of order $k \geq 1$ is an infinite tree whose each vertex has exactly $k+1$ edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y \rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1 \rangle, \ldots, \langle x_{d-1}, y \rangle$ is called a *path* from the point x to the point y. The distance $d(x, y), x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y. If we cut away an edge $\{x, y\}$ of the tree Γ^k, then Γ^k splits into connected components, called semi-infinite trees with roots x and y, which will be denoted respectively by $\Gamma^k(x)$ and $\Gamma^k(y)$. If we cut away from Γ^k the origin O together with all $k+1$ nearest neighbor vertices, in the result we obtain k semi-infinite $\Gamma^k(x)$ trees with $x \in S_0 = \{y \in \Gamma^k : d(O, y) = 1\}$. Hence we have

$$\Gamma^k = \bigcup_{x \in S_0} \Gamma^k(x) \cup \{O\}.$$

Therefore, in the sequel we will consider semi-infinite Cayley tree $\Gamma^k(x^0) = (L, E)$ with the root x^0, L is the set of vertices and E is the set of edges.

Now we are going to introduce a coordinate structure in $\Gamma^k(x^0)$ as follows: every vertex x (except for x^0) of $\Gamma^k(x^0)$ has coordinates $(i_1, \ldots, i_n)$, here $i_m \in \{1, \ldots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1, \ldots, i_n)$ form level n (i.e. $d(x^0, x) = n$) of the lattice (see Fig. 1).

Let us set

$$W_n = \{x \in L : d(x, x_0) = n\}, \quad \Lambda_n = \bigcup_{k=0}^n W_k, \quad \Lambda_{[n,m]} = \bigcup_{k=n}^m W_k, \quad (n < m)$$

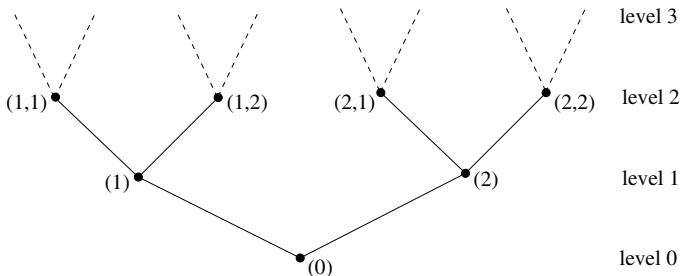$$E_n = \{\langle x, y \rangle \in E : x, y \in \Lambda_n\}, \quad \Lambda_n^c = \bigcup_{k=n}^\infty W_k.$$

Fig. 1.   The first levels of Γ^2.

For $x \in \Gamma^k(x^0)$, $x = (i_1, \ldots, i_n)$ denote $S(x) = \{(x, i) : 1 \le i \le k\}$, here (x, i) means that $(i_1, \ldots, i_n, i)$. This set is called a set of *direct successors* of x.

The algebra of observables $\mathcal{B}_x$ for any single site $x \in L$ will be taken as the algebra M_d of the complex $d \times d$ matrices. The algebra of observables localized in the finite volume $\Lambda \subset L$ is then given by $\mathcal{B}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}_x$. As usual if $\Lambda^1 \subset \Lambda^2 \subset L$, then $\mathcal{B}_{\Lambda^1}$ is identified as a subalgebra of $\mathcal{B}_{\Lambda^2}$ by tensoring with unit matrices on the sites $x \in \Lambda^2 \setminus \Lambda^1$. Note that, in the sequel, by $\mathcal{B}_{\Lambda,+}$ we denote positive part of $\mathcal{B}_\Lambda$. The full algebra $\mathcal{B}_L$ of the tree is obtained in the usual manner by an inductive limit

$$\mathcal{B}_L = \overline{\bigcup_{\Lambda_n} \mathcal{B}_{\Lambda_n}}.$$

In what follows, by $\mathcal{S}(\mathcal{B}_\Lambda)$ we will denote the set of all states defined on the algebra $\mathcal{B}_\Lambda$.

Consider a triplet $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras. Recall that a *quasi-conditional expectation* with respect to the given triplet is a completely positive (CP) identity preserving linear map $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{E}(ca) = c\mathcal{E}(a)$, $a \in \mathcal{A}$, $c \in \mathcal{C}$.

Definition 2.1. (Ref. 10) Let φ be a state on $\mathcal{B}_L$. Then φ is called:

(i) A forward quantum d-Markov chain (QMC), associated to $\{\Lambda_n\}$, if for each Λ_n, there exist a quasi-conditional expectation $\mathcal{E}_{\Lambda_n^c}$ with respect to the triplet

$$\mathcal{B}_{\Lambda_{n+1}^c} \subseteq \mathcal{B}_{\Lambda_n^c} \subseteq \mathcal{B}_{\Lambda_{n-1}^c} \tag{2.1}$$

and a state $\hat{\varphi}_{\Lambda_n^c} \in \mathcal{S}(\mathcal{B}_{\Lambda_n^c})$ such that for any $n \in \mathbb{N}$ one has

$$\hat{\varphi}_{\Lambda_n^c} |_{\mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}} = \hat{\varphi}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_{n+1}^c} |_{\mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}} \tag{2.2}$$

and

$$\varphi = \lim_{n \to \infty} \hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_{n-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c} \tag{2.3}$$

in the weak-* topology.

(ii) A backward quantum d-Markov chain, associated to $\{\Lambda_n\}$, if there exist a quasi-conditional expectation $\mathcal{E}_{\Lambda_n}$ with respect to the triple $\mathcal{B}_{\Lambda_{n-1}} \subseteq \mathcal{B}_{\Lambda_n} \subseteq \mathcal{B}_{\Lambda_{n+1}}$ for each $n \in \mathbb{N}$ and an initial state $\rho_0 \in S(B_{\Lambda_0})$ such that

$$\varphi = \lim_{n \to \infty} \rho_0 \circ \mathcal{E}_{\Lambda_0} \circ \mathcal{E}_{\Lambda_1} \circ \cdots \circ \mathcal{E}_{\Lambda_n}$$

in the weak-* topology.

In this definition, a forward QMC φ generated by $\mathcal{E}_{\Lambda_n^c}$ and $\varphi_{\Lambda_n^c}$, is well-defined. Indeed, we have

$$\hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c} |_{\mathcal{B}_{\Lambda_n}} = \hat{\varphi}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_n^c} |_{\mathcal{B}_{\Lambda_n}}$$

by (2.2) and a following remark so that, for $\Lambda \subset\subset \Lambda_k$ and $a \in \mathcal{B}_\Lambda$,

$$\lim_{n \to \infty} \hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_{n-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c}(a) = \hat{\varphi}_{\Lambda_k^c} \circ \mathcal{E}_{\Lambda_k^c} \circ \mathcal{E}_{\Lambda_{k-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c}(a).$$

Similarly, one can also demonstrate that backward QMC is well-defined.

Remark 2.2. Note that in Ref. 10 a forward QMC was called a generalized quantum Markov state.

Remark 2.3. We have to stress that in most well-known papers (see for example Refs. 4, 5, 18, 19 and 29) related to QMC, all such states were investigated as a backward QMC. Therefore, in the sequel we will be interested in forward QMC, which is less studied.

3. A Construction of Forward QMC on the Cayley Tree

In this section, we are going to provide a construction of forward quantum d-Markov chain. Note that a construction of backward QMC has been studied in Ref. 10.

Let us rewrite the elements of W_n in the following order, i.e.

$$\overrightarrow{W_n} := \left(x_{W_n}^{(1)}, x_{W_n}^{(2)}, \dots, x_{W_n}^{(|W_n|)} \right), \quad \overleftarrow{W_n} := \left(x_{W_n}^{(|W_n|)}, x_{W_n}^{(|W_n|-1)}, \dots, x_{W_n}^{(1)} \right).$$

Note that $|W_n| = k^n$. Vertices $x_{W_n}^{(1)}, x_{W_n}^{(2)}, \dots, x_{W_n}^{(|W_n|)}$ of W_n can be represented in terms of the coordinate system as follows:

$$x_{W_n}^{(1)} = (1, 1, \dots, 1, 1), \quad x_{W_n}^{(2)} = (1, 1, \dots, 1, 2), \dots, \quad x_{W_n}^{(k)} = (1, 1, \dots, 1, k),$$

$$x_{W_n}^{(k+1)} = (1, 1, \dots, 2, 1), \quad x_{W_n}^{(2)} = (1, 1, \dots, 2, 2), \dots, \quad x_{W_n}^{(2k)} = (1, 1, \dots, 2, k),$$

$$\vdots$$

$$x_{W_n}^{(|W_n|-k+1)} = (k, k, \dots, k, 1), \quad x_{W_n}^{(|W_n|-k+2)} = (k, k, \dots, k, 2), \dots,$$

$$x_{W_n}^{|W_n|} = (k, k, \dots, k, k).$$

Analogously, for a given vertex x, we shall use the following notation for the set of direct successors of x:

$$\overrightarrow{S(x)} := ((x, 1), (x, 2), \dots, (x, k)), \quad \overleftarrow{S(x)} := ((x, k), (x, k-1), \dots, (x, 1)).$$

In what follows, for the sake of simplicity, we will use notation $i \in \overrightarrow{S(x)}$ (respectively $i \in \overleftarrow{S(x)}$ instead of $(x, i) \in \overrightarrow{S(x)}$ (respectively $(x, i) \in \overleftarrow{S(x)}$).

Assume that for each edge $\langle x, y \rangle \in E$ of the tree, an operator $K_{\langle x,y \rangle} \in \mathcal{B}_{\{x,y\}}$ is assigned. We would like to define a state on $\mathcal{B}_{\Lambda_n}$ with boundary conditions $w_0 \in$

$\mathcal{B}_{(0),+}$ and $\mathbf{h} = \{h_x \in \mathcal{B}_{x,+}\}_{x \in L}$. To do this, we denote

$$K_{[m-1,m]} := \prod_{x \in \vec{W}_{m-1}} \prod_{y \in \overrightarrow{S(x)}} K_{\langle x,y \rangle}, \tag{3.1}$$

$$\mathbf{h}_n^{1/2} := \prod_{x \in \vec{W}_n} h_x^{1/2}, \quad \mathbf{h}_n := \mathbf{h}_n^{1/2}(\mathbf{h}_n^{1/2})^*, \tag{3.2}$$

$$K_n := w_0^{1/2} K_{[0,1]} K_{[1,2]} \cdots K_{[n-1,n]} \mathbf{h}_n^{1/2}, \tag{3.3}$$

$$\mathcal{W}_{n]} := K_n K_n^*. \tag{3.4}$$

It is clear that $\mathcal{W}_{n]}$ is positive.

In what follows, by $\mathrm{Tr}_\Lambda : \mathcal{B}_L \to \mathcal{B}_\Lambda$ we mean normalized partial trace, for any $\Lambda \subseteq_{\mathrm{fin}} L$. For the sake of shortness we put $\mathrm{Tr}_{n]} := \mathrm{Tr}_{\Lambda_n}$.

Let us define a positive functional $\varphi_{w_0,\mathbf{h}}^{(n,f)}$ on $\mathcal{B}_{\Lambda_n}$ by

$$\varphi_{w_0,\mathbf{h}}^{(n,f)}(a) = \mathrm{Tr}(\mathcal{W}_{n+1]}(a \otimes \mathbb{1}_{W_{n+1}})), \tag{3.5}$$

for every $a \in \mathcal{B}_{\Lambda_n}$, where $\mathbb{1}_{W_{n+1}} = \bigotimes_{y \in W_{n+1}} \mathbb{1}$. Note that here, Tr is a normalized trace on $\mathcal{B}_L$.

To get an infinite-volume state $\varphi^{(f)}$ on $\mathcal{B}_L$ such that $\varphi^{(f)}\lceil_{\mathcal{B}_{\Lambda_n}} = \varphi_{w_0,\mathbf{h}}^{(n,f)}$, we need to impose some constrains to the boundary conditions $\{w_0, \mathbf{h}\}$ so that the functionals $\{\varphi_{w_0,\mathbf{h}}^{(n,f)}\}$ satisfy the compatibility condition, i.e.

$$\varphi_{w_0,\mathbf{h}}^{(n+1,f)}\lceil_{\mathcal{B}_{\Lambda_n}} = \varphi_{w_0,\mathbf{h}}^{(n,f)}. \tag{3.6}$$

Theorem 3.1. *Let the boundary conditions* $w_0 \in \mathcal{B}_{(0),+}$ *and* $\mathbf{h} = \{h_x \in \mathcal{B}_{x,+}\}_{x \in L}$ *satisfy the following conditions:*

$$\mathrm{Tr}(w_0 h_0) = 1, \tag{3.7}$$

$$\mathrm{Tr}_{x]}\left[\prod_{y \in \overrightarrow{S(x)}} K_{\langle x,y \rangle} \prod_{y \in \overrightarrow{S(x)}} h_y \prod_{y \in \overleftarrow{S(x)}} K_{\langle x,y \rangle}^*\right] = h_x \quad \text{for every } x \in L. \tag{3.8}$$

Then the functionals $\{\varphi_{w_0,\mathbf{h}}^{(n,f)}\}$ *satisfy the compatibility condition* (3.6). *Moreover, there is a unique forward quantum d-Markov chain* $\varphi_{w_0,\mathbf{h}}^{(f)}$ *on* $\mathcal{B}_L$ *such that* $\varphi_{w_0,\mathbf{h}}^{(f)} = w - \lim_{n \to \infty} \varphi_{w_0,\mathbf{h}}^{(n,f)}$.

Proof. First note that Ref. 8 a family of states $\{\varphi_{w_0,\mathbf{h}}^{(n,f)}\}$ satisfy the compatibility condition if a sequence $\{\mathcal{W}_{n]}\}$ is *projective* with respect to $\mathrm{Tr}_{n]}$, i.e.

$$\mathrm{Tr}_{n-1]}(\mathcal{W}_{n]}) = \mathcal{W}_{n-1]}, \quad \forall n \in \mathbb{N}. \tag{3.9}$$

Now let us check the equality (3.9). From (3.1)–(3.4) one has

$$\mathcal{W}_{n]} = w_0^{1/2} \left(\prod_{m=1}^{n-1} K_{[m-1,m]} \right) K_{[n-1,n]} \mathbf{h}_n K_{[n-1,n]}^* \left(\prod_{m=1}^{n-1} K_{[m-1,m]} \right)^* w_0^{1/2}.$$

We know that for different x and x' taken from W_{n-1} the algebras $\mathcal{B}_{x \cup S(x)}$ and $\mathcal{B}_{x' \cup S(x')}$ commute, therefore from Eq. (3.2) one finds

$$K_{[n-1,n]} \mathbf{h}_n K_{[n-1,n]}^* = \prod_{x \in \overrightarrow{W}_{n-1}} \left(\prod_{y \in \overrightarrow{S(x)}} K_{\langle x,y \rangle} \right) \left(\prod_{y \in \overrightarrow{S(x)}} h_y \right) \left(\prod_{y \in \overleftarrow{S(x)}} K_{\langle x,y \rangle}^* \right).$$

Hence, from the last equality with (3.8) we get

$$\mathrm{Tr}_{n-1]}(\mathcal{W}_{n]}) = w_0^{1/2} \left(\prod_{m=1}^{n-1} K_{[m-1,m]} \right)$$

$$\cdot \prod_{x \in \overrightarrow{W}_{n-1}} \mathrm{Tr}_x \left(\prod_{y \in \overrightarrow{S(x)}} K_{\langle x,y \rangle} \prod_{y \in \overrightarrow{S(x)}} h_y \prod_{y \in \overleftarrow{S(x)}} K_{\langle x,y \rangle}^* \right)$$

$$\cdot \left(\prod_{m=1}^{n-1} K_{[m-1,m]} \right)^* w_0^{1/2}$$

$$= w_0^{1/2} \left(\prod_{m=1}^{n-1} K_{[m-1,m]} \right) \prod_{x \in \overrightarrow{W}_{n-1}} h_x \left(\prod_{m=1}^{n-1} K_{[m-1,m]} \right)^* w_0^{1/2}$$

$$= \mathcal{W}_{n-1]}.$$

From the above argument and (3.7), one can show that $\mathcal{W}_{n]}$ is density operator, i.e. $\mathrm{Tr}(\mathcal{W}_{n]}) = 1$.

Let us show that the defined state $\varphi_{w_0, \mathbf{h}}^{(f)}$ is a forward QMC. Indeed, define quasi-conditional expectations $\mathcal{E}_{\Lambda_n^c}$ as follows:

$$\hat{\mathcal{E}}_{\Lambda_1^c}(x_{[0]}) = \mathrm{Tr}_{[1}(K_{[0,1]} w_0^{1/2} x_{[0} w_0^{1/2} K_{[0,1]}^*), \quad x_{[0} \in \mathcal{B}_{\Lambda_0^c} \tag{3.10}$$

$$\mathcal{E}_{\Lambda_k^c}(x_{[k-1]}) = \mathrm{Tr}_{[n}(K_{[k-1,k]} x_{[k-1} K_{[k-1,k]}^*), \quad x_{[k-1} \in \mathcal{B}_{\Lambda_{k-1}^c},$$

$$k = 1, 2, \ldots, n+1, \tag{3.11}$$

here $\mathrm{Tr}_{[n} = \mathrm{Tr}_{\Lambda_n^c}$. Then for any monomial $a_{\Lambda_1} \otimes a_{W_2} \otimes \cdots \otimes a_{W_n} \otimes \mathbb{1}_{W_{n+1}}$, where $a_{\Lambda_1} \in \mathcal{B}_{\Lambda_1}, a_{W_k} \in \mathcal{B}_{W_k}, (k = 2, \ldots, n)$, we have

$$\varphi_{w_0, \mathbf{h}}^{(n,f)}(a_{\Lambda_1} \otimes a_{W_2} \otimes \cdots \otimes a_{W_n}) = \mathrm{Tr}(\mathbf{h}_{n+1} K_{[n,n+1]}^* \cdots K_{[0,1]}^* w_0^{1/2}(a_{\Lambda_1} \otimes a_{W_2}$$

$$\otimes \cdots \otimes a_{W_n}) w_0^{1/2} K_{[0,1]} \cdots K_{[n,n+1]})$$

$$= \mathrm{Tr}_{[1}\big(\mathbf{h}_{n+1}K^*_{[n,n+1]}\cdots K^*_{[1,2]}\hat{\mathcal{E}}_{\Lambda^c_1}(a_{\Lambda_1})a_{W_2}K_{[1,2]}$$

$$\cdots a_{W_n}K_{[n,n+1]}\big)$$

$$= \mathrm{Tr}_{[n+1}\big(\mathbf{h}_{n+1}\mathcal{E}_{\Lambda^c_{n+1}}\circ\mathcal{E}_{\Lambda^c_n}\circ\cdots$$

$$\mathcal{E}_{\Lambda^c_2}\circ\hat{\mathcal{E}}_{\Lambda^c_n}(a_{\Lambda_1}\otimes a_{W_2}\otimes\cdots\otimes a_{W_n})\big). \tag{3.12}$$

Hence, for any $a\in\Lambda\subset\Lambda_{n+1}$ from (3.5) with (3.1), (3.2)–(3.10) and (3.12) one can see that

$$\varphi^{(n,f)}_{w_0,\mathbf{h}}(a) = \mathrm{Tr}_{[n+1}\big(\mathbf{h}^{n+1}\mathcal{E}_{\Lambda^c_{n+1}}\circ\mathcal{E}_{\Lambda^c_n}\circ\cdots\circ\mathcal{E}_{\Lambda^c_2}\circ\hat{\mathcal{E}}_{\Lambda^c_n}(a)\big). \tag{3.13}$$

The projectivity of $\mathcal{W}_{n]}$ yields the equality (2.2) for $\varphi^{(n,f)}_{w_0,\mathbf{h}}$, therefore, from (3.13) we conclude that $\varphi^{(f)}_{w_0,\mathbf{h}}$ is a forward QMC. $\qquad\square$

Corollary 3.2. *If* (3.7), (3.8) *are satisfied, then one has* $\varphi^{(n,f)}_{w_0,\mathbf{h}}(a) = \mathrm{Tr}(\mathcal{W}_{n]}(a))$ *for any* $a\in\mathcal{B}_{\Lambda_n}$.

Remark 3.3. Note that if $k=1$ and $h_x=I$ for all $x\in L$, then we get conditional amplitudes introduced by Accardi.[8]

Observe that the state $\varphi^{(f)}_{w_0,\mathbf{h}}$ has a backward structure. Indeed, let us first define

$$T_k[X](y) = \mathrm{Tr}_{k]}(K_{[k,k+1]}XK^*_{[k,k+1]}y), \quad X\in\mathcal{B}_{\Lambda_{[k,k+1]}}, \quad y\in\mathcal{B}_{W_{k+1}}. \tag{3.14}$$

Then, using Corollary 3.2 one finds

$$\varphi^{(n,f)}_{w_0,\mathbf{h}}(a_{\Lambda_0}\otimes a_{W_1}\otimes\cdots\otimes a_{W_n}) = \mathrm{Tr}\big(w_0^{1/2}K_{[0,1]}\cdots K_{[n-1,n]}\mathbf{h}_nK^*_{[n-1,n]}\cdots K^*_{[0,1]}w_0^{1/2}$$

$$\cdot(a_{\Lambda_0}\otimes a_{W_1}\otimes\cdots\otimes a_{W_n})\big)$$

$$= \mathrm{Tr}\big(w_0^{1/2}K_{[0,1]}\cdots K_{[n-2,n-1]}\mathrm{Tr}_{n-1]}$$

$$\cdot(K_{[n-1,n]}\mathbf{h}_nK^*_{[n-1,n]}a_{W_n})$$

$$\cdot K^*_{[n-2,n-1]}a_{W_{n-1}}\cdots K^*_{[0,1]}a_{W_1}w_0^{1/2}a_{\Lambda_0}\big)$$

$$= \mathrm{Tr}\big(w_0^{1/2}K_{[0,1]}\cdots K_{[n-3,n-2]}T_{n-2}[T_{n-1}[\mathbf{h}_n](a_{W_n})]$$

$$\cdot(a_{W_{n-2}})K_{[n-3,n-2]}a_{W_{n-2}}\cdots K^*_{[0,1]}a_{W_1}w_0^{1/2}a_{\Lambda_0}\big)$$

$$= \mathrm{Tr}\big(w_0^{1/2}T_0[T_1[\cdots[T_{n-1}[\mathbf{h}_n](a_{W_n})]$$

$$\cdot(a_{W_{n-2}})\cdots](a_{W_1})]w_0^{1/2}a_{\Lambda_0}\big). \tag{3.15}$$

Remark 3.4. Formula (3.15) reminds the structure of a backward quantum Markov chain, however there is an important difference. For any positive $X\in$

$\mathcal{B}_{\Lambda_{[n-1,n]}}$ the maps

$$a_n \in \mathcal{B}_{W_n} \mapsto \mathrm{Tr}_{n-1]}(K_{[n-1,n]}X K^*_{[n-1,n]}a_n) =: E(a_n) \in \mathcal{B}_{W_{n-1}}$$

will be in general anti-CP rather than CP (i.e. the map $E^*(x) := E(x^*)$ is CP) (see Ref. 2). We will show elsewhere that there is indeed a deep connection between the present construction and Cecchini's λ-operator.[15, 16]

Remark 3.5. Note that the above construction has the advantage to work on arbitrary local algebras. It generalizes the construction in Ref. 2. Under additional assumptions, the local structure of the state becomes more transparent. It also exhibits a "forward" local structure which, however, is not backward Markovian in the sense of Definition 2.1, but rather in the sense of Cecchini.[15] The duality between a "forward" and "backward" Markovianity, emerging from (3.13), (3.15) is a nontrivial quantum extension of the fact that in a classical framework the two notions are equivalent and seems to deserve a deeper study.

4. Forward QMC Associated with XY-Model

In this section, we prove uniqueness of the quantum d-Markov chain associated with XY-model on a Cayley tree of order two. In what follows, we consider a semi-infinite Cayley tree $\Gamma^2 = (L, E)$ of order two. Our starting C^*-algebra is the same $\mathcal{B}_L$ but with $\mathcal{B}_x = M_2(\mathbb{C})$ for $x \in L$. By $\sigma_x^{(u)}, \sigma_y^{(u)}, \sigma_z^{(u)}$ we denote the Pauli spin operators for site $u \in L$. Here

$$\sigma_x^{(u)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y^{(u)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z^{(u)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.1}$$

For every edge $\langle u, v \rangle \in E$ put

$$K_{\langle u,v \rangle} = \exp\{\beta H_{\langle u,v \rangle}\}, \quad \beta > 0, \tag{4.2}$$

where

$$H_{\langle u,v \rangle} = \frac{1}{2}\big(\sigma_x^{(u)}\sigma_x^{(v)} + \sigma_y^{(u)}\sigma_y^{(v)}\big). \tag{4.3}$$

Now taking into account the following equalities

$$H_{\langle u,v \rangle}^{2m} = H_{\langle u,v \rangle}^2 = \frac{1}{2}\big(\mathbb{1} - \sigma_z^{(u)}\sigma_z^{(v)}\big), \quad H_{\langle u,v \rangle}^{2m-1} = H_{\langle u,v \rangle}, \quad m \in \mathbb{N},$$

one finds

$$K_{\langle u,v \rangle} = \mathbb{1} + \sinh\beta H_{\langle u,v \rangle} + (\cosh\beta - 1)H_{\langle u,v \rangle}^2.$$

We are going to describe all solutions $\mathbf{h} = \{h_x\}$ and w_0 of Eqs. (3.7) and (3.8). Furthermore, we shall assume that $h_x = h_y$ for every $x, y \in W_n$, $n \in \mathbb{N}$. Hence, we

denote $h_x^{(n)} := h_x$, if $x \in W_n$. Now from (4.2), (4.3) one can see that $K_{\langle u,u \rangle} = K_{\langle u,v \rangle}^*$, therefore, Eq. (3.8) can be rewritten as follows:

$$\text{Tr}_x(K_{\langle x,y \rangle} K_{\langle x,z \rangle} h_y^{(n+1)} h_z^{(n+1)} K_{\langle x,z \rangle} K_{\langle x,y \rangle}) = h_x^{(n)}, \quad \text{for every } x \in L. \quad (4.4)$$

After small calculations, Eq. (4.4) reduces to the following system of equations:

$$\begin{cases} \left(\dfrac{a_{11}^{(n+1)} + a_{22}^{(n+1)}}{2}\right)^2 \cosh^4 \beta + a_{12}^{(n+1)} a_{21}^{(n+1)} \sinh^2 \beta \cosh \beta = a_{11}^{(n)}, \\[4mm] a_{12}^{(n+1)} \dfrac{a_{11}^{(n+1)} + a_{22}^{(n+1)}}{2} \sinh \beta \cosh \beta (1 + \cosh \beta) = a_{12}^{(n)}, \\[4mm] a_{21}^{(n+1)} \dfrac{a_{11}^{(n+1)} + a_{22}^{(n+1)}}{2} \sinh \beta \cosh \beta (1 + \cosh \beta) = a_{21}^{(n)}, \\[4mm] \left(\dfrac{a_{11}^{(n+1)} + a_{22}^{(n+1)}}{2}\right)^2 \cosh^4 \beta + a_{12}^{(n+1)} a_{21}^{(n+1)} \sinh^2 \beta \cosh \beta = a_{22}^{(n)}, \end{cases} \quad (4.5)$$

here

$$h_x^{(n)} = \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} \\ a_{21}^{(n)} & a_{22}^{(n)} \end{pmatrix}, \quad h_y^{(n+1)} = h_z^{(n+1)} = \begin{pmatrix} a_{11}^{(n+1)} & a_{12}^{(n+1)} \\ a_{21}^{(n+1)} & a_{22}^{(n+1)} \end{pmatrix}.$$

From (4.5) we immediately get that $a_{11}^{(n)} = a_{22}^{(n)}$ for all $n \in \mathbb{N}$.

Self-adjointness of $h_x^{(n)}$, i.e. $\overline{a_{12}^{(n)}} = a_{21}^{(n)}$, for any $n \in \mathbb{N}$, allows us to reduce the system (4.5) to

$$\begin{cases} (a_{11}^{(n+1)})^2 \cosh^4 \beta + |a_{12}^{(n+1)}|^2 \sinh^2 \beta \cosh \beta = a_{11}^{(n)}, \\[2mm] a_{12}^{(n+1)} a_{11}^{(n+1)} \sinh \beta \cosh \beta (1 + \cosh \beta) = a_{12}^{(n)}. \end{cases} \quad (4.6)$$

Remark 4.1. Note that according to positivity and invertibility of $h_x^{(n)}$ we conclude that $a_{11}^{(n)} > |a_{12}^{(n)}|$ for all $n \in \mathbb{N}$.

Now we are going to investigate the derive system (4.6). To do this, let us consider the following system of equations:

$$\begin{cases} (x')^2 \cosh^4 \beta + |y'|^2 \sinh^2 \beta \cosh \beta = x, \\[2mm] x'y' \sinh \beta \cosh \beta (1 + \cosh \beta) = y, \end{cases} \quad (4.7)$$

here as before $\beta > 0$.

Taking from both sides of the second equation of (4.7) modules, we get

$$\begin{cases} (x')^2 \cosh^4 \beta + |y'|^2 \sinh^2 \beta \cosh \beta = x, \\ x'|y'| \sinh \beta \cosh \beta (1 + \cosh \beta) = |y|. \end{cases}$$

Therefore, in the sequel we shall consider the system given by

$$\begin{cases} (x')^2 \cosh^4 \beta + (y')^2 \sinh^2 \beta \cosh \beta = x \\ x'y' \sinh \beta \cosh \beta (1 + \cosh \beta) = y. \end{cases} \tag{4.8}$$

We want to show that the system (4.8) defines a dynamical system $f : (x, y) \in \Delta \to (x', y') \in \Delta$, where $\Delta = \{(x, y) \in \mathbb{R}_+^2 : x > y\}$. In fact, we should show that the system (4.8) has at most one solution with respect to x', y' in the domain Δ. From (4.8) we get

$$\begin{cases} (x')^2 \cosh^4 \beta + (y')^2 \sinh^2 \beta \cosh \beta = x, \\ (x')^2 (y')^2 \sinh^2 \beta \cosh^2 \beta (1 + \cosh \beta)^2 = y^2. \end{cases}$$

Letting $(x')^2 = u$ and $(y')^2 = v$, one finds

$$\begin{cases} u \cosh^4 \beta + v \sinh^2 \beta \cosh \beta = x, \\ uv \sinh^2 \beta \cosh^2 \beta (1 + \cosh \beta)^2 = y^2. \end{cases}$$

Then v can be represented by u as follows:

$$v = \frac{x - u \cosh^4 \beta}{\sinh^2 \beta \cosh \beta}. \tag{4.9}$$

Using this, we obtain the following quadratic equation

$$\cosh^5 \beta (1 + \cosh \beta)^2 \cdot u^2 - x \cosh \beta (1 + \cosh \beta)^2 \cdot u + y^2 = 0.$$

Solving such an equation w.r.t. u, we can find

$$u_\pm = \frac{x \pm \sqrt{x^2 - 4y^2 \frac{\cosh^3 \beta}{(1 + \cosh \beta)^2}}}{2 \cosh^4 \beta}.$$

Then from (4.9) one gets

$$v_\pm = \frac{x \mp \sqrt{x^2 - 4y^2 \frac{\cosh^3 \beta}{(1 + \cosh \beta)^2}}}{2 \sin h^2 \beta \cosh \beta}.$$

Since the point (x', y') belongs to the domain Δ, then u should be greater than v. Therefore, for every $(x, y) \in \Delta$ there exists at most one pair $(x', y') \in \Delta$ and the system (4.8) defines a dynamical system $f : \Delta \to \Delta$ with the following explicit form

$$\begin{cases} x' = \sqrt{\dfrac{x + \sqrt{x^2 - 4y^2 \frac{\cosh^3 \beta}{(1 + \cosh \beta)^2}}}{2 \cosh^4 \beta}}, \\[4mm] y' = \sqrt{\dfrac{x - \sqrt{x^2 - 4y^2 \frac{\cosh^3 \beta}{(1 + \cosh \beta)^2}}}{2 \sinh^2 \beta \cosh \beta}}. \end{cases} \tag{4.10}$$

Remark 4.2. Note that from (4.10) one can see that the map f is well defined if and only if x and y satisfy

$$x \geq 2y\sqrt{\frac{\cosh^3 \beta}{(1+\cosh \beta)^2}}. \tag{4.11}$$

In the sequel, we shall use interchangeably two forms (4.8), (4.10) of the dynamical system $f : \Delta \to \Delta$.

Further, we will need the following auxiliary fact:

Lemma 4.3. *If $\beta > 0$, then*

$$0 < \sinh \beta \cosh \beta (1 + \cosh \beta) < \cosh^4 \beta.$$

The proof is provided in the Appendix.

Let us first find all of the fixed points of the dynamical system $f : \Delta \to \Delta$.

Theorem 4.4. *Let $f : \Delta \to \Delta$ be a dynamical system given by (4.8). Then the following assertions hold true*:

(i) *The dynamical system $f : \Delta \to \Delta$ has a unique fixed point*;
(ii) *The dynamical system $f : \Delta \to \Delta$ does not have any k $(k \geq 2)$ periodic points.*

Proof. (i) Assume that (x, y) is a fixed point, i.e.

$$\begin{cases} x^2 \cosh^4 \beta + y^2 \sinh^2 \beta \cosh \beta = x, \\ xy \sinh \beta \cosh \beta (1 + \cosh \beta) = y. \end{cases} \tag{4.12}$$

Consider two different cases with respect to y.

Case (A). Let $y = 0$. Then one finds that either $x = 0$ or $x = \frac{1}{\cosh^4 \beta}$. But, only the point $(\frac{1}{\cosh^4 \beta}, 0)$ belongs to the domain Δ.

Case (B). Now suppose $y \neq 0$. Then from (4.12) one finds

$$x = \frac{1}{\sinh \beta \cosh \beta (1 + \cosh \beta)},$$

hence, we obtain

$$y^2 \sinh^2 \beta \cosh \beta = \frac{\sinh \beta \cosh \beta (1 + \cosh \beta) - \cosh^4 \beta}{\sinh^2 \beta \cosh^2 \beta (1 + \cosh \beta)^2}.$$

But, due to Lemma 4.3, we infer that

$$\frac{\sinh \beta \cosh \beta (1 + \cosh \beta) - \cosh^4 \beta}{\sinh^2 \beta \cosh^2 \beta (1 + \cosh \beta)^2} < 0$$

which is impossible. Therefore, in this case the dynamical system does not have any fixed point.

Consequently, the dynamical system has a unique fixed point which is equal to $(\frac{1}{\cosh^4 \beta}, 0)$.

(ii) Now let us turn to study periodic points of the dynamical system (4.8). Assume that the system has a periodic point $(x^{(0)}, y^{(0)})$ with a period of $k \geq 2$ in Δ. This means that there are points

$$(x^{(0)}, y^{(0)}), (x^{(1)}, y^{(1)}), \ldots, (x^{(k-1)}, y^{(k-1)}) \in \Delta,$$

such that they satisfy the following equalities

$$\begin{cases} (x^{(i+1)})^2 \cosh^4 \beta + (y^{(i+1)})^2 \sinh^2 \beta \cosh \beta = x^{(i)}, \\ x^{(i+1)} y^{(i+1)} \sinh \beta \cosh \beta (1 + \cosh \beta) = y^{(i)}, \end{cases} \tag{4.13}$$

where $i = \overline{0, k-1}$, i.e. $f(x^{(i)}, y^{(i)}) = (x^{(i+1)}, y^{(i+1)})$, with $x^{(k)} = x^{(0)}$, $y^{(k)} = y^{(0)}$.

Now again consider two different cases with respect to $y^{(0)}$.

Case (A). Let $y^{(0)} \neq 0$. Then $x^{(i)}, y^{(i)}$ should be positive for all $i = \overline{0, k-1}$. Therefore, we have

$$\frac{x^{(i)}}{y^{(i)}} = \frac{\left(\frac{x^{(i+1)}}{y^{(i+1)}}\right)^2 \cosh^4 \beta + \sinh^2 \beta \cosh \beta}{\frac{x^{(i+1)}}{y^{(i+1)}} \sinh \beta \cosh \beta (1 + \cosh \beta)}$$

$$= \frac{\cosh^3 \beta}{\sinh \beta (1 + \cosh \beta)} \cdot \frac{x^{(i+1)}}{y^{(i+1)}} + \frac{\sinh \beta}{1 + \cosh \beta} \cdot \frac{y^{(i+1)}}{x^{(i+1)}},$$

where $i = \overline{0, k-1}$.

Due to $x^{(i)}, y^{(i)} > 0$ for all $i = \overline{0, k-1}$, we obtain

$$\frac{x^{(i)}}{y^{(i)}} > \frac{\cosh^3 \beta}{\sinh \beta (1 + \cosh \beta)} \cdot \frac{x^{(i+1)}}{y^{(i+1)}}, \tag{4.14}$$

for all $i = \overline{0, k-1}$.

It then follows from (4.14) that

$$\frac{x^{(0)}}{y^{(0)}} > \left(\frac{\cosh^3 \beta}{\sinh \beta (1 + \cosh \beta)}\right)^k \cdot \frac{x^{(0)}}{y^{(0)}}.$$

But, the last inequality is impossible, since Lemma 4.3 implies

$$\frac{\cosh^3 \beta}{\sinh \beta (1 + \cosh \beta)} > 1.$$

Hence, in this case, the dynamical system (4.8) does not have any periodic point with $k \geq 2$.

Case (B). Now suppose that $y^{(0)} = 0$. Since $k \geq 2$ we have $x^{(0)} \neq \frac{1}{\cosh^4 \beta}$. So, from (4.13) we find that $y^{(i)} = 0$ for all $i = \overline{0, k-1}$. Then again (4.13) implies that

$$(x^{(i+1)})^2 \cosh^4 \beta = x^{(i)}, \quad \forall i = \overline{0, k-1},$$

which means

$$x^{(i+1)} = \frac{1}{\cosh^2 \beta} \sqrt{x^{(i)}}, \quad \forall i = \overline{0, k-1}.$$

Hence, we have

$$x^{(0)} = \frac{1}{\cosh^4 \beta} \sqrt[2^{k+1}]{x^{(0)} \cosh^4 \beta}.$$

This yields either $x^{(0)} = 0$ or $x^{(0)} = \frac{1}{\cosh^4 \beta}$, which is a contradiction. $\square$

Lemma 4.5. *Let $f : \Delta \to \Delta$ be a dynamical system given by (4.10). If x, y are positive and satisfy (4.11) then x', y' are positive and satisfy the following inequality*

$$\frac{x'}{y'} < \frac{\sinh \beta (1 + \cosh \beta)}{\cosh^3 \beta} \cdot \frac{x}{y}. \tag{4.15}$$

Proof. From (4.10) one can see that if x, y are positive and satisfy the condition (4.11), then x', y' are positive as well. From (4.10) we find

$$\frac{x'}{y'} = \frac{\sinh \beta (1 + \cosh \beta)}{\cosh^3 \beta} \cdot \frac{x + \sqrt{x^2 - 4y^2 \frac{\cosh^3 \beta}{(1 + \cosh \beta)^2}}}{2y}$$

$$< \frac{\sinh \beta (1 + \cosh \beta)}{\cosh^3 \beta} \cdot \frac{x}{y},$$

which is the desired inequality. $\square$

Now, we are going to study an asymptotical behavior of the trajectory of the dynamical system (4.10).

Theorem 4.6. *Let $f : \Delta \to \Delta$ be a dynamical system given by (4.10). Then the following assertions hold true*:

(i) *If $y^{(0)} > 0$ then finite terms of the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$, starting from the point $(x^{(0)}, y^{(0)})$ are contained in Δ, i.e. there exists a number $N_0 \in \mathbb{N}$ such that any term of the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$ does not belong to Δ except the first N_0 members.*

(ii) *If* $y^{(0)} = 0$ *then the trajectory* $\{(x^{(n)}, y^{(n)})\}_{n=1}^{\infty}$ *starting from the point* $(x^{(0)}, y^{(0)})$ *has the following form*

$$
\begin{cases}
x^{(n)} = \dfrac{\sqrt[2^n]{x^{(0)} \cosh^4 \beta}}{\cosh^4 \beta}, \\[2mm]
y^{(n)} = 0.
\end{cases}
$$

Proof. (i) Let $y^{(0)} > 0$ and assume that the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=1}^{\infty}$ of the dynamical system starting from the point $(x^{(0)}, y^{(0)})$ is infinite. This means that the points $(x^{(n)}, y^{(n)})$ are well defined for all $n \in \mathbb{N}$. Then according to Remark 4.2 and Lemma 4.5 we have

$$
\frac{x^{(n)}}{y^{(n)}} < \left(\frac{\sinh \beta (1 + \cosh \beta)}{\cosh^3 \beta} \right)^n \cdot \frac{x^{(0)}}{y^{(0)}} \tag{4.16}
$$

for all $n \in \mathbb{N}$.

On the other hand, according to Remark 4.2, $x^{(n)}$ and $y^{(n)}$ should satisfy the following inequality

$$
\frac{x^{(n)}}{y^{(n)}} \geq 2 \sqrt{\frac{\cosh^3 \beta}{(1 + \cosh \beta)^2}}, \tag{4.17}
$$

for all $n \in \mathbb{N}$. Due to Lemma 4.3 we find

$$
\left(\frac{\sinh \beta (1 + \cosh \beta)}{\cosh^3 \beta} \right)^n \to 0 \quad \text{as } n \to \infty,
$$

which with (4.16) implies that the inequality (4.17) is not satisfied starting from some number $N_0 \in \mathbb{N}$. This contradiction shows that the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=1}^{\infty}$ must be finite.

(ii) Now let $y^{(0)} = 0$, then (4.10) implies $y^{(n)} = 0$ for all $n \in \mathbb{N}$. Hence, from (4.10) one finds

$$
x^{(n)} = \sqrt{\frac{x^{(n-1)}}{\cosh^4 \beta}}.
$$

So iterating last equality we obtain

$$
x^{(n)} \cosh^4 \beta = \sqrt[2^n]{x^{(0)} \cosh^4 \beta},
$$

which yields the desired equality. $\qquad\square$

From the last Theorem 4.6, we infer that Eq. (4.4) has a lot of parametrical solutions $(w_0(\alpha), \{h_x(\alpha)\})$ given by

$$
w_0(\alpha) = \begin{pmatrix} \dfrac{1}{\alpha} & 0 \\[2mm] 0 & \dfrac{1}{\alpha} \end{pmatrix}, \quad h_x^{(n)}(\alpha) = \begin{pmatrix} \dfrac{\sqrt[2^n]{\alpha \cosh^4 \beta}}{\cosh^4 \beta} & 0 \\[3mm] 0 & \dfrac{\sqrt[2^n]{\alpha \cosh^4 \beta}}{\cosh^4 \beta} \end{pmatrix}, \tag{4.18}
$$

for every $x \in V$. Here α is any positive real number.

The boundary conditions corresponding to the fixed point of (4.8) are the following ones:

$$w_0 = \begin{pmatrix} \cosh^4 \beta & 0 \\ 0 & \cosh^4 \beta \end{pmatrix}, \quad h_x^{(n)} = \begin{pmatrix} \dfrac{1}{\cosh^4 \beta} & 0 \\ 0 & \dfrac{1}{\cosh^4 \beta} \end{pmatrix}, \quad \forall\, x \in V, \quad (4.19)$$

which correspond to the value of $\alpha_0 = \frac{1}{\cosh^4 \beta}$ in (4.18). Therefore, further, we denote such operators by $w_0(\alpha_0)$ and $h_x^{(n)}(\alpha_0)$.

Let us consider the states $\varphi_{w_0(\alpha),\mathbf{h}(\alpha)}^{(n,f)}$ corresponding to the solutions $(w_0(\alpha), \{h_x^{(n)}(\alpha)\})$. By definition we have

$$\varphi_{w_0(\alpha),\mathbf{h}(\alpha)}^{(n,f)}(x) = \mathrm{Tr}\left(w_0^{1/2}(\alpha) \prod_{i=0}^{n-1} K_{[i,i+1]} \prod_{x \in \vec{W}_n} h_x^{(n)}(\alpha) \prod_{i=1}^{n} K_{[n-i,n+1-i]} w_0^{1/2}(\alpha) x \right)$$

$$= \frac{(\sqrt[2^{n+1}]{\alpha \cosh^4 \beta})^{2^{n+1}}}{\alpha (\cosh^4 \beta)^{2^{n+1}}} \mathrm{Tr}\left(\prod_{i=0}^{n-1} K_{[i,i+1]} \prod_{i=1}^{n} K_{[n-i,n+1-i]} x \right)$$

$$= \frac{\alpha_0^{2^{n+1}}}{\alpha_0} \mathrm{Tr}\left(\prod_{i=0}^{n-1} K_{[i,i+1]} \prod_{i=1}^{n} K_{[n-i,n+1-i]} x \right)$$

$$= \mathrm{Tr}\left(w_0^{1/2}(\alpha_0) \prod_{i=0}^{n-1} K_{[i,i+1]} \prod_{x \in \vec{W}_n} h_x^{(n)}(\alpha_0) \right.$$

$$\left. \cdot \prod_{i=1}^{n} K_{[n-i,n+1-i]} w_0^{1/2}(\alpha_0) x \right) = \varphi_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}^{(n,f)}(x), \quad (4.20)$$

for any α. Hence, from the definition of forward QMC it follows that $\varphi_{w_0(\alpha),\mathbf{h}(\alpha)}^{(f)} = \varphi_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}^{(f)}$, which yields that the uniqueness of the forward QMC associated with the model (4.2). Therefore we have the following

Theorem 4.7. *There is a unique forward QMC for the model* (4.2).

Note that in Ref. 13 it was proved uniqueness of the ground state for the one-dimensional quantum XY-model

$$H = \beta \sum_{n \in \mathbb{Z}} \{ \sigma_x^{(n)} \sigma_x^{(n+1)} + \sigma_y^{(n)} \sigma_y^{(n+1)} \}. \quad (4.21)$$

The proved Theorem 4.7 suggests that similar result can be obtained for the Hamiltonian (4.21) in a Cayley tree of order two.

5. Observation

Let us denote

$$\tilde{K}_n(\alpha) = w_0^{1/2}(\alpha) \prod_{\{x,y\} \in E_1} K_{\langle x,y \rangle} \prod_{\{x,y\} \in E_2 \setminus E_1} K_{\langle x,y \rangle} \cdots \prod_{\{x,y\} \in E_{n+1} \setminus E_n} K_{\langle x,y \rangle}. \quad (5.1)$$

Define a function $F(\beta)$ by the following formula

$$\beta F(\beta) = \lim_{n \to \infty} \frac{1}{|V_n|} \log \mathrm{Tr}(\tilde{K}_n^*(\alpha) \tilde{K}_n(\alpha)). \quad (5.2)$$

Using the same argument as in (4.20) one gets

$$\frac{1}{|V_n|} \log \mathrm{Tr}(\tilde{K}_n^*(\alpha) \tilde{K}_n(\alpha))$$

$$= \frac{1}{|V_n|} \left(\log \left(\frac{\sqrt[2^{n+1}]{\alpha \cosh^4 \beta}}{\cosh^4 \beta} \right)^{-|W_{n+1}|} + \log \left(\varphi_{w_0, \mathbf{h}}^{(b)}(\mathbb{1}) \right) \right)$$

$$= -\frac{|W_{n+1}|}{2^{n+1}|V_n|} \log \left(\alpha \cosh^4 \beta \right) + \frac{|W_{n+1}|}{|V_n|} \log \left(\cosh^4 \beta \right). \quad (5.3)$$

So, taking into account $\lim_{n \to \infty} \frac{|W_{n+1}|}{|V_n|} = 1$ from (5.3), (5.2) we find

$$F(\beta) = \frac{4}{\beta} \log \cosh \beta.$$

One can see that $F(\beta)$ is an analytical function when $\beta > 0$. This corresponds to the fact that the free energy of a system is an analytical function. Of course, here the defined function F is not a free energy of the XY-model. On the other hand, for the same model in a Cayley tree of order three we shall show the existence of the quantum phase transition (see Ref. 11). Moreover, it will be established that derivative of certain thermodynamic function will have discontinuity at critical values of β.

6. Conclusions

Let us note that a first attempt of consideration of quantum Markov fields began in Refs. 4, 6 for the regular lattices (namely for $\mathbb{Z}^d$). But there, concrete examples of such fields were not given. In this paper, we have extended a notion of QMC to fields, i.e. to Cayley tree. Note that such a tree is the simplest hierarchical lattice with non-amenable graph structure. This means that the ratio of the number of boundary sites W_n to the number of interior sites V_n (see Sec. 2, for the definitions of W_n and V_n) of the tree tends to a nonzero constant in the thermodynamic limit of a large system. Here QMCs have been considered on discrete infinite tensor products of C^*-algebras over such a tree. A tree structure of graphs allowed us to give constructions of QMC, which generalizes the construction of Ref. 3 to trees. Namely, we have provided a construction of forward QMC defined on Cayley tree.

By means of such a construction we proved uniqueness of forward QMC associated with XY-model on the second-order Cayley tree. We have to stress here that the constructed QMC associated with XY-model, is different from thermal states of that model, since such states corresponds to the $\exp(-\beta \sum_{\langle x,y \rangle} H_{\langle x,y \rangle})$, which is different from a product of $\exp(-\beta H_{\langle x,y \rangle})$. Roughly speaking, if we consider the usual Hamiltonian system $H(\sigma) = -\beta \sum_{\langle x,y \rangle} h_{\langle x,y \rangle}(\sigma)$, then its Gibbs measure is defined by the fraction

$$\mu(\sigma) = \frac{e^{-H(\sigma)}}{\sum_{\sigma} e^{-H(\sigma)}}. \tag{6.1}$$

Such a measure can be viewed by another way as well. Namely,

$$\mu(\sigma) = \frac{\prod_{\langle x,y \rangle} e^{\beta h_{\langle x,y \rangle}(\sigma)}}{\sum_{\sigma} \prod_{\langle x,y \rangle} e^{\beta h_{\langle x,y \rangle}(\sigma)}}. \tag{6.2}$$

A usual quantum mechanical definition of the quantum Gibbs states is based on Eq. (6.1). But our approach is based on an alternative way (see (6.2)) of the definition of the quantum Gibbs states. Note that whether or not the resulting states have a physical interest is a question that cannot be solved on a purely mathematical ground.

Appendix A. Proof of Lemma 4.3

It is clear that, if $\beta > 0$, then

$$\sinh \beta \cosh \beta (1 + \cosh \beta) > 0.$$

Now we are going to show that

$$\sinh \beta (1 + \cosh \beta) < \cosh^3 \beta. \tag{A.1}$$

Noting

$$\sinh \beta = \frac{e^{\beta} - e^{-\beta}}{2}, \quad \cosh \beta = \frac{e^{\beta} + e^{-\beta}}{2},$$

and letting $t = e^{\beta}$, we reduce the last inequality (A.1) to

$$t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1 > 0. \tag{A.2}$$

Since $\beta > 0$, then $t > 1$. Therefore, we shall show that (A.2) is satisfied whenever $t > 1$. Now consider several cases with respect to t.

Case I. Let $t \geq 1 + \sqrt{2}$. Then we have

$$t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1 = t^4 \big(t - (1 + \sqrt{2})\big)\big(t - (1 - \sqrt{2})\big) + 7t^2 + 2t + 1 > 0.$$

Case II. Let $2 \le t \le 1 + \sqrt{2}$. Then it is clear that $t < \sqrt{7}$. Therefore,

$$t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1 = t^5(t - 2) + t^2(7 - t^2) + 2t + 1 > 0.$$

Case III. Let $\sqrt{\frac{7}{2}} \le t \le 2$. Then one gets

$$2(t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1) = 2t^4\left(t^2 - \frac{7}{2}\right) + \frac{5}{2}t^4(2 - t)$$
$$+ \frac{3}{2}t^2(8 - t^3) + 2t^2 + 4t + 2 > 0.$$

Case IV. Let $1 < t \le \sqrt{\frac{7}{2}}$. Then we have

$$t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1 = t^4(t - 1)^2 + t^2(7 - 2t^2) + 2t + 1 > 0.$$

Hence, the inequality (A.1) is satisfied for all $\beta > 0$.

Acknowledgments

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