PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 7, July 2010, Pages 2413–2423
S 0002-9939(10)10290-1
Article electronically published on March 15, 2010

BAIRE CATEGORY AND THE WEAK BANG-BANG PROPERTY FOR CONTINUOUS DIFFERENTIAL INCLUSIONS

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(Communicated by Yingfei Yi)

Abstract. For continuous differential inclusions the classical bang-bang property is known to fail, yet a weak form of it is established here, in the case where the right hand side is a multifunction whose values are closed convex and bounded sets with nonempty interior contained in a reflexive and separable Banach space. Our approach is based on the Baire category method.

1. Introduction

The bang-bang property for differential inclusions has been studied by many authors from different points of view. For recent contributions see, among others, Papageorgiou [14], Tolstonogov [18], Donchev, Farkhi, Mordukhovich [10]. A comprehensive account on differential inclusions and additional references can be found in the monographs by Aubin and Cellina [1], Hu and Papageorgiou [12], Mordukhovich [13], and Tolstonogov [17]. Usually in the investigation of the bang-bang property a crucial role is played by the assumption that the differential inclusion satisfies a globally Lipschitz condition. Recently, it has been shown that the bang-bang property remains valid even under a locally Lipschitz condition [9], while it is known to be false under the mere assumption of continuity in view of an example of Plis [16]. The aim of the present paper is to show that, under appropriate assumptions, a somewhat weaker form of the bang-bang property is valid for continuous differential inclusions (Theorem 1). Our method of approach is based on the Baire category as developed in [6]-[9]. To apply it we need some technical results, among which is a suitable infinite dimensional version of the classical Carathéodory theorem concerning compact convex sets in $\mathbb{R}^n$ (Proposition 4). For further details on the Baire method, see [4], [12], [15].

Let (M,ρ) be a metric space. The interior and the closure of a set $X \subset M$ are denoted by $\operatorname{int} X$ and $\overline{X}$. For $a \in M$ and $X \subset M$, $X \neq \emptyset$, we set $d(a,X) = \inf_{x\in X} \rho(a,X)$.

Throughout the paper $\mathbb{E}$ is a reflexive and separable real Banach space with norm $\|.\|$ and $\mathcal{B}(\mathbb{E})$ (resp. $\mathcal{C}(\mathbb{E})$) is the space of all subsets of $\mathbb{E}$ which are closed convex bounded with nonempty interior (resp. closed convex bounded nonempty). The spaces $\mathcal{B}(\mathbb{E})$, $\mathcal{C}(\mathbb{E})$ are equipped with the Pompeiu-Hausdorff metric

$$h(X,Y) = \max\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\}.$$

Received by the editors September 8, 2009.
2010 *Mathematics Subject Classification.* Primary 34AXX.

If $X \subset \mathbb{E}$, then coX denotes the convex hull of X. If $X \subset \mathbb{E}$ is convex, then $\operatorname{ext} X$ denotes the set of all extreme points of X. For $X \subset \mathbb{E}$, $X \neq \emptyset$, we put $\|X\| = \sup\{\|x\| \mid x \in X\}$. Moreover $B = \{x \in \mathbb{E} \mid \|x\| \leq 1\}$.

As usual $C(I, \mathbb{E})$, where $I \subset \mathbb{R}$ is a compact interval, denotes the space of all continuous functions $x : I \to \mathbb{E}$ with norm $\|x\|_I = \max\{t \in I \mid \|x(t)\|\}$.

Set $I = [t_o, t_1]$, $t_0 < t_1$. A multifunction $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$ is said to satisfy assumption (H) if :

(j) F is continuous on $I \times \mathbb{E}$,
(jj) F is bounded, i.e. $\|F(t, x)\| < M$ for every $(t, x) \in I \times \mathbb{E}$, M a positive constant.

For F satisfying (H) and $a \in \mathbb{E}$, consider the following Cauchy problems:

$$(C_{F,a}) \qquad \dot{x}(t) \in F(t, x(t)) , \quad x(t_0) = a,$$

$$(C_{\operatorname{int} F,a}) \qquad \dot{x}(t) \in \operatorname{int} F(t, x(t)) , \quad x(t_0) = a,$$

$$(C_{\operatorname{ext} F,a}) \qquad \dot{x}(t) \in \operatorname{ext} F(t, x(t)) , \quad x(t_0) = a.$$

A function $x : I \to \mathbb{E}$ is said to be a *solution* of the Cauchy problem $(C_{F,a})$ (resp. $(C_{\operatorname{int} F,a})$, $(C_{\operatorname{ext} F,a})$) if x is Lipschitzian on I, with $x(t_0) = a$, and satisfies $(C_{F,a})$ (resp. $(C_{\operatorname{int} F,a})$, $(C_{\operatorname{ext} F,a})$), $t \in I$ a.e.

For F satisfying (H) and $a \in \mathbb{E}$ set:

$$\mathcal{M}_{F,a} = \{x : I \to \mathbb{E} \mid x \text{ is a solution of } (C_{F,a})\},$$
$$\mathcal{M}_{\operatorname{int} F,a} = \{x : I \to \mathbb{E} \mid x \text{ is a solution of } (C_{\operatorname{int} F,a})\},$$
$$\mathcal{M}_{\operatorname{ext} F,a} = \{x : I \to \mathbb{E} \mid x \text{ is a solution of } (C_{\operatorname{ext} F,a})\}.$$

The space $\mathcal{M}_{F,a}$ is equipped with the metric induced by the norm of $C(I, E)$, i.e. the metric of uniform convergence.

It is evident that $\mathcal{M}_{\operatorname{ext} F,a}$ and $\mathcal{M}_{\operatorname{int} F,a}$ are contained in $\mathcal{M}_{F,a}$. Furthermore $\mathcal{M}_{F,a}$ can be empty if $\mathbb{E}$ is infinite dimensional and F is merely continuous.

Under the assumption that $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$ is continuous, locally Lipschitzian in the x-variable and bounded, the set $\mathcal{M}_{F,a}$ is nonempty and moreover the following bang-bang property holds (see [9]):

$$\overline{\mathcal{M}}_{\operatorname{ext} F,a} = \mathcal{M}_{F,a},$$

where the closure is in $C(I, \mathbb{E})$. Whenever F is only continuous, then the bang-bang property is no longer valid and one has

$$\overline{\mathcal{M}}_{\operatorname{ext} F,a} \subset \mathcal{M}_{F,a},$$

where the inclusion can be strict, in view of the Plis example [16].

The aim of this paper is to show that if $F : I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ is continuous and bounded, then $\mathcal{M}_{\operatorname{int} F,a} \neq \emptyset$ and the following weak form of the bang-bang property holds:

$$\overline{\mathcal{M}}_{\operatorname{ext} F,a} \supset \mathcal{M}_{\operatorname{int} F,a}. \tag{1.1}$$

Remark 1. If F takes on values in $\mathcal{C}(\mathbb{E})$ and $\mathbb{E}$ is infinite dimensional, then one can have $\mathcal{M}_{\operatorname{ext} F,a} = \mathcal{M}_{F,a} = \emptyset$, by virtue of Godunov's theorem [11].

Remark 2. The inclusion (1.1) can be strict, as is shown by an example presented in Section 3.

For any set $J \subset I$ we denote by $|J|$ and χ_J, respectively, the Lebesgue measure and the characteristic function of J.

By a *regular partition* of the interval $I = [t_0, t_1]$ we mean a finite or denumerable infinite family $\mathcal{P} = \{I_i\}$ of pairwise disjoint nondegenerate intervals $I_i \subset I$ such that, setting $N_0 = I \setminus \bigcup_i I_i$, one has $|N_0| = 0$.

Definition 1. A map $u : I \to \mathbb{E}$ is said to be *piecewise constant* if u is given by

$$u(t) = \sum_i u_i \chi_{I_i}(t) + u_0(t)\chi_{N_0}(t), \qquad t \in I, \tag{1.2}$$

where $\{I_i\}$ is a regular partition of I, $\{u_i\} \subset \mathbb{E}$ is a bounded sequence, and $u_0(t) \in \mathbb{E}$ for every $t \in N_0 = I \setminus \bigcup_i I_i$.

Definition 2. A solution $x \in \mathcal{M}_{F,a}$ is said to be *regular* if there exist a regular partition $\mathcal{P} = \{I_i\}$ of I and corresponding sequences $\{u_i\} \subset \mathbb{E}$ and $\{\sigma_i\} \subset (0, +\infty)$ such that, denoting by $u : I \to \mathbb{E}$ a piecewise constant map given by (1.2), one has:

$$x(t) = a + \int_{t_0}^{t} u(s)ds \text{ for each } t \in I, \tag{j}$$

$$\dot{x}(t) + \sigma_i B = u_i + \sigma_i B \subset F(t, x(t)) \text{ for each } t \in \operatorname{int} I_i \text{ and } I_i \in \mathcal{P}. \tag{jj}$$

Set

$$\mathcal{M}^0_{F,a} = \{x : I \to \mathbb{E} \mid x \text{ is a regular solution of } (C_{\operatorname{int} F,a})\},$$

and define

$$\mathcal{M} = \overline{\mathcal{M}}^0_{F,a},$$

where the closure is in $C(I, \mathbb{E})$. The space $\mathcal{M}$ is equipped with the metric induced by the norm of $C(I, \mathbb{E})$.

The Choquet function, which we now introduce, plays a crucial role in the proof of our main result.

Denote by $\mathbb{E}^*$ the topological dual of $\mathbb{E}$. Let $\{l_n\}$, $\|l_n\| = 1$, be a sequence dense in the unit sphere of $\mathbb{E}^*$. Let F satisfy assumption (H). Following Choquet [5], Vol. II, Ch. 6, we define $\varphi_F : I \times \mathbb{E} \times \mathbb{E} \to [0, +\infty]$ by

$$\varphi(t, x, v) = \begin{cases} \sum_{n=1}^{\infty} \frac{(l_n(v))^2}{2^n}, & v \in F(t, x), \\ +\infty, & v \in \mathbb{E} \setminus F(t, x). \end{cases}$$

Let $\mathcal{A}$ be the set of all continuous affine functions $a : \mathbb{E} \to \mathbb{R}$. Let $\overline{\varphi}_F : I \times \mathbb{E} \times \mathbb{E} \to [-\infty, +\infty)$ be given by

$$\overline{\varphi}_F(t, x, v) = \inf\{a(v) \mid a \in \mathcal{A} \text{ and } a(z) > \varphi_F(t, x, z) \text{ for every } z \in F(t, x)\}.$$

We define $d_F : I \times \mathbb{E} \times \mathbb{E} \to [-\infty, +\infty)$ by

$$d_F(t, x, v) = \overline{\varphi}_F(t, x, v) - \varphi_F(t, x, v).$$

In the next proposition we review some properties of d_F, the Choquet function associated to F (see Choquet [5], Castaing and Valadier [2]).

Proposition 1. *Let $F : I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ satisfy (H). Then:*
(i) *for each $(t, x) \in I \times \mathbb{E}$ and $v \in F(t, x)$ we have $0 \leq d_F(t, x, v) \leq M^2$. Moreover $d_F(t, x, v) = 0$ if and only if $v \in \operatorname{ext} F(t, x)$;*
(ii) *for each $(t, x) \in I \times \mathbb{E}$, the function $d_F(t, x, .)$ is concave on $\mathbb{E}$ and strictly concave on the set $F(t, x)$;*
(iii) *d_F is upper semicontinuous on $I \times \mathbb{E} \times \mathbb{E}$;*

(iv) *for each* $x \in \mathcal{M}_{F,a}$, *the function* $t \to d_F(t, x(t), \dot{x}(t))$ *is nonnegative, bounded and integrable on I;*
(v) *if* $\{x_n\} \subset \mathcal{M}_{F,a}$ *converges uniformly to* x, *then*

$$\limsup_{n\to\infty} \int_I d_F(t, x_n(t), \dot{x}_n(t))dt \le \int_I d_F(t, x(t), \dot{x}(t))dt.$$

2. Auxiliary results

In this section we prove some results which will be useful in what follows.

Proposition 2. *Let* $F : I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ *satisfy* (*H*). *Let* $x \in \mathcal{M}_{\mathrm{int}\, F,a}$ *and* $\varepsilon > 0$. *Then there exists a regular solution* $y : I \to \mathbb{E}$ *of the Cauchy problem* $(C_{\mathrm{int}\, F,a})$ *such that* $\|y - x\|_I < \varepsilon$.

Proof. Let $x \in \mathcal{M}_{\mathrm{int}\, F,a}$ and $\varepsilon > 0$ be given. Set

$$J = \{t \in (t_0, t_1) \mid \dot{x}(t) \text{ exists and } \dot{x}(t) \in \mathrm{int}\, F(t, x(t))\}.$$

Let $\tau \in J$ be arbitrary. Hence for some $\sigma > 0$ we have

$$\dot{x}(\tau) + 3\sigma B \subset F(\tau, x(\tau)).$$

Since F is continuous, there exists $\delta_\tau > 0$, with $[\tau, \tau + \delta_\tau] \subset (t_0, t_1)$, such that
(2.1)

$$0 < h < \delta_\tau,\ |t - \tau| < \delta_\tau,\ t \in I,\ \|z - x(\tau)\| < \delta_\tau \Rightarrow \frac{x(\tau + h) - x(\tau)}{h} + \sigma B \subset F(t, z).$$

For any δ with

$$0 < \delta < \min\{\frac{\delta_\tau}{M+1}, \frac{\varepsilon}{2M+1}\},$$

set $I_{\tau,\delta} = [\tau, \tau + \delta]$ and define $y_{\tau,\delta} : I_{\tau,\delta} \to \mathbb{E}$ by

$$y_{\tau,\delta}(t) = x(\tau) + \frac{x(\tau + \delta) - x(\tau)}{\delta}(t - \tau), \quad t \in I_{\tau,\delta}.$$

Claim 1. $y_{\tau,\delta} : I_{\tau,\delta} \to \mathbb{E}$ is a regular solution of the following boundary value problem:

$$(B_{\tau,\delta}) \qquad \dot{y}(t) = \mathrm{int}\, F(t, y(t)), \qquad y(\tau) = x(\tau), \quad y(\tau + \delta) = x(\tau + \delta).$$

Evidently $y(\tau) = x(\tau)$ and $y(\tau + \delta) = x(\tau + \delta)$. Moreover for $t \in I_{\tau,\delta}$ we have

$$\|y_{\tau,\delta}(t) - x(\tau)\| = \frac{\|x(\tau + \delta) - x(\tau)\|}{\delta}(t - \tau) < \delta_\tau, \tag{2.2}$$

because $\|x(\tau + \delta) - x(\tau)\| < M\delta$ and $t - \tau \le \delta < \delta_\tau/(M + 1)$. From (2.1) (with $h = \delta$), in view of (2.2) it follows that

$$\dot{y}_{\tau,\delta}(t) + \sigma B = \frac{x(\tau + \delta) - x(\tau)}{\delta} + \sigma B \subset F(t, y_{\tau,\delta}(t)), \quad t \in (\tau, \tau + \delta),$$

and hence Claim 1 holds.

Now the family

$$\mathcal{F} = \{I_{\tau,\delta}\ ,\ \tau \in J\ ,\ 0 < \delta < \delta_\tau\}$$

of closed intervals $I_{\tau,\delta}$ covers J in the sense of Vitali. Hence there exists a finite or denumerable infinite family $\mathcal{F}_0 = \{I_{\tau_j,\delta_j}\} \subset \mathcal{F}$ of pairwise disjoint closed intervals

$I_{\tau_j,\delta_j} \in \mathcal{F}$ such that $J \setminus \bigcup_j I_{\tau_j,\delta_j}$ has measure zero. As $I \setminus J$ has measure zero it follows that $N_0 = I \setminus \bigcup_j I_{\tau_j,\delta_j}$ has measure zero. Evidently

$$I = \bigcup_j I_{\tau_j,\delta_j} \cup N_0, \tag{2.3}$$

and thus $\mathcal{F}_0$ is a regular partition of I. Now define $u : I \to \mathbb{E}$ and $y : I \to \mathbb{E}$ as follows:

$$u(t) = \sum_j \dot{y}_{\tau_j,\delta_j}(t)\chi_{I_{\tau_j,\delta_j}}(t)\ , \quad t \in I,$$

$$y(t) = a + \int_{t_0}^{t} u(s)ds\ , \quad t \in I.$$

Claim 2. $y : I \to \mathbb{E}$ is a regular solution of the Cauchy problem $(C_{\operatorname{int} F,a})$ satisfying $\|y - x\|_I < \varepsilon$.

Since the functions y_{τ_j,δ_j} and x agree at τ_j and $\tau_j + \delta_j$, the end points of I_{τ_j,δ_j}, then in view of the definition of y it is easy to show that $y(t) = y_{\tau_j,\delta_j}(t)$ for each $t \in I_{\tau_j,\delta_j}$ and $I_{\tau_j,\delta_j} \in \mathcal{F}_0$. By virtue of Claim 1 and (2.3) it follows that y is a regular solution of the Cauchy problem $(C_{\operatorname{int} F,a})$. Furthermore for any $I_{\tau_j,\delta_j} \in \mathcal{F}_0$ and all $t \in I_{\tau_j,\delta_j}$ we have

$$\|y(t) - x(t)\| \leq \|y(t) - y(\tau_j)\| + \|x(\tau_j) - x(t)\| < 2M\delta_j.$$

As $\delta_j < \varepsilon/(2M+1)$, in view of (2.3) it follows that $\|y-x\|_I < \varepsilon$. Therefore Claim 2 holds. This completes the proof. □

Proposition 3. *Let $F : I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ satisfy (H). Then $\mathcal{M}$ is a nonempty complete metric space (under the induced metric of $C(I, \mathbb{E})$) and $\mathcal{M} \subset \mathcal{M}_{F,a}$.*

Proof. Since the multifunction $\operatorname{int} F$ admits locally Lipschitzian selections, we have $\mathcal{M}_{\operatorname{int} F,a} \neq \emptyset$ and thus $\mathcal{M}^0_{F,a} \neq \emptyset$, by virtue of Proposition 2. Hence $\mathcal{M} \neq \emptyset$. Evidently $\mathcal{M}$ is complete for $C(I, \mathbb{E})$ is so. As F is a continuous and bounded multifunction with closed convex values contained in $\mathbb{E}$, a reflexive Banach space, the uniform limit of solutions is also a solution, and hence $\mathcal{M} \subset \mathcal{M}_{F,a}$. □

Proposition 4. *Let $F : I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ satisfy (H). Let $(t, x) \in I \times \mathbb{E}$ and let $u \in \operatorname{int} F(t, x)$ and $\alpha > 0$ be given. Then for some $n \in \mathbb{N}$ there exist points $a_i \in \operatorname{int} F(t, x)$, with $d_F(t, x, a_i) < \alpha$, $i = 1, \dots, n$, and numbers $\lambda_i > 0$, with $\lambda_1 + \cdots + \lambda_n = 1$, such that*

$$\sum_{i=1}^{n} \lambda_i a_i = u.$$

Proof. Fix $\theta > 0$ so that $u + \theta B \subset F(t, x)$. For some $p \in \mathbb{N}$ there exist points $e_i \in \operatorname{ext} F(t, x)$ and numbers $\mu_i > 0$ with $\mu_1 + \cdots + \mu_p = 1$, such that, setting $c = \mu_1 e_1 + \cdots + \mu_p e_p$, one has

$$0 < \|c - u\| < \frac{\theta}{2}\ .$$

By Proposition 1, $d_F(t, x, e_i) = 0,\ i = 1, \dots, p$, and thus sufficiently close to each e_i there exists a point $a_i \in \operatorname{int} F(t, x)$ such that

$$d_F(t, x, a_i) < \frac{\alpha}{2}\ ,\ i = 1, \dots, p \quad \text{and} \quad 0 < r = \|a - u\| < \frac{\theta}{2}\ , \tag{2.4}$$

where $a = \mu_1 a_1 + \cdots + \mu_p a_p$. By virtue of Proposition 1 and (2.4) there exists an $\varepsilon > 0$ such that each ball $B_i = a_i + \varepsilon B$, $i = 1, \ldots, p$, is contained in $F(t,x)$ and moreover,

$$d_F(t,x,v) < \alpha \quad \text{for every } v \in B_i \ , \ i = 1, \ldots, p. \tag{2.5}$$

It is evident that

$$\sum_{i=1}^{p} \mu_i B_i = B_\varepsilon(a), \quad \text{where } B_\varepsilon(a) = a + \varepsilon B. \tag{2.6}$$

Let $b \in \operatorname{int} F(t,x)$ be such that $u = \frac{a+b}{2}$. As above, for some $q \in \mathbb{N}$ there exist points $a_i' \in \operatorname{int} F(t,x)$, with $d_F(t,x,a_i') < \alpha$, $i = 1, \ldots, q$, and numbers $\nu_i > 0$, with $\nu_1 + \cdots + \nu_q = 1$, such that setting

$$b' = \sum_{i=1}^{q} \nu_i a_i' \tag{2.7}$$

one has $\|b' - b\| < \frac{\varepsilon}{4}$. As $u + \frac{\varepsilon}{2}B \subset co\{b, B_\varepsilon(a)\}$ and $h(co\{b, B_\varepsilon(a)\}, co\{b', B_\varepsilon(a)\}) \leq \|b' - b\| < \frac{\varepsilon}{4}$, it follows that $u + \frac{\varepsilon}{2}B \subset co\{b', B_\varepsilon(a)\} + \frac{\varepsilon}{4}B$, which implies $u + \frac{\varepsilon}{4}B \subset co\{b', B_\varepsilon(a)\}$. Hence there exist $d \in B_\varepsilon(a)$ and $t \in [0,1]$ so that

$$u = tb' + (1-t)d. \tag{2.8}$$

As $d \in B_\varepsilon(a)$, by virtue of (2.6) and (2.5) there exist points $d_i' \in B_i$, with $d_F(t,x,d_i') < \alpha$, $i = 1, \ldots, p$, such that

$$d = \sum_{i=1}^{p} \mu_i d_i'. \tag{2.9}$$

From (2.8), in view of (2.7) and (2.9), it follows that

$$u = t \sum_{i=1}^{q} \nu_i a_i' + (1-t) \sum_{i=1}^{p} \mu_i d_i'.$$

This completes the proof. □

3. The weak bang-bang property

In this section we shall prove the weak bang-bang property. To this end, for $F : I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ satisfying (H) and $\alpha > 0$, we set

$$\mathcal{N}_\alpha = \{x \in \mathcal{M} \ : \ \int_I d_F(t, x(t), \dot{x}(t))dt < \alpha |I|\}.$$

Proposition 5. *$\mathcal{N}_\alpha$ is dense in $\mathcal{M}$.*

Proof. It suffices to show that given $x \in \mathcal{M}^0_{F,a}$ and $\varepsilon > 0$ there exists $y \in \mathcal{N}_\alpha$ such that $\|y - x\|_I \leq \varepsilon$.

By hypothesis x is a regular solution of $(C_{\operatorname{int} F,a})$ and thus, with the notation of Definition 2, for some piecewise constant map $u : I \to \mathbb{E}$, given by (1.2), we have:

$$x(t) = a + \int_{t_0}^{t} u(s)ds \ , \quad t \in I,$$

$$\dot{x}(t) + \sigma_i B = u_i + \sigma_i B \subset F(t, x(t)), \qquad t \in \operatorname{int} I_i \ , \ I_i \in \mathcal{P}.$$

Consider an interval $I_i \in \mathcal{P}$ with end points $\alpha_i < \beta_i$ and let $\tau \in (\alpha_i, \beta_i)$ be arbitrary. Evidently

$$u_i + \sigma_i B \subset F(\tau, x(\tau)),$$

and thus by Proposition 4, for some $n \in \mathbb{N}$ there exist points $e_k \in \operatorname{int} F(\tau, x(\tau))$, with $d_F(\tau, x(\tau), \dot{x}(\tau)) < \alpha$, $k = 1, \ldots, n$, and corresponding numbers $\lambda_k > 0$, with $\lambda_1 + \cdots + \lambda_n = 1$, such that

$$u_i = \sum_{k=1}^{n} \lambda_k e_k. \tag{3.1}$$

Clearly for some $\gamma_\tau > 0$,

$$\bigcup_{k=1}^{n} (e_k + \gamma_\tau B) \subset F(\tau, x(\tau)). \tag{3.2}$$

Since F is continuous and satisfies (3.2) there exists ρ_0, with

$$0 < \rho_0 < \min\{\varepsilon, \tau - \alpha_i, \beta_i - \tau\}, \tag{3.3}$$

such that

$$t \in [\tau - \rho_0, \tau + \rho_0] \ , \ \|y - x(\tau)\| < \rho_0 \quad \Rightarrow \quad \bigcup_{k=1}^{n} (e_k + \frac{\gamma_\tau}{2} B) \subset F(t, y). \tag{3.4}$$

Furthermore as d_F is upper semicontinuous at $(\tau, x(\tau), e_k)$ and $d_F(\tau, x(\tau), e_k) < \alpha$ there exists a ρ, with $0 < \rho < \min\{\rho_0, \gamma_\tau/2\}$, such that for $k = 1, \ldots, n$ we have

$$t \in [\tau - \rho, \tau + \rho] \ , \ \|y - x(\tau)\| < \rho \ , \ \|v - e_k\| < \rho \quad \Rightarrow \quad d_F(t, y, v) < \alpha. \tag{3.5}$$

Let δ_τ satisfy

$$0 < \delta_\tau < \frac{\rho}{4M + 1} \tag{3.6}$$

and, for $0 < \delta < \delta_\tau$, set

$$I_{\tau,\delta} = [\tau - \delta, \tau + \delta].$$

Let $\{J_{\tau,\delta}^k\}_{k=1}^n$ be a partition of $I_{\tau,\delta}$ into n pairwise disjoint nondegenerate subintervals $J_{\tau,\delta}^k$ of length

$$|J_{\tau,\delta}^k| = \lambda_k |I_{\tau,\delta}| \ \ , \ k = 1, \ldots, n. \tag{3.7}$$

Now define $u_{\tau,\delta} : I \to \mathbb{E}$ and $y_{\tau,\delta} : I_{\tau,\delta} \to \mathbb{E}$ as follows:

$$u_{\tau,\delta}(t) = \sum_{k=1}^{n} e_k \chi_{J_{\tau,\delta}^k}(t) \ , \ t \in I_{\tau,\delta}, \tag{3.8}$$

$$y_{\tau,\delta}(t) = x(\tau - \delta) + \int_{\tau-\delta}^{t} u_{\tau,\delta}(s) ds \ , \ t \in I_{\tau,\delta}. \tag{3.9}$$

We have

$$(j) \qquad \dot{y}_{\tau,\delta}(t) + \frac{\gamma_\tau}{2}B \subset F(t, y_{\tau,\delta}(t)), \quad t \in \bigcup_{k=1}^{n} \operatorname{int} J^k_{\tau,\delta},$$

$$(jj) \qquad y_{\tau,\delta}(\tau \pm \delta) = x(\tau \pm \delta),$$

$$(jjj) \qquad \|y_{\tau,\delta}(t) - x(t)\| < \varepsilon\ ,\ t \in I_{\tau,\delta},$$

$$(jv) \qquad d_F(t, y_{\tau,\delta}(t), \dot{y}_{\tau,\delta}(t)) < \alpha\ , \quad t \in \bigcup_{k=1}^{n} \operatorname{int} J^k_{\tau,\delta}.$$

(j) Let $t \in \operatorname{int} J^k_{\tau,\delta}$, where $1 \leq k \leq n$. Clearly

$$t \in [\tau - \rho, \tau + \rho] \tag{3.10}$$

since $|t - \tau| \leq \delta < \delta_\tau < \rho$ by (3.6). Moreover,

$$\begin{aligned} \|y_{\tau,\delta}(t) - x(\tau)\| &= \|x(\tau - \delta) + \int_{\tau-\delta}^{t} u_{\tau,\delta}(s)ds - a - \int_{t_0}^{\tau} u(s)ds\| \\ &= \| \int_{\tau-\delta}^{t} u_{\tau,\delta}(s)ds - \int_{\tau-\delta}^{\tau} u(s)ds\| \\ &\leq \int_{J^k_{\tau,\delta}} (\|u_{\tau,\delta}(s)\| + \|u(s)\|)ds \leq 4M\delta_\tau, \end{aligned}$$

for $|J^k_{\tau,\delta}| \leq |I_{\tau,\delta}| = 2\delta < 2\delta_\tau$ and $u_{\tau,\delta}$ and u are bounded by M. Since $\delta_\tau < \rho/(4M+1)$ by (3.6), it follows that

$$\|y_{\tau,\delta}(t) - x(\tau)\| < \rho. \tag{3.11}$$

From (3.4), in view of (3.10) and (3.11), as $\rho < \rho_0$ and $\dot{y}_{\tau,\delta}(t) = e_k$, one has

$$\dot{y}_{\tau,\delta}(t) + \frac{\gamma_\tau}{2}B \subset F(t, y_{\tau,\delta}(t))$$

and thus (j) holds.

(jj) From (3.9), in view of (3.8), (3.7) and (3.1) we have:

$$\begin{aligned} y_{\tau,\delta}(\tau + \delta) &= x(\tau - \delta) + \int_{\tau-\delta}^{\tau+\delta} (\sum_{k=1}^{n} e_k \chi_{J^k_{\tau,\delta}}(s))ds \\ &= x(\tau - \delta) + \sum_{k=1}^{n} e_k |J^k_{\tau,\delta}| = x(\tau - \delta) + \sum_{k=1}^{n} \lambda_k e_k |I_{\tau,\delta}| \\ &= x(\tau - \delta) + u_i |I_{\tau,\delta}| = x(\tau - \delta) + \int_{\tau-\delta}^{\tau+\delta} u(s)ds = x(\tau + \delta). \end{aligned}$$

Clearly $y_{\tau,\delta}(\tau - \delta) = x(\tau - \delta)$, and thus (jj) holds.

(jjj) For any $t \in I_{\tau,\delta}$ we have:

$$\begin{aligned} \|y_{\tau,\delta}(t) - x(t)\| &= \|x(\tau - \delta) + \int_{\tau-\delta}^{t} u_{\tau,\delta}(s)ds - a - \int_{t_0}^{t} u(s)ds\| \\ &= \| \int_{\tau-\delta}^{t} u_{\tau,\delta}(s)ds - \int_{\tau-\delta}^{t} u(s)ds\| \leq \int_{I_{\tau,\delta}} (\|u_{\tau,\delta}(s)\| + \|u(s)\|)ds \leq 4M\delta. \end{aligned}$$

From the latter, (jjj) follows at once since $\delta < \delta_\tau < \rho/(4M+1)$ by (3.6), and $\rho < \rho_0 < \varepsilon$ by (3.3).

(jv) Let $t \in \operatorname{int} J^k_{\tau,\delta}$ be arbitrary, where $1 \le k \le n$. From (3.5), in view of (3.10) and (3.11), as $\dot y_{\tau,\delta}(t) = e_k \in F(t, y_{\tau,\delta}(t))$, one has $d_F(t, y_{\tau,\delta}(t), \dot y_{\tau,\delta}(t)) < \alpha$, and thus (jv) holds.

It is evident that the family

$$\mathcal{F} = \{I_{\tau,\delta} \mid \tau \in \bigcup_i \operatorname{int} I_i \ , \ 0 < \delta < \delta_\tau\}$$

of closed intervals $I_{\tau,\delta} \subset I$ covers the set $\bigcup_i \operatorname{int} I_i$ in the sense of Vitali. Hence there is a finite or denumerable infinite family $\mathcal{F}_0 = \{I_{\tau_j,\delta_j}\} \subset \mathcal{F}$ of pairwise disjoint closed intervals I_{τ_j,δ_j} such that

$$I = (\bigcup_j I_{\tau_j,\delta_j}) \cup N_0, \tag{3.12}$$

where $N_0 = I \setminus \bigcup_j I_{\tau_j,\delta_j}$ has measure zero. Now define $v : I \to \mathbb{E}$ and $y : I \to \mathbb{E}$ as follows:

$$v(t) = \sum_j u_{\tau_j,\delta_j}(t)\chi_{I_{\tau_j,\delta_j}}(t) \ , \quad t \in I,$$

$$y(t) = a + \int_{t_0}^t v(s)ds \ , \quad t \in I.$$

Since by (jj) $y_{\tau_j,\delta_j}(\tau_j \pm \delta_j) = x(\tau_j \pm \delta_j)$, it is easy to see that for each $I_{\tau_j,\delta_j} \in \mathcal{F}_0$ we have

$$y(t) = y_{\tau_j,\delta_j}(t) \quad \text{for every} \ \ t \in I_{\tau_j,\delta_j}. \tag{3.13}$$

In view of (3.13), (3.12) and (j) it follows that $y : I \to \mathbb{E}$ is a regular solution of the Cauchy problem $(C_{\operatorname{int} F,a})$; hence $y \in \mathcal{M}^0_{\operatorname{int} F,a}$ and so a fortiori $y \in \mathcal{M}$. Furthermore, by virtue of (3.13), (3.12) and (jv),

$$\int_I d_F(t, y(t), \dot y(t))dt < \alpha|I|$$

and hence $y \in \mathcal{N}_\alpha$. Finally from (3.13), (3.12) and (jjj) it follows that $\|y - x\|_I \le \varepsilon$. This completes the proof. ☐

Proposition 6. *$\mathcal{N}_\alpha$ is open in $\mathcal{M}$.*

Proof. Let $\{x_n\} \subset \mathcal{M} \setminus \mathcal{N}_\alpha$ be a sequence which converges uniformly to $x \in \mathcal{M}$. Then, by Proposition 1,

$$\int_I d_F(t, x(t), \dot x(t))dt \ge \limsup_{n\to\infty} \int_I d_F(t, x_n(t), \dot x_n(t))dt \ge \alpha|I|.$$

Thus $x \in \mathcal{M} \setminus \mathcal{N}_\alpha$, completing the proof. ☐

We are now ready to prove the following weak form of the bang-bang property.

Theorem 1. *Let $F : I \times \mathbb{E} \to \mathcal{B}(\mathbb{E})$ satisfy assumption (H). Then $\mathcal{M}_{\operatorname{int} F,a} \neq \emptyset$ and*

$$\overline{\mathcal{M}}_{\operatorname{ext} F,a} \supset \mathcal{M}_{\operatorname{int} F,a}. \tag{3.14}$$

Proof. Under our assumptions, $\mathcal{M}_{\operatorname{int} F,a} \neq \emptyset$. To prove (3.14) set

$$\mathcal{M}^* = \bigcap_{n\in\mathbb{N}} \mathcal{N}_{1/n}.$$

By virtue of Propositions 5 and 6 each $\mathcal{N}_{1/n}$ is open and dense in $\mathcal{M}$, which is a complete metric space, by Proposition 3. Consequently $\mathcal{M}^*$ is dense in $\mathcal{M}$.

Let $x \in \mathcal{M}^*$. Then for every $n \in \mathbb{N}$,

$$\int_I d_F(t, x(t), \dot{x}(t))dt < (1/n)|I|,$$

and thus, by Proposition 1, $d_F(t, y(t), \dot{y}(t)) = 0$, $t \in I$ a.e., which implies that $x \in \mathcal{M}_{\text{ext}\, F,a}$. Therefore $\mathcal{M}_{\text{ext}\, F,a} \supset \mathcal{M}^*$ and hence,

$$\overline{\mathcal{M}}_{\text{ext}\, F,a} \supset \mathcal{M}.$$

Since, by Proposition 2, $\mathcal{M} = \overline{\mathcal{M}}^0_{F,a} \supset \mathcal{M}_{\text{int}\, F,a}$, it follows that $\overline{\mathcal{M}}_{\text{ext}\, F,a} \supset \mathcal{M}_{\text{int}\, F,a}$. This completes the proof. □

The following example shows that in (3.14) the inclusion can be strict.

Example 3.1. Set $f(y) = \sqrt{|y|}$ if $|y| \leq 1$ and $f(y) = 1$ if $|y| > 1$. For $F : \mathbb{R}^2 \to \mathcal{B}(\mathbb{R}^2)$ given by

$$F(x, y) = \{(u, v) \in \mathbb{R}^2 \mid u \in [0, 1]\ ,\ v \in [f(y), 2]\}, \quad (x, y) \in \mathbb{R}^2,$$

consider the following Cauchy problems:

$$(C_{\text{int}\, F,0}) \qquad (\dot{x}(t), \dot{y}(t)) \in \operatorname{int} F(x(t), y(t)), \quad (x(0), y(0)) = (0, 0),$$
$$(C_{\text{ext}\, F,0}) \qquad (\dot{x}(t), \dot{y}(t)) \in \operatorname{ext} F(x(t), y(t)), \quad (x(0), y(0)) = (0, 0).$$

Put $I = [0, 1/2]$. Clearly

$$\operatorname{ext} F(x, y) = \{(0, f(y))\ ,\ (0, 2)\ ,\ (1, f(y))\ ,\ (1, 2)\}$$

and thus $(0, 0) \in \operatorname{ext} F(0, 0)$, which shows that $(x_0(t), y_0(t)) = (0, 0)$, $t \in I$, is a solution of $(C_{\text{ext}\, F,0})$. Let $(x(t), y(t))$, $t \in I$, be an arbitrary solution of $(C_{\text{int}\, F,0})$. As $x(0) = y(0) = 0$ and $x(t)$ and $y(t)$ satisfy

$$\dot{x}(t) \in (0, 1), \quad \dot{y}(t) \in (f(y(t)), 2), \tag{3.15}$$

it follows that $x(t)$ and $y(t)$ are strictly positive for every $t \in (0, 1/2]$. Clearly $\dot{y}(t) \in (\sqrt{y(t)}, 2)$ for $t \in I$ a.e. since, by (3.15), $y(t) \leq 1$ for every $t \in I$. Let $0 < \varepsilon < 1/4$. Then for $t \in [\varepsilon, 1/2]$ a.e. we have $\dot{y}(t) > \sqrt{y(t)}$, and hence $\sqrt{y(t)} > \sqrt{y(\varepsilon)} + \frac{t-\varepsilon}{2}$, $t \in [\varepsilon, 1/2]$, from which letting $\varepsilon \to 0$ one has

$$y(t) \geq \frac{t^2}{4}, \quad t \in I.$$

Consequently

$$d((x_0(.), y_0(.)), \mathcal{M}_{\text{int}\, F,0}) \geq \frac{1}{8},$$

as $(x(t), y(t))$, $t \in I$, is an arbitrary solution of $(C_{\text{int}\, F,0})$. This shows that the inclusion $\overline{\mathcal{M}}_{\text{ext}\, F,0} \supset \mathcal{M}_{\text{int}\, F,0}$ is strict.

References

[1] J.P. Aubin, A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984. MR755330 (85j:49010)

[2] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin, 1977. MR0467310 (57:7169)

[3] A. Cellina, "On the differential inclusion $\dot{x} \in [-1, 1]$", *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **(8) 69** (1980), 1-6. MR641583 (83g:34012)

[4] A. Cellina, "A view on differential inclusions", *Rend. Semin. Mat. Univ. Politec. Torino* **63** (2005), 197-209. MR2201565 (2006i:34026)

[5] G. Choquet, *Lectures on Analysis*, Vols. I, II, III, Mathematics Lecture Notes Series, Benjamin, New York, Amsterdam, 1969. MR0250011 (40:3252), MR0250012 (40:3253), MR0250013 (40:3254)

[6] F.S. De Blasi, G. Pianigiani, "A Baire category approach to the existence of solutions of multivalued differential equations in Banach spaces", *Funkcial. Ekvac.* **25** (1982), 153-162. MR694909 (84e:34019)

[7] F.S. De Blasi, G. Pianigiani, "Differential inclusions in Banach spaces", *J. Differential Equations* **66** (1987), 208-229. MR871995 (88a:34096)

[8] F.S. De Blasi, G. Pianigiani, "Non-convex-valued differential inclusions in Banach spaces", *J. Math. Anal. Appl.* **157** (1991), 469-494. MR1112329 (92h:34037)

[9] F.S. De Blasi, G. Pianigiani, "Baire category and relaxation problems for locally Lipschitzian differential inclusions on finite and infinite time intervals", *Nonlinear Analysis TMA*, DOI 10.1016/j.na.2009.06.110

[10] T. Donchev, E. Farkhi and B.S. Mordukhovich, "Discrete approximations, relaxation, and optimization of one-sided Lipschitzian differential inclusions in Hilbert spaces", *J. Differential Equations* **243** (2007), 301-328. MR2371790 (2008j:49040)

[11] A.N. Godunov, "The Peano theorem in Banach spaces", *Functional Anal. Appl.* **9** (1975), 53-55. MR0364797 (51:1051)

[12] S. Hu, N.S. Papageorgiou, *Handbook of multivalued analysis,* Vols. I, II, Kluwer, Dordrecht, 1998. MR1485775 (98k:47001)

[13] B.S. Mordukhovich, *Variational analysis and generalized differentiation,* Vols. I, II, Springer-Verlag, Berlin, 2006. MR2191744 (2007b:49003a), MR2191745 (2007b:49003b)

[14] N.S. Papageorgiou, "On the bang-bang principle for nonlinear evolution inclusions", *Aequationes Math.* **45** (1993), 267-280. MR1212391 (94d:49010)

[15] G. Pianigiani, "Differential inclusions. The Baire category method", in *Methods of nonconvex analysis*, edited by A. Cellina, Lecture Notes in Math., 1446, Springer-Verlag, Berlin, 1990, 104-136. MR1079761 (91k:49007)

[16] A. Plis, "Trajectories and quasitrajectories of an orientor field", *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys.* **11** (1963), 369-370. MR0155072 (27:5014)

[17] A.A. Tolstonogov, *Differential inclusions in a Banach space*, Kluwer, Dordrecht, 2000. MR1888331 (2003g:34129)

[18] A.A. Tolstonogov, "The bang-bang principle for controlled systems of subdifferential type", *Proc. Steklov Inst. Math.* (2005), Dynamical Systems and Control Problems, 222-233. MR2157141 (2006f:49007)

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