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REPRESENTATIONS OF THE BRAUER ALGEBRA AND LITTLEWOOD'S RESTRICTION RULES

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Abstract. Let G be either $Sp(V)$ or $O(V)$. Using an action of the Brauer algebra, we describe the subspace $T^k\left(V^{\otimes m}\right) \subseteq V^{\otimes m}$ of tensors of valence k as an induced representation. As an application, we recover a special case of Littlewood's restriction rule, affording the decomposition of an irreducible $GL(V)$–module when restricted to G. Moreover we get an explicit realization of the irreducible representations of the Brauer algebra.

§1 Introduction.

Let V be a complex vector space of dimension $2n$, endowed with a symplectic (i.e. non-degenerate bilinear skew-symmetric) form $\langle\ ,\ \rangle$. Consider the symplectic group $Sp(V)$ of linear automorphisms of V preserving the symplectic form $\langle\ ,\ \rangle$. It is well known that all irreducible finite dimensional representations of $Sp(V)$ can be realized as subrepresentations of tensor powers $V^{\otimes m}$ $(m \in \mathbb{N})$; on the other hand, consider the centralizer of the $Sp(V)$–action on $V^{\otimes m}$, which is a quotient of the so-called Brauer algebra $\mathbb{B}_m^{-2n}$: Schur duality tells us that the algebra of operators generated by $Sp(V)$ and the above quotient of the Brauer algebra are mutual centralizer, and establishes a bijective correspondence between the representations of either of these algebras.

The $Sp(V)$–module $V^{\otimes m}$ splits as $V^{\otimes m} = \bigoplus_{k=0}^{\left[\frac{m}{2}\right]} T^k\left(V^{\otimes m}\right)$, the subspace $T^k\left(V^{\otimes m}\right)$ being the sum of the $Sp(V)$–isotypic components of $V^{\otimes m}$ which appear for the first time in tensor power $m-2k$; more directly, if $\Psi_{pq} : V^{\otimes m} \longrightarrow V^{\otimes (m+2)}$ is the extension operator which inserts in the positions p, q the canonical element of the skew-form $\langle\ ,\ \rangle$, $T^k\left(V^{\otimes m}\right)$ is the vector space generated by k–fold extensions of the traceless tensors in $V^{\otimes (m-2k)}$ (i.e.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX tensors killed by any contraction). Note that, if S_m denotes the symmetric group on m letters, $T^k\left(V^{\otimes m}\right)$ has a natural structure of $Sp(V)\times S_m$–module (even more, of $Sp(V)\times \mathbb{B}_m^{-2n}$–module).

In this paper we show (Theorem 4.1) that, for $n \geq m$ (i.e. in the "stable case"), $T^k\left(V^{\otimes m}\right)$ is obtained by inducing the S_m–module structure from a representation of $S_{m-2k}\times S_{2k}$ built up by taking the tensor product of traceless tensors in $V^{\otimes(m-2k)}$ and $Sp(V)$–invariants in $V^{\otimes(2k)}$. This is proved by considering two actions of the Brauer algebra: the natural action of $\mathbb{B}_m^{-2n}$ on $T^k\left(V^{\otimes m}\right)$ and an action on the induced representation, which we directly define in §3; relating and comparing these actions we will be able to show that $\mathbb{B}_m^{-2n}$ is the whole centralizer of the $Sp(V)$–action on the induced representation: this fact — whose proof is reduced to a combinatorial calculation — allows us to apply symplectic Schur duality and to get the desired isomorphism using elementary representation theory.

A first application is a proof of Littlewood's restriction rule in the stable case. Namely, let V_λ be an irreducible finite dimensional polynomial $GL(V)$–module indexed by a partition λ of m; its restriction to $Sp(V)$ is no longer irreducible in general: in [L] Littlewood furnished a formula describing the decomposition of V_λ into irreducible $Sp(V)$–modules under the assumption that λ has at most n parts; note that this condition is always satisfied in the stable case. Using the description of $T^k\left(V^{\otimes m}\right)$ we gave, it is not difficult to recover Littlewood's rule using standard techniques of classical invariant theory (cf. §5).

The previous arguments can be repeated almost word-by-word for the orthogonal group; in §6 we point out the few modifications needed.

Finally, in §7, we recover from our main result an explicit realization, inside $V^{\otimes m}$, of the irreducible representations of the Brauer algebra in the stable case, and describe the relation among our results and the combinatorial description of these representations (due to Kerov [K]).

In §2 we introduce the basic definitions and recollect well-known results of representation theory which will be needed in the sequel; almost all the results of this section can be found in Weyl's fundamental book [W].

We adopt the following notational conventions: if $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k \geq 0)$ is a partition of m, (i.e. $m = \sum_{i=1}^k \lambda_i \equiv |\lambda|$) we will write $\lambda \vdash m$. Moreover we define the *depth* of λ as:

$$l(\lambda) := \text{number of non-zero parts in } \lambda.$$

In the following we will freely use the following canonical isomorphisms:

(1) $\varphi : V \xrightarrow{\cong} V^*,\ \varphi(v)(w) = \langle v, w\rangle,\ v, w \in V.$

(2) $\alpha : V^* \otimes V \xrightarrow{\cong} End(V),\ \alpha(\phi\otimes w)(v) = \phi(v)w,\ v, w \in V, \Phi \in V^*.$

(3) $V \otimes V \xrightarrow{\varphi\otimes Id} V^*\otimes V \xrightarrow{\alpha} End(V).$

(4) $V \otimes V \xrightarrow{\varphi\otimes\varphi} V^* \otimes V^* \xrightarrow{\cong} (V \otimes V)^*$.

For V a symplectic vector space of dimension $2n$, we fix a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ such that $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$, $\langle e_i, f_j\rangle = \delta_{ij}$ and we consider the element $\psi = \sum_{i=1}^n (e_i \otimes f_i - f_i \otimes e_i) \in V \otimes V$. It is easy to show that the image of ψ is Id_V under the identification (3) and the skew-form $\langle\ ,\ \rangle$ under the identification (4); in particular, ψ does not depend on the choice of the basis, hence it will be referred to as the *canonical element* for the form $\langle\ ,\ \rangle$.

In the orthogonal case, we denote by $(\ ,\ )$ the bilinear symmetric non-degenerate form; if $\{e_1, \dots, e_n\}$ is an orthonormal basis, then the canonical element is $\psi = \sum_{i=1}^n e_i \otimes e_i \in V \otimes V$.

§2 Symplectic invariants and Schur duality.

In this section we recall the first and second fundamental theorems of invariant theory for the symplectic group and then we recollect some related results which will be needed in the sequel.

Let $\mathcal{P}(V^{\oplus m})$ denote the space of polynomial functions of m vector variables, each of dimension $2n$.

Theorem 2.1. *([D-P], Th. 6.7)*

(1)
$$\left(\mathcal{P}\left(V^{\oplus m}\right)\right)^{Sp(V)} = \mathbb{C}[\langle v_i, v_j\rangle].$$

(2) *The ideal of relations between the generators $\langle v_i, v_j\rangle$ is generated by the Pfaffians of order $2n+2$ of the $m \times m$ skew-symmetric matrix $\left(\langle v_i, v_j\rangle\right)_{i,j=1,\dots,m}$.*

Since we are working in characteristic 0, we can reduce to consider multilinear invariants, that is (having identified V with V^*) elements of $\left(V^{\otimes 2m}\right)^{Sp(V)}$. Such elements can be encoded as follows. Consider the polynomial ring

$$A = \mathbb{C}[x_{ij}]_{i,j=1,i\neq j}^{2m} \Big/ (x_{ij} = -x_{ji})$$

in $m(2m-1)$ skew-symmetric variables.

Definition. *The space A_m of multilinear elements in A is the linear span over $\mathbb{C}$ of monomials of degree m*

$$x_{i_1 j_1} \dots x_{i_m j_m}$$

where $(i_1, j_1, \dots, i_m, j_m)$ is a permutation of $\{1, 2, \dots, 2m\}$.

In other words, in each multilinear monomial all indexes appear and they appear exactly once.

It is clear that A_m is a representation of S_{2m}. Every multilinear monomial is up to sign in the orbit of $m_0 := x_{1\,m+1}x_{2\,m+2}\ldots x_{m\,2m}$; the S_{2m} stabilizer of the line through m_0 is the hyperoctahedral group $K_m := S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$, which induces on this line the sign representation. We can therefore deduce the first assertion in the following statement (see [L-P], Prop. 3.3).

Proposition 2.2. *The representation of S_{2m} on A_m is induced by the sign representation of K_m. Moreover, as S_{2m}–modules,*

$$\bigoplus_{\substack{\sigma\vdash 2m\\ \sigma \text{ has even columns}}} M_\sigma \cong A_m,$$

M_σ being the irreducible representation of S_{2m} associated to the partition σ.

The fundamental theorems can be restated as follows:

Theorem 2.3. *([L-P], Th. 3.8) The map of S_{2m}–modules*

$$\alpha : A_m \longrightarrow \left((V^{\otimes 2m})^*\right)^{Sp(V)}$$

defined by extending linearly $\alpha(x_{i_1j_1}\ldots x_{i_mj_m}) = \eta_{i_1j_1\ldots i_mj_m}$ where

$$\eta_{i_1j_1\ldots i_mj_m}(v_1\otimes v_2\otimes\ldots\otimes v_{2m}) := \prod_{k=1}^{m}\langle v_{i_k}, v_{j_k}\rangle$$

is a surjective homomorphism of S_{2m}–modules. Its kernel is the intersection of A_m with the ideal $Pf_{(2n+2)}$ of A generated by the Pfaffians of order $2n+2$ of the skew-symmetric matrix (x_{ij}) and it corresponds, in the isomorphism of the Proposition 2.2, to the S_{2m}–submodule

$$\bigoplus_{\substack{\sigma\vdash 2m\\ l(\sigma)>2n\\ \sigma \text{ has even columns}}} M_\sigma$$

Remark. Let $\sigma = (i_1, j_1, i_2, j_2, \ldots, i_m, j_m) \in S_{2m}$; then set

$$\prod_{k=1}^{m}\psi_{i_kj_k} := \sigma.(\psi^{\otimes m}) \in (V^{\otimes(2m)})^{Sp(V)}.$$

It is easy to verify that under the canonical isomorphism $\left(V^{\otimes 2m}\right)^* \cong V^{\otimes 2m}$ described in §1, $\eta_{i_1j_1\ldots i_mj_m}$ maps to $\prod_{k=1}^{m}\psi_{i_kj_k}$. In particular any invariant in $V^{\otimes 2m}$ is a linear combination of elements $\prod_{k=1}^{m}\psi_{i_kj_k}$.

We will need the first fundamental theorem for the linear group too; it will be stated in two forms equivalent to the standard polynomial version: Schur duality and the mutual commutant theorem.

It is well-known that finite-dimensional representations of $GL(V)$ are completely reducible and irreducible (polynomial) representations are indexed by highest weights ω or equivalently by partitions λ with $l(\lambda) \leq dim(V)$. Let V_λ be the irreducible representation of $GL(V)$ relative to the partition λ, $(l(\lambda) \leq 2n)$ and let M_σ denote the irreducible representation of S_m, $(m = |\sigma|)$ relative to the partition σ.

Consider now $V^{\otimes m}$; there is a standard tensor product action of $GL(V)$ on it and also a natural action of S_m given on decomposable tensors by permutation of the factors:

$$\pi(v_1 \otimes \ldots \otimes v_m) = v_{\pi^{-1}(1)} \otimes \ldots \otimes v_{\pi^{-1}(m)}.$$

Theorem 2.4 (Schur Duality). *Let V be an n–dimensional complex vector space. For $m \in \mathbb{N}$, $V^{\otimes m}$ is a $GL(V) \times S_m$–module whose decomposition into irreducibles is:*

$$V^{\otimes m} \cong \bigoplus_{\substack{\lambda \vdash m \\ l(\lambda) \leq n}} V_\lambda \otimes M_\lambda.$$

Theorem 2.5. *The algebras spanned by the images of $GL(V)$ and of S_m, each acting on $V^{\otimes m}$ as described above, are mutual centralizers in $End(V^{\otimes m})$.*

The situation is no longer the same if we consider the action on $V^{\otimes m}$ of $Sp(V)$; in this case the centralizer of the $Sp(V)$–action properly contains the group algebra of the symmetric group: it is easy to see that also the operatorsτ_{pq} introduced in the next definition commute with the $Sp(V)$–action.

Definitions. *Fix $m \in \mathbb{N}$; for each pair p, q of integers between 1 and m we define*

(1) *a contraction operator $\Phi_{pq} : V^{\otimes m} \longrightarrow V^{\otimes (m-2)}$ (for $p < q$, say),*

$$\Phi_{pq}(v_1 \otimes \ldots \otimes v_m) = \langle v_p, v_q \rangle\, v_1 \otimes \ldots \widehat{v_p} \otimes \ldots \otimes \widehat{v_q} \otimes \ldots \otimes v_m$$

(2) *an insertion operator $\Psi_{pq} : V^{\otimes m} \longrightarrow V^{\otimes (m+2)}$, obtained inserting the element ψ in the positions p, q;*

(3) *an operator $\tau_{pq} : V^{\otimes m} \longrightarrow V^{\otimes m}$ defined by $\tau_{pq} := \Psi_{pq}\Phi_{pq}$.*

The same definition can be given in the orthogonal case, with the symmetric form $(\ ,\ )$ in place of $\langle\ ,\ \rangle$.

Definition. *We call Brauer algebra $\mathbb{B}_m^{-2n}$ of $Sp(V)$ the associative $\mathbb{C}$–algebra with 1 defined by generators σ in bijection with elements of S_m and τ_{pq} $(p, q = 1, \dots, m)$ and relations (assume all the index sets disjoint)*

$$\tau_{pq} = \tau_{qp} \qquad \sigma\tau_{pq}\sigma^{-1} = \tau_{\sigma(p)\sigma(q)} \qquad \tau_{pq}\tau_{hk} = \tau_{hk}\tau_{pq}$$
$$\tau_{pq}\tau_{qr} = \tau_{pq}\,(p\,r) \qquad \tau_{pq}^2 = 2n\tau_{pq} \qquad \tau_{pq} = -\tau_{pq}\,(p\,q)$$

as well as all relations of the symmetric group S_m among the σ's.

For the case of the orthogonal group we introduce similarly a Brauer algebra, with the same presentation except for the last two relations, which are replaced by the following ones:

$$\tau_{pq}^2 = n\tau_{pq} \qquad \tau_{pq} = \tau_{pq}\,(p\,q)$$

Finally, we call "Brauer algebra with formal parameter" $\mathcal{B}_m^x$ the associative $\mathbb{C}(x)$–algebra with 1 defined by generators σ in bijection with elements of S_m and T_{pq} $(p, q = 1, \dots, m,\ p \neq q)$ and relations (assume all the index sets disjoint)

$$T_{pq} = T_{qp} \qquad \sigma T_{pq}\sigma^{-1} = T_{\sigma(p)\sigma(q)} \qquad T_{pq}T_{hk} = T_{hk}T_{pq}$$
$$T_{pq}T_{qr} = T_{pq}\,(p\,r) \qquad T_{pq}^2 = xT_{pq} \qquad T_{pq} = T_{pq}\,(p\,q)$$

as well as all relations of the symmetric group S_m among the σ's.

The same presentation with $N \in \mathbb{C}$ instead of x defines the $\mathbb{C}$–algebra $\mathcal{B}_m^N$.

Remark. *There exist $\mathbb{C}$–algebra isomorphisms*

$$\Delta^-\colon \mathcal{B}_m^{-2n} \xrightarrow{\cong} \mathbb{B}_m^{-2n}(Sp(2n))\,, \qquad \Delta^+\colon \mathcal{B}_m^{n} \xrightarrow{\cong} \mathbb{B}_m^{n}(O(n))$$

given by

$$\Delta^-\colon T_{pq} \mapsto -\tau_{pq}\,,\ \sigma \mapsto sgn(\sigma)\sigma\,, \qquad \Delta^+\colon T_{pq} \mapsto \tau_{pq}\,,\ \sigma \mapsto \sigma$$

Theorem 2.6. *[Br] The natural map $\rho\colon \mathbb{B}_m^{-2n} \to End_{Sp(V)}\,(V^{\otimes m})$ is surjective; if $n \geq m$, then ρ is an isomorphism.*

Definition. *The space*

$$T^0(V^{\otimes m}) := \bigcap_{p<q} Ker\,\Phi_{pq}$$

is called the space of traceless tensors. More generally set, for $k = 0, \dots, [\frac{m}{2}]$:

$$T^k(V^{\otimes m}) := \sum_{i_1<j_1,\dots,i_k<j_k} \Psi_{i_1j_1}\dots\Psi_{i_kj_k}\left(T^0(V^{\otimes(m-2k)})\right).$$

The same definition can be given for the orthogonal case.

Remark. Weyl calls the elements of $T^k(V^{\otimes m})$ *tensors of valence k*, since they become traceless ("of valence 0") after k (but not less) contractions.

Lemma 2.7.

$$V^{\otimes m} = \bigoplus_{k=0}^{[\frac{m}{2}]} T^k(V^{\otimes m})$$

The space of traceless tensors plays a fundamental role in the construction of irreducible $Sp(V)$–modules; more precisely:

Proposition 2.8. *Let V be a $2n$–dimensional complex vector space, and let V_λ be the irreducible representation of $GL(V)$ indexed by the partition λ, $(l(\lambda) \leq 2n)$, with $|\lambda| = d$. Fix a Young symmetrizer $e_\lambda \in \mathbb{C}[S_d]$, so that $e_\lambda . V^{\otimes d} \cong V_\lambda$. Then*

$$W_\lambda := e_\lambda . V^{\otimes d} \cap T^0(V^{\otimes d})$$

is nonzero if and only if $l(\lambda) \leq n$. In this case, W_λ is the irreducible representation of $Sp(V)$ of highest weight[1] $\sum_{i=1}^n \lambda_i \varepsilon_i$.

When acting on traceless tensors, the operators τ_{pq} vanish, so the centralizer of the action of the symplectic group is exactly the group algebra of the symmetric group. We can therefore deduce the following fact.

Proposition 2.9. *The following decomposition of $T^0(V^{\otimes m})$ into isotypic components with respect to the action of $Sp(V) \times S_m$ holds:*

$$T^0(V^{\otimes m}) \cong \bigoplus_{\substack{\mu \vdash m \\ l(\mu) \leq n}} W_\mu \otimes M_\mu .$$

Moreover the following theorem holds:

Theorem 2.10. *([D-S] Theorem 3.5) The natural map*

$$\mathbb{C}[S_m] \to End_{Sp(V)} \left(T^0(V^{\otimes m}) \right)$$

is surjective.
Its kernel is the ideal generated by the antisymmetrizer $\sum_{\sigma \in S_{n+1}} (-1)^{sgn\, \sigma} \sigma$ on $n+1$ elements, which is

$$\bigoplus_{\substack{\lambda \vdash m \\ l(\lambda) > n}} I_\lambda$$

I_λ being the minimal two-sided ideal in $\mathbb{C}[S_m]$ associated to the partition λ (i.e. $I_\lambda = \mathbb{C}[S_m] e_\lambda \mathbb{C}[S_m]$, where e_λ is a Young symmetrizer associated to λ).

[1] $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the standard basis in $\mathbb{R}^n$.

§3 A $\mathbb{B}_m^{-2n}$–action on $Ind_{S_{m-2k}\times S_{2k}}^{S_m}\left(T^0\left(V^{\otimes(m-2k)}\right)\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}\right)$.

In this section we define an action of the Brauer algebra $\mathbb{B}_m^{-2n}$ on the space $Ind_{S_{m-2k}\times S_{2k}}^{S_m}\left(T^0\left(V^{\otimes(m-2k)}\right)\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}\right)$: besides being interesting on its own — as it will be explained in §7 — this will allow us to prove, in the stable, case our main result, namely

$$Ind_{S_{m-2k}\times S_{2k}}^{S_m}\left(T^0\left(V^{\otimes(m-2k)}\right)\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}\right)\cong T^k\left(V^{\otimes m}\right).$$

Set $H_k := S_{m-2k}\times S_{2k}$, $U_k := T^0\left(V^{\otimes(m-2k)}\right)\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}$. Since $Ind_{H_k}^{S_m}(U_k)$ has a natural structure of S_m–module, we need only to define an action of the elements τ_{pq} compatible with such an action of S_m ; we will first define auxiliary operators t_{pq} acting on U_k with values in $Ind_{H_k}^{S_m}(U_k)$, then we will extend their definition to the whole induced representation. Finally we will check that these extended operators τ_{pq} satisfy the defining relations of the Brauer algebra.

Set $A := \{1,\dots,m-2k\}$, $B := \{m-2k+1,\dots,m\}$. We remark that the natural $\mathbb{B}_m^{-2n}$–action on $V^{\otimes m}$ restricts to $T^k\left(V^{\otimes m}\right)$, affording a representation $\overline{\rho}_k$ of $\mathbb{B}_m^{-2n}$; so we are led to give the following definition.

Definition. *Define a map* $t_{pq} : U_k \to \mathbb{C}[S_m]\otimes_{\mathbb{C}[H_k]} U_k$, $p,q\in\{1,\dots,m\}$, $p<q$, *as follows.*
(1) For $p,q\in A$ *set*

$$t_{pq}(u)=0 \qquad \forall u\in U_k.$$

(2) For $p,q\in B$ *set*

$$t_{pq}(u)=\begin{cases} 2n(1\otimes_{\mathbb{C}[H_k]}u), \ \textit{if } u=x\otimes\psi_{pq}\left(\prod_{s=1}^{k-1}\psi_{i_sj_s}\right), \ x\in T^0\left(V^{\otimes(m-2k)}\right)\\ (pr)\otimes_{\mathbb{C}[H_k]}u, \ \textit{if } u=x\otimes\psi_{pt}\psi_{qr}\left(\prod_{s=1}^{k-2}\psi_{a_sb_s}\right), x\in T^0\left(V^{\otimes(m-2k)}\right)\end{cases}$$

where $\{i_1,j_1,\dots,i_{k-1},j_{k-1}\}$ *is a permutation of* $B\setminus\{p,q\}$ *in the first case and* $\{a_1,b_1,\dots,a_{k-2},b_{k-2}\}$ *is a permutation of* $B\setminus\{p,q,t,r\}$ *in the second one.*
(3) For $p\in A$, $q\in B$ *set*

$$t_{pq}(u)=(pr)\otimes_{\mathbb{C}[H_k]}u, \quad \textit{if } u=x\otimes\psi_{qr}\left(\prod_{s=1}^{k-1}\psi_{u_sv_s}\right), \ x\in T^0\left(V^{\otimes(m-2k)}\right)$$

where $\{u_1,v_1,\dots,u_{k-1},v_{k-1}\}$ *is a permutation of* $B\setminus\{r,q\}$.

Remarks.

(1) In cases (2) and (3) we have defined t_{pq} only on a set of special elements of U_k; by the remark after Theorem 2.3 these elements are linear generators for $(V^{\otimes(2k)})^{Sp(V)}$; this suffices to determine t_{pq} completely since the action just defined is induced by $\overline{\rho}_k$, so in particular $t_{ij}(Pf_{(2n+2)}) = 0 \,\forall\, i, j$.

(2) To simplify notation, in the following we will denote an element $u = x \otimes \psi_{pq} \left(\prod_{s=1}^{k-1} \psi_{i_s j_s} \right)$ as $u = x \otimes \psi_{pq} R$ writing down only the relevant indexes.

(3) Notice that the symbols ψ_{ij} are skew-symmetric; so the following useful identity in the lower subcase of (2) holds:

$$\begin{aligned}(pr) \otimes_{\mathbb{C}[H_k]} u &= (qt) \otimes_{\mathbb{C}[H_k]} x \otimes \psi_{tp}\psi_{rq} R = (qt) \otimes_{\mathbb{C}[H_k]} x \otimes (qt)(pr)\psi_{pt}\psi_{qr} R \\ &= (qt)(pr)(pr) \otimes_{\mathbb{C}[H_k]} u = (qt) \otimes_{\mathbb{C}[H_k]} u.\end{aligned}$$

(4) If we set formally $t_{qp} := t_{pq}$ for $p \in A, q \in B$ we may extend the definition of the operators to any pair of indexes; moreover it follows immediately from the definitions and from the previous remark that $t_{ij} = t_{ij}$ if both the indexes are in A or in B, so for any pair of indexes in $\{1, \dots, m\}$ the relation $t_{ij} = t_{ji}$ holds.

Lemma 3.1. $ht_{ij} = t_{h(i)h(j)} h \quad \forall\, h \in H_k$.

Proof. If $i, j \in A$, then $ht_{ij} \equiv 0$; on the other hand H_k preserves A so $h(i), h(j) \in A$ and $t_{h(i)h(j)} = t_{h(i)h(j)} h = 0$.

Suppose $i, j \in B$; in the first case we have

$$ht_{ij}(u) = 2n(h(u)) = t_{h(i)h(j)} h(u)$$

since $h(u) = h(x) \otimes \psi_{h(i)h(j)} h(R)$; for the other case suppose $u = x \otimes \psi_{ip}\psi_{jq} R$; we have

$$ht_{ij}(u) = h(pj)u = (h(p)h(j))hu = t_{h(i)h(j)} h(u).$$

Finally suppose $i \in A, j \in B$; if $u = x \otimes \psi_{jr} R$

$$ht_{ij}(u) = h(ir)u = h(ir)h^{-1}h(u) = (h(i)h(r))\, h(u) = t_{h(i)h(j)} h(u). \qquad \square$$

Definition. *Define* $\tau_{ij} : C[S_m] \otimes_{\mathbb{C}[H_k]} U_k \to C[S_m] \otimes_{\mathbb{C}[H_k]} U_k$ *by the formula*

$$\tau_{ij}(g \otimes u) := g t_{g^{-1}(i) g^{-1}(j)}(u).$$

By the previous lemma, τ_{ij} is well-defined.

Lemma 3.2. *For any $\sigma \in S_m$ we have $\sigma\tau_{ij}\sigma^{-1} = \tau_{\sigma(i)\sigma(j)}$.*

Proof. Just compute:

$$\begin{aligned}\sigma\tau_{ij}\sigma^{-1}(g \otimes u) = \sigma\tau_{ij}(\sigma^{-1}g \otimes u) = \sigma(\sigma^{-1}gt_{g^{-1}\sigma(i)g^{-1}\sigma(j)}(u)) = \\ = gt_{g^{-1}\sigma(i)g^{-1}\sigma(j)}(u) = \tau_{\sigma(i)\sigma(j)}(g \otimes u) \quad \square\end{aligned}$$

Theorem 3.3. *The operators $\sigma \in S_m$ and τ_{ij} define a representation ρ_k of $\mathbb{B}_m^{-2n}$ on $Ind_{H_k}^{S_m}(U_k)$.*

Proof. By the previous lemma and remark (4), we need only to verify the last four defining relations. We will work out explicitly the proof only for the first relation; the other ones can be treated similarly.

Consider the relation $\tau_{ij}^2 = 2n\,\tau_{ij}$; evaluating both sides on $g \otimes u$ we are immediately reduced to prove the following identity on U_k:

$$\tau_{ij}t_{ij} = 2n\,t_{ij}.$$

We have to examine three cases, according whether $i, j \in A$, $i, j \in B$ or $i \in A, j \in B$. The first case is trivial, so let us consider the second one; we have two subcases: if u is an eigenvector for t_{ij} then the relation is immediate; otherwise, assume $u = x \otimes \psi_{ip}\psi_{jq}R$; we have

$$\tau_{ij}t_{ij}(u) = \tau_{ij}(pj)u = (pj)t_{ip}(u) = 2n\,(pj)u = 2n\,t_{ij}(u).$$

Finally suppose $i \in A, j \in B$ and $u = x \otimes \psi_{jr}R$:

$$\tau_{ij}t_{ij}(u) = \tau_{ij}(ir)u = (ir)t_{jr}(u) = 2n(ir)u = 2n\,t_{ij}(u). \quad \square$$

Proposition 3.4. *Let $k \in \{0, 1, \dots, [\frac{m}{2}]\}$. The map*

$$\Theta_k : Ind_{H_k}^{S_m}\left(T^0\left(V^{\otimes(m-2k)}\right) \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right) \longrightarrow T^k\left(V^{\otimes m}\right)$$

given, for all $\alpha \in S_m$, $a \in T^0\left(V^{\otimes(m-2k)}\right)$, $b \in \left(V^{\otimes(2k)}\right)^{Sp(V)}$, by

$$\Theta_k\big(\alpha \otimes_{\mathbb{C}[H_k]} (a \otimes b)\big) = \alpha(a \otimes b)$$

is a morphism of $Sp(V) \times S_m$–modules.

Moreover, Θ_k is an epimorphism of $Sp(V) \times \mathbb{B}_m^{-2n}$–modules.

Proof. By the very definition, it is clear that the map Θ_k is a well-defined $Sp(V) \times S_m$–module homomorphism.

Consider the subspace $Z_k \subset T^k\left(V^{\otimes m}\right)$ defined by

$$\begin{aligned} Z_k &:= \Psi_{m-1m} \dots \Psi_{m-2k+1m-2k}\left(T^0\left(V^{\otimes(m-2k)}\right)\right) \\ &= \{v \otimes \psi^{\otimes k} \mid v \in T^0\left(V^{\otimes(m-2k)}\right)\}; \end{aligned}$$

then Z_k generates $T^k\left(V^{\otimes m}\right)$ as an S_m–module; moreover we have $\psi^{\otimes k} \in (V^{\otimes(2k)})^{Sp(V)}$, hence Z_k can be canonically identified with a subspace of U_k and, in turn, with a subspace $\tilde{Z}_k$ of $Ind_{H_k}^{S_m}(U_k)$. By S_m–equivariance of Θ_k we get

$$\Theta_k(S_m.\tilde{Z}_k) = S_m.\Theta_k(\tilde{Z}_k) = S_m.Z_k = T^k\left(V^{\otimes m}\right)$$

so that Θ_k is onto. Finally, the very definition of the $\mathbb{B}_m^{-2n}$–actions on both sides immediately implies that Θ_k is $\mathbb{B}_m^{-2n}$–equivariant too. $\square$

Remark. *Due to the reductivity of $Sp(V)$, the equivariant epimorphism $\Theta_k : Ind_{H_k}^{S_m}(U_k) \longrightarrow T^k\left(V^{\otimes m}\right)$ induces an epimorphism*

$$End'_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right) \longrightarrow End_{Sp(V)}\left(T^k\left(V^{\otimes m}\right)\right),$$

where $End'_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right)$ is the subalgebra of $End_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right)$ stabilizing the kernel of Θ_k. In particular:

$$dim\left(End'_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right)\right) \geq dim\left(End_{Sp(V)}\left(T^k(V^{\otimes m})\right)\right).$$

The following is standard.

Lemma 3.5. *Let G be a group, H a subgroup of G and M a finite dimensional G–module. Then there exists a $G \times G$–equivariant canonical isomorphism of $G \times G$–modules:*

$$End\left(Ind_H^G(M)\right) \cong Ind_{H\times H}^{G\times G}\left(End(M)\right).$$

Corollary. *As $S_m \times S_m$–modules*

$$End_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right) \cong Ind_{H_k\times H_k}^{S_m\times S_m}\left(\frac{\mathbb{C}[S_{m-2k}]}{\bigoplus_{\substack{\lambda\vdash(m-2k)\\ l(\lambda)>n}} I_\lambda} \otimes End\left(\frac{A_k}{Pf_{(2n+2)}}\right)\right).$$

Proof. The claim follows from the following chain of $S_m \times S_m$–isomorphisms:

$$End_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right) = \left(End\left(Ind_{H_k}^{S_m}(U_k)\right)\right)^{Sp(V)} =$$

$$= \left(Ind_{H_k \times H_k}^{S_m \times S_m} End\left(T^0\left(V^{\otimes(m-2k)}\right) \otimes \left(V^{\otimes(2k)}\right)^{Sp(V)}\right)\right)^{Sp(V)} =$$

$$= Ind_{H_k \times H_k}^{S_m \times S_m}\left(End\left(T^0\left(V^{\otimes(m-2k)}\right)\right) \otimes End\left(\left(V^{\otimes(2k)}\right)^{Sp(V)}\right)\right)^{Sp(V)} =$$

$$= Ind_{H_k \times H_k}^{S_m \times S_m}\left(End_{Sp(V)}\left(T^0\left(V^{\otimes(m-2k)}\right)\right) \otimes End\left(\left(V^{\otimes(2k)}\right)^{Sp(V)}\right)\right) =$$

$$= Ind_{H_k \times H_k}^{S_m \times S_m}\left(\frac{\mathbb{C}[S_{m-2k}]}{\bigoplus_{\substack{\lambda \vdash (m-2k) \\ l(\lambda) > n}} I_\lambda} \otimes End\left(\frac{A_k}{Pf_{(2n+2)}}\right)\right).$$

The second equality follows from the previous lemma, the third from the fact that taking invariants commutes with induction, the fourth from the fact that the $Sp(V)$ action is trivial on $End\left(\left(V^{\otimes(2k)}\right)^{Sp(V)}\right)$ and the last from Theorems 2.3, 2.10. □

Now we can prove the following key fact.

Proposition 3.6. *If $n \geq m$, then the centralizer of the $Sp(V)$–action on $Ind_{H_k}^{S_m}(U_k)$ is $\rho_k(\mathbb{B}_m^{-2n})$.*

Proof. Since $\rho_k(\mathbb{B}_m^{-2n}) \subseteq End_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right)$ we can prove equality by comparing dimensions. First recall that $dim\, \mathbb{B}_m^{-2n} = (2m-1)!!$ (cf. [Wz] or

§7 below); then

$$\begin{aligned}(2m-1)!! &= dim\left(\mathbb{B}_m^{-2n}\right) = dim\left(\rho\left(\mathbb{B}_m^{-2n}\right)\right) = dim\left(End_{Sp(V)}\left(V^{\otimes m}\right)\right) = \\ &= dim\left(End_{Sp(V)}\left(\bigoplus_{k=0}^{[m/2]} T^k(V^{\otimes m})\right)\right) = \\ &= \sum_{k=0}^{[m/2]} dim\left(End_{Sp(V)}\left(T^k\left(V^{\otimes m}\right)\right)\right) \leq \\ &\leq \sum_{k=0}^{[m/2]} dim\left(End'_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right)\right) \leq \\ &\leq \sum_{k=0}^{[m/2]} dim\left(End_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right)\right) = \\ &= \sum_{k=0}^{[m/2]} dim\left(Ind_{H_k\times H_k}^{S_m\times S_m}\left(End_{Sp(V)}(U_k)\right)\right) = \\ &= \sum_{k=0}^{[m/2]} \left(\frac{m!}{(m-2k)!(2k)!}\right)^2 (m-2k)!\,(2k-1)!!^2\,;\end{aligned}$$

in fact, since $n \geq m$, ρ is faithful, whence the second equality, and the previous Corollary gives $End_{Sp(V)}(U_k) = \mathbb{C}[S_{m-2k}] \otimes End(A_k)$; we also used the remark after Proposition 3.4. In Lemma 3.7 below we prove that

$$\sum_{k=0}^{[m/2]} \left(\frac{m!}{(m-2k)!(2k)!}\right)^2 (m-2k)!\,(2k-1)!!^2 = (2m-1)!!$$

so we can deduce that

$$End_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right) = End'_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right) = End_{Sp(V)}\left(T^k\left(V^{\otimes m}\right)\right).$$

Since $\overline{\rho}_k$ is surjective and the following diagram commutes

$$\begin{array}{ccc} \mathbb{B}_m^{-2n} & = & \mathbb{B}_m^{-2n} \\ {\scriptstyle \rho_k}\downarrow & & \downarrow{\scriptstyle \overline{\rho}_k} \\ End_{Sp(V)}\left(Ind_{H_k}^{S_m}(U_k)\right) & \xrightarrow{\cong} & End_{Sp(V)}\left(T^k\left(V^{\otimes m}\right)\right) \end{array}$$

the proof is completed. □

Lemma 3.7.

$$\sum_{k=0}^{[m/2]} \left(\frac{m!}{(m-2k)!(2k)!} \right)^2 (m-2k)!\,(2k-1)!!^2 = (2m-1)!!.$$

Proof. Consider $2m$ elements partitioned into two sets A and B of m elements each. It is well-known that the number of pairings between elements of $A \cup B$ is $(2m-1)!!$; we want to show that the previous formula expresses a way of counting such pairings. We may select a pairing in $A \cup B$ giving the following data:

(1) $m-2k$ elements in A;
(2) $m-2k$ elements in B;
(3) a pairing between the $m-2k$ elements selected in A and the $m-2k$ elements selected in B;
(4) a pairing of the remaining $2k$ elements in A;
(5) a pairing of the remaining $2k$ elements in B.

The reader will easily convince himself that when k runs from 0 to $[m/2]$ the procedure above displays all possible pairings of $A \cup B$ without repetitions. Now we have only to count the possible choices for a fixed configuration and to sum over k from 0 to $[m/2]$: there is a contribution $\binom{m}{m-2k}^2$ for (1),(2), a contribution $(m-2k)!$ for (3) and finally a contribution $(2k-1)!!^2$ for (4), (5) so our claim follows. $\square$

§4 The isomorphism $T^k\left(V^{\otimes m}\right) \cong Ind_{H_k}^{S_m}\left(U_k\right)$ (stable case).

In Proposition 3.4 we showed the existence of an $Sp(V) \times \mathbb{B}_m^{-2n}$–epimorphism $\Theta_k : Ind_{H_k}^{S_m}\left(U_k\right) \longrightarrow T^k\left(V^{\otimes m}\right)$; in this section we will prove — using the constructions and results of §3 — that in the stable case Θ_k is an isomorphism: this is done by comparing the $\mathbb{B}_m^{-2n}$–actions on both sides.

Theorem 4.1. *Let $n \geq m$. Then*

$$\Theta_k : Ind_{H_k}^{S_m}\left(T^0\left(V^{\otimes(m-2k)}\right) \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right) \longrightarrow T^k\left(V^{\otimes m}\right)$$

is an isomorphism of $Sp(V) \times \mathbb{B}_m^{-2n}$–modules.

Proof. From §3, we have actions ρ_k, $\overline{\rho}_k$ of $\mathbb{B}_m^{-2n}$ on $Ind_{H_k}^{S_m}\left(U_k\right)$, $T^k\left(V^{\otimes m}\right)$ respectively, so we can decompose both $Ind_{H_k}^{S_m}\left(U_k\right)$ and $T^k\left(V^{\otimes m}\right)$ into isotypic component with respect to $Sp(V) \times \mathbb{B}_m^{-2n}$; furthermore, the (image of the) Brauer algebra is the whole $Sp(V)$–centralizer for these actions, hence by the double centralizer theorem these decompositions are multiplicity free, i.e. of the form $\bigoplus_\mu W_\mu \otimes N_\mu$, N_μ being an irreducible $\mathbb{B}_m^{-2n}$–module. Now remark that the irreducible representations of $Sp(V)$ occurring in the two decompositions are the same. Moreover Θ_k is equivariant, thus it maps isotypic components into isotypic components. By Schur lemma we have $\Theta_k(W_\mu \otimes N_\mu) = 0$ or $\Theta_k(W_\mu \otimes N_\mu) = W_\mu \otimes N_\mu$: since $W_\mu \otimes N_\mu$ *does* occur in $T^k\left(V^{\otimes m}\right) = Im(\Theta_k)$, the map Θ_k must be injective, whence the thesis follows. $\square$

§5 Littlewood's restriction rule.

As a corollary of Theorem 4.1 we get a representation-theoretic proof of Littlewood's restriction rule (Littlewood, [L]), in the following slightly weaker version (Littelwood makes the assumption $\lambda \vdash m$, $l(\lambda) \leq n$, which is a weaker condition than $n \geq m$, because $l(\lambda) \leq m$ for all $\lambda \vdash m$):

Proposition 5.1. *If $n \geq m$ and $\lambda \vdash m$, then*

$$V_\lambda \downarrow_{Sp(V)}^{GL(V)} \cong \bigoplus_\mu \left(\sum_{\substack{\sigma \vdash (|\lambda| - |\mu|) \\ \sigma \text{ has even columns}}} c_{\mu,\sigma}^\lambda \right) W_\mu$$

where $c_{\mu,\sigma}^\lambda$ is the Littlewood-Richardson coefficient expressing the multiplicity of M_λ in the decomposition into irreducibles of $Ind_{S_{|\mu|} \times S_{|\sigma|}}^{S_{|\lambda|}} (M_\mu \otimes M_\sigma)$.

Proof. We will compute the "branching rules" of $T^k(V^{\otimes m})$ and $Ind_{H_k}^{S_m}(U_k)$. By Propositions 2.2 and 2.9 and Theorem 2.3 (with $n \geq m$) we have

$$\begin{aligned}
Ind_{H_k}^{S_m}(U_k) &= Ind_{H_k}^{S_m}\left(T^0\left(V^{\otimes(m-2k)}\right) \otimes \left(V^{\otimes(2k)}\right)^{Sp(V)} \right) \cong \\
&\cong \bigoplus_{\mu \vdash (m-2k)} \bigoplus_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even columns}}} Ind_{H_k}^{S_m}(W_\mu \otimes M_\mu \otimes M_\sigma) \cong \\
&\cong \bigoplus_{\mu \vdash (m-2k)} \bigoplus_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even columns}}} W_\mu \otimes Ind_{H_k}^{S_m}(M_\mu \otimes M_\sigma) \cong \\
&\cong \bigoplus_{\mu \vdash (m-2k)} \bigoplus_{\lambda \vdash m} \left(\sum_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even columns}}} c_{\mu,\sigma}^\lambda \right) W_\mu \otimes M_\lambda.
\end{aligned}$$

Since $n \geq m$, Theorem 4.1 and the previous computation give:

$$T^k\left(V^{\otimes m}\right) \cong Ind_{H_k}^{S_m}(U_k) \cong \bigoplus_{\mu \vdash (m-2k)} \bigoplus_{\lambda \vdash m} \left(\sum_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even columns}}} c^{\lambda}_{\mu,\sigma} \right) W_\mu \otimes M_\lambda.$$

Now remark that, as $GL(V) \times S_m$–modules,

$$\bigoplus_{k=0}^{[\frac{m}{2}]} T^k\left(V^{\otimes m}\right) = V^{\otimes m} \cong \bigoplus_{\substack{\lambda \vdash m \\ l(\lambda) \leq 2n}} V_\lambda \otimes M_\lambda = \bigoplus_{\lambda \vdash m} V_\lambda \otimes M_\lambda.$$

Restrict V_λ to the symplectic group; then $V_\lambda \downarrow_{Sp(V)}^{GL(V)} \cong \oplus_\mu n^\lambda_\mu W_\mu$ for some coefficients n^λ_μ; thanks to Theorem 2.2 the part relative to T^k of the previous decomposition of $V^{\otimes m}$ can be expressed as follows:

$$T^k\left(V^{\otimes m}\right) \cong \bigoplus_{\mu \vdash (m-2k)} \bigoplus_{\lambda \vdash m} n^\lambda_\mu W_\mu \otimes M_\lambda.$$

Comparing the decompositions gives the thesis, for

$$\bigoplus_{\lambda \vdash m} V_\lambda \downarrow_{Sp(V)}^{GL(V)} \otimes M_\lambda \cong \bigoplus_{\mu \vdash (m-2k)} \bigoplus_{\lambda \vdash m} n^\lambda_\mu W_\mu \otimes M_\lambda \cong$$

$$\cong \bigoplus_{\mu \vdash (m-2k)} \bigoplus_{\lambda \vdash m} \left(\sum_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even columns}}} c^{\lambda}_{\mu,\sigma} \right) W_\mu \otimes M_\lambda. \quad \square$$

§6 The orthogonal case.

Let now V be an n–dimensional complex vector vector space endowed with a symmetric non-degenerate bilinear form $(\ ,\ )$. We can adapt the previous results to the case of $O(V)$, pointing out some few changes to be made.

Invariants on $V^{\oplus m}$ are generated by scalar products (v_i, v_j) and the ideal $\mathcal{M}_{(n+1)}$ of relations among them is generated by minors of order $n+1$ of the $m \times m$ matrix $\big((v_i, v_j)\big)_{i,j=1,\dots,m}$; so we can encode invariants as elements of the polynomial ring $B = \mathbb{C}[x_{ij}]_{i,j=1,i\neq j}^{2m} \Big/ (x_{ij} = x_{ji})$ in $m(2m+1)$ symmetric variables, and multilinear invariants B_m as linear combination of monomials $x_{i_1 j_1} \dots x_{i_m j_m}$ where $(i_1, j_1, \dots, i_m, j_m)$ is a permutation of $\{1, 2, \dots, 2m\}$. We have the following description of the S_{2m}–module structure on B_m.

Proposition 6.1. *The representation of S_{2m} on B_m is induced by the trivial representation of S_m. Moreover, as S_{2m}–module,*

$$\bigoplus_{\substack{\sigma \vdash 2m \\ \sigma \text{ has even rows}}} M_\sigma \cong B_m .$$

When dealing with traceless tensors, Proposition 2.9 and Theorem 2.10 can be formulated as follows. Let D_λ the irreducible representation of $O(V)$ corresponding the partition λ of m, and denote by $\mu^t = (\mu_1^t, \mu_2^t, \dots)$ the transposed partition of $\mu = (\mu_1, \mu_2, \dots)$.

Proposition 6.2. *The following decomposition of $T^0(V^{\otimes m})$ into isotypic components with respect to the action of $O(V) \times S_m$ holds:*

$$T^0(V^{\otimes m}) = \bigoplus_{\substack{\mu \vdash m \\ \mu_1^t + \mu_2^t \leq n}} D_\mu \otimes M_\mu .$$

Theorem 6.3. *The natural map $\mathbb{C}[S_m] \to End_{O(V)}\left(T^0(V^{\otimes m})\right)$ is surjective; its kernel is the ideal generated by all the Young symmetrizers relative to diagrams with the first two columns adding to a number $\geq n+1$, i.e.*

$$\bigoplus_{\substack{\lambda \vdash m \\ \lambda_1^t + \lambda_2^t > n}} I_\lambda .$$

Now the arguments used in the symplectic case work again, *mutatis mutandis*, in the orthogonal case as well: in particular, there exists an action of the Brauer algebra (of the orthogonal group) on $Ind_{H_k}^{S_m}(U_k)$, with $U_k := T^0\left(V^{\otimes(m-2k)}\right) \otimes \left(V^{2k}\right)^{O(V)}$, defined as in §3. Then everything goes through in a similar fashion, up to the obvious changes: for instance the statement of the corollary to Lemma 3.5 in the orthogonal case turns into the following:

$$End_{O(V)}\Big(Ind_{H_k}^{S_m}(U_k)\Big) \cong Ind_{H_k \times H_k}^{S_m \times S_m}\left(\frac{\mathbb{C}[S_{m-2k}]}{\bigoplus_{\substack{\lambda \vdash (m-2k) \\ \lambda_1^t + \lambda_2^t > n}} I_\lambda} \otimes End\left(\frac{B_k}{\mathcal{M}_{(n+1)}}\right)\right).$$

In particular *in the stable case, i.e. for $n \geq m$*, we have the analog of Theorem 4.1, namely

There exists an isomorphism of $O(V)\times\mathbb{B}_m^{-2n}$*–modules*

$$\Theta_k : Ind_{H_k}^{S_m}\left(T^0\left(V^{\otimes(m-2k)}\right)\otimes\left(V^{\otimes 2k}\right)^{O(V)}\right)\longrightarrow T^k\left(V^{\otimes m}\right)$$

and Littlewood's restriction rule as its corollary, that is

$$V_\lambda\downarrow_{O(V)}^{GL(V)}\cong\bigoplus_{\mu}\left(\sum_{\substack{\sigma\vdash(|\lambda|-|\mu|)\\ \sigma\text{ has even rows}}} c_{\mu,\sigma}^{\lambda}\right)D_\mu\,.$$

§7 The irreducible $\mathbb{B}_m^{-2n}$–modules (stable case).

By general theory, in the stable case all irreducible representations of $\mathbb{B}_m^{-2n}$ can be realized in tensor spaces $V^{\otimes m}$; in this section we show that from our main result we can also deduce a complete description of such representations: in fact we produce an explicit realization of the irreducible representations which are described by Kerov (cf. [K]) for the Brauer algebra with formal parameter; in particular, all results therein stated are here proved from a representation-theoretic viewpoint.

Consider the space $Ind_{H_k}^{S_m}\left(T^0\left(V^{\otimes(m-2k)}\right)\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}\right)$: this is a $Sp(V)\times\mathbb{B}_m^{-2n}$–module. Since $T^0\left(V^{\otimes(m-2k)}\right)\cong\bigoplus_{\substack{\mu\vdash(m-2k)\\ l(\mu)\le n}} W_\mu\otimes M_\mu$ as a module over $Sp(V)\times S_{m-2k}$, and S_m centralizes $Sp(V)$, the previous module splits into direct sum of $Sp(V)\times S_m$–modules as follows

$$\begin{aligned}Ind_{H_k}^{S_m}&\left(T^0\left(V^{\otimes(m-2k)}\right)\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}\right)\cong\\ &\cong\bigoplus_{\substack{\mu\vdash(m-2k)\\ l(\mu)\le n}} W_\mu\otimes Ind_{H_k}^{S_m}\left(M_\mu\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}\right)\,;\end{aligned}$$

moreover, as $\mathbb{B}_m^{-2n}$ centralizes $Sp(V)$, the spaces $Ind_{H_k}^{S_m}\left(M_\mu\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}\right)$ are in fact $\mathbb{B}_m^{-2n}$–modules: the action of $\mathbb{B}_m^{-2n}$ is properly described as in §3.

Now assume we are in the stable case: then $Sp(V)$ and $\mathbb{B}_m^{-2n}$ are mutual centralizer (Proposition 3.6) and $Ind_{H_k}^{S_m}\left(T^0\left(V^{\otimes(m-2k)}\right)\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}\right)\cong T^k\left(V^{\otimes m}\right)$, hence from $V^{\otimes m}\cong\bigoplus_{k=0}^{[m/2]}T^k\left(V^{\otimes m}\right)$ we get

$$V^{\otimes m}=\bigoplus_{k=0}^{[m/2]}\bigoplus_{\mu\vdash(m-2k)} W_\mu\otimes Ind_{H_k}^{S_m}\left(M_\mu\otimes\left(V^{\otimes 2k}\right)^{Sp(V)}\right)\,;$$

then, by the double centralizer theorem, the $Ind_{H_k}^{S_m}\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right)$'s are irreducible modules; moreover, the representation of $\mathbb{B}_m^{-2n}$ on $V^{\otimes m}$ is faithful, hence the $Ind_{H_k}^{S_m}\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right)$'s ($\mu \vdash (m-2k)$, $k \in \{0, 1, \dots, [m/2]\}$) are all of the irreducible representations of $\mathbb{B}_m^{-2n}$. The same situation occurs when dealing with the orthogonal group (and the associated Brauer algebra) instead of the symplectic group.

To describe $Ind_{H_k}^{S_m}\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right)$, note that, in the stable case, $\left(V^{\otimes 2k}\right)^{Sp(V)}$ has a basis of antisymmetric "monomials" $\prod_{h=1}^{k} \psi_{i_h j_h}$: using this basis one recognizes that $\left(V^{\otimes 2k}\right)^{Sp(V)}$ is $Ind_{S_2^{\times k}}^{S_{2k}}\left(\Sigma_2^{\otimes k}\right)$ (as an S_{2k}–module), where Σ_2 is the sign representation of S_2. The orthogonal case is slightly simpler, for the trivial representation T_2 of S_2 occurs in place of Σ_2.

Now we recall an alternative (mostly used, indeed) description of the algebra $\mathcal{B}_m^x$ (cf. for instance [Wz], §2). Consider graphs with $2m$ vertices and m edges such that each edge joins exactly two vertices and each vertex belongs to exactly one edge; represent the vertices with spots arranged in two lines, one upon the other: the picture below shows an example for $m = 5$.

We call such graphs m*–diagrams*; they are as many as the pairings of $2m$ elements, hence $(2m-1)!!$ in number. We define a product of m–diagrams a and b by the following rule:

(1) draw b below a;
(2) connect the i–th lower vertex of a with the i–th upper vertex of b;
(3) let d be the number of cycles in the new graph obtained in (2) and let c be this graph without the cycles; then c is an m–diagram, and we set $ab := x^d c$.

It is well-known that the $\mathbb{C}(x)$–algebra with basis the set of m–diagrams and product defined by linear extension of the rule above is canonically isomorphic to $\mathcal{B}_m^x$.

Using the description of $\mathcal{B}_m^x$ just recalled, Wenzl shows the following:

Proposition 7.1. *([Wz], §3) If $N \in \mathbb{C}$ is not an integer such that $\left[\frac{|N|}{2}\right] < m$, then $\mathcal{B}_m^N$ is semisimple, and its decomposition into direct sum of full matrix rings is the same of $\mathcal{B}_m^x$. In fact $\mathcal{B}_m^x \cong \mathbb{C}(x) \otimes \mathcal{B}_m^N$.*

As a consequence, the representation theory of the Brauer algebra is the same in the formal parameter case and in the stable case.

Now we sketch the construction of irreducible representations provided by Kerov.

Definition. *Let $m \in \mathbb{N}_+$, $k \in \{1, \dots, [m/2]\}$. We call (m,k)–junction any graph with m vertices and k edges such that each edge joins exactly two vertices and each vertex belongs to at most one edge. We denote the set of (m,k)–junctions by $X_{m,k}$, and by $H_{m,k}$ the $\mathbb{C}$–vector space with basis $X_{m,k}$.*

It is clear that $dim(H_{m,k}) = \left|X_{m,k}\right| = \binom{m}{2k}(2k-1)!!$. The following is an example of $(8,2)$–junction:

Let a be an m–diagram, and let v be an (m,k)–junction; for all $i = 1, \dots, m$, connect the i–th lower vertex of a with the i–th vertex of v: let d be the number of loops occurring in the new graph $\Gamma(a,v)$ obtained in this way, and let $a * v$ be the graph made of the points of the upper line of a, connected by an edge iff they are connected (by an edge or a path) in the new graph $\Gamma(a,v)$; then $a * v \in X_{m,k'}$, with $k' \geq k$ and $k' = k$ iff each pair of vertices of v which are connected by a path in $\Gamma(a,v)$ are in fact linked by an edge in v: in this cases we say that the (m,k)–junction v is *admissible* for the m–diagram a.

We set

$$\begin{aligned} a.v &:= x^d a * v \quad \text{if } v \text{ is admissible for } a \\ a.v &:= 0 \quad \text{otherwise.} \end{aligned}$$

Proposition 7.2. *([K]) Linear extension of the previous rule endows $H_{m,k}$ with a well-defined structure of $\mathcal{B}_m^x$–module, which is irreducible.*

To any pair (a, v) of an m–diagram and an (m, k)–junction we can also attach an element $\pi(a, v) \in S_{m-2k}$: this is the permutation which carries — through the graph $\Gamma(a, v)$ — the isolated vertices of v into the isolated vertices of $a * v$ (one keeps into account only the relative position of the isolated vertices in v, $a * v$) in case v is admissible for a, and is id otherwise. In the previous example we have $\pi(a, v) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$.

Given $H_{m,k}$, let $\mu \vdash (m - 2k)$ and let M_μ be the associated irreducible representation of S_{m-2k} : we set

$$H_{m,k}^\mu := M_\mu \otimes H_{m,k} .$$

Proposition 7.3. *([K]) Linear extension of the rule*

$$a.(u \otimes v) := \pi(a, v).u \otimes a.v$$

(for every m–diagram a and every (m, k)–junction v) endows $H_{m,k}^\mu$ with a well-defined structure of $\mathcal{B}_m^x$–module, which is irreducible. Conversely, any irreducible representation of $\mathcal{B}_m^x$ is isomorphic to one of the $H_{m,k}^\mu$.

From now on, for all $\mu \vdash (m-2k)$ we denote by μ' the transposed partition of μ; then we fix any identification $M_{\mu'} \cong \Sigma_{m-2k} \otimes M_\mu$ and define $\phi''\colon M_\mu \to M_{\mu'} \cong \Sigma_{m-2k} \otimes M_\mu$ by $\phi''(m) := 1 \otimes m$. Finally set $N_k := Ind_{S_2^{\times k}}^{S_{2k}} \left(T_2^{\otimes k}\right)$.

Our goal is to show that $H_{m,k}^{\mu'}$ is isomorphic, as $\mathbb{B}_m^{-2n}$–module, to the $Ind_{H_k}^{S_m} \left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right)$. We need the following easy lemma:

Lemma 7.4. *There exist isomorphisms of S_{2k}–modules*

$$\phi'\colon \left(V^{\otimes 2k}\right)^{Sp(V)} \otimes \Sigma_{2k} \xrightarrow{\cong} N_k$$

($dim(V) = 2n$) where Σ_{2k} is the sign representation of S_{2k}, and

$$\phi'\colon \left(V^{\otimes 2k}\right)^{O(V)} \xrightarrow{\cong} N_k$$

($dim(V) = n$).

Theorem 7.5. *There exists a $\mathcal{B}_m^{-2n}$–isomorphism*

$$\phi\colon Ind_{H_k}^{S_m}\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right) \xrightarrow{\cong} H_{m,k}^{\mu'}$$

(symplectic case) and a $\mathcal{B}_m^{n}$–isomorphism

$$\phi\colon Ind_{H_k}^{S_m}\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{O(V)}\right) \xrightarrow{\cong} H_{m,k}^{\mu}$$

(orthogonal case).

Proof. The very definitions give

$$H_{m,k}^{\mu'} := M_{\mu'} \otimes Ind_{H_k}^{S_m}(N_k) \cong M_{\mu'} \otimes \left(\bigoplus_{\bar{\sigma}\in S_m/H_k} \sigma.N_k\right) \cong \bigoplus_{\bar{\sigma}\in S_m/H_k} M_{\mu'} \otimes \sigma.N_k$$

and

$$Ind_{H_k}^{S_m}\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right) = \bigoplus_{\bar{\sigma}\in S_m/H_k} \sigma.\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right) ;$$

by Lemma 7.4 these two spaces have the same dimension. Now the assignment

$$\tilde{\phi}\colon\, m \otimes v \mapsto \phi''(m) \otimes \bar{1}.\phi'(v)$$

provides a linear map

$$\tilde{\phi}\colon M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)} \longrightarrow M_{\mu'} \otimes Ind_{H_k}^{S_m}(N_k) =: H_{m,k}^{\mu'} ;$$

moreover, this is a morphism of H_k–modules: in fact

$$\begin{aligned}
(\tau,\nu).\tilde{\phi}\big(u \otimes v\big) &= (\tau,\nu).\big(\phi''(u) \otimes \bar{1}.\phi'(v)\big) = \\
&= \pi\big(\tau\cdot\nu, \bar{1}.\phi'(v)\big).\phi''(u) \otimes (\tau\cdot\nu).\phi'(v) = \tau.\phi''(u) \otimes \bar{1}.\phi'(\nu.v) = \\
&= \tilde{\phi}\big(sgn(\tau)\tau.u \otimes sgn(\nu)\nu.v\big) = \tilde{\phi}\big((\tau,\nu).(u\otimes v)\big)
\end{aligned}$$

for all $(\tau,\nu) \in H_k = S_{m-2k} \times S_{2k}$ and $u \otimes v \in M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}$ (recall that $\mathcal{B}_m^{-2n}$ acts on $Ind_{H_k}^{S_m}\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right)$ through the action of $\mathbb{B}_m^{-2n}$ and the isomorphism Δ^-). Therefore $\tilde{\phi}$ has a unique S_m–invariant extension ϕ to $Ind_{S_{m-2k}\times S_{2k}}^{S_m}(M_\mu \otimes N_k)$, i.e. we have a morphism of S_m–modules $Ind_{H_k}^{S_m}\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right) \longrightarrow H_{m,k}^{\mu'}$. Clearly $\phi\left(M_\mu \otimes \left(V^{\otimes 2k}\right)^{Sp(V)}\right) =$

$M_{\mu'} \otimes \bar{1}.N_k$, so by S_m–equivariance we deduce that ϕ is onto, and finally for dimension reasons it is a linear isomorphism.

As for operators T_{pq}, thanks to the S_m–action it is enough to check that $\phi\big(T_{pq}.(u \otimes v)\big) = T_{pq}.\phi(u \otimes v)$ for all $u \otimes v \in M_\mu \otimes \big(V^{\otimes 2k}\big)^{Sp(V)}$, $p, q \in \{1, \dots, m\} (p < q)$. We have three cases: $p, q \in \{1, \dots, m-2k\}$, $p, q \in \{m-2k+1, \dots, m\}$, and $p \in \{1, \dots, m-2k\}$, $q \in \{m-2k+1, \dots, m\}$.

The first two cases are trivial at all; as for the third one, we can assume that v is a monomial, so that $\phi'(v)$ is a junction: in this junction, let the q–th spot be linked with the r–th one; then we have:

$$\begin{aligned} \phi\big(T_{pq}.(u \otimes v)\big) &= \phi\big(-\tau_{pq}.(u \otimes v)\big) = \phi\big(-(pr).(u \otimes v)\big) = \\ &= (pr).\phi(u \otimes v) = (pr).\phi''(u) \otimes \bar{1}.\phi'(v) \\ T_{pq}.\big(\phi(u \otimes v)\big) &= T_{pq}.(\phi''(u) \otimes \bar{1}.\phi'(v)) = \\ &= \pi(T_{pq}, \phi'(v)).\phi''(u) \otimes T_{pq}.\phi'(v) = (pr).\phi''(u) \otimes \bar{1}.\phi'(v) \end{aligned}$$

i.e. $\phi\big(T_{pq}.(u \otimes v)\big) = T_{pq}.\big(\phi(u \otimes v)\big)$.

Once again the same arguments work in the orthogonal case too, with some shortcuts and simplifications. $\square$

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References

[Br] R. Brauer, *On algebras which are concerned with semisimple continous groups*, Ann. of Math. **38** (1937), 854–872.

[D-P] C. De Concini, C. Procesi, *A characteristic free approach to invariant theory*, Advances in Math. **21** (1976), 330–354.

[D-S] C. De Concini, E. Strickland, *Traceless tensors and the symmetric group*, J. of Alg. **61** (1979), 112-128.

[K] S. V. Kerov, *Realizations of representations of the Brauer semigroup*, J. of Soviet Math. **47** (1989), 2503–2507.

[L] D.E. Littlewood, *On invariant theory under restricted groups*, Phil. Trans. Roy. Soc. A **239** (1944), 387–417.

[L-P] J-L. Loday, C. Procesi, *Homology of symplectic and orthogonal algebras*, Adv. Math. **21** (1976), 93–108.

[W] H. Weyl, *The classical groups*, Princeton University Press, New York, 1946.

[Wz] H. Wenzl,, *On the structure of Brauer's centralizer algebras*, Ann. Math. **188** (1988), 173–193.

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