

## THE $R$ -MATRIX ACTION OF UNTWISTED AFFINE QUANTUM GROUPS AT ROOTS OF 1

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ABSTRACT. Let  $\hat{\mathfrak{g}}$  be an untwisted affine Kac-Moody algebra. The quantum group  $U_q(\hat{\mathfrak{g}})$  is known to be a quasitriangular Hopf algebra (to be precise, a braided Hopf algebra). Here we prove that its unrestricted specializations at odd roots of 1 are braided too: in particular, specializing  $q$  at 1 we have that the function algebra  $F[\hat{H}]$  of the Poisson proalgebraic group  $\hat{H}$  dual of  $\hat{G}$  (a Kac-Moody group with Lie algebra  $\hat{\mathfrak{g}}$ ) is braided. This in turn implies also that the action of the universal  $R$ -matrix on the tensor products of pairs of Verma modules can be specialized at odd roots of 1.

### Introduction

*”Oh, quant’è affine alla sua genitrice!  
Osserva come anch’ella ha belle trecce  
ch’ha ereditate dalla sua matrice”*

*N. Barbecue, ”Scholia”*

A Hopf algebra  $H$  is called quasitriangular (cf. [Dr], [C-P]) if there exists an invertible element  $R \in H \otimes H$  (or an element of an appropriate completion of  $H \otimes H$ ) such that

$$\begin{aligned} \text{Ad}(R)(\Delta(a)) &= \Delta^{\text{op}}(a) \quad \forall a \in H \\ (\Delta \otimes \text{id})(R) &= R_{13}R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{13}R_{12} \end{aligned}$$

where  $\text{Ad}(R)(x) := R \cdot x \cdot R^{-1}$ ,  $\Delta^{\text{op}}$  is the opposite comultiplication (i. e.  $\Delta^{\text{op}}(a) = \sigma \circ \Delta(a)$  with  $\sigma: A^{\otimes 2} \rightarrow A^{\otimes 2}$ ,  $a \otimes b \mapsto b \otimes a$ ), and  $R_{12}, R_{13}, R_{23} \in H^{\otimes 3}$  (or the appropriate completion of  $H^{\otimes 3}$ ),  $R_{12} = R \otimes 1$ ,  $R_{23} = 1 \otimes R$ ,  $R_{13} = (\sigma \otimes \text{id})(R_{23}) = (\text{id} \otimes \sigma)(R_{12})$ .

As a corollary of this definition,  $R$  satisfies the Yang-Baxter equation in  $H^{\otimes 3}$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

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so that a braid group action can be defined on tensor products of  $H$ -modules (whence applications to knot theory arise). If  $\hat{\mathfrak{g}}$  is an untwisted affine Kac-Moody algebra, the quantum universal enveloping algebra  $U_h(\hat{\mathfrak{g}})$ , over  $\mathbb{C}[[h]]$ , is quasitriangular (cf. [Dr], [C-P]). On the other hand, this is not true — strictly speaking — for its "polynomial version", the  $\mathbb{C}(q)$ -algebra  $U_q(\hat{\mathfrak{g}})$ : nonetheless, it is a braided algebra, in the sense of the following

**Definition.** (cf. [Re1], Definition 2) A Hopf algebra  $H$  is called braided if there exists an automorphism  $\mathcal{R}$  of  $H \otimes H$  (or of an appropriate completion of  $H \otimes H$ ) distinct from  $\sigma: a \otimes b \mapsto b \otimes a$  such that

$$\begin{aligned} \mathcal{R} \circ \Delta &= \Delta^{\text{op}} \\ (\Delta \otimes \text{id}) \circ \mathcal{R} &= \mathcal{R}_{13} \circ \mathcal{R}_{23} \circ (\Delta \otimes \text{id}), \quad (\text{id} \otimes \Delta) \circ \mathcal{R} = \mathcal{R}_{13} \circ \mathcal{R}_{12} \circ (\text{id} \otimes \Delta) \end{aligned}$$

where  $\mathcal{R}_{12} := \mathcal{R} \otimes \text{id}$ ,  $\mathcal{R}_{23} = \text{id} \otimes \mathcal{R}$ ,  $\mathcal{R}_{13} = (\sigma \otimes \text{id}) \circ (\text{id} \otimes \mathcal{R}) \circ (\sigma \otimes \text{id}) \in \text{Aut}(H \otimes H \otimes H)$ .

It follows from this definition that  $\mathcal{R}$  satisfies the Yang-Baxter equation in  $\text{End}(H^{\otimes 3})$ :

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} = \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}$$

which yields a braid group action on tensor powers of  $H$ , which is still important for applications. Notice that if  $(H, R)$  is quasitriangular, then  $(H, \text{Ad}(R))$  is braided.

In this paper we prove that the unrestricted specializations of  $U_q(\hat{\mathfrak{g}})$  at odd roots of 1 are braided too: indeed, we show that the braiding automorphism of  $U_q(\hat{\mathfrak{g}})$  — which is, roughly speaking, the conjugation by its universal  $R$ -matrix — does leave stable the integer form — of  $U_q(\hat{\mathfrak{g}})$  — which is to be "specialized". This extends to the present case a result due to Reshetikhin (cf. [Re1]) for the case of the quantum group  $U_q(\mathfrak{sl}(2))$ , and to Reshetikhin (cf. [Re2]) and the author (cf. [Ga1]) for  $U_q(\mathfrak{g})$ , with  $\mathfrak{g}$  finite dimensional semisimple. The most general case is developed in [G-H]. As a consequence, we get that the action of the universal  $R$ -matrix of  $U_q(\hat{\mathfrak{g}})$  on tensor products of pairs of Verma modules does specialize at odd roots of 1 as well.

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## § 1 Definitions

**1.1 The classical data.** Let  $\mathfrak{g}$  be a simple finite dimensional Lie algebra over the field  $\mathbb{C}$  of complex numbers, and consider the following data.

The set  $I_0 = \{1, \dots, n\}$ , of the vertices of the Dynkin diagram of  $\mathfrak{g}$  (see [Bo], [Ka] for the identification between  $I_0$  and  $\{1, \dots, n\}$ ); a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ ; the root system  $\Phi_0(\subseteq \mathfrak{h}^*)$  of  $\mathfrak{g}$ ; the set of simple roots  $\{\alpha_i \mid i \in I_0\}$ ; the Killing form  $(\cdot | \cdot)$  of  $\mathfrak{g}$ , normalized so that short roots have square length 2. For all  $i \in I_0$ , we set  $d_i := \frac{(\alpha_i | \alpha_i)}{2}$ .

We denote  $\hat{\mathfrak{g}}$  the untwisted affine Kac-Moody algebra associated to  $\mathfrak{g}$ , which can be realized as  $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \cdot c \oplus \mathbb{C} \cdot \partial$ , with the Lie bracket given by:  $[c, z] = 0$ ,  $[\partial, x \otimes t^m] = m x \otimes t^m$ ,  $[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + \delta_{r, -s} r(x|y) c$  (for all  $z \in \hat{\mathfrak{g}}$ ,  $x, y \in \mathfrak{g}$ ,  $m \in \mathbb{Z}$ ).

For  $\hat{\mathfrak{g}}$  we consider:  $I := I_0 \cup \{0\}$ ,  $I_\infty := I \cup \{\infty\}$ , and  $d_0 := 1$ ; the (generalized) Cartan matrix  $A = (a_{ij})_{i, j \in I}$  (after [Ka]); the maximal abelian subalgebra  $\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C} \cdot c \oplus \mathbb{C} \cdot \partial$  ( $\subseteq \hat{\mathfrak{g}}$ ); the root system  $\Phi = \Phi_+ \cup (-\Phi_+)$  ( $\subset (\mathfrak{h} \oplus \mathbb{C} \cdot c)^* \subset \hat{\mathfrak{h}}^*$ ),  $\Phi_+ = \Phi_+^{\text{re}} \cup \Phi_+^{\text{im}}$  being the set of positive roots, with  $\Phi_+^{\text{im}} = \{m\delta \mid m \in \mathbb{N}_+\}$  the set of imaginary positive roots and  $\Phi_+^{\text{re}}$  the set of real positive roots. Then  $\hat{\mathfrak{g}}$  splits as  $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus (\oplus_{\alpha \in \Phi} \hat{\mathfrak{g}}_\alpha)$ , and  $\dim_{\mathbb{C}}(\hat{\mathfrak{g}}_\alpha) = 1 \forall \alpha \in \pm \Phi_+^{\text{re}}$ ,  $\dim_{\mathbb{C}}(\hat{\mathfrak{g}}_\alpha) = \#(I_0) = n \forall \alpha \in \pm \Phi_+^{\text{im}}$ ; so we define the set  $\tilde{\Phi}_+$  of "positive roots with multiplicity" as  $\tilde{\Phi}_+ := \Phi_+^{\text{re}} \cup \tilde{\Phi}_+^{\text{im}}$ , where  $\tilde{\Phi}_+^{\text{im}} := \Phi_+^{\text{im}} \times I_0$ . We let  $\alpha_\infty$  be the unique element of  $\hat{\mathfrak{h}}^*$  such that  $\langle \alpha_\infty, c \rangle = 1$ ,  $\langle \alpha_\infty, \mathfrak{h} \rangle = 0$ ,  $\langle \alpha_\infty, \partial \rangle = 0$ . Furthermore, we set:  $Q := \sum_{i \in I} \mathbb{Z} \cdot \alpha_i$ ,  $Q_\infty := \sum_{i \in I_\infty} \mathbb{Z} \cdot \alpha_i \subset \hat{\mathfrak{h}}^*$ ; for any  $\beta = \sum_{i \in I} z_i \alpha_i \in Q$  ( $z_i \in \mathbb{Z}$  for all  $i$ ) we set  $|\beta| := \sum_{i \in I} z_i$ . Finally, we define the non-degenerate symmetric bilinear form on  $\mathbb{R} \otimes_{\mathbb{Z}} Q_\infty$  given by:  $(\alpha_i | \alpha_j) := d_i a_{ij}$  ( $\forall i, j \in I$ ),  $(\alpha_\infty | \alpha_j) := \delta_{0, j}$  ( $\forall j \in I_\infty$ ).

**1.2 Some  $q$ -tools.** For all  $m, n, k, s \in \mathbb{N}_+$ ,  $n \leq m$ , we define:  $(s)_q := \frac{q^s - 1}{q - 1}$ ,  $[s]_q := \frac{q^s - q^{-s}}{q - q^{-1}}$ ,  $(k)_q! := \prod_{s=1}^k (s)_q$ ,  $[k]_q! := \prod_{s=1}^k [s]_q$ ,  $\binom{m}{n}_q := \frac{(m)_q!}{(m-n)_q! (n)_q!}$ ,  $\left[ \begin{matrix} m \\ n \end{matrix} \right]_q := \frac{[m]_q!}{[m-n]_q! [n]_q!}$  (all belonging to  $\mathbb{Z}[q, q^{-1}]$ ). For later use, we define also:  $q_\alpha := q^{\frac{(\alpha| \alpha)}{2}}$  for all  $\alpha \in \Phi_+^{\text{re}}$ ,  $q_\alpha := q^{d_i}$  for all  $\alpha = (r\delta, i) \in \tilde{\Phi}_+^{\text{im}}$ ,  $q_i := q_{\alpha_i} = q^{d_i}$  for all  $i \in I$ .

Second, we define the symbol  $(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k)$ , for  $n \in \mathbb{N}$ ,  $a \in \mathbb{C}$ . Now consider the function of  $z$ :  $(z; q)_\infty := \prod_{n=0}^{\infty} (1 - zq^n)$  to be thought of as an element of  $\mathbb{C}(q)[[z]]$ : if  $q$  is a complex number such that  $|q| < 1$ , the infinite product expressing  $(z; q)_\infty$  converges to an analytic function of  $z$  in any finite part of  $\mathbb{C}$ ; its Taylor series is then  $(z; q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} z^n$ . Define also  $\exp_q(z) := \sum_{n=0}^{\infty} \frac{1}{(n)_{q^2}!} z^n$ ; then one has

$$\exp_q(z) = e_{q^2} \left( (1 - q^2)z \right) = \left( (1 - q^2)z; q^2 \right)_\infty^{-1}.$$

The following lemma describes the behavior of  $(z; q)_\infty$  for  $q \rightarrow \varepsilon$ ,  $\varepsilon$  a root of 1.

**Lemma 1.3.** ([Re1], Lemma 3.4.1; [Ga], Lemma 2.2) *Let  $\varepsilon$  be a primitive  $\ell$ -th root of 1, with  $\ell$  odd. The asymptotic behavior of the function (of  $q$ )  $(z; q)_\infty$  for  $q \rightarrow \varepsilon$  is given by*

$$(z; q)_\infty = \exp \left( \frac{1}{q^{\ell^2} - 1} \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot z^{\ell n} \right) \cdot (1 - z^\ell)^{-1/2} \cdot \prod_{k=0}^{\ell-1} (1 - \varepsilon^k z)^{k/\ell} \cdot (1 + \mathcal{O}(q - \varepsilon)). \quad \square$$

**1.4 The quantum group  $U_q(\hat{\mathfrak{g}})$ .** The quantized universal enveloping algebra  $U_q(\hat{\mathfrak{g}})$  (cf. e.g. [Dr]) is the unital associative  $\mathbb{C}(q)$ -algebra with generators  $F_i, K_\mu, E_i$  ( $i \in I, \mu \in Q_\infty$ ) and relations (for all  $\mu, \nu \in Q_\infty, i, j, h \in I, i \neq j$ )

$$\begin{aligned} K_\mu K_\nu &= K_{\mu+\nu} = K_\nu K_\mu, & K_0 &= 1 \\ K_\mu E_i &= q^{(\mu|\alpha_i)} E_i K_\mu, & K_\mu F_i &= q^{-(\mu|\alpha_i)} F_i K_\mu, & E_i F_h - F_h E_i &= \delta_{ih} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q_i - q_i^{-1}} \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k &= 0, & \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k &= 0 \end{aligned}$$

A Hopf algebra structure on  $U_q(\hat{\mathfrak{g}})$  is defined by ( $i \in I; \mu \in Q_\infty$ )

$$\begin{aligned} \Delta(F_i) &:= F_i \otimes K_{-\alpha_i} + 1 \otimes F_i, & S(F_i) &:= -F_i K_{\alpha_i}, & \epsilon(F_i) &:= 0 \\ \Delta(K_\mu) &:= K_\mu \otimes K_\mu, & S(K_\mu) &:= K_{-\mu}, & \epsilon(K_\mu) &:= 1 \\ \Delta(E_i) &:= E_i \otimes 1 + K_{\alpha_i} \otimes E_i, & S(E_i) &:= -K_{-\alpha_i} E_i, & \epsilon(E_i) &:= 0. \end{aligned}$$

Moreover,  $U_q(\hat{\mathfrak{g}})$  has a natural Hopf algebra  $Q$ -grading,  $U_q(\hat{\mathfrak{g}}) = \bigoplus_{\eta \in Q} U_q(\hat{\mathfrak{g}})_\eta$ .

Let  $U_q^+, U_q^0, U_q^-$  be the subalgebras of  $U_q(\hat{\mathfrak{g}})$  respectively generated by  $\{E_i \mid i \in I\}, \{K_\nu \mid \nu \in Q_\infty\}, \{F_i \mid i \in I\}$ ; then  $U_q^+$  and  $U_q^-$  are both naturally graded by  $Q_+ := \sum_{i \in I} \mathbb{N} \cdot \alpha_i (\subset Q)$ . Finally, let  $U_q^\geq := U_q^+ \cdot U_q^0 = U_q^0 \cdot U_q^+, U_q^\leq := U_q^- \cdot U_q^0 = U_q^0 \cdot U_q^-$ , to be called *quantum Borel (sub)algebras*: these are *Hopf subalgebras* of  $U_q(\hat{\mathfrak{g}})$ .

*Remark:* In the definition of  $U_q(\hat{\mathfrak{g}})$  several choices for the "toral part"  $U_q^0$  are possible, mainly depending on the choice of any lattice  $M$  such that  $Q_0 \leq M \leq P_0$ ,  $P_0$  being the weight lattice of  $\mathfrak{g}$  (cf. for instance [B-K]). All what follows holds as well for every such choice, up to suitably adapting the statements involving the toral part.

**1.5 Quantum root vectors.** It is known (cf. [Be1], [Be2]) that one can define a total ordering on  $\tilde{\Phi}_+$ , and accordingly define quantum root vectors: from now on, we assume a total ordering be fixed and quantum root vectors be defined as in [Ga2], § 2, so that  $E_\alpha$ , resp.  $F_\alpha$ , is the quantum root vector in  $U_q^+$ , resp. in  $U_q^-$ , attached to the positive, resp. negative, root (with multiplicity)  $\alpha$ , resp.  $-\alpha$  (for any  $\alpha \in \tilde{\Phi}_+$ ).

**1.6 Integer forms.** The main interest of quantum groups is to specialize them at roots of 1: thus we need suitable integer forms of them.

First, let  $\mathfrak{R}$  be the set of all roots of 1 (in  $\mathbb{C}$ ) whose order is either 1 or an odd number  $\ell$  with  $g.c.d.(\ell, n+1) = 1$  if  $\mathfrak{g}$  is of type  $A_n$ ,  $\ell \notin 3\mathbb{N}_+$  if  $\mathfrak{g}$  is of type  $E_6$  or  $G_2$ ; then let  $\mathcal{A}$  be the subset of  $\mathbb{C}(q)$  of rational functions of  $q$  with no poles in  $\mathfrak{R}$ . Second, define renormalized root vectors by  $\tilde{E}_\alpha := (q_\alpha - q_\alpha^{-1}) E_\alpha, \tilde{F}_\alpha := (q_\alpha^{-1} - q_\alpha) F_\alpha$ , for all  $\alpha \in \tilde{\Phi}_+$ , and let  $\mathcal{U}_q(\hat{\mathfrak{g}})$  be the  $\mathcal{A}$ -subalgebra of  $U_q(\hat{\mathfrak{g}})$  generated by  $\{\tilde{F}_\alpha, K_\mu, \tilde{E}_\alpha \mid \alpha \in \tilde{\Phi}_+, \mu \in Q_\infty\}$ : this is a  $Q$ -graded Hopf subalgebra (and an  $\mathcal{A}$ -form) of  $U_q(\hat{\mathfrak{g}})$  (cf. [B-K]). We define also  $\mathcal{U}_q^\leq := \mathcal{U}_q(\hat{\mathfrak{g}}) \cap U_q^\leq, \mathcal{U}_q^\geq := \mathcal{U}_q(\hat{\mathfrak{g}}) \cap U_q^\geq$ .

## § 2 Braiding of quantum enveloping algebras

**2.1 More notation.** As we said, it is well known (cf. [Dr]) that the quantum algebra  $U_h(\hat{\mathfrak{g}})$  (defined over the ring  $\mathbb{C}[[h]]$ ) is quasitriangular; this is proved by means of Drinfeld's method of the "quantum double". On the other hand, for the  $\mathbb{C}(q)$ -algebras  $U_q(\hat{\mathfrak{g}})$  the correct statement is that they are braided. To see this, we define a suitable completion of  $U_q(\hat{\mathfrak{g}})^{\otimes 2}$ , namely  $U_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2} := \left\{ \sum_{n=0}^{+\infty} \mathcal{E}_n \cdot P_n^- \otimes P_n^+ \cdot \mathcal{F}_n \right\}$  where  $P_n^- \in U_q^{\leq}$ ,  $P_n^+ \in U_q^{\geq}$ ,  $\mathcal{E}_n \in \sum_{|\beta|=n} (U_q(\hat{\mathfrak{g}}))_{\beta}$ ,  $\mathcal{F}_n \in \sum_{|\beta|=-n} (U_q(\hat{\mathfrak{g}}))_{\beta}$ . It is clear that  $U_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  is a completion of  $U_q(\hat{\mathfrak{g}})^{\otimes 2}$  as a Hopf algebra.

Define new quantum root vectors  $\dot{F}_{\alpha}$  and  $\dot{E}_{\alpha}$  ( $\alpha \in \widetilde{\Phi}_+$ ) as follows (like in the proof of Proposition 4.6 in [Ga2]). For all  $\alpha \in \Phi_+^{\text{re}}$ , set  $\dot{F}_{\alpha} := \dot{F}_{\alpha}$ . For all  $\alpha = (r\delta, i) \in \widetilde{\Phi}_+^{\text{im}}$ , consider the matrix

$$M_r := \left( (o(i)o(j))^r [a_{ij}]_{q_i^r} \right)_{i,j \in I_0}$$

where  $o(i) = \pm 1 (i \in I_0)$  is defined in such a way that  $o(h)o(k) = -1$  whenever  $a_{hk} < 0$ ; then  $\det(M_r)$  is an invertible element of  $\mathcal{A}$  (see [Ga2] for the exact value), so the inverse matrix  $M_r^{-1} = (\mu_{ij})_{i,j \in I_0}$  has all its entries in  $\mathcal{A}$ ; now define

$$\dot{F}_{(r\delta, i)} := \sum_{j \in I_0} \mu_{ji} F_{(r\delta, j)}.$$

Similarly we define positive root vectors  $\dot{E}_{\alpha}$ , for all  $\alpha \in \widetilde{\Phi}_+$ .

Now set  $\exp_{\alpha} := \exp_{q_{\alpha}}$ , for  $\alpha \in \Phi_+^{\text{re}}$ , and  $\exp_{\alpha} := \exp$ , for  $\alpha \in \widetilde{\Phi}_+^{\text{im}}$ ; set also  $a_{\alpha} := 1$  for  $\alpha \in \Phi_+^{\text{re}}$  and  $a_{\alpha} := \frac{r}{[r]_q [d_i]_q}$  for  $\alpha = (r\delta, i) \in \widetilde{\Phi}_+^{\text{im}}$ .

**Theorem 2.2.** *Let  $\mathcal{R}^{(0)}$  be the algebra automorphism of  $U_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  defined by*

$$\begin{aligned} \mathcal{R}^{(0)}(K_{\mu} \otimes 1) &:= K_{\mu} \otimes 1, & \mathcal{R}^{(0)}(1 \otimes K_{\mu}) &:= 1 \otimes K_{\mu} \\ \mathcal{R}^{(0)}(E_i \otimes 1) &:= E_i \otimes K_{-\alpha_i}, & \mathcal{R}^{(0)}(1 \otimes E_i) &:= K_{-\alpha_i} \otimes E_i \\ \mathcal{R}^{(0)}(F_i \otimes 1) &:= F_i \otimes K_{\alpha_i}, & \mathcal{R}^{(0)}(1 \otimes F_i) &:= K_{\alpha_i} \otimes F_i \end{aligned}$$

( $i \in I$ ,  $\mu \in Q_{\infty}$ ) and let  $R^{(1)} \in U_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  be defined as the ordered product

$$R^{(1)} := \prod_{\alpha \in \widetilde{\Phi}_+} \exp_{\alpha} \left( a_{\alpha} (q_{\alpha}^{-1} - q_{\alpha}) E_{\alpha} \otimes \dot{F}_{\alpha} \right) = \prod_{\alpha \in \widetilde{\Phi}_+} \exp_{\alpha} \left( a_{\alpha} (q_{\alpha}^{-1} - q_{\alpha}) \dot{E}_{\alpha} \otimes F_{\alpha} \right)$$

Then  $(U_q(\hat{\mathfrak{g}}), \text{Ad}(R^{(1)}) \circ \mathcal{R}^{(0)})$  is a braided Hopf algebra (with  $R^{(1)}$  as R-matrix, in the sense of [Re1], Definition 3).

*Proof.* This is essentially proved in [Da2]: to get exactly the present claim, one just has to take into account the following. The formula for the  $R$ -matrix given in [Da2] is obtained by computing bases (of PBW type), in quantum Borel subalgebras of opposite sign, which are orthogonal to each other with respect to a certain perfect Hopf pairing: Lemma 2.4 in [Ga2] extends the result of [Da2] to other possible bases, specifying which choices of quantum root vectors are "admissible", i.e. are such that starting from them the construction in [Da2] still works and gives similar orthogonal bases; finally, the remarks in the proof of Proposition 4.6 in [Ga2] show that both the choice of quantum root vectors  $E_\beta$  and  $\dot{F}_\gamma$  ( $\beta, \gamma \in \tilde{\Phi}_+$ ) and the choice of  $\dot{E}_\beta$  and  $F_\gamma$  ( $\beta, \gamma \in \tilde{\Phi}_+$ ) are admissible (in the previous sense).  $\square$

**2.3 The braiding structure at roots of 1.** Our goal now is to show that  $\mathcal{U}_q(\hat{\mathfrak{g}})$  is braided: to be precise, we could say that the braiding structure of  $U_q(\hat{\mathfrak{g}})$  gives by restriction a braiding structure for  $\mathcal{U}_q(\hat{\mathfrak{g}})$ . To begin with, we define a suitable completion of  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\otimes 2}$  (mimicking §2.1), namely

$$\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2} := \left\{ \sum_{n=0}^{+\infty} \bar{\mathcal{E}}_n \cdot P_n^- \otimes P_n^+ \cdot \bar{\mathcal{F}}_n \right\}$$

where  $P_n^- \in \mathcal{U}_q^{\leq}$ ,  $P_n^+ \in \mathcal{U}_q^{\geq}$ ,  $\bar{\mathcal{E}}_n \in \sum_{|\beta|=n} (\mathcal{U}_q(\hat{\mathfrak{g}}))_\beta$ ,  $\bar{\mathcal{F}}_n \in \sum_{|\beta|=-n} (\mathcal{U}_q(\hat{\mathfrak{g}}))_\beta$ . It is clear that  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  is a completion of  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\otimes 2}$  as Hopf algebra, and that  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2} \subseteq U_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  via the natural embedding  $\mathcal{U}_q(\hat{\mathfrak{g}}) \hookrightarrow U_q(\hat{\mathfrak{g}})$ .

Moreover, for all  $\alpha \in \tilde{\Phi}_+$  we define  $\dot{F}_\alpha := (q_\alpha - q_\alpha^{-1})\dot{F}_\alpha$ ,  $\dot{E}_\alpha := (q_\alpha - q_\alpha^{-1})\dot{E}_\alpha \in \mathcal{U}_q(\hat{\mathfrak{g}})$ .

For any  $\varepsilon \in \mathfrak{R}$ , we call  $\mathcal{U}_\varepsilon(\hat{\mathfrak{g}})$  the specialization of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  at  $q = \varepsilon$ , that is

$$\mathcal{U}_\varepsilon(\hat{\mathfrak{g}}) := \mathcal{U}_q(\hat{\mathfrak{g}}) / (q - \varepsilon)\mathcal{U}_q(\hat{\mathfrak{g}}).$$

**Theorem 2.4.** *The restriction of  $\mathcal{R}^{(0)}$  (cf. Theorem 2.2) to  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  is given by*

$$\begin{aligned} \check{\mathcal{R}}^{(0)}(K_\mu \otimes 1) &= K_\mu \otimes 1, & \check{\mathcal{R}}^{(0)}(1 \otimes K_\mu) &= 1 \otimes K_\mu \\ \check{\mathcal{R}}^{(0)}(\dot{E}_\alpha \otimes 1) &= \dot{E}_\alpha \otimes K_{-\alpha}, & \check{\mathcal{R}}^{(0)}(1 \otimes \dot{E}_\alpha) &= K_{-\alpha} \otimes \dot{E}_\alpha \\ \check{\mathcal{R}}^{(0)}(\dot{F}_\alpha \otimes 1) &= \dot{F}_\alpha \otimes K_\alpha, & \check{\mathcal{R}}^{(0)}(1 \otimes \dot{F}_\alpha) &= K_\alpha \otimes \dot{F}_\alpha \end{aligned}$$

( $\mu \in Q_\infty$ ,  $\alpha \in \tilde{\Phi}_+$ ) thus  $\mathcal{R}^{(0)}$  restricts to an algebra automorphism  $\check{\mathcal{R}}^{(0)}$  of  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$ . Moreover, let  $R^{(1)} \in U_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  be given (as in Theorem 2.2) by

$$R^{(1)} := \prod_{\alpha \in \tilde{\Phi}_+} \exp_\alpha \left( a_\alpha (q_\alpha^{-1} - q_\alpha) E_\alpha \otimes \dot{F}_\alpha \right) = \prod_{\alpha \in \tilde{\Phi}_+} \exp_\alpha \left( a_\alpha (q_\alpha^{-1} - q_\alpha) \dot{E}_\alpha \otimes F_\alpha \right).$$

Then the adjoint action by  $R^{(1)}$  leaves  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  stable; thus  $\text{Ad}(R^{(1)})$  restricts to an automorphism  $\check{\mathcal{R}}^{(1)}$  of  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$ , and  $(\mathcal{U}_q(\hat{\mathfrak{g}}), \check{\mathcal{R}})$  — with  $\check{\mathcal{R}} := \check{\mathcal{R}}^{(1)} \circ \check{\mathcal{R}}^{(0)}$  — is a braided Hopf algebra.

*Proof.* The first part of the statement is trivial, and the third is a direct consequence of the first, the second, and Theorem 2.2. To prove that  $\text{Ad}(R^{(1)})$  stabilizes  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$ , we apply an idea of Reshetikhin.

We look at the different factors  $R_\alpha^{(1)}$  (for  $\alpha \in \widetilde{\Phi}_+$ ) in the product defining  $R^{(1)}$ . Notice that  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  has a natural "Q-pseudograding", extending that of  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\otimes 2}$ : so we can look at the homogeneous summands of  $R^{(1)}$ , and then we find that each of them is given by the product of *finitely many* factors  $R_\alpha^{(1)}$ . Therefore, to prove the claim we have only to show that the adjoint action by every factor  $R_\alpha^{(1)}$  leaves  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  stable.

For a real root  $\alpha \in \Phi_+^{\text{re}}$ , the factor  $R_\alpha^{(1)}$  is of type

$$R_\alpha^{(1)} := \exp_\alpha \left( a_\alpha (q_\alpha^{-1} - q_\alpha) E_\alpha \otimes \dot{F}_\alpha \right) = \exp_{q_\alpha} \left( (q_\alpha^{-1} - q_\alpha)^{-1} \check{E}_\alpha \otimes \check{F}_\alpha \right).$$

Using the identity in §1.2 this reads

$$R_\alpha^{(1)} = \left( q_\alpha \cdot \check{E}_\alpha \otimes \check{F}_\alpha ; q_\alpha^2 \right)_\infty^{-1}.$$

Now apply Lemma 1.3: it gives

$$\begin{aligned} R_\alpha^{(1)} = \exp \left( \frac{-1}{q^{\ell^2} - 1} \cdot \frac{1}{2d_\alpha} \cdot \varphi \left( \check{E}_\alpha^\ell \otimes \check{F}_\alpha^\ell \right) \right) \cdot \\ \cdot \left( 1 - \check{E}_\alpha^\ell \otimes \check{F}_\alpha^\ell \right)^{-1/2} \cdot \prod_{k=0}^{\ell-1} \left( 1 - \varepsilon^k \cdot \check{E}_\alpha \otimes \check{F}_\alpha \right)^{k/\ell} + \mathcal{O}(q - \varepsilon) \end{aligned}$$

where we set  $\varphi(z) := \sum_{n=1}^{\infty} \frac{1}{n^2} z^n$ . Thus  $R_\alpha^{(1)}$  — modulo a "tail" vanishing at  $q = \varepsilon$  — contains the factor  $\left( 1 - \check{E}_\alpha^\ell \otimes \check{F}_\alpha^\ell \right)^{-1/2} \cdot \prod_{k=0}^{\ell-1} \left( 1 - \varepsilon^k \cdot \check{E}_\alpha \otimes \check{F}_\alpha \right)^{k/\ell}$ , which is "harmless" (and is trivial if  $\ell = 1$ , that is  $\varepsilon = 1$ ), but also the factor  $\exp \left( \frac{1}{q^{\ell^2} - 1} \frac{1}{2d_\alpha} \varphi \left( \varepsilon^{d_\alpha} \check{E}_\alpha^\ell \otimes \check{F}_\alpha^\ell \right) \right)$ , which has a pole at  $q = \varepsilon$ .

Here we can act as in the proof of the finite case (cf. [Ga1], Proposition 4.2). Recall that  $\text{Ad}(\exp(x)) = \exp(\text{ad}(x))$ , where  $(\text{ad}(x))(y) := [x, y] = xy - yx$ . Moreover, it is known (see [B-K], §2, or [Da1], §3) that the images of  $\check{E}_\alpha^\ell$  and of  $\check{F}_\alpha^\ell$  belong to the centre of the specialized algebra  $\mathcal{U}_\varepsilon(\hat{\mathfrak{g}}) := \mathcal{U}_q(\hat{\mathfrak{g}}) / (q - \varepsilon)\mathcal{U}_q(\hat{\mathfrak{g}})$ ; therefore  $\check{E}_\alpha^\ell \otimes \check{F}_\alpha^\ell$  belong to the centre of  $\mathcal{U}_\varepsilon(\hat{\mathfrak{g}}) \otimes \mathcal{U}_\varepsilon(\hat{\mathfrak{g}})$ . This implies that

$$[\check{E}_\alpha^\ell \otimes \check{F}_\alpha^\ell, y \otimes z] \in (q - \varepsilon) \cdot \mathcal{U}_\varepsilon(\hat{\mathfrak{g}}) \otimes \mathcal{U}_\varepsilon(\hat{\mathfrak{g}})$$

hence

$$(q - \varepsilon)^{-1} [\check{E}_\alpha^\ell \otimes \check{F}_\alpha^\ell, y \otimes z] \in \mathcal{U}_\varepsilon(\hat{\mathfrak{g}}) \otimes \mathcal{U}_\varepsilon(\hat{\mathfrak{g}})$$

and this clearly implies  $\text{Ad} \left( R_\alpha^{(1)} \right) \left( \mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2} \right) \subseteq \mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$ , q.e.d.

Now consider the factor  $R_\alpha^{(1)}$  associated to any imaginary root  $\alpha = (r\delta, i) \in \tilde{\Phi}_+^{\text{im}}$ : by definition,

$$R_\alpha^{(1)} := \exp_\alpha \left( a_\alpha (q_\alpha^{-1} - q_\alpha) E_\alpha \otimes \dot{F}_\alpha \right) = \exp \left( \frac{-r}{[r]_q [d_i]_q} \frac{1}{(q_\alpha^{-1} - q_\alpha)} \check{E}_\alpha \otimes \dot{F}_\alpha \right).$$

In this case, we have to distinguish the cases  $\ell > 1$  and  $\ell = 1$ .

If  $\ell > 1$ , when  $\ell \nmid r$  the coefficient of  $\check{E}_\alpha \otimes \dot{F}_\alpha$  in the right-hand-side expression above is regular at  $q = \varepsilon$ , thus no problem arises. On the other hand, if  $\ell \mid r$  that coefficient has a pole (in the factor  $[r]_q^{-1}$ ) at  $q = \varepsilon$ ; but then (see again [B-K], §2, and [Da1], §3) the root vectors  $\check{E}_\alpha$  and  $\dot{F}_\alpha$  are again central modulo  $(q - \varepsilon)$ , and we can conclude as in the case of real roots.

If  $\ell = 1$ , the coefficient of  $\check{E}_\alpha \otimes \dot{F}_\alpha$  has a pole at  $q = 1$  (in the factor  $(q_\alpha^{-1} - q_\alpha)^{-1}$ ). Now, from [B-K], §3, we know that  $\mathcal{U}_1(\hat{\mathfrak{g}}) := \mathcal{U}_q(\hat{\mathfrak{g}}) / (q - 1)\mathcal{U}_q(\hat{\mathfrak{g}})$  is commutative, so we can apply once more the same argument than before to get that  $\text{Ad}(R_\alpha^{(1)}) \left( \mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2} \right) \subseteq \mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$ .  $\square$

Let  $\widehat{G}$  be a connected Kac-Moody group with Lie algebra  $\hat{\mathfrak{g}}$ , and let  $\widehat{H}$  be the Poisson proalgebraic group dual of  $\widehat{G}$  (in the sense of [B-K]): so  $\widehat{G}$  is a Poisson proalgebraic group whose tangent Lie bialgebra is  $\hat{\mathfrak{g}}^*$ . We denote by  $F[\widehat{H}]$  the Poisson Hopf algebra of (algebraic) regular functions on  $\widehat{H}$ .

### Corollary 2.5.

(a) For any  $\varepsilon \in \mathfrak{R}$ , let  $\mathcal{R}_\varepsilon$  be the algebra automorphism of  $\mathcal{U}_\varepsilon(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  given by specialization of  $\check{\mathcal{R}}$  at  $q = \varepsilon$ . Then  $(\mathcal{U}_\varepsilon(\hat{\mathfrak{g}}), \mathcal{R}_\varepsilon)$  is a braided Hopf algebra.

(b) The algebra  $F[\widehat{H}]$  is braided, by a braiding automorphism which is one of Poisson algebra.

*Proof.* Claim (a) is a direct consequence of Theorem 2.4. As for claim (b), first we recall — from [B-K], §4 — that there exists a Poisson Hopf algebra isomorphism

$$\mathcal{U}_1(\hat{\mathfrak{g}}) \cong F[\widehat{H}]$$

thus the first part of claim (b) is nothing but a special case of (a).

In addition, the Poisson bracket on  $\mathcal{U}_1(\hat{\mathfrak{g}})$  is defined, as usual, by

$$\{x, y\} := \frac{x'y' - y'x'}{q - 1} \Big|_{q=1} \quad (\star)$$

for all  $x, y \in \mathcal{U}_1(\hat{\mathfrak{g}})$ , with  $x', y' \in \mathcal{U}_q(\hat{\mathfrak{g}})$  such that  $x = x' \Big|_{q=1}$ ,  $y = y' \Big|_{q=1}$ ; of course a like formula defines the Poisson bracket on the completion  $\mathcal{U}_1(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$ . Now, since  $\check{\mathcal{R}}$  is



an algebra automorphism (of  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\otimes 2}$ ) its specialization  $\mathcal{R}_1$  automatically preserves the Poisson bracket  $(\star)$ , i.e. it is a Poisson algebra automorphism, q.e.d.  $\square$

*Remark:* The results in Theorem 2.4 and Corollary 2.5 was first proved for the finite dimensional case of the Lie algebra  $\mathfrak{sl}(2)$  in [Re1]; the case of any finite dimensional semisimple Lie algebra was developed (and solved) in [Re2] and in [Ga1]. The (affine) case of  $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2)$  has been done in [H-S]. The most general situation, dealing with any quasitriangular Lie bialgebra — giving rise to a quantized universal enveloping algebra which is quasitriangular as a Hopf algebra — is treated in [G-H].

### 2.6 The geometry of the $R$ -matrix action: comparison with the finite case.

In [Ga1], §4, the geometrical meaning of the braiding of quantum groups of finite type at roots of 1 is explained. The main points are Theorems 4.4–5, where one shows that the braiding automorphism  $\mathcal{R}_1$  (in the present notation) is more than a formal object — defined on some completion made of formal series — for it maps rational functions (on the dual Poisson group times itself) onto rational functions: hence it defines a birational automorphism of the square of this dual group (as a complex variety), which enjoys nice properties. The key step in the proof of such a result exploits the fact that the Hamiltonian vector fields associated to the functions  $\check{E}_\alpha$  and  $\check{F}_\alpha$ , for all positive roots  $\alpha$  (real, since we are in the finite case), are integrable (we consider root vectors  $\check{E}_\alpha, \check{F}_\alpha$  at  $q = 1$  as holomorphic functions on the dual Poisson group  $H \cong \text{Spec}(\mathcal{U}_1(\mathfrak{g}))$ ).

In the affine case, the situation is in part similar: one can treat the factors of the  $R$ -matrix associated to real positive roots exactly as in the finite case (compare the first part of the proof of Theorem 2.4 above with the proof of Proposition 3.7 in [Ga1]), and everything works (as in the proof of Proposition 4.2 in [Ga1]) because the Hamiltonian vector fields which occur — associated to real root vectors — are again integrable; but in the case of imaginary positive roots one has to deal with Hamiltonian vector fields — associated to  $\check{E}_\alpha$  and  $\check{F}_\alpha$ , for imaginary  $\alpha$  — which are *not integrable*.

So in the affine case the most one gets is that the braiding defines an automorphism (with “nice” properties) of the dual *formal* Poisson group associated to  $\widehat{H} \times \widehat{H}$ . This last result extends, via a different approach, to the general case of any quasitriangular Lie bialgebra: see [G-H].

**2.7 The  $R$ -matrix action on Verma modules.** For any commutative unital ring  $A$ , denote by  $A^\star$  the group of invertible elements of  $A$ . Given  $\lambda = (\lambda_i)_{i \in I_\infty} \in (\mathbb{C}(q)^\star)^{n+2}$ , let  $V_q(\lambda)$  be the Verma module (for  $U_q(\hat{\mathfrak{g}})$ ) of highest weight  $\lambda$ . We recall that it is defined as follows: define on the line  $\mathbb{C}(q) \cdot v_\lambda$  a structure of  $U_q^\geq$ -module by

$$E_i \cdot v_\lambda := 0, \quad K_j \cdot v_\lambda := \lambda_j v_\lambda \quad \forall i \in I, j \in I_\infty;$$

then  $V_q(\lambda)$  is by definition the  $U_q(\hat{\mathfrak{g}})$ -module induced by  $\mathbb{C}(q) \cdot v_\lambda$ ; in particular, it is a free  $U_q^-$ -module of rank 1, hence it is  $Q_+$ -graded:  $V_q(\lambda) = \bigoplus_{\eta \in Q_+} (V_q(\lambda))_\eta$ , with  $K_i \cdot v = \lambda_i q^{-(\alpha_i|\eta)} \cdot v$  for all  $i \in I_\infty$ ,  $v \in (V_q(\lambda))_\eta$ ,  $\eta \in Q_+$ .

Now assume  $\lambda = (\lambda_i)_{i \in I_\infty} \in (\mathcal{A}^*)^{n+2}$ : then  $V_q(\lambda)$  is also an  $\mathcal{U}_q(\hat{\mathfrak{g}})$ -module, and  $\mathcal{V}_q(\lambda) := \mathcal{U}_q(\hat{\mathfrak{g}}).v_\lambda$  is an  $\mathcal{U}_q(\hat{\mathfrak{g}})$ -module. It is also clear that  $\mathcal{V}_q(\lambda)$  is a free  $\mathcal{U}_q^-$ -module of rank 1, and it is of course  $Q_+$ -graded as well.

For any  $\varepsilon \in \mathfrak{R}$ , we denote  $\mathcal{V}_\varepsilon(\lambda)$  the specialization of  $\mathcal{V}_q(\lambda)$  at  $q = \varepsilon$ , i.e.

$$\mathcal{V}_\varepsilon(\lambda) := \mathcal{V}_q(\lambda) / (q - \varepsilon)\mathcal{V}_q(\lambda).$$

Consider on the Cartan subalgebra  $\hat{\mathfrak{h}}$  the Killing form — which is dual of the form  $(\mid)$  on  $\hat{\mathfrak{h}}^*$  defined in §1.1 — and let  $T$  be its canonical element: i.e.,  $T = \sum_{i \in I_\infty} u_i \otimes w_i$  where  $\{u_i\}_{i \in I_\infty}$  and  $\{w_i\}_{i \in I_\infty}$  are basis of  $\hat{\mathfrak{h}}$  dual of each other with respect to the Killing form. Let  $\lambda, \mu \in (\mathcal{A}^*)^{n+2}$ : for simplicity, we assume  $\lambda$  and  $\mu$  to be of the form

$$\lambda = (q^{l_i})_{i \in I_\infty}, \quad \mu = (q^{m_i})_{i \in I_\infty}$$

for some integers  $l_i, m_i$  ( $i \in I_\infty$ ), and we set  $l := \sum_{i \in I_\infty} l_i \omega_i$ ,  $m := \sum_{i \in I_\infty} m_i \omega_i$ , where  $\{\omega_i\}_{i \in I_\infty}$  is a basis of  $\hat{\mathfrak{h}}^*$  dual of  $\{\alpha_i\}_{i \in I_\infty}$  with respect to the Killing form (i.e.  $(\alpha_i \mid \omega_j) = \delta_{ij} \forall i, j \in I_\infty$ ). We define a linear operator

$$q^{-T}: \mathcal{V}_q(\lambda) \otimes \mathcal{V}_q(\mu) \longrightarrow \mathcal{V}_q(\lambda) \otimes \mathcal{V}_q(\mu)$$

$$\text{by } q^{-T} \cdot (v' \otimes v'') := q^{-(l-\eta \mid m-\xi)} \cdot v' \otimes v'', \quad \forall v' \in (\mathcal{V}_q(\lambda))_\eta, v'' \in (\mathcal{V}_q(\mu))_\xi.$$

For any pair of Verma modules  $V_q(\lambda)$  and  $V_q(\mu)$ , the algebra  $U_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  acts on  $V_q(\lambda) \otimes V_q(\mu)$ : in fact since  $V_q(\lambda)$  is highest weight, it is clear that only finitely many summands in the expansion of any element of  $U_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  act non-trivially; similarly, the algebra  $\mathcal{U}_q(\hat{\mathfrak{g}})^{\widehat{\otimes} 2}$  acts on  $\mathcal{V}_q(\lambda) \otimes \mathcal{V}_q(\mu)$ . As a consequence,  $R^{(1)}$  acts as a well-defined operator on  $V_q(\lambda) \otimes V_q(\mu)$ .

We call *universal R-matrix* of  $U_q(\hat{\mathfrak{g}})$  the formal element

$$\begin{aligned} R &:= R^{(1)} \cdot q^{-T} = \prod_{\alpha \in \tilde{\Phi}_+} \exp_\alpha \left( a_\alpha (q_\alpha^{-1} - q_\alpha) E_\alpha \otimes \dot{F}_\alpha \right) \cdot q^{-T} = \\ &= \prod_{\alpha \in \tilde{\Phi}_+} \exp_\alpha \left( a_\alpha (q_\alpha^{-1} - q_\alpha) \dot{E}_\alpha \otimes F_\alpha \right) \cdot q^{-T}; \end{aligned}$$

this is a universal  $R$ -matrix for  $U_q(\hat{\mathfrak{g}})$  in the sense of [Da2].

*Remark:* notice that, if we deal with the quantum group  $U_h(\hat{\mathfrak{g}})$  over the ring  $\mathbb{C}[[h]]$ , the  $R$ -matrix takes the simpler form

$$R := \prod_{\alpha \in \tilde{\Phi}_+} \exp_\alpha \left( a_\alpha (\exp^{-d_\alpha h} - \exp^{d_\alpha h}) E_\alpha \otimes \dot{F}_\alpha \right) \cdot \exp(-hT)$$

which is an element of the topological ( $h$ -adically complete) tensor product  $U_h(\hat{\mathfrak{g}}) \otimes U_h(\hat{\mathfrak{g}})$ .

For any pair of Verma modules  $V_q(\lambda)$  and  $V_q(\mu)$ , the  $R$ -matrix acts as a well-defined operator on  $V_q(\lambda) \otimes V_q(\mu)$ . Our previous results tell us that this action can be specialized at roots of 1.

**Theorem 2.8.** *The action of the universal  $R$ -matrix on  $V_q(\lambda) \otimes V_q(\mu)$  restricts to an action on  $\mathcal{V}_q(\lambda) \otimes \mathcal{V}_q(\mu)$ , hence it specializes to an action on  $\mathcal{V}_\varepsilon(\lambda) \otimes \mathcal{V}_\varepsilon(\mu)$  for any  $\varepsilon \in \mathfrak{R}$ .*

*Proof.* The second part of the claim is a direct consequence of the first.

Since  $\mathcal{V}_q(\lambda) = \mathcal{U}_q(\hat{\mathfrak{g}}).v_\lambda$  and  $\mathcal{V}_q(\mu) = \mathcal{U}_q(\hat{\mathfrak{g}}).v_\mu$ , we just need to look at elements such as  $R(x.v_\lambda \otimes y.v_\mu)$  with  $x, y \in \mathcal{U}_q(\hat{\mathfrak{g}})$ . We have

$$\begin{aligned} R(x.v_\lambda \otimes y.v_\mu) &= R((x \otimes y).(v_\lambda \otimes v_\mu)) = \\ &= (R(x \otimes y)R^{-1}) \cdot R.(v_\lambda \otimes v_\mu) = \text{Ad}(R)(x \otimes y).(R.(v_\lambda \otimes v_\mu)) \end{aligned}$$

where  $R^{-1}$  denotes a formal inverse to  $R$ , (which induces the inverse operator on tensor product of Vermas modules). Now, definitions are given in such a way that  $\text{Ad}(R)$  coincides with the braiding automorphism  $\text{Ad}(R^{(1)}) \circ \mathcal{R}^{(0)}$  of  $U_q(\hat{\mathfrak{g}})$  (cf. Theorem 2.2, and the *Remark* above): then by Theorem 2.4 we get that  $\text{Ad}(R)(x \otimes y) \in \mathcal{U}_q(\hat{\mathfrak{g}})$ , so we only need to show that  $R.(v_\lambda \otimes v_\mu) \in \mathcal{V}_q(\lambda) \otimes \mathcal{V}_q(\mu)$ . It is clear that  $q^{-T}(v_\lambda \otimes v_\mu) = q^{(l|m)}.v_\lambda \otimes v_\mu \in \mathcal{V}_q(\lambda) \otimes \mathcal{V}_q(\mu)$ . Moreover, from  $E_i.v_\lambda = 0$  for all  $i \in I$  we have also  $E_\alpha.v_\lambda = 0$  for all  $\alpha \in \tilde{\Phi}_+$ : then by definition of  $R^{(1)}$  we have  $R^{(1)}.(v_\lambda \otimes v_\mu) = v_\lambda \otimes v_\mu$  in  $V_q(\lambda) \otimes V_q(\mu)$ , hence also in  $\mathcal{V}_q(\lambda) \otimes \mathcal{V}_q(\mu)$ , q.e.d.  $\square$

**Remarks 2.9:** (a) As it is clear from the proof, the previous result holds as well for *lowest weight* modules, and even for pair of modules in which only the first one is highest weight or the second is lowest weight.

(b) The analogues of Theorem 2.4 and Corollary 2.5 also hold for *finite type* quantum groups (cf. [Ga1], §§3–4): therefore Theorem 2.8 holds as well in the finite case (with the same proof). In the case of  $\mathfrak{g} = \mathfrak{sl}(2)$ , such a result is complementary to another one — due to Date *et al.*, cf. [D-J-M-M] and [C-P], Proposition 11.1.17 — which concern cyclic (or periodic) representations.

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