Journal of Algebra **315** (2007), 761–800 — DOI: 10.1006/jabr.1996.7003

$F_q[M_n]$, $F_q[GL_n]$ AND $F_q[SL_n]$ AS QUANTIZED HYPERALGEBRAS

Fabio Gavarini[†], Zoran Rakić[‡]

[†]Università di Roma "Tor Vergata" — Dipartimento di Matematica
Via della Ricerca Scientifica 1, I-00133 Roma — ITALY

[‡]Univerzitet u Beogradu — Matematički fakultet
Studentski trg 16, 11000 Beograd — SERBIA

Abstract. Within the quantum function algebra $F_q[GL_n]$, we study the subset $\mathcal{F}_q[GL_n]$ — introduced in [Ga1] — of all elements of $F_q[GL_n]$ which are $\mathbb{Z}\big[q,q^{-1}\big]$–valued when paired with $\mathcal{U}_q(\mathfrak{gl}_n)$, the unrestricted $\mathbb{Z}\big[q,q^{-1}\big]$–integral form of $U_q(\mathfrak{gl}_n)$ introduced by De Concini, Kac and Procesi. In particular we obtain a presentation of it by generators and relations, and a PBW-like theorem. Moreover, we give a direct proof that $\mathcal{F}_q[GL_n]$ is a Hopf subalgebra of $F_q[GL_n]$, and that $\mathcal{F}_q[GL_n]\Big|_{q=1} \cong U_{\mathbb{Z}}({\mathfrak{gl}_n}^*)$. We describe explicitly its specializations at roots of 1, say ε, and the associated quantum Frobenius (epi)morphism from $\mathcal{F}_\varepsilon[GL_n]$ to $\mathcal{F}_1[GL_n] \cong U_{\mathbb{Z}}({\mathfrak{gl}_n}^*)$, also introduced in [Ga1]. The same analysis is done for $\mathcal{F}_q[SL_n]$ and (as key step) for $\mathcal{F}_q[M_n]$.

Introduction

Let G be a semisimple, connected, simply connected affine algebraic group over $\mathbb{Q}$, and $\mathfrak{g}$ its tangent Lie algebra. Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group over $\mathfrak{g}$ defined (after Jimbo) as a Hopf algebra over the field $\mathbb{Q}(q)$, where q is an indeterminate. After Lusztig, one has an integral form over $\mathbb{Z}\big[q,q^{-1}\big]$, say $\mathfrak{U}_q(\mathfrak{g})$, which for $q \to 1$ specializes to $U_{\mathbb{Z}}(\mathfrak{g})$, the integral $\mathbb{Z}$–form of $U(\mathfrak{g})$ defined by Kostant (see [CP], §9.3, and references therein, and [DL], §§2–3). As $U_{\mathbb{Z}}(\mathfrak{g})$ is usually called "hyperalgebra", we call $\mathfrak{U}_q(\mathfrak{g})$ *"quantum"* (or *quantized) hyperalgebra"*. In particular, as $U_{\mathbb{Z}}(\mathfrak{g})$ is generated by divided powers (in the simple root vectors) and binomial coefficients (in the simple coroots) so $\mathfrak{U}_q(\mathfrak{g})$ is generated by quantum analogues of divided powers and of binomial coefficients. Moreover, if ε is a root of 1 with

Keywords: *Hopf Algebras, Quantum Groups.*

2000 *Mathematics Subject Classification:* Primary 16W30, 17B37; Secondary 81R50.

Partially supported by a cooperation agreement between the Department of Mathematics of the University of Rome "Tor Vergata" and the Faculty of Mathematics of the University of Belgrade (2000–2003). The first author was also supported by the Italian Ministry of Teaching, University and Research (MIUR), within the project COFIN 2003 "Group actions: algebraic and geometric aspects". The second author was also supported by the Serbian Ministry of Science and Environmental Protection, project No. 144032D.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

odd order ℓ, if $\mathbb{Z}_\varepsilon$ is the formal extension of $\mathbb{Z}$ by ϵ (see §1.4) and $\mathfrak{U}_\varepsilon(\mathfrak{g})$ is the corresponding specialization of $\mathfrak{U}_q(\mathfrak{g})$, there is a Hopf algebra epimorphism $\mathfrak{Fr}_{\mathfrak{g}}^{\mathbb{Z}} : \mathfrak{U}_\varepsilon(\mathfrak{g}) \relbar\joinrel\twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g})$, described as an "$\ell$-th root operation" on generators. This is a quantum analogue of the Frobenius morphism in positive characteristic, so is called *quantum Frobenius morphism for* $\mathfrak{g}$.

In a Hopf-dual setting, one constructs ([DL], §§4–6) a Hopf algebra $F_q[G]$ of matrix coefficients of $U_q(\mathfrak{g})$, and a $\mathbb{Z}\big[q,q^{-1}\big]$–form $\mathfrak{F}_q[G]$ of it which specializes to $F_{\mathbb{Z}}[G]$ (the algebra of regular functions over $G_{\mathbb{Z}}$, the algebraic scheme of $\mathbb{Z}$–points of G) as a Hopf algebra, for $q \to 1$. In particular, $\mathfrak{F}_q[G]$ is nothing but the set of "functions" in $F_q[G]$ which take values in $\mathbb{Z}\big[q,q^{-1}\big]$ when "evaluated" on $\mathfrak{U}_q(\mathfrak{g})$: in a word, the $\mathbb{Z}\big[q,q^{-1}\big]$–valued functions on $\mathfrak{U}_q(\mathfrak{g})$. When specializing at roots of 1 (with notation as above) there is a Hopf algebra monomorphism $\mathfrak{Fr}_G^{\mathbb{Z}} : F_{\mathbb{Z}}[G] \longhookrightarrow \mathfrak{F}_\varepsilon[G]$ dual to the above epimorphism and described, roughly, as an "ℓ-th power operation" on generators. This also is a quantum analogue of the classical Frobenius morphism, which is therefore called the *quantum Frobenius morphism for* G.

The quantization $\mathfrak{U}_q(\mathfrak{g})$ of $U_{\mathbb{Z}}(\mathfrak{g})$ endows the latter with a co-Poisson (Hopf) algebra structure which makes $\mathfrak{g}$ into a Lie bialgebra; similarly, $\mathfrak{F}_q[G]$ endows $F_{\mathbb{Z}}[G]$ with a Poisson (Hopf) algebra structure which makes G into a Poisson group. The Lie bialgebra structure on $\mathfrak{g}$ is exactly the one induced by the Poisson structure on G. Then one can consider the dual Lie bialgebra $\mathfrak{g}^*$, and dual Poisson groups $G^\star$ having $\mathfrak{g}^*$ as tangent Lie bialgebra.

Lusztig's $\mathbb{Z}\big[q,q^{-1}\big]$–integral forms $\mathfrak{U}_q(\mathfrak{g})$ and $\mathfrak{F}_q[G]$ are said to be *restricted*. On the other hand, another $\mathbb{Z}\big[q,q^{-1}\big]$–integral form of $U_q(\mathfrak{g})$, say $\mathcal{U}_q(\mathfrak{g})$, has been introduced by De Concini and Procesi (cf. [CP], §9.2, and [DP], §12.1 — the original construction is over $\mathbb{C}\big[q,q^{-1}\big]$, but it works the same over $\mathbb{Z}\big[q,q^{-1}\big]$ too), called *unrestricted*. It is generated by suitably rescaled quantum root vectors and by toral quantum analogues of simple root vectors, and for $q \to 1$ specializes to $F_{\mathbb{Z}}[G^*]$ (notation as before). Moreover, at roots of 1 there is a Hopf algebra monomorphism $\mathcal{F}r_{\mathfrak{g}}^{\mathbb{Z}} : F_{\mathbb{Z}}[G^*] = \mathcal{U}_1(\mathfrak{g}) \hookrightarrow \mathcal{U}_\varepsilon(\mathfrak{g})$, defined on generators as an "ℓ-th power operation". This is a quantum analogue of the classical Frobenius morphisms (for G^*), strictly parallel to $\mathcal{F}r_{\mathfrak{g}}^{\mathbb{Z}}$ above, so is called the *quantum Frobenius morphism for* G^*. In the Hopf-dual setting, one can consider — [Ga1], §§4, 7 — as "dual" of $\mathcal{U}_q(\mathfrak{g})$ the subset $\mathcal{F}_q[G]$ of "functions" in $F_q[G]$ which take values in $\mathbb{Z}\big[q,q^{-1}\big]$ when "evaluated" on $\mathcal{U}_q(\mathfrak{g})$; this subset is a Hopf subalgebra, which for $q \to 1$ specializes to $U_{\mathbb{Z}}(\mathfrak{g}^*)$. When specializing at roots of 1 there is a Hopf epimorphism $\mathcal{F}r_G^{\mathbb{Z}_\varepsilon} : \mathcal{F}_\varepsilon[G] \relbar\joinrel\twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}^*)$, dual to the previous monomorphism; this again is a quantum analogue of the classical Frobenius morphism (for $\mathfrak{g}^*$), hence it is called the *quantum Frobenius morphism for* $\mathfrak{g}^*$.

In this paper we provide an explicit description of $\mathcal{F}_q[G]$, its specializations at roots of 1 and its quantum Frobenius morphisms $\mathcal{F}r_G^{\mathbb{Z}_\varepsilon}$ for $G = SL_n$. The whole construction makes sense for $G = GL_n$ and $G = M_n$ (the Poisson algebraic monoid of square matrices of size n) too, and we find similar results for them. In fact, we first approach the case of $\mathcal{F}_q[M_n]$, for which the strongest results are found; then from these we get those for $\mathcal{F}_q[GL_n]$ and $\mathcal{F}_q[SL_n]$.

Our starting point is the well-known description of $\mathfrak{F}_q[M_n]$ by generators and relations, as a $\mathbb{Z}\big[q,q^{-1}\big]$–algebra generated by the entries of a quantum $(n \times n)$–matrix (see [APW], Appendix). In particular, this is an algebra of skew-commutative polynomials, much like $\mathcal{U}_q(\mathfrak{g})$ is just an algebra of skew-commutative polynomials (which are Laurent in some variables). Dually, this leads us to expect that, like $\mathfrak{U}_q(\mathfrak{gl}_n)$, also $\mathcal{F}_q[M_n]$ be generated by quantum divided powers and quantum binomial coefficients: also, we expect that a suitable PBW-like theorem holds for $\mathcal{F}_q[M_n]$, like for $\mathcal{U}_q(\mathfrak{gl}_n)$. Similarly, as $\mathfrak{Fr}_{M_n}^{\mathbb{Z}} : F_{\mathbb{Z}}[M_n] \longhookrightarrow \mathfrak{F}_\varepsilon[M_n]$ is defined on generators as an "ℓ-th power operation", dually we expect that $\mathcal{F}r_{M_n}^{\mathbb{Z}_\varepsilon} : \mathcal{F}_\varepsilon[M_n] \relbar\joinrel\twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{gl}_n^*\big)$

be given by an "ℓ–th root operation", much like $\mathfrak{Fr}_{\mathfrak{gl}_n}^{\mathbb{Z}_\varepsilon} : \mathfrak{U}_\varepsilon(\mathfrak{gl}_n) \twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{gl}_n)$.

In fact, all these conjectural expectations turn out to be true. From this we get similar (yet slightly weaker) results for $\mathcal{F}_q[GL_n]$ and $\mathcal{F}_q[SL_n]$. On the way, we also improve the (already known) above mentioned results about specializations and quantum Frobenius epimorphisms.

The intermediate step is the quantum group $U_q(\mathfrak{g}^*)$, analogue for $\mathfrak{g}^*$ of what $U_q(\mathfrak{g})$ is for $\mathfrak{g}$ (see [Ga1], §6). In particular, there are integral $\mathbb{Z}\big[q,q^{-1}\big]$–forms $\mathfrak{U}_q(\mathfrak{g}^*)$ and $\mathcal{U}_q(\mathfrak{g}^*)$ of $U_q(\mathfrak{g}^*)$ for which PBW theorems and presentations hold. Moreover, a Hopf algebra embedding $\mathcal{F}_q[M_n] \hookrightarrow \mathfrak{U}_q\big(\mathfrak{gl}_n^*\big)$ exists, via which we "pull back" a PBW-like basis and a presentation from $\mathfrak{U}_q\big(\mathfrak{gl}_n^*\big)$ to $\mathcal{F}_q[M_n]$. These arguments work, *mutatis mutandis*, for GL_n and SL_n as well. As aside results, we provide (in §3) explicit descriptions of these embeddings, and related results which turn useful in studying specializations at roots of 1.

The present work bases upon the analysis of the case $n=2$, which is treated in [GR].

Warning: at the end of the paper, we report a short list of the main symbols we use.

ACKNOWLEDGMENTS

The authors thank the referee for his/her careful analysis of the work, and the number of valuable suggestions he/she offered to improve the paper.

DEDICATORY

This work grew out of a cooperation supported by an official agreement between the Department of Mathematics of the University of Rome "Tor Vergata" and the Faculty of Mathematics of the University of Belgrade in the period 2000–2003. This agreement was the outcome of a common wish of peaceful, fruitful partnership, as an answer to the military aggression of NATO countries to the Federal Republic of Yugoslavia, which started in spring of 1999.

This paper is dedicated to the memory of all victims of that war.

§ 1 Geometrical background and q–numbers

1.1 Poisson structures on linear groups. Let $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{Q})$, with its natural structure of Lie algebra; it has basis given by the elementary matrices $\mathrm{m}_{i,j} := \big(\delta_{\ell,i}\,\delta_{j,k}\big)_{\ell,=1,\dots,n}^{k=1,\dots,n}$ ($\forall\, i,j=1,\dots,n$). Define $e_i := \mathrm{m}_{i,i+1}$, $g_j := \mathrm{m}_{j,j}$, $f_i := \mathrm{m}_{i+1,i}$, and also $h_i := g_i - g_{i+1}$ ($i=1,\dots,n-1$, $j=1,\dots,n$): then $\big\{ e_i\,, g_k\,, f_i \,\big|\, i=1,\dots,n-1,\, k=1,\dots,n \big\}$ is a set of Lie algebra generators of $\mathfrak{g}$. Moreover, a Lie cobracket is defined on $\mathfrak{g}$ by $\delta(e_i) = h_i \otimes e_i - e_i \otimes h_i$, $\delta(g_k) = 0$, $\delta(f_i) = h_i \otimes f_i - f_i \otimes h_i$ (for all i and k), which makes $\mathfrak{g}$ itself into a *Lie bialgebra* ([CP], §1.3.8). Then $U(\mathfrak{g})$ is naturally a co-Poisson Hopf algebra, whose co-Poisson bracket is the extension of the Lie cobracket of $\mathfrak{g}$ (the Hopf structure being standard). Finally, Kostant's $\mathbb{Z}$–integral form of $U(\mathfrak{g})$ — also called *hyperalgebra* — is the unital $\mathbb{Z}$–subalgebra $U_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$ generated by the divided powers $f_i^{(m)}$, $e_i^{(m)}$ and the binomial coefficients $\binom{g_k}{m}$ (for all i, k, and all $m \in \mathbb{N}$), where hereafter we use notation $x^{(m)} := x^m/m!$ and $\binom{t}{m} := \frac{t(t-1)\cdots(t-m+1)}{m!}$. This again is a co-Poisson Hopf algebra (over $\mathbb{Z}$); it is free as a $\mathbb{Z}$–module, with PBW-like $\mathbb{Z}$–basis the set of ordered monomials $\Big\{ \prod_{i<j} e_{i,j}^{(\eta_{i,j})} \prod_{k=1}^n \binom{g_k}{\gamma_k} \prod_{i>j} f_{i,j}^{(\varphi_{i,j})} \,\Big|\, \eta_{i,j}, \gamma_k, \varphi_{i,j} \in \mathbb{N} \Big\}$ (w.r.t. any total order of the pairs (i,j) with $i \neq j$), where $e_{i,j} := \mathrm{m}_{i,j}$ and $f_{j,i} := \mathrm{m}_{j,i}$ for all $i<j$; see e.g. [Hu], Ch. VII.

As $\mathfrak{gl}_n(\mathbb{Q})$ is a Lie bialgebra, by general theory $G := GL_n(\mathbb{Q})$ is then a *Poisson* group, its Poisson structure being the unique one which induces on $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{Q})$ the Lie bialgebra structure mentioned above. Explicitly, the algebra $F[G]$ of regular functions on G is the unital associative commutative $\mathbb{Q}$–algebra with generators $\bar{t}_{i,j}$ $(i,j = 1,\dots,n)$ and D^{-1}, where $D := det\big(\bar{t}_{i,j}\big)_{i,j=1,\dots,n}$ is the determinant. The group structure on G yields on $F[G]$ the natural Hopf structure given by matrix product; in particular D^{-1} is group-like. The Poisson structure — usually referred to as "standard" — is given by (see, e.g., [BGY], §§1.4–5)

$$\big\{\bar{t}_{i,j},\bar{t}_{i,k}\big\} = \bar{t}_{i,j}\,\bar{t}_{i,k} \quad \forall\, j<k\,, \qquad \big\{\bar{t}_{i,j},\bar{t}_{\ell,j}\big\} = \bar{t}_{i,j}\,\bar{t}_{\ell,j} \quad \forall\, i<\ell$$
$$\big\{\bar{t}_{i,j},\bar{t}_{\ell,k}\big\} = 0 \quad \forall\, i<\ell, k<j\,, \qquad \big\{\bar{t}_{i,j},\bar{t}_{\ell,k}\big\} = 2\,\bar{t}_{i,k}\,\bar{t}_{\ell,j} \quad \forall\, i<\ell, j<k$$
$$\big\{D^{-1},\bar{t}_{i,j}\big\} = -\big\{D,\bar{t}_{i,j}\big\} \quad \forall\, i,j=1,\dots,n\,.$$

We shall also consider the Poisson group-scheme $G_{\mathbb{Z}}$ associated to GL_n, for which a like analysis applies: in particular, its function algebra $F[G_{\mathbb{Z}}]$ is a Poisson Hopf $\mathbb{Z}$–algebra with the same presentation as $F[G]$ but over the ring $\mathbb{Z}$.

Similar constructions hold with $\mathfrak{sl}_n$ replacing $\mathfrak{gl}_n$ and with SL_n instead of GL_n: one simply replaces the g_k's with the h_i's ($i = 1,\dots,n-1$). In particular, $\mathfrak{sl}_n(\mathbb{Z})$ is a Lie $\mathbb{Z}$–subbialgebra of $\mathfrak{gl}_n(\mathbb{Z})$, and $U_{\mathbb{Z}}(\mathfrak{sl}_n)$ is a co-Poisson Hopf subalgebra (over $\mathbb{Z}$) of $U_{\mathbb{Z}}(\mathfrak{gl}_n)$, with a PBW-like $\mathbb{Z}$–basis as above but with the h_i's instead of the g_k's; moreover, $F\big[(SL_n)_{\mathbb{Z}}\big]$ is the quotient Poisson-Hopf algebra of $F\big[(GL_n)_{\mathbb{Z}}\big]$ modulo the principal ideal $(D-1)$.

Finally, the subalgebra of $F\big[(GL_n)_{\mathbb{Z}}\big]$ generated by the $\bar{t}_{i,j}$'s alone clearly is a Poisson subbialgebra of $F\big[(GL_n)_{\mathbb{Z}}\big]$: indeed, it is the algebra $F\big[(M_n)_{\mathbb{Z}}\big]$ of regular functions of the $\mathbb{Z}$-scheme associated to the Poisson algebraic monoid M_n of all $(n \times n)$–matrices.

1.2 Dual Lie bialgebras and dual Poisson groups. Let $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{Q})$ be a Lie bialgebra structure as in §1.1; then its dual $\mathfrak{g}^*$ is a Lie algebra too. Let $\big\{ e_{i,j}^*, g_k^*, f_{j,i}^* \,\big|\, \forall\, k, \forall\, i<j \big\}$ be the dual basis to the basis of elementary matrices for $\mathfrak{g}$. Set $\mathrm{e}_{i,j} := e_{i,j}^*\big/2$, $\mathrm{g}_k := g_k^*$, $\mathrm{f}_{j,i} := f_{j,i}^*\big/2$ (for all k and all $i<j$), and $\mathrm{e}_i := \mathrm{e}_{i,i+1}$, $\mathrm{f}_i := \mathrm{f}_{i,i+1}$ for all $1 \leq i < n$. Then $\mathfrak{g}^*$ is the Lie algebra with generators $\mathrm{e}_1,\dots,\mathrm{e}_{n-1},\mathrm{g}_1,\dots,\mathrm{g}_n,\mathrm{f}_1,\dots,\mathrm{f}_{n-1}$ and relations

$$\big[\mathrm{g}_i\,,\,\mathrm{f}_j\big] = \big(\delta_{i,j}-\delta_{i-1,j}\big)\,\mathrm{f}_j\,, \quad \big[\mathrm{f}_k\,,\,\mathrm{e}_\ell\big] = 0\,, \quad \big[\mathrm{g}_i\,,\,\mathrm{e}_j\big] = \big(\delta_{i,j}-\delta_{i-1,j}\big)\,\mathrm{e}_j \qquad \forall\, k,\ell,i,j$$
$$\big[\mathrm{g}_k\,,\,\mathrm{g}_\ell\big] = 0\,, \quad \big[\mathrm{f}_i\,,\,\mathrm{f}_j\big] = 0\,, \quad \big[\mathrm{e}_i\,,\,\mathrm{e}_j\big] = 0 \qquad \forall\, k,\ell, \quad \forall\, i,j : |i-j|>1$$
$$\big[\mathrm{f}_i\,,\big[\mathrm{f}_i\,,\,\mathrm{f}_j\big]\big] = 0\,, \qquad \big[\mathrm{e}_i\,,\big[\mathrm{e}_i\,,\,\mathrm{e}_j\big]\big] = 0 \qquad \forall\, i,j : |i-j|=1$$

(just compute! See also [Ga2], §1). The Lie cobracket is given (for $x \wedge y := x \otimes y - y \otimes x$) by

$$\delta\big(\mathrm{f}_i\big) = \mathrm{f}_i \wedge \big(\mathrm{g}_i - \mathrm{g}_{i+1}\big) + 2 \cdot \big(\textstyle\sum_{j=1}^{i-1} \mathrm{f}_{i+1,j} \wedge \mathrm{e}_{j,i} + \sum_{j=i+2}^{n+1} \mathrm{e}_{i+1,j} \wedge \mathrm{f}_{j,i}\big)\,, \quad 1 \leq i \leq n-1$$
$$\delta\big(\mathrm{g}_k\big) = 4 \cdot \textstyle\sum_{\ell=1}^{k-1} \mathrm{f}_{k,\ell} \wedge \mathrm{e}_{\ell,k} + 4 \cdot \sum_{\ell=k+1}^{n+1} \mathrm{e}_{k,\ell} \wedge \mathrm{f}_{\ell,k}\,, \qquad 1 \leq k \leq n$$
$$\delta\big(\mathrm{e}_i\big) = \big(\mathrm{g}_i - \mathrm{g}_{i+1}\big) \wedge \mathrm{e}_i + 2 \cdot \big(\textstyle\sum_{\ell=1}^{i-1} \mathrm{f}_{i,\ell} \wedge \mathrm{e}_{\ell,i+1} + \sum_{\ell=i+2}^{n+1} \mathrm{e}_{i,\ell} \wedge \mathrm{f}_{\ell,i+1}\big)\,, \quad 1 \leq i \leq n-1\,.$$

All these formulæ also provide a presentation of $U(\mathfrak{g}^*)$ as a co-Poisson Hopf algebra. Finally, we define the Kostant's $\mathbb{Z}$–integral form, or hyperalgebra, $U_{\mathbb{Z}}(\mathfrak{g}^*)$ of $U(\mathfrak{g}^*)$ as the unital $\mathbb{Z}$–subalgebra generated by the divided powers $\mathrm{f}_i^{(m)}$, $\mathrm{e}_i^{(m)}$ and binomial coefficients $\binom{\mathrm{g}_k}{m}$ (for all $m \in \mathbb{N}$ and all i, k). This again is a co-Poisson Hopf $\mathbb{Z}$–algebra, with PBW-like $\mathbb{Z}$–basis the set $\Big\{ \prod_{i<j} \mathrm{e}_{i,j}^{(\eta_{i,j})} \prod_{k=1}^{n} \binom{\mathrm{g}_k}{\gamma_k} \prod_{i>j} \mathrm{f}_{i,j}^{(\varphi_{i,j})} \,\Big|\, \eta_{i,j}, \gamma_k, \varphi_{i,j} \in \mathbb{N} \Big\}$ of ordered monomials (w.r.t. any total order as before). Alternatively, one can take the elements $\mathrm{l}_k := \mathrm{g}_1 + \cdots + \mathrm{g}_k$ ($k = 1,\dots,n$) instead of the g_k's in the definition of $U_{\mathbb{Z}}(\mathfrak{g}^*)$; the presentation changes accordingly, and $\Big\{ \prod_{i<j} \mathrm{e}_{i,j}^{(\eta_{i,j})} \prod_{k=1}^{n} \binom{\mathrm{l}_k}{\lambda_k} \prod_{i>j} \mathrm{f}_{i,j}^{(\varphi_{i,j})} \,\Big|\, \eta_{i,j}, \lambda_k, \varphi_{i,j} \in \mathbb{N} \Big\}$ is another PBW-like basis.

A like description holds for $\mathfrak{sl}_n(\mathbb{Q})^*$. In fact, $\mathfrak{sl}_n(\mathbb{Q})^* = \mathfrak{gl}_n(\mathbb{Q})^* \big/ \mathbb{Q} \cdot \mathrm{l}_n$, dually to the embedding $\mathfrak{sl}_n(\mathbb{Q}) \longrightarrow \mathfrak{gl}_n(\mathbb{Q})$; thus one simply has to add to the presentation of $\mathfrak{gl}_n(\mathbb{Q})^*$ the additional relation $\mathrm{g}_1 + \cdots + \mathrm{g}_n = 0$, or $\mathrm{l}_n = 0$. Then $U_{\mathbb{Z}}(\mathfrak{sl}_n^*)$ is the $\mathbb{Z}$–subalgebra of $U(\mathfrak{sl}_n(\mathbb{Q})^*)$ generated by divided powers and binomial coefficients as above, but taking care of the additional relation); also, $U_{\mathbb{Z}}(\mathfrak{sl}_n^*)$ is a quotient co-Poisson Hopf $\mathbb{Z}$–algebra of $U_{\mathbb{Z}}(\mathfrak{gl}_n^*)$. Again, admits the set $\Big\{ \prod_{i<j} \mathrm{e}_{i,j}^{(\eta_{i,j})} \prod_{k=1}^{n-1} \binom{\mathrm{l}_h}{\lambda_h} \prod_{i>j} \mathrm{f}_{i,j}^{(\varphi_{i,j})} \,\Big|\, \eta_{i,j}, \lambda_k, \varphi_{i,j} \in \mathbb{N} \Big\}$ as $\mathbb{Z}$–basis. Finally, note that $\Big\{ \mathrm{e}_{i,j} := e_{i,j}^*\big/2 \, , \, \mathrm{l}_k \, , \, \mathrm{f}_{j,i} := f_{j,i}^*\big/2 \;\Big|\; 1 \leq k \leq n-1, \, 1 \leq i < j \leq n \Big\}$ is the basis of $\mathfrak{sl}_n^*$ dual to the basis $\big\{ e_{i,j} \, , h_k \, , f_{j,i} \,\big|\, 1 \leq k \leq n-1, \, 1 \leq i < j \leq n \big\}$ of $\mathfrak{sl}_n$.

If $\mathfrak{g} = \mathfrak{gl}_n$, a simply connected algebraic Poisson group with tangent Lie bialgebra $\mathfrak{g}^*$ is the subgroup ${}_sG^*$ of $G \times G$ made of all pairs $(L, U) \in G \times G$ such that L is lower triangular, U is upper triangular, and their diagonals are inverse to each other (i.e. their product is the identity matrix). This is a Poisson subgroup for the natural Poisson structure of $G \times G$. Its centre is $Z := \big\{ (zI, z^{-1}I) \,\big|\, z \in \mathbb{Q} \setminus \{0\} \big\}$, hence the associated adjoint group is ${}_aG^* := {}_sG^* \big/ Z$. The same construction defines Poisson group-schemes ${}_sG_{\mathbb{Z}}^*$ and ${}_aG_{\mathbb{Z}}^*$.

If $\mathfrak{g} = \mathfrak{sl}_n$ the construction of dual Poisson group-schemes ${}_sG_{\mathbb{Z}}^*$ and ${}_aG_{\mathbb{Z}}^*$ is entirely similar, just taking SL_n instead of GL_n in the previous recipe.

1.3 q–numbers, q–divided powers and q–binomial coefficients. Let q be an indeterminate. For all $n, s, k_1, \dots, k_r \in \mathbb{N}$, let $(n)_q := \frac{q^n - 1}{q - 1}$, $(n)_q! := \prod_{r=1}^n (r)_q$, $\binom{n}{s}_q := \frac{(n)_q!}{(s)_q!(n-s)_q!}$, $\binom{n}{k_1, \dots, k_r}_q := \frac{(n)_q!}{(k_1)_q! \cdots (k_r)_q!}$ $(\in \mathbb{Z}[q])$, then $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$, $[n]_q! := \prod_{r=1}^n [r]_q$, $\begin{bmatrix} n \\ s \end{bmatrix}_q := \frac{[n]_q!}{[s]_q![n-s]_q!}$ $(\in \mathbb{Z}[q, q^{-1}])$, and also $\binom{-n}{s}_q := (-1)^s q^{-ns - \binom{s}{2}} \frac{(n-1+s)_q!}{(s)_q!}$ $(\in \mathbb{Z}[q])$.

If A is any $\mathbb{Q}(q)$–algebra we define q–divided powers and q–binomial coefficients as follows: for every $X \in A$, each $n \in \mathbb{N}$ and each $c \in \mathbb{Z}$ we set $X^{(n)} := X^n \big/ [n]_q!$ and $\binom{X;c}{n} := \prod_{s=1}^n \frac{q^{c+1-s} X - 1}{q^s - 1}$. Furthermore, if $Z \in A$ is *invertible* we define also $\begin{bmatrix} X;c \\ n \end{bmatrix} := \prod_{s=1}^n \frac{q^{c+1-s} Z^{+1} - q^{s-1-c} Z^{-1}}{q^{+s} - q^{-s}}$ for every $n \in \mathbb{N}$ and $c \in \mathbb{Z}$. We shall also consider the elements $\left\{ \begin{matrix} X;c \\ n,r \end{matrix} \right\} := \sum_{s=0}^r q^{\binom{s+1}{2}} \binom{r}{s}_q \cdot \binom{X;c+s}{n-r}$ for every $X \in A$, $n, r \in \mathbb{N}$, $c \in \mathbb{Z}$.

For later use, we remark that the q–divided powers in $X \in A$ satisfy the (obvious) relations

$$X^{(r)} X^{(s)} = \begin{bmatrix} r+s \\ s \end{bmatrix}_q X^{(r+s)} \, , \qquad X^{(0)} = 1 \tag{1.1}$$

Similarly, the q–binomial coefficients in $X \in A$ satisfy the relations

$$\begin{gathered}
\binom{X;c}{t}\binom{X;c-t}{s} = \binom{t+s}{t}_q \binom{X;c}{t+s} \, , \qquad \binom{X;c+1}{t} - q^t \binom{X;c}{t} = \binom{X;c}{t-1} \\
\binom{X;c}{m}\binom{X;s}{n} = \binom{X;s}{n}\binom{X;c}{m} \, , \qquad \binom{X;c}{t} = \sum_{p \geq 0}^{p \leq c,t} q^{(c-p)(t-p)} \binom{c}{p}_q \binom{X;0}{t-p} \\
\binom{X;c}{0} = 1 \, , \qquad \binom{X;-c}{t} = \sum_{p=0}^{t} (-1)^p q^{-t(c+p)+p(p+1)/2} \binom{p+c-1}{p}_q \binom{X;0}{t-p} \\
\binom{X;c+1}{t} - \binom{X;c}{t} = q^{c-t+1} \left(1 + (q-1) \binom{X;0}{1} \right) \binom{X;c}{t-1}
\end{gathered} \tag{1.2}$$

1.4 Roots of 1 and specializations. Let $\ell \in \mathbb{N}_+$ be odd, set $\mathbb{Z}_\varepsilon := Z[q] \big/ \big(\phi_\ell(q)\big)$ where $\phi_\ell(q)$ is the ℓ-th cyclotomic polynomial in q, and let $\varepsilon := \overline{q}$, a (formal) primitive ℓ-th root of 1 in $\mathbb{Z}_\varepsilon$. Similarly, let $\mathbb{Q}_\varepsilon := \mathbb{Q}[q] \big/ \big(\phi_\ell(q)\big)$, the field of quotients of $\mathbb{Z}_\varepsilon$. If M is a module over $\mathbb{Z}\big[q,q^{-1}\big]$ or $\mathbb{Q}\big[q,q^{-1}\big]$ we set $M_\varepsilon := M \big/ \big(\phi_\ell(q)\big) M$, which is a module over $\mathbb{Z}_\varepsilon$ or over $\mathbb{Q}_\varepsilon$, called the *specialization of* M *at* $q=\varepsilon$. In fact, we have $M_\varepsilon \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}[q,q^{-1}]} M$ or $M_\varepsilon \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}[q,q^{-1}]} M$ as modules over $\mathbb{Z}_\varepsilon$ or $\mathbb{Q}_\varepsilon$ (in either case).

For later use, we recall now some results about q-numbers and their specializations:

Lemma 1.5. *Let A be any $\mathbb{Q}\big[p,p^{-1}\big]$–algebra, p being any indeterminate.*
(a) If $x_1, \dots, x_s \in A$ are such that $x_i\,x_j = p\,x_j\,x_i$ for all $i<j$, then for all $m \in \mathbb{N}$

$$\big(\textstyle\sum_{k=1}^s x_k\big)^m \;=\; \sum_{k_1+\cdots+k_s=m} \binom{m}{k_1,\dots,k_s}_{p^{-1}} x_1^{k_1}\cdots x_s^{k_s}$$

(b) If $y_1, \dots, y_s \in A$ are such that $y_i\,y_j = p^2\,y_j\,y_i$ for all $i<j$, then for all $m \in \mathbb{N}$

$$\big(\textstyle\sum_{k=1}^s y_k\big)^{(m)} \;=\; \sum_{k_1+\cdots+k_s=m} p^{\binom{k_1+1}{2}+\cdots+\binom{k_s+1}{2}-\binom{m+1}{2}} \cdot y_1^{(k_1)}\cdots y_s^{(k_s)}$$

Proof. *(a)* is the well-known, generalized q–Leibniz' identity, and *(b)* follows trivially. □

Lemma 1.6. *Let A be any $\mathbb{Q}(q)$–algebra, and let $x,y,z,w \in A$ be such that $x\,w = q^2\,w\,x$, $x\,y = q\,y\,x$, $x\,z = q\,z\,x$ and $y\,z = z\,y$. Then for all $m \in \mathbb{N}$ and $t \in \mathbb{Z}$ we have*

(a-1) $$\binom{x+\big(q-q^{-1}\big)^2 w\,;\,t}{m} = \sum_{r=0}^m q^{r(t-m)} \big(q-q^{-1}\big)^r \cdot w^{(r)} \left\{ \begin{matrix} x\,;\,t \\ m\,,\,r \end{matrix} \right\} =$$

(a-2) $$= \sum_{r=0}^m q^{r(t-m)} \big(q-q^{-1}\big)^r \cdot \left\{ \begin{matrix} x\,;\,t-2\,r \\ m\,,\,r \end{matrix} \right\} w^{(r)}$$

(b) $$\binom{x+\big(q-q^{-1}\big)^2 y\,z\,;\,t}{m} = \sum_{r=0}^m q^{r(t-m)} \big(q-q^{-1}\big)^r\, [r]_q! \cdot y^{(r)} \left\{ \begin{matrix} x\,;\,t-r \\ m\,,\,r \end{matrix} \right\} z^{(r)}$$

If instead $w\,x = q^2\,x\,w$, $y\,x = q\,x\,y$, $z\,x = q\,x\,z$ and $z\,y = y\,z$, then ($\forall\, m \in \mathbb{N}\,,\, t \in \mathbb{Z}$)

(c-1) $$\binom{x+\big(q-q^{-1}\big)^2 w\,;\,t}{m} = \sum_{r=0}^m q^{r(t-m)} \big(q-q^{-1}\big)^r \cdot \left\{ \begin{matrix} x\,;\,t \\ m\,,\,r \end{matrix} \right\} w^{(r)} =$$

(c-2) $$= \sum_{r=0}^m q^{r(t-m)} \big(q-q^{-1}\big)^r \cdot w^{(r)} \left\{ \begin{matrix} x\,;\,t-2\,r \\ m\,,\,r \end{matrix} \right\}$$

(d) $$\binom{x+\big(q-q^{-1}\big)^2 y\,z\,;\,t}{m} = \sum_{r=0}^m q^{r(t-m)} \big(q-q^{-1}\big)^r\, [r]_q! \cdot z^{(r)} \left\{ \begin{matrix} x\,;\,t-r \\ m\,,\,r \end{matrix} \right\} y^{(r)}$$

Proof. Claims *(a-1/2)* and *(b)* are proved in Lemma 6.2 of [GR]. Instead, claims *(c-1/2)* and *(d)* follow directly from claims *(a-1/2)* and *(b)* when one applies them to the algebra A endowed with the *opposite* product. □

Lemma 1.7. *([GR], Lemma 6.3) Let $\varOmega$ be any $\mathbb{Z}\big[q,q^{-1}\big]$–algebra, let ε be a (formal) primitive ℓ-th root of 1, with $\ell \in \mathbb{N}_+$, and let $\varOmega_\varepsilon := \varOmega \Big/ (q-\varepsilon)\,\varOmega$ be the specialization of $\varOmega$ at $q=\varepsilon$ (a $\mathbb{Z}_\varepsilon$–algebra). Then for each $x, y \in \varOmega$ we have*

$$\varOmega \ni \binom{x\,;\,0}{\ell} \implies \big(x\big|_{q=\varepsilon}\big)^\ell = 1 \;\; in \;\; \varOmega_\varepsilon \,, \qquad \varOmega \ni y^{(\ell)} \implies \big(y\big|_{q=\varepsilon}\big)^\ell = 0 \;\; in \;\; \varOmega_\varepsilon \,. \qquad \square$$

§ 2 Quantum groups

2.1 The quantum groups $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{sl}_n)$. Let $U_q(\mathfrak{gl}_n)$ be the Drinfeld-Jimbo quantization of $U(\mathfrak{gl}_n)$: following [Ji], §2 (with normalizations of [No], §1.2), we define it as the unital $\mathbb{Q}(q)$–algebra with generators F_i, $G_j^{\pm 1}$, E_i $(1 \leq i \leq n{-}1\,;\, 1 \leq j \leq j\,)$ and relations

$$\begin{aligned}
G_i\,G_i^{-1} \,=\, 1 \,=\, G_i^{-1}\,G_i\,, \qquad G_i^{\pm 1}\,G_j^{\pm 1} \,&=\, G_j^{\pm 1}\,G_i^{\pm 1} && \forall\, i,j \\
G_i\,F_j\,G_i^{-1} \,=\, q^{\delta_{i,j+1}-\delta_{i,j}}\,F_j\,, \qquad G_i\,E_j\,G_i^{-1} \,&=\, q^{\delta_{i,j}-\delta_{i,j+1}}\,E_j && \forall\, i,j \\
E_i\,F_j \,-\, F_j\,E_i \,&=\, \delta_{i,j}\,\frac{G_i\,G_{i+1}^{-1} \,-\, G_i^{-1}\,G_{i+1}}{q-q^{-1}} && \forall\, i,j \\
E_i\,E_j \,=\, E_j\,E_i\,, \qquad F_i\,F_j \,&=\, F_j\,F_i && \forall\;\, i,j : |i-j|>1 \\
E_i^2\,E_j \,-\, \big(q+q^{-1}\big)\,E_i\,E_j\,E_i \,+\, E_j\,E_i^2 \,&=\, 0 && \forall\;\, i,j : |i-j|=1 \\
F_i^2\,F_j \,-\, \big(q+q^{-1}\big)\,F_i\,F_j\,F_i \,+\, F_j\,F_i^2 \,&=\, 0 && \forall\;\, i,j : |i-j|=1\,.
\end{aligned}$$

Moreover, $U_q(\mathfrak{gl}_n)$ has a Hopf algebra structure, given by

$$\begin{aligned}
&\Delta\,(F_i) \,=\, F_i \otimes G_i^{-1}\,G_{i+1} \,+\, 1 \otimes F_i\,, && S\,(F_i) \,=\, -F_i\,G_i\,G_{i+1}^{-1}\,, && \epsilon\,(F_i) \,=\, 0 && \forall\;\, i \\
&\Delta\,\big(G_i^{\pm 1}\big) \,=\, G_i^{\pm 1} \otimes G_j^{\pm 1}\,, && S\,\big(G_j^{\pm 1}\big) \,=\, G_j^{\mp 1}\,, && \epsilon\,\big(G_j^{\pm 1}\big) \,=\, 1 && \forall\;\, j \\
&\Delta\,(E_i) \,=\, E_i \otimes 1 \,+\, G_i G_{i+1}^{-1} \otimes E_i\,, && S\,(E_i) \,=\, -G_i^{-1}\,G_{i+1}\,E_i\,, && \epsilon\,(E_i) \,=\, 0 && \forall\;\, i\,.
\end{aligned}$$

Consider also the elements $K_i^{\pm 1} := G_i^{\pm 1} G_{i+1}^{\mp 1}$ (for all $i = 1, \dots, n-1$). The unital $\mathbb{Q}(q)$–subalgebra of $U_q(\mathfrak{gl}_n)$ generated by $\big\{F_i\,, K_i^{\pm 1}, E_i\big\}_{i=1,\dots,n-1}$ is the well known Drinfeld-Jimbo's quantum algebra $U_q(\mathfrak{sl}_n)$. From the presentation of $U_q(\mathfrak{gl}_n)$ one deduces one of $U_q(\mathfrak{sl}_n)$ too, and also sees that the latter is even a Hopf subalgebra of the former.

2.2 Quantum root vectors and quantum PBW theorem. Now we introduce quantum analogues of root vectors for both $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{sl}_n)$, following an idea due to Jimbo (as in [Ji], §2, up to changing $q^{\pm 1} \longleftrightarrow q^{\mp 1}$). Set

$$\begin{aligned}
&E_{i,i+1} := E_i\,, && E_{i,j} := \big[\,E_{i,k}\,, E_{k,j}\,\big]_{q^{-1}} \equiv\, E_{i,k}\,E_{k,j} - q^{-1}\,E_{k,j}\,E_{i,k} && \forall\, i<k<j \\
&F_{i+1,i} := F_i\,, && F_{j,i} := \big[\,F_{j,k}\,, F_{k,i}\,\big]_{q^{+1}} \equiv\, F_{j,k}\,F_{k,i} - q^{+1}\,F_{k,i}\,F_{j,k} && \forall\, j>k>i\,.
\end{aligned}$$

Then all these are quantum root vectors, in $U_q(\mathfrak{gl}_n)$ or $U_q(\mathfrak{sl}_n)$, and the very definition gives a complete set of commutation relations among them. For later use, we point out that they can be obtained also via a general method given by Lusztig.

Namely, let $\{\alpha_1, \dots, \alpha_{n-1}\}$ be a basis of simple roots of $\mathfrak{sl}_n$ (or of $\mathfrak{gl}_n$), and let $s_1, \dots, s_{n-1}$ be generators of its Weyl group W. Let $w_0 = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_{N-1}} s_{i_N}$ be a reduced expression for the longest element w_0 of W, where $N := \binom{n}{2}$. Then $\big\{ \alpha^k := s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) \,\big|\, k = 1, \dots, N \big\}$ is the set of all positive roots of $\mathfrak{sl}_n$. Now, Lusztig defines an action of the braid group associated to W, with generators $T_1, \dots, T_{n-1}$: we consider it normalized as in [Lu], §3. Lusztig's result (see [CP], §8.1, and references therein) is that the elements $E_{\alpha^k} := T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(E_{i_k})$ and $F_{\alpha^k} := T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(F_{i_k})$ $(k = 1, \dots, N)$ are quantum analogues of root vectors, respectively of weight $+\alpha^k$ and $-\alpha^k$. Now consider the sequence $(i_1, \dots, i_N) := (1, 2, \dots, n-1, n, 1, 2, \dots, n-2, n-1, 1, 2, \dots, 3, 4, 1, 2, 3, 1, 2, 1)$. Then $w_0 = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_{N-1}} s_{i_N}$ is a reduced expression for w_0, and the previous recipe applies. It is easy to check ([Ga2], §5.3) that, with this choice of the sequence $(i_1, \dots, i_N)$, one has $E_{\alpha^k} = E_{i,j}$, $F_{\alpha^k} = F_{j,i}$ for $\alpha^k = \sum_{t=i}^{j-1} \alpha_t$, for all k (a key fact is that $E_{\alpha^k} = E_h$ and $F_{\alpha^k} = F_h$ if $\alpha^k = \alpha_h$). So our choice of quantum root vectors is a special case of Lusztig's.

The quantum version of the PBW theorem claims that the set of ordered monomials $B^g := \Big\{ \prod_{i<j} E_{i,j}^{\eta_{i,j}} \prod_{k=1}^n G_k^{\gamma_k} \prod_{i<j} F_{j,i}^{\varphi_{j,i}} \,\Big|\, \eta_{i,j}, \gamma_k, \varphi_{j,i} \in \mathbb{N} \Big\}$ (w.r.t. any total order of the pairs (i,j) with $i \neq j$) is a $\mathbb{Q}(q)$–basis of $U_q(\mathfrak{gl}_n)$, and the set $B^s := \Big\{ \prod_{i<j} E_{i,j}^{\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} F_{j,i}^{\varphi_{j,i}} \,\Big|\, \eta_{i,j}, \kappa_k, \varphi_{j,i} \in \mathbb{N} \Big\}$ is a $\mathbb{Q}(q)$–basis of $U_q(\mathfrak{sl}_n)$: see [CP], §8.1, and references therein.

2.3 Integral forms of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{sl}_n)$. After Lusztig's idea for semisimple Lie algebras, adapted to the case of $\mathfrak{gl}_n$, one can define (see [CP], §9.3, and references therein) a suitable *restricted* integral form of $U_q(\mathfrak{gl}_n)$ over $\mathbb{Z}\big[q, q^{-1}\big]$, say $\mathfrak{U}_q(\mathfrak{gl}_n)$: following [DL], §3, it is the unital $\mathbb{Z}\big[q, q^{-1}\big]$–subalgebra of $U_q(\mathfrak{gl}_n)$ generated by $F_1^{(m)}, \dots, F_{n-1}^{(m)}, G_1^{\pm 1}, \binom{G_1\,;\,c}{m}, \dots, G_n^{\pm 1}, \binom{G_n\,;\,c}{m}, E_1^{(m)}, \dots, E_{n-1}^{(m)}$, for all $m \in \mathbb{N}$ and $c \in \mathbb{Z}$. This $\mathfrak{U}_q(\mathfrak{gl}_n)$ is a $\mathbb{Z}\big[q, q^{-1}\big]$–integral form of $U_q(\mathfrak{gl}_n)$ as a Hopf algebra, and specializes to $U_{\mathbb{Z}}(\mathfrak{gl}_n)$ for $q \mapsto 1$, that is $\mathfrak{U}_q(\mathfrak{gl}_n) \Big/ (q-1)\, \mathfrak{U}_q(\mathfrak{gl}_n) \cong U_{\mathbb{Z}}(\mathfrak{gl}_n)$ as co-Poisson Hopf algebras; therefore we call $\mathfrak{U}_q(\mathfrak{gl}_n)$ a *quantum (*or *quantized) hyperalgebra*. Moreover, a suitable PBW-like theorem holds for $\mathfrak{U}_q(\mathfrak{gl}_n)$. Finally, for every root of 1, say ε, of odd order ℓ, a well defined *quantum Frobenius morphism* $\mathfrak{Fr}_{\mathfrak{gl}_n}^{\mathbb{Z}_\varepsilon} : \mathfrak{U}_\varepsilon(\mathfrak{gl}_n) \twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{gl}_n)$ exists (a Hopf algebra epimorphism): here $\mathfrak{U}_\varepsilon(\mathfrak{gl}_n) := \big(\mathfrak{U}_q(\mathfrak{gl}_n)\big)_\varepsilon$ — as in §1.4 — and $\mathfrak{Fr}_{\mathfrak{gl}_n}^{\mathbb{Z}_\varepsilon}$ is defined on generators by dividing out the order of each quantum divided power and each quantum binomial coefficient by ℓ, if this makes sense, and mapping to zero otherwise. The obvious parallel construction — just replacing the $G_k^{\pm 1}$'s by the $K_i^{\pm 1}$'s everywhere — provides $\mathfrak{U}_q(\mathfrak{sl}_n)$, the *restricted* $\mathbb{Z}\big[q, q^{-1}\big]$–integral form of $U_q(\mathfrak{sl}_n)$: it specializes to $U_{\mathbb{Z}}(\mathfrak{sl}_n)$ for $q \mapsto 1$, and *quantum Frobenius morphisms* $\mathfrak{Fr}_{\mathfrak{sl}_n}^{\mathbb{Z}_\varepsilon} : \mathfrak{U}_\varepsilon(\mathfrak{sl}_n) \twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{sl}_n)$ exist for every ϵ as above. See [DL], §§3, 6, or [CP], §9.3.

However, we are mainly interested now in the *unrestricted* integral forms and their specializations. First set, as notation, $\overline{X} := \big(q - q^{-1}\big) X$. Following [DP], §12.1, or [Ga1], §3.3 — whose constructions apply to semisimple Lie algebras, but also work for $\mathfrak{gl}_n$ *and* with $\mathbb{Z}$ instead of $\mathbb{C}$ — we define $\mathcal{U}_q(\mathfrak{gl}_n)$ to be the unital $\mathbb{Z}\big[q, q^{-1}\big]$–subalgebra of $U_q(\mathfrak{gl}_n)$ generated by $\big\{ \overline{F}_{j,i}, G_k^{\pm 1}, \overline{E}_{i,j} \big\}_{1 \leq i < j \leq n}^{k=1,\dots,n}$. This $\mathcal{U}_q(\mathfrak{gl}_n)$ is a Hopf $\mathbb{Z}\big[q, q^{-1}\big]$–subalgebra of $U_q(\mathfrak{gl}_n)$, and is a free $\mathbb{Z}\big[q, q^{-1}\big]$–module with PBW-like basis $\mathcal{B}^g := \Big\{ \prod_{i<j} \overline{E}_{i,j}^{\,\eta_{i,j}} \prod_{k=1}^n G_k^{\gamma_k} \prod_{i<j} \overline{F}_{j,i}^{\,\varphi_{j,i}} \,\Big|\, \eta_{i,j}, \gamma_k, \varphi_{j,i} \in \mathbb{N} \Big\}$ (monomials being ordered w.r.t. any total order as before). Thus $\mathcal{U}_q(\mathfrak{gl}_n)$

is another $\mathbb{Z}\big[q,q^{-1}\big]$–integral form of $U_q(\mathfrak{gl}_n)$ — after $\mathfrak{U}_q(\mathfrak{gl}_n)$ — as a Hopf algebra, in that it is a Hopf $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra such that $\mathbb{Q}(q)\otimes_{\mathbb{Z}[q,q^{-1}]}\mathcal{U}_q(\mathfrak{gl}_n)\cong U_q(\mathfrak{gl}_n)$.

From [DP], §14.1, or [Ga1], Theorem 7.4, one has that $\mathcal{U}_q(\mathfrak{gl}_n)$ is a *quantization* of $F\big[({}_sGL_n{}^*)_{\mathbb{Z}}\big]$, i.e. $\mathcal{U}_1(\mathfrak{gl}_n):=\big(\mathcal{U}_q(\mathfrak{gl}_n)\big)_1\cong F\big[({}_sGL_n{}^*)_{\mathbb{Z}}\big]$ as Poisson Hopf algebras (notation of §§1.1/4), where on left-hand side we consider the standard Poisson structure inherited from $\mathcal{U}_q(\mathfrak{gl}_n)$. Finally, let ℓ, ε and $\mathbb{Z}_\varepsilon$ be as in §1.4, and $\mathcal{U}_\varepsilon(\mathfrak{gl}_n):=\big(\mathcal{U}_q(\mathfrak{gl}_n)\big)_\varepsilon$: then there is a Hopf $\mathbb{Z}$–algebra embedding $\mathcal{F}r_{\mathfrak{gl}_n}^{\mathbb{Z}}\colon F\big[({}_sGL_n{}^*)_{\mathbb{Z}}\big]\cong\mathcal{U}_1(\mathfrak{gl}_n)\longhookrightarrow\mathcal{U}_\varepsilon(\mathfrak{gl}_n)$, called the *quantum Frobenius morphism* for ${}_sGL_n{}^*$, given by ℓ–th power on generators (see [Ga3], §§3.6–7, for details). It also extends to a $\mathbb{Z}_\varepsilon$–linear map $\mathcal{F}r_{\mathfrak{gl}_n}^{\mathbb{Z}_\varepsilon}$ defined on $F\big[({}_sGL_n{}^*)_{\mathbb{Z}_\varepsilon}\big]\cong\mathbb{Z}_\varepsilon\otimes_{\mathbb{Z}}\mathcal{U}_1(\mathfrak{gl}_n)$.

Once again, the same constructions can be done with $\mathfrak{sl}_n$ instead of $\mathfrak{gl}_n$. In this way one construct $\mathcal{U}_q(\mathfrak{sl}_n)$ simply following the recipe above but replacing the $G_k^{\pm1}$'s with the $K_i^{\pm1}$'s. Then $\mathcal{U}_q(\mathfrak{sl}_n)$ is another $\mathbb{Z}\big[q,q^{-1}\big]$–integral form of $U_q(\mathfrak{sl}_n)$ (after $\mathfrak{U}_q(\mathfrak{sl}_n)$) as a Hopf algebra, and it is free as a $\mathbb{Z}\big[q,q^{-1}\big]$–module with (PBW-like) $\mathbb{Z}\big[q,q^{-1}\big]$–basis the set of ordered monomials $\mathcal{B}^s:=\Big\{\prod_{i<j}\overline{E}_{i,j}^{\,\eta_{i,j}}\prod_{k=1}^{n-1}K_k^{\kappa_k}\prod_{i<j}\overline{F}_{j,i}^{\,\varphi_{j,i}}\,\Big|\,\eta_{i,j},\kappa_k,\varphi_{j,i}\in\mathbb{N}\Big\}$. Entirely similar results to those for $\mathcal{U}_q(\mathfrak{gl}_n)$ then hold for $\mathcal{U}_q(\mathfrak{sl}_n)$ too, also regarding specializations at roots of 1 (including 1 itself) and quantum Frobenius morphisms $\mathcal{F}r_{\mathfrak{sl}_n}^{\mathbb{Z}}$ and $\mathcal{F}r_{\mathfrak{sl}_n}^{\mathbb{Z}_\varepsilon}$ (see [Ga3], §4).

The embedding $U_q(\mathfrak{sl}_n)\longhookrightarrow U_q(\mathfrak{gl}_n)$ restricts to Hopf embeddings $\mathfrak{U}_q(\mathfrak{sl}_n)\longhookrightarrow\mathfrak{U}_q(\mathfrak{gl}_n)$ and $\mathcal{U}_q(\mathfrak{sl}_n)\longhookrightarrow\mathcal{U}_q(\mathfrak{gl}_n)$. The specializations of the latter ones at $q=\varepsilon$ and at $q=1$ are compatible (in the obvious sense) with the quantum Frobenius morphisms.

2.4 Quantum function algebras, their integral forms and quantum Frobenius morphisms. Let $F_q[M_n]$ be the quantum function algebra over M_n introduced by Manin. Namely (cf. [No], §1.1, and references therein), $F_q[M_n]$ is the unital associative $\mathbb{Q}(q)$–algebra with generators $t_{i,j}$ $(i,j=1,\dots,n)$ and relations

$$t_{i,j}\,t_{i,k}=q\,t_{i,k}\,t_{i,j}\ ,\qquad t_{i,k}\,t_{h,k}=q\,t_{h,k}\,t_{i,k}\qquad\qquad\forall\quad j<k\,,\ i<h\,,$$
$$t_{i,l}\,t_{j,k}=t_{j,k}\,t_{i,l}\ ,\qquad t_{i,k}\,t_{j,l}-t_{j,l}\,t_{i,k}=\big(q-q^{-1}\big)\ t_{i,l}\,t_{j,k}\qquad\forall\quad i<j\,,\ k<l\,.$$

The $(n\times n)$–matrix $T:=\big(t_{i,j}\big)_{i,j=1,\dots,n}$ will be called a *q–matrix*. We call *"quantum determinant"* the element $D_q\equiv det_q\Big(\big(t_{k,\ell}\big)_{k,\ell=1,\dots,n}\Big):=\sum_{\sigma\in\mathcal{S}_n}(-q)^{l(\sigma)}t_{1\,\sigma(1)}\,t_{2,\sigma(2)}\cdots t_{n,\sigma(n)}$, belonging to the center of $F_q[M_n]$. This $F_q[M_n]$ is a bialgebra, with $\Delta(t_{i,j})=\sum_{k=1}^{n}t_{i,k}\otimes t_{k,j}$ and $\epsilon(t_{i,j})=\delta_{ij}$ (for all i, j); then D_q is group-like, i.e. $\Delta(D_q)=D_q\otimes D_q$ and $\epsilon(D_q)=1$.

Finally, $F_q[M_n]$ admits the set $B_{M_n}:=\Big\{\prod_{i<j}t_{i,j}^{\tau_{i,j}}\prod_{k=1}^{n}t_{k,k}^{\tau_{k,k}}\prod_{i>j}t_{i,j}^{\tau_{i,j}}\;\Big|\;\tau_{i,j}\in\mathbb{N}\ \forall\,i,j\Big\}$ as $\mathbb{Q}(q)$–basis, where again factors in monomials are ordered w.r.t. any fixed total order of the pairs (i,j) with $i\neq j$ (see [Ga4], Theorem 2.1*(a)*, and references therein).

We define $\mathfrak{F}_q[M_n]$ to be the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $F_q[M_n]$ generated by the $t_{i,j}$'s (for all i, j); this is also a *sub-bialgebra*, and admits *the same presentation* as $F_q[M_n]$ but over $\mathbb{Z}\big[q,q^{-1}\big]$. Therefore $D_q\in\mathfrak{F}_q[M_n]$, the latter is a free $\mathbb{Z}\big[q,q^{-1}\big]$–module with $\mathbb{Z}\big[q,q^{-1}\big]$–basis B_{M_n} ([Ga4], Theorem 2.1*(a)*), and $\mathfrak{F}_q[M_n]$ is a *$\mathbb{Z}\big[q,q^{-1}\big]$–integral form of $F_q[M_n]$*. The presentation shows that $\mathfrak{F}_q[M_n]$ is a *quantization* of $F\big[(M_n)_{\mathbb{Z}}\big]$, i.e. $\mathfrak{F}_1[M_n]:=\big(\mathfrak{F}_q[M_n]\big)_1\cong F\big[(M_n)_{\mathbb{Z}}\big]$ as Poisson bialgebras (where on left-hand side we consider the standard Poisson structure inherited from $\mathfrak{F}_q[M_n]$), the isomorphism being given by $t_{i,j}\big|_{q=1}\cong\bar{t}_{i,j}$ for all i,j. Finally let ℓ, ε, $\mathbb{Z}_\varepsilon$ be as in §1.4, $\mathfrak{F}_\varepsilon[M_n]:=\big(\mathfrak{F}_q[M_n]\big)_\varepsilon$. Then there is a bialgebra embedding $\mathfrak{Fr}_{M_n}^{\mathbb{Z}}:F\big[(M_n)_{\mathbb{Z}}\big]\longhookrightarrow\mathfrak{F}_\varepsilon[M_n]$ $\Big(\bar{t}_{i,j}\mapsto t_{i,j}^{\ell}\big|_{q=\varepsilon}\Big)$, the *quantum Frobenius morphism* for M_n.

Besides the presentation given above, another description of $\mathfrak{F}_q[G]$ is available. In fact, using a characterization as algebra of matrix coefficients (cf. [APW], Appendix), $F_q[M_n]$ embeds into $U_q(\mathfrak{gl}_n)^*$ as a dense subalgebra (w.r.t. the weak topology). Hence the natural evaluation pairing $\big(\langle f, u\rangle \mapsto f(u)\big)$ restricts to a perfect (= non-degenerate) Hopf pairing $\langle\ ,\ \rangle : F_q[M_n] \times U_q(\mathfrak{gl}_n) \longrightarrow \mathbb{Q}(q)$ (see e.g. [No], §1.3, for details). Then, by the analysis in [DL], §4 (adapted to M_n), one has $\mathfrak{F}_q[M_n] = \Big\{ f \in F_q[M_n] \,\Big|\, \big\langle f, \mathfrak{U}_q(\mathfrak{gl}_n)\big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \Big\}$.

All this gives the idea, like in [Ga1], §4.3, for semisimple algebraic groups, to *define*

$$\mathcal{F}_q[M_n] \;:=\; \Big\{ f \in F_q[M_n] \,\Big|\, \big\langle f, \mathcal{U}_q(\mathfrak{gl}_n)\big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \Big\} \quad .$$

The arguments in [Ga1], Proposition 5.12, prove also that $\mathbb{Q} \cdot \mathcal{F}_q[M_n]$ is a $\mathbb{Q}\big[q,q^{-1}\big]$–integral form of $F_q[M_n]$. Moreover, the analysis therein together with [Ga2], §6, proves that $\mathbb{Q} \cdot \mathcal{F}_q[M_n]$ is a *quantization* of $U\big(\mathfrak{gl}_n{}^*\big)$, i.e. $\big(\mathbb{Q} \cdot \mathcal{F}_q[M_n]\big)_1 \cong U\big(\mathfrak{gl}_n{}^*\big)$ as co-Poisson bialgebras (taking on left-hand side the co-Poisson structure inherited from $\mathbb{Q} \cdot \mathcal{F}_q[M_n]$).

Finally, let ℓ, ε and $\mathbb{Z}_\varepsilon$ be as in §1.4, and set $\mathcal{F}_\varepsilon[M_n] := \big(\mathcal{F}_q[M_n]\big)_\varepsilon$. Then again the arguments in [Ga1], §7, show that there is an epimorphism

$$\mathcal{F}r_{M_n}^{\mathbb{Q}_\varepsilon} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathcal{F}_\varepsilon[M_n] \relbar\joinrel\twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{gl}_n^{\,*}\big) \;=\; \mathbb{Q}_\varepsilon \otimes_{\mathbb{Q}} U\big(\mathfrak{gl}_n^{\,*}\big) \tag{2.1}$$

of $\mathbb{Q}_\varepsilon$–bialgebras, which will be called (again) the *quantum Frobenius morphism* for M_n .

All the previous construction can be repeated for GL_n instead of M_n (see again [No], §1.1, and references therein). One defines $F_q[GL_n] := \big(F_q[M_n]\big)\big[D_q{}^{-1}\big]$, the extension of $F_q[M_n]$ by a formal inverse to D_q ; thus the presentation of $F_q[M_n]$ induces one of $F_q[GL_n]$ as well. This $F_q[GL_n]$ is a Hopf algebra: the coproduct and counit on the $t_{i,j}$'s are given by the formulæ for $F_q[M_n]$ (which is then a sub-bialgebra) plus the formulæ saying that $D_q{}^{-1}$ is group-like, and the antipode by $S(t_{i,j}) = (-1)^{i+j}\, det_q\Big(\big(t_{k,\ell}\big)_{k,\ell=1,\dots,n}^{k\neq i;\ell\neq j}\Big)\, D_q^{-1}$, $S\big(D_q{}^{-1}\big) = D_q$.

We define $\mathfrak{F}_q[GL_n]$ to be the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $F_q[GL_n]$ generated by $t_{i,j}$ (for all i, j) and $D_q{}^{-1}$. This is a $\mathbb{Z}\big[q,q^{-1}\big]$*–integral form of* $F_q[GL_n]$, as a Hopf algebra, and it is a *quantization* of $F\big[(GL_n)_{\mathbb{Z}}\big]$, with $t_{i,j}\big|_{q=1} \cong \bar{t}_{i,j}$ (for all i, j) and $D_q^{-1}\big|_{q=1} \cong D^{-1}$.

It is known (cf. [Ga4], Theorem 2.1*(a)*, and references therein), that $\mathfrak{F}_q[GL_n]$ is free over $\mathbb{Z}\big[q,q^{-1}\big]$, with (PBW-like) basis the set of ordered (w.r.t. any total order) monomials

$$B_{GL_n} := \Big\{ \prod_{i<j} t_{i,j}^{\tau_{i,j}} \prod_{k=1}^{n} t_{k,k}^{\tau_{k,k}} \prod_{i>j} t_{i,j}^{\tau_{i,j}}\, D_q^{-m} \,\Big|\, \tau_{i,j}, m \in \mathbb{N}\ \forall\, i,j\,;\ \min\big(\{\tau_{i,i}\}_{1\leq i\leq n} \cup \{m\}\big) = 0 \Big\}$$

The perfect Hopf pairing $\langle\ ,\ \rangle : F_q[M_n] \times U_q(\mathfrak{gl}_n) \longrightarrow \mathbb{Q}(q)$ extends (on left-hand side) to $F_q[GL_n]$; then one deduces that $\mathfrak{F}_q[GL_n] = \Big\{ f \in F_q[GL_n] \,\Big|\, \big\langle f, \mathfrak{U}_q(\mathfrak{gl}_n)\big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \Big\}$ (just like for $F_q[M_n]$). This gives the idea — as in [Ga1], §4.3 — to *define*

$$\mathcal{F}_q[GL_n] \;:=\; \Big\{ f \in F_q[GL_n] \,\Big|\, \big\langle f, \mathcal{U}_q(\mathfrak{gl}_n)\big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \Big\} \quad .$$

Again, one proves that $\mathbb{Q} \cdot \mathcal{F}_q[GL_n]$ is a $\mathbb{Q}\big[q,q^{-1}\big]$–integral form of $F_q[GL_n]$ (as a Hopf algebra), and also that $\mathbb{Q} \cdot \mathcal{F}_q[GL_n]$ is a *quantization* of $U\big(\mathfrak{gl}_n{}^*\big)$.

Finally, for ℓ and ε as in §1.4, let $\mathfrak{F}_\varepsilon[GL_n] := \big(\mathfrak{F}_q[GL_n]\big)_\varepsilon$ and $\mathcal{F}_\varepsilon[GL_n] := \big(\mathcal{F}_q[GL_n]\big)_\varepsilon$. Then there are two *quantum Frobenius morphisms* for GL_n — both extending the corresponding ones for M_n — namely $\mathfrak{Fr}_{GL_n}^{\mathbb{Z}} : F\big[(GL_n)_{\mathbb{Z}}\big] \cong \mathfrak{F}_1[GL_n] \lhook\joinrel\longrightarrow \mathfrak{F}_\varepsilon[GL_n]$, a Hopf algebra monomorphism (given by $\bar{t}_{i,j} \mapsto t_{i,j}^{\ell}\big|_{q=\varepsilon}$, $D^{-1} \mapsto D_q^{-\ell}$), and a Hopf algebra epimorphism

$$\mathcal{F}r_{GL_n}^{\mathbb{Q}_\varepsilon} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathcal{F}_\varepsilon[GL_n] \relbar\joinrel\twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{F}_1[GL_n] \cong \mathbb{Q}_\varepsilon \otimes_{\mathbb{Q}} U\big(\mathfrak{gl}_n{}^*\big) \quad . \tag{2.2}$$

As to SL_n (see [*loc. cit.*]), one defines $F_q[SL_n] := F_q[GL_n]\big/\big(D_q - 1\big) \cong F_q[M_n]\big/\big(D_q - 1\big)$; here $\big(D_q - 1\big)$ is the two-sided ideal of $F_q[GL_n]$ or of $F_q[M_n]$ generated by the central element $D_q - 1$: it is clearly a Hopf ideal, so $F_q[SL_n]$ is a Hopf algebra on its own. Explicitly, $F_q[SL_n]$ admits the same presentation as $F_q[M_n]$ or $F_q[GL_n]$ but with the additional relation $D_q = 1$ (in either case); its Hopf structure is given by the same formulæ, but for setting $D_q^{-1} = 1$.

Like before, we define $\mathfrak{F}_q[SL_n]$ as the unital $\mathbb{Z}\big[q, q^{-1}\big]$–subalgebra of $F_q[SL_n]$ generated by the $t_{i,j}$'s: it has *the same presentation* as $F_q[SL_n]$ *but over* $\mathbb{Z}\big[q, q^{-1}\big]$, and is a Hopf subalgebra, hence a $\mathbb{Z}\big[q, q^{-1}\big]$–integral form of $F_q[SL_n]$. In addition we have isomorphisms $\mathfrak{F}_q[SL_n] \cong \mathfrak{F}_q[GL_n]\big/\big(D_q - 1\big) \cong \mathfrak{F}_q[M_n]\big/\big(D_q - 1\big)$ of Hopf $\mathbb{Z}\big[q, q^{-1}\big]$–algebras, and $\mathfrak{F}_q[SL_n]$ is a *quantization* of $F\big[(SL_n)_{\mathbb{Z}}\big]$, i.e. $\mathfrak{F}_1[SL_n] := \big(\mathfrak{F}_q[SL_n]\big)_1 \cong F\big[(SL_n)_{\mathbb{Z}}\big]$. It is proved in [Ga4], Theorem 2.1*(a)*, that $\mathfrak{F}_q[SL_n]$ is free over $\mathbb{Z}\big[q, q^{-1}\big]$ with (PBW-like) basis the set

$$B_{SL_n} := \Big\{ \prod_{i<j} t_{i,j}^{\tau_{i,j}} \prod_{k=1}^{n} t_{k,k}^{\tau_{k,k}} \prod_{i>j} t_{i,j}^{\tau_{i,j}} \;\Big|\; \tau_{i,j} \in \mathbb{N} \;\forall\, i, j \,;\, \min\{\tau_{k,k}\}_{k=1,\dots,n} = 0 \Big\}$$

of ordered monomials (w.r.t. any total order).

Like for $F_q[M_n]$ and $F_q[GL_n]$, also $F_q[SL_n]$ can be naturally embedded as a dense subalgebra of $U_q\big(\mathfrak{sl}_n^{\,*}\big) := U_q(\mathfrak{sl}_n)^*$. Then the natural evaluation pairing between $U_q(\mathfrak{sl}_n)^*$ and $U_q(\mathfrak{sl}_n)$ restricts to a perfect Hopf pairing $\langle\ ,\ \rangle : F_q[SL_n] \times U_q(\mathfrak{sl}_n) \longrightarrow \mathbb{Q}(q)$; then (like for $F_q[M_n]$ and $F_q[GL_n]$) one finds that $\mathfrak{F}_q[SL_n] = \Big\{ f \in F_q[SL_n] \;\Big|\; \big\langle f, \mathfrak{U}_q(\mathfrak{sl}_n) \big\rangle \subseteq \mathbb{Z}\big[q, q^{-1}\big] \Big\}$. As in [Ga1], §4.3, this leads us to *define*

$$\mathcal{F}_q[SL_n] := \Big\{ f \in F_q[SL_n] \;\Big|\; \big\langle f, \mathcal{U}_q(\mathfrak{sl}_n) \big\rangle \subseteq \mathbb{Z}\big[q, q^{-1}\big] \Big\} \quad .$$

Then $\mathbb{Q} \cdot \mathcal{F}_q[SL_n]$ is a $\mathbb{Q}\big[q, q^{-1}\big]$–integral form of $F_q[SL_n]$ (as a Hopf algebra), and it is a *quantization* of $U\big(\mathfrak{sl}_n^{\,*}\big)$. Also, a bialgebra epimorphism $\mathcal{F}_q[M_n] \twoheadrightarrow \mathcal{F}_q[SL_n]$ and a Hopf algebra epimorphism $\mathcal{F}_q[GL_n] \twoheadrightarrow \mathcal{F}_q[SL_n]$ exist, both dual to $\mathcal{U}_q(\mathfrak{sl}_n) \hookrightarrow \mathcal{U}_q(\mathfrak{gl}_n)$.

Finally, for any root of 1 of odd order ℓ , say ε , there are two *quantum Frobenius morphisms* for SL_n : a Hopf algebra monomorphism $\mathfrak{Fr}_{SL_n}^{\mathbb{Z}} : F\big[(SL_n)_{\mathbb{Z}}\big] \cong \mathfrak{F}_1[SL_n] \hookrightarrow \mathfrak{F}_\varepsilon[SL_n]$ (given by $\bar{t}_{i,j} \mapsto t_{i,j}^{\ell}\big|_{q=\varepsilon}$), and a Hopf algebra epimorphism

$$\mathcal{F}r_{SL_n}^{\mathbb{Q}_\varepsilon} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathcal{F}_\varepsilon[SL_n] \twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{F}_1[SL_n] \cong \mathbb{Q}_\varepsilon \otimes_{\mathbb{Q}} U\big(\mathfrak{sl}_n^{\,*}\big) \quad . \tag{2.3}$$

2.5 $U_q\big(\mathfrak{gl}_n^*\big)$, $U_q\big(\mathfrak{sl}_n^*\big)$, integral forms and quantum Frobenius morphisms. It is shown in [Ga1], §§5–6, that the linear dual $U_q(\mathfrak{sl}_n)^*$ of $U_q(\mathfrak{sl}_n)$ can be seen again as a quantum group on its own: namely, we can set $U_q\big(\mathfrak{sl}_n^*\big) := U_q(\mathfrak{sl}_n)^*$, such a notation being motivated by the fact that $U_q\big(\mathfrak{sl}_n^*\big)$ stands for the Lie bialgebra $\mathfrak{sl}_n^*$ exactly like $U_q(\mathfrak{sl}_n)$ stands for $\mathfrak{sl}_n$. Indeed, $U_q\big(\mathfrak{sl}_n^*\big)$ bears a natural structure of *topological* Hopf $\mathbb{Q}(q)$–algebra, and has two integral forms $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$ and $\mathcal{U}_q\big(\mathfrak{sl}_n^*\big)$ which play for $U_q\big(\mathfrak{sl}_n^*\big)$ the same rôle as $\mathfrak{U}_q(\mathfrak{sl}_n)$ and $\mathcal{U}_q(\mathfrak{sl}_n)$ for $U_q(\mathfrak{sl}_n)$. We shall here mainly consider the first integral form (and less the second), after [Ga1], §§5–6, and apply the same procedure to construct $U_q\big(\mathfrak{gl}_n^{\,*}\big)$ and $\mathfrak{U}_q\big(\mathfrak{gl}_n^{\,*}\big)$ too.

First we define $\mathbf{H}_q^g$ as the unital associative $\mathbb{Q}(q)$–algebra with generators $F_1, \dots, F_{n-1}$, $\Lambda_1^{\pm 1}, \dots, \Lambda_{n-1}^{\pm 1}, \Lambda_n^{\pm 1}, E_1, \dots, E_{n-1}$ and relations

$$\begin{aligned}
\Lambda_i \Lambda_i^{-1} = 1 = \Lambda_i^{-1} \Lambda_i , \qquad & \Lambda_i^{\pm 1} \Lambda_j^{\pm 1} = \Lambda_j^{\pm 1} \Lambda_i^{\pm 1} & \forall\, i, j \\
\Lambda_i F_j \Lambda_i^{-1} = q^{\delta_{i,j} - \delta_{i,j+1}} F_j , \qquad & \Lambda_i E_j \Lambda_i^{-1} = q^{\delta_{i,j} - \delta_{i,j+1}} E_j & \forall\, i, j \\
E_i F_j - F_j E_i = 0 & & \forall\, i, j
\end{aligned}$$

$$E_i\,E_j \;=\; E_j\,E_i\,, \qquad F_i\,F_j \;=\; F_j\,F_i \qquad\qquad \forall\;\, i,j: |i-j|>1$$
$$E_i^2\,E_j \;-\; \big(q+q^{-1}\big)\;E_i\,E_j\,E_i \;+\; E_j\,E_i^2 \;=\; 0 \qquad\qquad \forall\;\, i,j: |i-j|=1$$
$$F_i^2\,F_j \;-\; \big(q+q^{-1}\big)\;F_i\,F_j\,F_i \;+\; F_j\,F_i^2 \;=\; 0 \qquad\qquad \forall\;\, i,j: |i-j|=1\,.$$

Consider also the elements $L_h^{\pm 1} := \prod_{k=1}^h \Lambda_k^{\pm 1}$ (for all $h=1,\dots,n$), and denote $\mathbf{H}_q^s$ the unital $\mathbb{Q}(q)$–subalgebra[1] of $\mathbf{H}_q^g$ generated by $\big\{F_i\,,L_i^{\pm 1},E_i\big\}_{i=1,\dots,n-1}$. From the presentation of $\mathbf{H}_q^g$ one deduces one of $\mathbf{H}_q^s$ too. Note also that both $\mathbf{H}_q^g$ and $\mathbf{H}_q^s$ contain quantum root vectors $E_{i,j}$ and $F_{j,i}$ (for all $i<j$) as in §2.2, and the sets of PBW-like ordered monomials $B_*^g := \Big\{ \prod_{i<j} E_{i,j}^{\eta_{i,j}} \prod_{k=1}^n \Lambda_k^{\lambda_k} \prod_{i<j} F_{j,i}^{\varphi_{j,i}} \,\Big|\, \varphi_{j,i}, \lambda_k, \eta_{i,j} \in \mathbb{N} \Big\}$ for $\mathbf{H}_q^g$ and $B_*^s := \Big\{ \prod_{i<j} E_{i,j}^{\eta_{i,j}} \prod_{k=1}^{n-1} L_k^{l_k} \prod_{i<j} F_{j,i}^{\varphi_{j,i}} \,\Big|\, \varphi_{j,i}, l_k, \eta_{i,j} \in \mathbb{N} \Big\}$ for $\mathbf{H}_q^s$ are bases over $\mathbb{Q}(q)$.

One defines $U_q\big(\mathfrak{sl}_n^*\big)$ as a suitable completion of $\mathbf{H}_q^s$, so that $U_q\big(\mathfrak{sl}_n^*\big)$ is a topological $\mathbb{Q}(q)$–algebra topologically generated by $\mathbf{H}_q^s$, and B_*^s is a $\mathbb{Q}(q)$–basis of $U_q\big(\mathfrak{sl}_n^*\big)$ in topological sense. Then $U_q\big(\mathfrak{sl}_n^*\big)$ is also a topological Hopf $\mathbb{Q}(q)$–algebra (see [Ga1], §6). The same construction makes sense with $\mathbf{H}_q^g$ instead of $\mathbf{H}_q^s$ and yields the definition of $U_q\big(\mathfrak{gl}_n^*\big)$, a topological Hopf algebra with B_*^s as (topological) $\mathbb{Q}(q)$–basis. By construction $\mathbf{H}_q^s$ is a subalgebra of $\mathbf{H}_q^g$ but also a quotient via $\mathbf{H}_q^g\Big/\big(L_n-1\big) \cong \mathbf{H}_q^s$. Similarly $U_q\big(\mathfrak{sl}_n^*\big)$ is a topological Hopf subalgebra of $U_q\big(\mathfrak{gl}_n^*\big)$ but also a quotient via $U_q\big(\mathfrak{gl}_n^*\big)\Big/\big(L_n-1\big) \cong U_q\big(\mathfrak{sl}_n^*\big)_q$.

Let $\langle\ ,\ \rangle : U_q\big(\mathfrak{sl}_n^*\big) \times U_q(\mathfrak{sl}_n) \longrightarrow \mathbb{Q}(q)$ be the natural evaluation pairing, given by $\langle f,u\rangle := f(u)$ for all $u \in U_q(\mathfrak{sl}_n)$, $f \in U_q\big(\mathfrak{sl}_n^*\big) := U_q(\mathfrak{sl}_n)^*$. Using it, one defines (see [Ga1], §6) $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big) := \Big\{ \eta \in U_q\big(\mathfrak{sl}_n^*\big) \,\Big|\, \big\langle \eta\,, \mathcal{U}_q(\mathfrak{sl}_n)\big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \textit{ and } \Phi(\gamma) \textit{ holds} \Big\}$, where $\Phi(\gamma)$ is a suitable "growth condition" for elements in $U_q\big(\mathfrak{sl}_n^*\big)$. One proves that $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$ is a $\mathbb{Z}\big[q,q^{-1}\big]$–integral form (as a topological Hopf subalgebra) of $U_q\big(\mathfrak{sl}_n^*\big)$. To describe $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$, let $\mathfrak{H}_q^s$ be the $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbf{H}_q^s$ generated by all $F_i^{(m)}$'s, $E_i^{(m)}$'s, $L_i^{\pm 1}$'s and $\binom{L_i\,;\,c}{n}$'s ($m \in \mathbb{N}$, $c \in \mathbb{Z}$, $1 \leq i < n$): then $\mathfrak{B}_*^s := \Big\{ \prod_{i<j} E_{i,j}^{(\eta_{i,j})} \prod_{k=1}^{n-1} \binom{L_k\,;\,0}{l_k} L_k^{-Ent(l_k/2)} \prod_{i<j} F_{j,i}^{(\varphi_{j,i})} \,\Big|\, \eta_{i,j}, l_k, \varphi_{j,i} \in \mathbb{N} \Big\}$ is a basis of $\mathfrak{H}_q^s$ over $\mathbb{Z}\big[q,q^{-1}\big]$, and $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$ is the topological closure of $\mathfrak{H}_q^s$, so that $\mathfrak{B}_*^s$ is a topological $\mathbb{Z}\big[q,q^{-1}\big]$–basis of $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$.

Similarly, starting from the pairing $\langle\ ,\ \rangle : U_q\big(\mathfrak{gl}_n^*\big) \times U_q(\mathfrak{gl}_n) \longrightarrow \mathbb{Q}(q)$ one can define $\mathfrak{U}_q\big(\mathfrak{gl}_n^*\big) := \Big\{ \eta \in U_q\big(\mathfrak{gl}_n^*\big) \,\Big|\, \big\langle \eta\,, \mathcal{U}_q(\mathfrak{gl}_n)\big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \textit{ and } \Psi(\gamma) \textit{ holds} \Big\}$, where $\Psi(\gamma)$ is a suitable "growth condition" in $U_q\big(\mathfrak{gl}_n^*\big)$. Then $\mathfrak{U}_q\big(\mathfrak{gl}_n^*\big)$ is a (topological) $\mathbb{Z}\big[q,q^{-1}\big]$–integral form of $U_q\big(\mathfrak{gl}_n^*\big)$, which can be described explicitly: everything goes like for $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$ but replacing the symbols L_k with the Λ_h's ($h=1,\dots,n$) and the $\mathbb{Z}\big[q,q^{-1}\big]$–basis $\mathfrak{B}_*^s$ with $\mathfrak{B}_*^g := \Big\{ \prod_{i<j} E_{i,j}^{(\eta_{i,j})} \prod_{h=1}^{n} \binom{\Lambda_h\,;\,0}{\lambda_h} \Lambda_h^{-Ent(\lambda_h/2)} \prod_{i<j} F_{j,i}^{(\varphi_{j,i})} \,\Big|\, \eta_{i,j}, \lambda_h, \varphi_{j,i} \in \mathbb{N} \Big\}$. We call $\mathfrak{H}_q^g$ the $\mathbb{Z}\big[q,q^{-1}\big]$–span of the latter, and $\mathfrak{U}_q\big(\mathfrak{gl}_n^*\big)$ is just the completion of $\mathfrak{H}_q^g$. Again by construction $\mathfrak{H}_q^s$ is a subalgebra of $\mathfrak{H}_q^g$ but also a quotient (restricting $\mathbf{H}_q^g\Big/\big(L_n-1\big) \cong \mathbf{H}_q^s$ to $\mathfrak{H}_q^g$), so $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$ is a topological Hopf subalgebra of $\mathfrak{U}_q\big(\mathfrak{gl}_n^*\big)$, and also quotient of the latter.

We can describe $\mathfrak{H}_q^s$ rather explicitly: it is the unital associative $\mathbb{Z}\big[q,q^{-1}\big]$–algebra with generators $F_i^{(m)}$, $E_i^{(m)}$, $L_i^{\pm 1}$, $\binom{L_i\,;\,c}{m}$ (for $m \in \mathbb{N}$, $c \in \mathbb{Z}$, $i=1,\dots,n-1$) and relations

[1] The corresponding notation in [Ga1], §6, is $\mathbf{H}_\varphi^P$, and similarly for integral forms.

$$\prod_{s=1}^{m}(q^s-1)\binom{L_k;c}{m}=\prod_{s=1}^{m}\big(q^{1-s+c}L_k-1\big),\quad L_kL_k^{-1}=1=L_k^{-1}L_k\,,\quad \binom{L_h;c}{m}L_k^{\pm1}=L_k^{\pm1}\binom{L_h;c}{m}$$

relations (1.2) *for all* $X\in\{L_k\}_{k=1,\dots,n-1}$, *relations* (1.1) *for all* $X\in\{F_i,E_i\}_{i=1,\dots,n-1}$

$$L_h^{\pm1}F_k^{(m)}=q^{\pm\delta_{h,k}m}F_k^{(m)}L_h^{\pm1}\,,\qquad E_i^{(r)}F_j^{(s)}=F_j^{(s)}E_i^{(r)}\,,\qquad L_h^{\pm1}E_k^{(m)}=q^{\pm\delta_{h,k}m}E_k^{(m)}L_h^{\pm1}$$

$$\binom{L_h\,;\,c}{t}E_k^{(m)}=E_k^{(m)}\binom{L_h\,;\,c+\delta_{h,k}\,m}{t}\,,\qquad \binom{L_h\,;\,c}{t}F_k^{(m)}=F_k^{(m)}\binom{L_h\,;\,c+\delta_{h,k}\,m}{t}\,.$$

The Hopf structure of $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$ then can be described explicitly, but we do not need it.

Similarly, $\mathfrak{H}_q^g$ is the unital associative $\mathbb{Z}\big[q,q^{-1}\big]$–algebra with generators $F_i^{(m)}$, $E_i^{(m)}$, $\Lambda_k^{\pm1}$, $\binom{\Lambda_k\,;\,c}{m}$ (for $m\in\mathbb{N}$, $c\in\mathbb{Z}$, $i=1,\dots,n-1$, $k=1,\dots,n$) and relations

$$\prod_{s=1}^{m}(q^s-1)\binom{\Lambda_k;c}{m}=\prod_{s=1}^{m}\big(q^{1-s+c}\Lambda_k-1\big),\quad \Lambda_k\Lambda_k^{-1}=1=\Lambda_k^{-1}\Lambda_k\,,\quad \binom{\Lambda_h;c}{m}\Lambda_k^{\pm1}=\Lambda_k^{\pm1}\binom{\Lambda_h;c}{m}$$

relations (1.2) *for all* $X\in\{\Lambda_k\}_{k=1,\dots,n}$, *relations* (1.1) *for all* $X\in\{F_i,E_i\}_{i=1,\dots,n-1}$

$$E_i^{(r)}F_j^{(s)}=F_j^{(s)}E_i^{(r)}\,,\qquad \Lambda_h^{\pm1}Y_k^{(m)}=q^{\pm(\delta_{h,k}-\delta_{h,k+1})m}\,Y_k^{(m)}\Lambda_h^{\pm1}\qquad\forall\ Y\in\{F,E\}$$

$$\binom{\Lambda_h\,;\,c}{t}Y_k^{(m)}=Y_k^{(m)}\binom{\Lambda_h\,;\,c+(\delta_{h,k}-\delta_{h,k+1})\,m}{t}\qquad\forall\ Y\in\{F,E\}\,.$$

The Hopf structure of $\mathfrak{U}_q\big(\mathfrak{gl}_n^*\big)$ can also be given explicitly, yet we do not need that.

$\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$ is a *quantization* of $U_{\mathbb{Z}}\big(\mathfrak{sl}_n^*\big)$, for $\mathfrak{U}_1\big(\mathfrak{sl}_n^*\big):=\big(\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)\big)_1\cong U_{\mathbb{Z}}\big(\mathfrak{sl}_n^*\big)$ as co-Poisson Hopf algebras, with on left-hand side the co-Poisson structure inherited from $\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$. In terms of generators (notation of §1.2) this reads $F_i^{(m)}\Big|_{q=1}\cong \mathrm{f}_i^{(m)}$, $\binom{L_i\,;\,0}{m}\Big|_{q=1}\cong\binom{\mathrm{h}_i}{m}$, $L_i^{\pm1}\Big|_{q=1}\cong 1$, $E_i^{(m)}\Big|_{q=1}\cong \mathrm{e}_i^{(m)}$ for all $i=1,\dots,n-1$ and $m\in\mathbb{N}$. Similarly $\mathfrak{U}_1\big(\mathfrak{gl}_n^*\big):=\big(\mathfrak{U}_q\big(\mathfrak{gl}_n^*\big)\big)_1\cong U_{\mathbb{Z}}\big(\mathfrak{gl}_n^*\big)$ as co-Poisson Hopf algebras, with $F_i^{(m)}\Big|_{q=1}\cong \mathrm{f}_i^{(m)}$, $\binom{\Lambda_h\,;\,0}{m}\Big|_{q=1}\cong\binom{\mathrm{l}_h}{m}$, $\Lambda_h^{\pm1}\Big|_{q=1}\cong 1$, $E_i^{(m)}\Big|_{q=1}\cong \mathrm{e}_i^{(m)}$ for $m\in\mathbb{N}$, $i=1,\dots,n-1$, $h=1,\dots,n$. The map $\mathfrak{U}_q\big(\mathfrak{gl}_n^*\big)\twoheadrightarrow\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$ then is a quantization of the natural epimorphism $\mathfrak{gl}_n^*\twoheadrightarrow\mathfrak{sl}_n^*$.

Finally, let ℓ and ε be as in §1.4. Set $\mathfrak{U}_\varepsilon\big(\mathfrak{sl}_n^*\big):=\big(\mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)\big)_\varepsilon$ and $\mathfrak{H}_\varepsilon^s:=\big(\mathfrak{H}^s\big)_\varepsilon$. Then (cf. [Ga1], §7.7) the embedding $\mathfrak{H}_\varepsilon^s\hookrightarrow\mathfrak{U}_\varepsilon\big(\mathfrak{sl}_n^*\big)$ is an isomorphism, thus $\mathfrak{U}_\varepsilon\big(\mathfrak{sl}_n^*\big)=\mathfrak{H}_\varepsilon^s$. Similarly (with like notation) $\mathfrak{U}_\varepsilon\big(\mathfrak{gl}_n^*\big)=\mathfrak{H}_\varepsilon^g$. Also, there are Hopf algebra epimorphisms

$$\mathfrak{Fr}_{\mathfrak{sl}_n^*}^{\mathbb{Z}_\varepsilon}:\mathfrak{U}_\varepsilon\big(\mathfrak{sl}_n^*\big)=\mathfrak{H}_\varepsilon^s\twoheadrightarrow\mathbb{Z}_\varepsilon\otimes_{\mathbb{Z}}\mathfrak{H}_1^s=\mathbb{Z}_\varepsilon\otimes_{\mathbb{Z}}\mathfrak{U}_1\big(\mathfrak{sl}_n^*\big)\cong\mathbb{Z}_\varepsilon\otimes_{\mathbb{Z}}U_{\mathbb{Z}}\big(\mathfrak{sl}_n^*\big)\tag{2.4}$$

$$\mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}:\mathfrak{U}_\varepsilon\big(\mathfrak{gl}_n^*\big)=\mathfrak{H}_\varepsilon^g\twoheadrightarrow\mathbb{Z}_\varepsilon\otimes_{\mathbb{Z}}\mathfrak{H}_1^g=\mathbb{Z}_\varepsilon\otimes_{\mathbb{Z}}\mathfrak{U}_1\big(\mathfrak{gl}_n^*\big)\cong\mathbb{Z}_\varepsilon\otimes_{\mathbb{Z}}U_{\mathbb{Z}}\big(\mathfrak{gl}_n^*\big)\tag{2.5}$$

defined by $X_i^{(s)}\Big|_{q=\varepsilon}\mapsto \mathrm{x}_i^{(s/\ell)}$, $\binom{Y_j\,;\,0}{s}\Big|_{q=\varepsilon}\mapsto\binom{\mathrm{k}_j}{s/\ell}$ if $\ell\,\big|\,s$, $X_i^{(s)}\Big|_{q=\varepsilon}\mapsto 0$, $\binom{Y_j\,;\,0}{s}\Big|_{q=\varepsilon}\mapsto 0$ if $\ell\!\not|\,s$, and $Y_j^{\pm1}\Big|_{q=1}\mapsto 1$, with $(X,\mathrm{x})\in\big\{(F,\mathrm{f}),(E,\mathrm{e})\big\}$, and $(Y,\mathrm{k}):=(L,\mathrm{h})$ in the $\mathfrak{sl}_n$ case, $(Y,\mathrm{k}):=(\Lambda,\mathrm{g})$ in the $\mathfrak{gl}_n$ case. These are *quantum Frobenius morphism* for $\mathfrak{sl}_n^*$ and $\mathfrak{gl}_n^*$.

The epimorphism $\pi_q : \mathfrak{U}_q\big(\mathfrak{gl}_n^*\big) \twoheadrightarrow \mathfrak{U}_q\big(\mathfrak{sl}_n^*\big)$ of topological Hopf $\mathbb{Z}\big[q,q^{-1}\big]$–algebras mentioned above is compatible with the quantum Frobenius morphisms, that is to say $\pi_1 \circ \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon} = \mathfrak{Fr}_{\mathfrak{sl}_n^*}^{\mathbb{Z}_\varepsilon} \circ \pi_\varepsilon$ (where π_1 and π_ε have the obvious meaning).

We remark that $\mathfrak{H}_q^s = \big\{ \eta \in \mathbf{H}_q^s \,\big|\, \big\langle \eta \, , \mathcal{U}_q(\mathfrak{sl}_n) \big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \big\}$, and similarly for $\mathfrak{H}_q^g$. Then one can consider ([Ga1], §§6–7) $\mathcal{H}_q^s := \big\{ \eta \in \mathbf{H}_q^s \,\big|\, \big\langle \eta \, , \mathfrak{U}_q(\mathfrak{sl}_n) \big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \big\}$. This is the $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbf{H}_q^s$ generated by $\big\{ \overline{E}_{i,j} \, , L_h^{\pm 1} \, , \overline{F}_{j,i} \big\}_{1 \leq i<j \leq n}^{1 \leq h \leq n-1}$, with a PBW-like basis over $\mathbb{Z}\big[q,q^{-1}\big]$ given by the ordered monomials in these generators. The unrestricted $\mathbb{Z}\big[q,q^{-1}\big]$–integral form $\mathcal{U}_q\big(\mathfrak{sl}_n^*\big)$ is the (suitable) topological completion of $\mathcal{H}_q^s$ inside $U_q\big(\mathfrak{sl}_n^*\big)$; one also has another *quantum Frobenius morphism* $\mathcal{F}r_{\mathfrak{gl}_n^*}^{\mathbb{Z}} : \mathcal{U}_1(\mathfrak{g}^*) \hookrightarrow \mathcal{U}_\varepsilon(\mathfrak{g}^*)$, defined on generators as an "ℓ-th power operation". Similarly holds for $\mathcal{H}_q^g := \big\{ \eta \in \mathbf{H}_q^g \,\big|\, \big\langle \eta \, , \mathfrak{U}_q(\mathfrak{gl}_n) \big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big] \big\}$ and $\mathcal{U}_q\big(\mathfrak{gl}_n^*\big)$, with Λ_k's instead of L_h's, and a suitable quantum Frobenius $\mathcal{F}r_{\mathfrak{sl}_n^*}^{\mathbb{Z}}$.

<u>*Warnings:*</u> *(a)* In [Ga1] integral forms are over $\mathbb{Q}\big[q,q^{-1}\big]$, but now we work over $\mathbb{Z}\big[q,q^{-1}\big]$. Some results here for $\mathcal{F}_q[SL_n]$ are improvements of those for $\mathbb{Q}\big[q,q^{-1}\big] \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{F}_q[SL_n]$ in [Ga1], and similarly for GL_n (see §§4.8–10). For $U_q(\mathfrak{g}^*)$ too, the results in [Ga1] are over $\mathbb{Q}\big[q,q^{-1}\big]$: yet the arguments therein also apply over $\mathbb{Z}\big[q,q^{-1}\big]$, giving the results of §2.5.

(b) In this work we deal with integral forms $\mathcal{F}_q[X]$ of $F_q[X]$ (for $X \in \big\{M_n \, , GL_n \, , SL_n\big\}$), passing through integral forms $\mathfrak{U}_q\big(\mathfrak{g}^*\big)$ of $U_q(\mathfrak{g}^*)$ (for $\mathfrak{g} \in \big\{\mathfrak{gl}_n \, , \mathfrak{sl}_n\big\}$). The same methods and ideas apply also to the forms $\mathfrak{F}_q[X]$, passing through the forms $\mathcal{U}_q\big(\mathfrak{g}^*\big)$: this yields new proofs of the results for them and their quantum Frobenius morphisms mentioned in §§2.4.

§ 3 Embedding quantum function algebras into dual quantum groups.

3.1 Construction of $U_q(\mathfrak{g}^*)$ and the natural evaluation pairing. The description of $U_q(\mathfrak{g}^*) := U_q(\mathfrak{g})^*$, for $\mathfrak{g} \in \big\{\mathfrak{gl}_n, \mathfrak{sl}_n\big\}$, given above comes from the following construction (cf. [Ga1], §§5–6). Let $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ be the *quantum Borel subalgebras* of $U_q(\mathfrak{g})$: namely, $U_q(\mathfrak{b}_+)$, resp. $U_q(\mathfrak{b}_-)$, is the subalgebra of $U_q(\mathfrak{g})$ obtained discarding the F_i's, resp. the E_i's, from the set of generators; similarly one defines integral forms $\mathfrak{U}_q(\mathfrak{b}_\pm)$ and $\mathcal{U}_q(\mathfrak{b}_\pm)$ of $U_q(\mathfrak{b}_\pm)$ as well. Then $U_q(\mathfrak{g})$ as a coalgebra is a quotient of $U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)$. Therefore $U_q(\mathfrak{g}^*) := U_q(\mathfrak{g})^*$ is a subalgebra of $\big(U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)\big)^* \cong U_q(\mathfrak{b}_+)^* \widehat{\otimes}\, U_q(\mathfrak{b}_-)^*$, where $\widehat{\otimes}$ denotes topological tensor product (completion of $\otimes$ w.r.t. weak topology). In [Ga1], §5, it was observed that $U_q(\mathfrak{b}_\pm)^*$ as an algebra is the completion of $U_q(\mathfrak{b}_\mp)$, so that $\big(U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)\big)^* \cong U_q(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q(\mathfrak{b}_+)$. Then $U_q(\mathfrak{g}^*)$ can be described as topologically generated by elements $F_i \otimes 1$, $T_k^{\mp 1} \otimes T_k^{\pm 1}$, $1 \otimes E_i \in U_q(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q(\mathfrak{b}_+) \cong \big(U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)\big)^*$ (with $T_k^{\pm 1} \in \big\{G_j^{\pm 1}\big\}_{1 \leq j \leq n}$ if $\mathfrak{g} = \mathfrak{gl}_n$ and $T_k^{\pm 1} \in \big\{K_h^{\pm 1}\big\}_{1 \leq h \leq n-1}$ if $\mathfrak{g} = \mathfrak{sl}_n$).

Now we observe that $U_q(\mathfrak{b}_+) \cong U_q(\mathfrak{b}_-)$ as algebras, via $E_i \mapsto F_i$, $T_k^{\pm 1} \mapsto T_k^{\mp 1}$. Then we have also $\big(U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)\big)^* \cong U_q(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q(\mathfrak{b}_+) \cong U_q(\mathfrak{b}_+) \,\widehat{\otimes}\, U_q(\mathfrak{b}_-)$. Thus $U_q(\mathfrak{g}^*) := U_q(\mathfrak{g})^*$ can be described as topologically generated by elements $E_i \otimes 1$, $T_k^{\pm 1} \otimes T_k^{\mp 1}$, $1 \otimes F_i \in U_q(\mathfrak{b}_+) \,\widehat{\otimes}\, U_q(\mathfrak{b}_-)$, which by abuse of notation we shall call E_i, $T_k^{\pm 1}$ and F_i. The relations among them are those for E_i, $T_k^{\pm 1}$ (in $\big\{ G_j^{\pm 1} \big\}_{1 \leq j \leq n}$ or in $\big\{ K_h^{\pm 1} \big\}_{1 \leq h \leq n-1}$ according to whether $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{g} = \mathfrak{sl}_n$) and F_i in §2.5. Using this description for $U_q(\mathfrak{g}^*)$, the evaluation pairing $\langle \ , \ \rangle : U_q(\mathfrak{g}^*) \times U_q(\mathfrak{g}) \longrightarrow \mathbb{Q}(q)$ is uniquely determined by its values on PBW bases. It is described explicitly in [Ga1] (§2.4 and §6), but before identifying $U_q(\mathfrak{b}_+) \cong U_q(\mathfrak{b}_-)$ and $U_q(\mathfrak{b}_-) \cong U_q(\mathfrak{b}_+)$. Once we adopt these last identifications, which map $E_{i,j} \mapsto (-q)^{i-j+1} F_{j,i}$ and $F_{j,i} \mapsto (-q)^{j-i-1} E_{i,j}$ (for $i<j$), the results in [Ga1], §6, eventually read

$$\left\langle \prod_{i<j} E_{i,j}^{(\eta_{i,j})} \prod_{r=1}^{l} \binom{Y_r;0}{\chi_r} Y_r^{-Ent(\chi_r/2)} \prod_{i<j} F_{j,i}^{(\varphi_{j,i})} \, , \, \prod_{i<j} \overline{E}_{i,j}^{\,e_{i,j}} \prod_{s=1}^{l} Z_s^{z_s} \prod_{i<j} \overline{F}_{j,i}^{\,f_{j,i}} \right\rangle = \\ = \prod_{i<j} (-1)^{\eta_{i,j}} \, \delta_{f_{j,i},\varphi_{j,i}} \, \delta_{e_{i,j},\eta_{i,j}} \, (-q)^{(j-i-1)(\varphi_{j,i}-\eta_{i,j})} \prod_{t=1}^{l} \binom{z_t}{\chi_t}_q q^{-z_t Ent(\chi_t/2)} \tag{3.1}$$

where $\big(Y,Z,l\big)$ stands for $\big(L,K,n-1\big)$ in the $\mathfrak{sl}_n$ case and for $\big(\Lambda,G,n\big)$ in the $\mathfrak{gl}_n$ case.

3.2 The embedding $\xi \colon F_q[G] \longrightarrow U_q(\mathfrak{g}^*)$ and its restrictions to integral forms. Let $G \in \big\{SL_n, GL_n\big\}$ and $\mathfrak{g} := \mathcal{L}ie(G)$. Definitions embed $F_q[G]$ into $U_q(\mathfrak{g})^* =: U_q(\mathfrak{g}^*)$, via a monomorphism $\xi \colon F_q[G] \longrightarrow U_q(\mathfrak{g}^*)$ of topological Hopf $\mathbb{Q}(q)$–algebras: they also imply $\mathcal{F}_q[G] = \xi^{-1}\big(\mathfrak{U}_q(\mathfrak{g}^*)\big)$, thus ξ restricts to a monomorphism $\widehat{\xi} \colon \mathcal{F}_q[G] \longrightarrow \mathfrak{U}_q(\mathfrak{g}^*)$, and similarly to $\widetilde{\xi} \colon \mathfrak{F}_q[G] \longrightarrow \mathcal{U}_q(\mathfrak{g}^*)$ too. These verify $F_q[G] = \xi^{-1}(\mathbf{H}_q^x)$, $\mathcal{F}_q[G] = \widehat{\xi}^{-1}\big(\mathfrak{H}_q^x\big)$ and $\mathfrak{F}_q[G] = \widetilde{\xi}^{-1}\big(\mathcal{H}_q^x\big)$ (with $x = s$ or $x = g$ according to the type of G). As $F_q[M_n]$ embeds into $F_q[GL_n]$ — and similarly for integer forms — one has also bialgebra embeddings $\xi \colon F_q[M_n] \longrightarrow U_q\big(\mathfrak{gl}_n^*\big)$, $\widehat{\xi} \colon \mathcal{F}_q[M_n] \longrightarrow \mathfrak{U}_q\big(\mathfrak{gl}_n^*\big)$ and $\widetilde{\xi} \colon \mathfrak{F}_q[M_n] \longrightarrow \mathcal{U}_q\big(\mathfrak{gl}_n^*\big)$.

Moreover, by construction $\widehat{\xi}$ and $\widetilde{\xi}$ are compatible with specializations and quantum Frobenius morphisms, i.e. $\Big(\mathrm{id}_{\mathbb{Q}_\varepsilon} \otimes_{\mathbb{Z}} \widehat{\xi}\,\big|_{q=1}\Big) \circ \mathcal{F}r_G^{\mathbb{Q}_\varepsilon} = \Big(\mathrm{id}_{\mathbb{Q}_\varepsilon} \otimes_{\mathbb{Z}_\varepsilon} \mathfrak{Fr}_{\mathfrak{g}^*}^{\mathbb{Z}_\varepsilon}\Big) \circ \Big(\mathrm{id}_{\mathbb{Q}_\varepsilon} \otimes_{\mathbb{Z}_\varepsilon} \widehat{\xi}\,\big|_{q=\varepsilon}\Big)$ for $\widehat{\xi}$, and similarly for $\widetilde{\xi}$ (for any root of unity ε of odd order). Finally, all the embeddings ξ and their restrictions to integral forms are compatible (in the obvious sense) with the epimorphisms $U_q\big(\mathfrak{gl}_n^*\big) \overset{\pi}{\twoheadrightarrow} U_q\big(\mathfrak{sl}_n^*\big)$, $F_q[GL_n] \overset{\pi}{\twoheadrightarrow} F_q[SL_n]$ or $F_q[M_n] \overset{\pi}{\twoheadrightarrow} F_q[SL_n]$ and their restrictions to integral forms, see Proposition 3.7*(b)*. All this follows from definitions (see also [Ga1], §6).

The following gives an explicit description of all the embeddings mentioned above:

Proposition 3.3. *The embeddings* $\widetilde{\xi} \colon \mathfrak{F}_q[M_n] \longrightarrow \mathcal{U}_q\big(\mathfrak{gl}_n^*\big)$, $\widetilde{\xi} \colon \mathfrak{F}_q[GL_n] \longrightarrow \mathcal{U}_q\big(\mathfrak{gl}_n^*\big)$, $\xi \colon F_q[M_n] \longrightarrow U_q\big(\mathfrak{gl}_n^*\big)$ *and* $\xi \colon F_q[GL_n] \longrightarrow U_q\big(\mathfrak{gl}_n^*\big)$ *are uniquely determined by*

$$t_{i,j} \mapsto -(-q)^{j-i-1}\, \overline{E}_{i,j}\, \Lambda_j - (-q)^{j-i-2} \sum_{k=j+1}^{n} \overline{E}_{i,k}\, \Lambda_k\, \overline{F}_{k,j} \qquad \forall\ i<j$$

$$t_{l,l} \mapsto \Lambda_l - q^{-2} \sum_{k=l+1}^{n} \overline{E}_{l,k}\, \Lambda_k\, \overline{F}_{k,l} \qquad \forall\ i=l=j$$

$$t_{i,j} \mapsto +(-q)^{j-i-1}\, \Lambda_i\, \overline{F}_{i,j} - (-q)^{j-i-2} \sum_{k=i+1}^{n} \overline{E}_{i,k}\, \Lambda_k\, \overline{F}_{k,j} \qquad \forall\ i>j$$

With $\Lambda_k^{\pm 1} = L_k^{\pm 1} L_{k-1}^{\mp 1}$, *this describes* $\widetilde{\xi} \colon \mathfrak{F}_q[SL_n] \hookrightarrow \mathcal{U}_q\big(\mathfrak{sl}_n^*\big)$ *and* $\xi \colon F_q[SL_n] \hookrightarrow U_q\big(\mathfrak{sl}_n^*\big)$.

Proof. It is enough to prove the claims for the $\widetilde{\xi}$'s: the rest follows by scalar extension.

Let $G := GL_n$, let $\mathfrak{F}_q[B_+]$ and $\mathfrak{F}_q[B_-]$ be the quotient Hopf algebras of $\mathfrak{F}_q[G]$ obtained factoring out the generators $t_{i,j}$ with $i>j$ and $i<j$ respectively. Let $\pi_+ : \mathfrak{F}_q[G] \twoheadrightarrow \mathfrak{F}_q[B_+]$ and $\pi_- : \mathfrak{F}_q[G] \twoheadrightarrow \mathfrak{F}_q[B_-]$ be the corresponding epimorphisms.

In [DL], §4.3, an embedding $\widetilde{\xi} \colon \mathfrak{F}_q[G] \longrightarrow \mathcal{U}_q(\mathfrak{g}^*)$ is given by the composition

$$\mathfrak{F}_q[G] \xrightarrow{\ \Delta\ } \mathfrak{F}_q[G] \otimes \mathfrak{F}_q[G] \xrightarrow{\ \pi_+ \otimes \pi_-\ } \mathfrak{F}_q[B_+] \otimes \mathfrak{F}_q[B_-] \xrightarrow{\ \vartheta_+ \otimes \vartheta_-\ } \mathcal{U}_q(\mathfrak{b}_-) \otimes \mathcal{U}_q(\mathfrak{b}_+)$$

whose image is contained in the subalgebra of $\mathcal{U}_q(\mathfrak{b}_-)\otimes\mathcal{U}_q(\mathfrak{b}_+)$ which is isomorphic (through the identifications explained in §3.1) to the subalgebra $\mathcal{H}_q^g$ of $\mathcal{U}_q(\mathfrak{g}^*)$ considered at the end of §2.5. The last step in this composition is given by isomorphisms $\vartheta_+ : \mathfrak{F}_q[B_+] \cong \mathcal{U}_q(\mathfrak{b}_-)$ and $\vartheta_- : \mathfrak{F}_q[B_-] \cong \mathcal{U}_q(\mathfrak{b}_+)$. We modify this construction (as sketched in §3.1) by changing the last step with $\mathfrak{F}_q[B_+]\otimes\mathfrak{F}_q[B_-] \xrightarrow{\vartheta_+\otimes\vartheta_-} \mathcal{U}_q(\mathfrak{b}_+)\otimes\mathcal{U}_q(\mathfrak{b}_-)$, where $\vartheta_+ : \mathfrak{F}_q[B_+] \cong \mathcal{U}_q(\mathfrak{b}_+)$ and $\vartheta_- : \mathfrak{F}_q[B_-] \cong \mathcal{U}_q(\mathfrak{b}_-)$ are the algebra isomorphisms given by

$$\begin{aligned} \vartheta_+ :&\quad t_{i,k} \mapsto +(-q)^{k-i}\Lambda_k^{+1}\overline{E}_{i,k} \quad \big(\forall\ i<k\big), \qquad t_{l,l} \mapsto \Lambda_l^{+1} \quad \big(\forall\ l\big) \\ \vartheta_- :&\quad t_{k,j} \mapsto -(-q)^{j-k}\overline{F}_{k,j}\,\Lambda_k^{-1} \quad \big(\forall\ k>j\big), \qquad t_{l,l} \mapsto \Lambda_l^{-1} \quad \big(\forall\ l\big) \end{aligned}$$

Therefore, explicit computation gives (using notation $\delta_{i\sim j}:=1$ or $\delta_{i\sim j}:=0$ according to whether $i\sim j$ or not, for any relation $\sim$)

$$\begin{aligned} \big((\vartheta_+\otimes\vartheta_-)\circ(\pi_+\otimes\pi_-)\circ\Delta\big)(t_{i,j}) \;=\; \delta_{i=j}\,\Lambda_i^{+1}\otimes\Lambda_i^{-1} \,-\, \delta_{i>j}\,(-q)^{j-i}\,\Lambda_i^{+1}\otimes\overline{F}_{i,j}\,\Lambda_i^{-1} \,+ \\ +\,\delta_{i<j}\,(-q)^{j-i}\,\Lambda_j^{+1}\,\overline{E}_{i,j}\otimes\Lambda_j^{-1} \,-\, \textstyle\sum_{k=(i\vee j)+1}^{n}(-q)^{j-i}\,\Lambda_k^{+1}\,\overline{E}_{i,k}\otimes\overline{F}_{k,j}\,\Lambda_k^{-1} \end{aligned}$$

Exploiting the commutation relations in $\mathcal{U}_q(\mathfrak{b}_\pm)$ and using notation $E_{i,\ell}:=E_{i,\ell}\otimes 1$, $\Lambda_\ell^{\pm1}:=\Lambda_\ell^{\pm1}\otimes\Lambda_\ell^{\mp1}$ and $F_{\ell,j}:=1\otimes F_{\ell,j}$ as in §3.1, the above result gives the claim. □

3.4 Remarks: *(a)* The embedding considered in [GR] for $n=2$ is

$$\widetilde{\xi}:\quad t_{1,1}\mapsto \Lambda_1-\overline{F}\Lambda_2\overline{E}\,,\quad t_{1,2}\mapsto -\overline{F}\Lambda_2\,,\quad t_{2,1}\mapsto +\Lambda_2\overline{E}\,,\quad t_{2,2}\mapsto\Lambda_2\,,$$

after the recipe of [DL]. In the sequel we exploit some results from [GR], but these only use intrinsic properties of $F_q[SL_2]$, independent of any embedding of the latter into $U_q\big(\mathfrak{sl}_2\big)$. Thus the present work can be applied (in a non self-contradicting way) for $n=2$ too.

(b) In the following to simplify notation we shall identify $F_q[M_n]$ and $F_q[G]$ with their isomorphic images in $U_q(\mathfrak{g}^*)$ via the embeddings ξ, and the same for integral forms.

3.5 From inclusions to identities. Let $G:=SL_n$. By [DL], §4, or [Ga1], Theorem 5.14, one knows that the monomorphisms $\widetilde{\xi}\colon \mathfrak{F}_q[G]\hookrightarrow\mathcal{U}_q(\mathfrak{g}^*)$, $\xi\colon F_q[G]\hookrightarrow U_q(\mathfrak{g}^*)$ and $\widehat{\xi}\colon\mathcal{F}_q[G]\hookrightarrow\mathfrak{U}_q(\mathfrak{g}^*)$ extend to isomorphisms (of algebras) $\widetilde{\xi}\colon\mathfrak{F}_q[G]\big[\phi^{-1}\big]\xrightarrow{\cong}\mathcal{U}_q(\mathfrak{g}^*)$, $\xi\colon F_q[G]\big[\phi^{-1}\big]\xrightarrow{\cong}U_q(\mathfrak{g}^*)$ and $\widehat{\xi}\colon\mathcal{F}_q[G]\big[\phi^{-1}\big]\xrightarrow{\cong}\mathfrak{U}_q(\mathfrak{g}^*)$, for some $\phi\in\mathfrak{F}_q[G]$.

We shall now improve this result, with an independent approach, and extend it to $G=GL_n$ and M_n. Mimicking [DL], §4, the would-be element $\phi\in\mathfrak{F}_q[GL_n]$ should be given by

$$\phi \;=\; T_1\,T_2\cdots T_{n-1}\,T_n \;=\; (\Lambda_1\cdots\Lambda_n)\,(\Lambda_2\cdots\Lambda_n)\cdots(\Lambda_{n-1}\,\Lambda_n)\,\Lambda_n \;=\; \Lambda_1\,\Lambda_2^2\,\Lambda_3^3\cdots\Lambda_{n-1}^{n-1}\,\Lambda_n^n$$

where $T_l:=\Lambda_l\,\Lambda_{l+1}\cdots\Lambda_{n-1}\,\Lambda_n\in\mathcal{H}_q^g$ (see §2.5) for $l=1,2,\dots,n$, and we identify quantum function algebras with their images in $U_q\big(\mathfrak{gl}_n^*\big)$ as in Proposition 3.3. The key step is

Lemma 3.6. *Let's identify $F_q[M_n]$ with its copy $\xi\big(F_q[M_n]\big)$ inside $U_q\big(\mathfrak{gl}_n^*\big)$. Then*

$$\phi \,:=\, T_1\,T_2\cdots T_{n-1}T_n \;=\; \Lambda_1\Lambda_2^2\cdots\Lambda_{n-1}^{n-1}\Lambda_n^n \;\in\; \mathfrak{F}_q[M_n] \;\;\big(\subset\mathcal{F}_q[M_n]\subset F_q[M_n]\big)$$

Proof. Exploiting carefully the formulæ of Proposition 3.3, one easily proves that

$$T_l \;:=\; \Lambda_l\,\Lambda_{l+1}\cdots\Lambda_{n-1}\,\Lambda_n \in \mathfrak{F}_q[M_n] \qquad \text{for all } l=1,2,\dots,n,n+1 \tag{3.2}$$

by a simple induction, whence the claim follows at once. □

Now using this lemma (and Corollary 3.8 later on) we get our "improved result":

Theorem 3.7. *The embeddings* $\widetilde{\xi}$, $\widehat{\xi}$ *and* ξ *extend to the following identifications:*

$$\begin{array}{llll} (a) & \mathfrak{F}_q[M_n]\big[\phi^{-1}\big] = \mathcal{H}_q^g \, , & \mathcal{F}_q[M_n]\big[\phi^{-1}\big] = \mathfrak{H}_q^g \, , & F_q[M_n]\big[\phi^{-1}\big] = \mathbf{H}_q^g \\ (b) & \mathfrak{F}_q[GL_n]\big[\phi^{-1}\big] = \mathcal{H}_q^g \, , & \mathcal{F}_q[GL_n]\big[\phi^{-1}\big] = \mathfrak{H}_q^g \, , & F_q[GL_n]\big[\phi^{-1}\big] = \mathbf{H}_q^g \\ (c) & \mathfrak{F}_q[SL_n]\big[\phi^{-1}\big] = \mathcal{H}_q^s \, , & \mathcal{F}_q[SL_n]\big[\phi^{-1}\big] = \mathfrak{H}_q^s \, , & F_q[SL_n]\big[\phi^{-1}\big] = \mathbf{H}_q^s \end{array}$$

where in the last row we write again ϕ *for the image of* $\phi \in F_q[M_n]$ *via* $F_q[M_n] \overset{\pi}{\twoheadrightarrow} F_q[SL_n]$.

Proof. *(a)* It is clear that having $T_1, T_2, \dots, T_{n-1}, T_n \in \mathfrak{F}_q[M_n] \subseteq \mathfrak{F}_q[M_n]\big[\phi^{-1}\big]$ together with $\phi^{-1} = T_1^{-1} T_2^{-1} \cdots T_{n-1}^{-1} T_n^{-1} \in \mathfrak{F}_q[M_n]\big[\phi^{-1}\big]$ imply

$$T_l^{-1} = \Lambda_l^{-1} \Lambda_{l+1}^{-1} \cdots \Lambda_{n-1}^{-1} \Lambda_n^{-1} \in \mathfrak{F}_q[M_n]\big[\phi^{-1}\big] \qquad \forall \; l = 1, 2, \dots, n, n+1 \; . \tag{3.3}$$

From formulæ (3.2–3) we easily find that $\Lambda_1^{\pm 1}, \Lambda_2^{\pm 1}, \cdots, \Lambda_{n-1}^{\pm 1}, \Lambda_n^{\pm 1} \in \mathfrak{F}_q[M_n]\big[\phi^{-1}\big]$. Finally, using this and the formulæ for the $t_{i,j}$'s in Proposition 3.3 we eventually find also $\overline{E}_{i,j}$, $\overline{F}_{i,j} \in \mathfrak{F}_q[M_n]\big[\phi^{-1}\big]$ for all $1 \leq i < j \leq n$. All this says that $\mathfrak{F}_q[M_n]\big[\phi^{-1}\big]$ *contains all generators of* $\mathcal{H}_q^g$, so (as $\mathfrak{F}_q[M_n]\big[\phi^{-1}\big] \subseteq \mathcal{H}_q^g$ by construction) we conclude that $\mathfrak{F}_q[M_n]\big[\phi^{-1}\big] = \mathcal{H}_q^g$, q.e.d. By scalar extension it follows that $F_q[M_n]\big[\phi^{-1}\big] = \mathbf{H}_q^g$ too.

Finally, recall that $\mathcal{F}_q[M_n] = F_q[M_n] \cap \mathfrak{H}_q^g$. Then given $f \in \mathfrak{H}_q^g \subset \mathbf{H}_q^g = F_q[M_n]\big[\phi^{-1}\big]$, there is $k \in \mathbb{N}$ such that $f\phi^k \in F_q[M_n] \cap \phi^k \mathcal{F}_q[M_n] \subseteq F_q[M_n] \cap \mathfrak{H}_q^g = \mathcal{F}_q[M_n]$, whence $f \in \mathcal{F}_q[M_n]\big[\phi^{-1}\big]$. Thus $\mathfrak{H}_q^g \subseteq \mathcal{F}_q[M_n]\big[\phi^{-1}\big]$, and the converse is clear. This proves *(a)*.

(b) By construction we have $F_q[M_n] \longhookrightarrow F_q[GL_n] \longhookrightarrow \mathbf{H}_q^g$, which clearly induces $F_q[M_n]\big[\phi^{-1}\big] \longhookrightarrow F_q[GL_n]\big[\phi^{-1}\big] \longhookrightarrow \mathbf{H}_q^g$. But then by *(a)* we get also $F_q[GL_n]\big[\phi^{-1}\big] = \mathbf{H}_q^g$. The same argument works for integral forms as well.

(c) By construction we have $\mathbf{H}_q^s = \pi\big(\mathbf{H}_q^g\big)$ and $F_q[SL_n] = \pi\big(F_q[M_n]\big)$, where in both cases π denotes the natural epimorphism of §3.2. Now, by Corollary 3.8 below — whose proof needs only claim *(a)* above, so we are *not* using a circular argument! — we have

$$F_q[SL_n]\big[\phi^{-1}\big] = \pi\big(F_q[M_n]\big)\big[\pi(\phi)^{-1}\big] = \pi\Big(F_q[M_n]\big[\phi^{-1}\big]\Big) = \pi\big(\mathbf{H}_q^g\big) = \mathbf{H}_q^s \, .$$

The same argument works for integral forms too, thus proving *(c)*. □

As a byproduct of Theorem 3.7*(a)*, we have the following consequence:

Corollary 3.8. *Let's identify* $F_q[M_n]$ *with its copy* $\xi\big(F_q[M_n]\big)$ *inside* $U_q\big(\mathfrak{gl}_n^{\,*}\big)$. *Then*

$$D_q = \Lambda_1 \Lambda_2 \cdots \Lambda_{n-1} \Lambda_n =: T_1$$

Therefore, the various embeddings ξ *of quantum function algebras into dual quantum groups, and their restrictions to integral forms (cf. §3.2), are compatible with the canonical epimorphisms* π, *and their restrictions to integral forms (cf. §3.2), namely* $\pi \circ \xi = \xi \circ \pi$. *In other words,* $F_q[GL_n] \overset{\pi}{\twoheadrightarrow} F_q[SL_n]$ *and* $F_q[M_n] \overset{\pi}{\twoheadrightarrow} F_q[SL_n]$ *are the restrictions of* $U_q\big(\mathfrak{gl}_n^{\,*}\big) \overset{\pi}{\twoheadrightarrow} U_q\big(\mathfrak{sl}_n^{\,*}\big)$, *and similarly for their restrictions to integral forms.*

Proof. Let $Z(A)$ denote the centre of any algebra A. We know that $D_q \in Z\big(F_q[M_n]\big)$; by Theorem 3.7*(a)*, this gives also $D_q \in Z\big(F_q[M_n]\big[\phi^{-1}\big]\big) = Z\big(\mathbf{H}_q^g\big)$. On the other hand, one sees (via direct computation) that $Z\big(\mathbf{H}_q^g\big) = \mathbb{Q}(q)\big[T_1, T_1^{-1}\big]$, where $T_1 := \Lambda_1 \Lambda_2 \cdots \Lambda_{n-1} \Lambda_n$. Therefore $D_q = P\big(T_1, T_1^{-1}\big)$, for some Laurent polynomial P.

Now, since D_q and $T_1^{\pm 1}$ are both group-like, it must necessarily be $D_q = T_1^{\,z}$, for some $z \in \mathbb{Z}$. Finally, the pairing with $U_q(\mathfrak{gl}_n)$ gives $\big\langle D_q, G_i \big\rangle = q$ and $\big\langle T_1^{\,z}, G_i \big\rangle = q^z$ (for all $i = 1, \dots, n$) for any $z \in \mathbb{Z}$. This forces $z = 1$, hence $D_q = T_1$. □

§ 4 The structure of $\mathcal{F}_q[M_n]$, $\mathcal{F}_q[GL_n]$ and $\mathcal{F}_q[SL_n]$, specializations and quantum Frobenius epimorphisms.

4.1 Summary. In this section we present our main results. First, we prove that $\mathcal{F}_q[M_n]$ is a free $\mathbb{Z}\big[q,q^{-1}\big]$–module, providing a basis of Poincaré-Birkhoff-Witt type; we give a presentation by generators and relations, and we show that $\mathcal{F}_q[M_n]$ is a $\mathbb{Z}\big[q,q^{-1}\big]$–subbialgebra, hence a $\mathbb{Z}\big[q,q^{-1}\big]$–integral form, of $F_q[M_n]$. Also, we prove that the specialization of $\mathcal{F}_q[M_n]$ at $q=1$ is just $U_{\mathbb{Z}}\big(\mathfrak{gl}_n^{\,*}\big)$; in particular, it is a *Hopf* $\mathbb{Z}$–algebra, which is true also for any other specialization at a root of 1 but needs other arguments to be proved (see Corollary 5.7). Finally, we show that the quantum Frobenius morphism (2.1) for $\mathfrak{gl}_n^{\,*}$ is defined over $\mathbb{Z}_\varepsilon$, and we describe it in terms of the previously mentioned presentation.

All this has direct consequences regarding $\mathcal{F}_q[GL_n]$ and $\mathcal{F}_q[SL_n]$. Namely, we provide a $\mathbb{Z}\big[q,q^{-1}\big]$–spanning set for $\mathcal{F}_q[GL_n]$, we give a presentation by generators and relations, and we show that $\mathcal{F}_q[GL_n]$ is a $\mathbb{Z}\big[q,q^{-1}\big]$–integral form, as a Hopf algebra, of $F_q[GL_n]$. As a consequence, we prove that the specialization of $\mathcal{F}_q[GL_n]$ at $q=1$ is $U_{\mathbb{Z}}\big(\mathfrak{gl}_n^{\,*}\big)$. Another result is that for any root of unity ε of odd order we have an isomorphism $\mathcal{F}_\varepsilon[GL_n]\cong\mathcal{F}_\varepsilon[M_n]$ as $\mathbb{Z}_\varepsilon$–bialgebras: in particular, $\mathcal{F}_\varepsilon[M_n]$ is a Hopf $\mathbb{Z}_\varepsilon$–algebra. Finally, we show that the quantum Frobenius morphism (2.2) for $\mathfrak{gl}_n^{\,*}$ is defined over $\mathbb{Z}_\varepsilon$, and we describe it in terms of the above mentioned presentation. The same results also are proved, up to details, for $\mathcal{F}_q[SL_n]$ too.

The results about $\mathcal{F}_q[GL_n]$ are proved from those about $\mathcal{F}_q[M_n]$, via a monomorphism $\mathcal{F}_q[M_n]\lhook\joinrel\longrightarrow\mathcal{F}_q[GL_n]$ (over $\mathbb{Z}\big[q,q^{-1}\big]$) induced by the monomorphism $F_q[M_n]\lhook\joinrel\longrightarrow F_q[GL_n]$ (over $\mathbb{Q}(q)$). Instead, all results about $\mathcal{F}_q[SL_n]$ come out of those for $\mathcal{F}_q[M_n]$ via the epimorphism $\mathcal{F}_q[M_n]\relbar\joinrel\twoheadrightarrow\mathcal{F}_q[M_n]\Big/\big(D_q-1\big)\cong\mathcal{F}_q[SL_n]$ induced by the natural map (a $\mathbb{Z}\big[q,q^{-1}\big]$–bialgebra epimorphism!) $F_q[M_n]\relbar\joinrel\twoheadrightarrow F_q[M_n]\Big/\big(D_q-1\big)\cong F_q[SL_n]$.

We need more notation. Set $\mathbf{t}_{i,j}:=\big(q-q^{-1}\big)^{-1}t_{i,j}$ for all $i\neq j$, and $\mathbf{t}_{i,j}^{(m)}:=\mathbf{t}_{i,j}^{\,m}\Big/[m]_q!$ for all $m\in\mathbb{N}$, like in §1.3. Next result will be basic in the following:

Lemma 4.2. *For all $m,k\in\mathbb{N}$ and $c\in\mathbb{Z}$, the embedding $\xi\colon F_q[M_n]\lhook\joinrel\longrightarrow U_q\big(\mathfrak{gl}_n^{\,*}\big)$ gives*

$$\xi\Big(\mathbf{t}_{i,j}^{(m)}\Big)=(-1)^{m(j-i+1)}\,q^{m(j-i-2)-\binom{m}{2}}\sum_{\sum_{k=0}^{n-j}e_k=m}q^{e_0+\sum_{s=1}^{n-j}\binom{e_s}{2}}\,(-1)^{-e_0}\,\big(q-q^{-1}\big)^{m-e_0}\times$$
$$\times\,[e_1]_q!\cdots[e_{n-j}]_q!\cdot E_{i,j}^{(e_0)}E_{i,j+1}^{(e_1)}\cdots E_{i,n}^{(e_{n-j})}\,\Lambda_j^{e_0}\Lambda_{j+1}^{e_1}\cdots\Lambda_n^{e_{n-j}}\,F_{j+1,j}^{(e_1)}\cdots F_{n,j}^{(e_{n-j})}\quad(i<j)$$

$$\xi\left(\begin{pmatrix}t_{h,h}\,;c\\k\end{pmatrix}\right)=\sum_{r=0}^{k}q^{r(c-k-2)-\binom{r}{2}}\big(q-q^{-1}\big)^r\sum_{\sum_{s=1}^{n-h}e_s=r}q^{\sum_{s=1}^{n-h}\binom{e_s}{2}}\,[e_1]_q!\cdots[e_{n-h}]_q!\times$$
$$\times\,E_{h,h+1}^{(e_1)}\cdots E_{h,n}^{(e_{n-h})}\begin{Bmatrix}\Lambda_h\,;c-r\\k\,,r\end{Bmatrix}\Lambda_{h+1}^{e_1}\cdots\Lambda_n^{e_{n-h}}\,F_{h+1,h}^{(e_1)}\cdots F_{n,h}^{(e_{n-h})}\quad(\forall\,h)$$

$$\xi\Big(\mathbf{t}_{i,j}^{(m)}\Big)=(-1)^{m(j-i+1)}\,q^{m(j-i-2)-\binom{m}{2}}\sum_{\sum_{k=0}^{n-i}e_k=m}q^{me_0+\sum_{s=1}^{n-i}\binom{e_s}{2}}\,\big(q-q^{-1}\big)^{m-e_0}\times$$
$$\times\,[e_1]_q!\cdots[e_{n-i}]_q!\cdot E_{i,i+1}^{(e_1)}\cdots E_{i,n}^{(e_{n-i})}\,\Lambda_i^{e_0}\Lambda_{i+1}^{e_1}\cdots\Lambda_n^{e_{n-i}}\,F_{i,j}^{(e_0)}F_{i+1,j}^{(e_1)}\cdots F_{n,j}^{(e_{n-i})}\quad(i>j)$$

Proof. Let's look at $\xi\Big(\mathbf{t}_{i,j}^{(m)}\Big)$ for $i<j$. By Proposition 3.3 we have

$$\xi\Big(\mathbf{t}_{i,j}^{(m)}\Big)=\Big(-(-q)^{j-i-1}E_{i,j}\,\Lambda_j-(-q)^{j-i-2}\sum_{k=j+1}^{n}\big(q-q^{-1}\big)E_{i,k}\,\Lambda_k\,F_{k,j}\Big)^{(m)}.$$

The right-hand side above fulfills the hypotheses of Lemma 1.5*(b)* with $p = q$, and the claim follows at once by brute computation. The case $i > j$ is entirely similar.

As for $\xi\left(\binom{t_{h,h}\,;c}{k}\right)$, Proposition 3.3 again gives

$$\xi\left(\binom{t_{h,h}\,;c}{k}\right) \;=\; \binom{\xi(t_{h,h})\,;c}{k} \;=\; \binom{\Lambda_h \;-\; \left(q-q^{-1}\right)^2 q^{-2} \sum_{s=h+1}^{n} E_{h,s}\,\Lambda_s\,F_{s,h} \;\;;\;\; c}{k}$$

Applying Lemma 1.6*(a)*, with $x := \Lambda_h$ and $w := -q^{-2} \sum_{s=h+1}^{n} E_{h,s}\,\Lambda_s\,F_{s,h}$, we get

$$\xi\left(\binom{t_{h,h}\,;c}{k}\right) \;=\; \sum_{r=0}^{k} q^{r(c-k)} \left(q-q^{-1}\right)^r q^{-2r} \cdot \left(\sum_{s=h+1}^{n} E_{h,s}\,\Lambda_s\,F_{s,h}\right)^{(r)} \left\{ \begin{matrix} \Lambda_h\,;c \\ k\,,r \end{matrix} \right\}$$

Now we apply Lemma 1.5*(b)* to expand $\left(\sum_{s=h+1}^{n} E_{h,s}\,\Lambda_s\,F_{s,h}\right)^{(r)}$. Eventually, one rearranges factors using the commutation relations in $U_q\big(\mathfrak{gl}_n^{\,*}\big)$, and the result is there. □

Theorem 4.3.

(a) $\mathcal{F}_q[M_n]$ *is a free* $\mathbb{Z}\big[q,q^{-1}\big]$*–module, with basis the set of ordered (PBW-like) monomials*

$$\mathcal{B}_{M_n} := \left\{ \mathcal{M}_{\underline{\tau}} := \prod_{i<j} \mathbf{t}_{i,j}^{(\tau_{i,j})} \prod_{k=1}^{n} \binom{t_{k,k}\,;0}{\tau_{k,k}} \prod_{i>j} \mathbf{t}_{i,j}^{(\tau_{i,j})} \;\middle|\; \underline{\tau} = \left(\tau_{i,j}\right)_{i,j=1}^{n} \in M_n(\mathbb{N}) \right\}$$

where monomials are ordered w.r.t. any total order of the pairs (i,j) *with* $i \neq j$*. Similarly, any other set obtained from* $\mathcal{B}_{M_n}$ *via permutations of factors is a* $\mathbb{Z}\big[q,q^{-1}\big]$*–basis as well.*

(b) $\mathcal{F}_q[GL_n]$ *is the* $\mathbb{Z}\big[q,q^{-1}\big]$*–span of the set of ordered (PBW-like) monomials*

$$\mathcal{S}_{GL_n} := \left\{ \prod_{i<j} \mathbf{t}_{i,j}^{(\tau_{i,j})} \prod_{k=1}^{n} \binom{t_{k,k}\,;0}{\tau_{k,k}} \prod_{i>j} \mathbf{t}_{i,j}^{(\tau_{i,j})} D_q^{-\delta} \;\middle|\; \underline{\tau} = \left(\tau_{i,j}\right)_{i,j=1}^{n} \in M_n(\mathbb{N})\,,\; \delta \in \mathbb{N} \right\}.$$

where monomials are ordered w.r.t. any total order of the pairs (i,j) *with* $i \neq j$*. Similarly, any other set obtained from* $\mathcal{S}_{GL_n}$ *via permutations of factors (of the monomials in* $\mathcal{S}_{GL_n}$*) is a* $\mathbb{Z}\big[q,q^{-1}\big]$*–spanning set for* $\mathcal{F}_q[GL_n]$ *as well. Moreover, if* $f \in \mathcal{F}_q[GL_n]$ *then* f *can be expanded into a* $\mathbb{Z}\big[q,q^{-1}\big]$*–linear combination of elements of* $\mathcal{S}_{GL_n}$ *which all bear the same exponent* δ*; similarly for the other spanning sets mentioned above.*

(c) $\mathcal{F}_q[SL_n]$ *is the* $\mathbb{Z}\big[q,q^{-1}\big]$*–span of the set of ordered (PBW-like) monomials*

$$\mathcal{S}_{SL_n} := \left\{ \prod_{i<j} \mathbf{t}_{i,j}^{(\tau_{i,j})} \prod_{k=1}^{n} \binom{t_{k,k}\,;0}{\tau_{k,k}} \prod_{i>j} \mathbf{t}_{i,j}^{(\tau_{i,j})} \;\middle|\; \underline{\tau} = \left(\tau_{i,j}\right)_{i,j=1}^{n} \in M_n(\mathbb{N}) \right\}.$$

where monomials are ordered w.r.t. any total order of the pairs (i,j) *with* $i \neq j$*. Similarly, any other set obtained from* $\mathcal{S}_{SL_n}$ *via permutations of factors (of the monomials in* $\mathcal{S}_{SL_n}$*) is a* $\mathbb{Z}\big[q,q^{-1}\big]$*–spanning set for* $\mathcal{F}_q[SL_n]$ *as well.*

Proof. (a) We know from §2.4 that B_{M_n} is a $\mathbb{Q}(q)$–basis of $F_q[M_n]$. Then it is clear that the same is true for $\mathcal{B}_{M_n}$, hence its $\mathbb{Z}\big[q,q^{-1}\big]$–span is a free $\mathbb{Z}\big[q,q^{-1}\big]$–submodule of $F_q[M_n]$.

Thanks to Lemma 4.2, each of the factors $\mathbf{t}_{i,j}^{(\tau_{i,j})}$ or $\prod_{k=1}^{n}\binom{t_{k,k}\,;0}{\tau_k}$ of the monomials in $\mathcal{B}_{M_n}$ is mapped by $\xi : F_q[M_n] \longrightarrow U_q\big(\mathfrak{gl}_n^{\,*}\big)$ into $\mathfrak{H}_q^g$. Since the latter is a subalgebra, each of the monomials also is mapped into $\mathfrak{H}_q^g$. Thus (cf. §3.2) we deduce that $\mathcal{B}_{M_n}$ is contained in $\mathcal{F}_q[M_n]$, hence the same holds for its (free) $\mathbb{Z}\big[q,q^{-1}\big]$–span.

Consider in $\mathfrak{H}_q^g$ the unique $\mathbb{Z}\big[q,q^{-1}\big]$–algebra grading such that the generic monomial in the PBW basis $\mathfrak{B}_*^g$ of $\mathfrak{H}_q^g$, say $\prod\limits_{i<j} E_{i,j}^{(\eta_{i,j})} \prod\limits_{h=1}^{n} \binom{\Lambda_h\,;\,0}{\lambda_h} \Lambda_h^{-Ent(\lambda_h/2)} \prod\limits_{i<j} F_{j,i}^{(\varphi_{j,i})}$ (cf. §2.5), has degree $\sum\limits_{i<j} \big(\eta_{i,j}+\varphi_{i,j}\big)$. Then Lemma 4.2 reads

$$\begin{aligned}
\xi\Big(\mathbf{t}_{i,j}^{(m)}\Big) &= (-1)^{m(j-i+1)}\, q^{m(j-i-2)-\binom{m}{2}} q^m \,(-1)^{-m} \cdot E_{i,j}^{(m)}\, \Lambda_j^m \;+\; h.d.t. && \forall\ i<j \\
\xi\Bigg(\binom{t_{h,h}\,;0}{k}\Bigg) &= \begin{Bmatrix} \Lambda_h\,;0 \\ k\,,0 \end{Bmatrix} +\; h.d.t. \;=\; \binom{\Lambda_h\,;0}{k} +\; h.d.t. && \forall\ h \\
\xi\Big(\mathbf{t}_{i,j}^{(m)}\Big) &= (-1)^{m(j-i+1)}\, q^{m(j-i-2)-\binom{m}{2}} q^{m^2} \cdot \Lambda_i^m\, F_{i,j}^{(m)} \;+\; h.d.t. && \forall\ i>j
\end{aligned}$$

where "*h.d.t.*" stands for "*higher degree terms*". It follows that

$$\begin{aligned}
\xi(\mathcal{M}_{\underline{\tau}}) \;&=\; \prod_{i<j} \xi\Big(\mathbf{t}_{i,j}^{(\tau_{i,j})}\Big) \prod_{k=1}^{n} \xi\Bigg(\binom{t_{k,k}\,;0}{\tau_{k,k}}\Bigg) \prod_{i>j} \xi\Big(\mathbf{t}_{i,j}^{(\tau_{i,j})}\Big) \;= \\
&=\; \varepsilon\, q^{\zeta} \prod_{i<j} E_{i,j}^{(\tau_{i,j})} \prod_{i<j} \Lambda_j^{\tau_{i,j}} \prod_{k=1}^{n} \binom{\Lambda_k\,;0}{\tau_k} \prod_{i<j} \Lambda_i^{\tau_{i,j}} \prod_{i<j} F_{i,j}^{(\tau_{i,j})} \;+\; h.d.t.
\end{aligned}$$

for some sign $\varepsilon = \pm 1$ and some integers $z, \zeta \in \mathbb{Z}$.

Now pick $f \in \mathcal{F}_q[M_n]$. Since $\mathcal{B}_{M_n}$ is a $\mathbb{Q}(q)$–basis of $F_q[M_n]$, there exists a unique expansion $f = \sum_{\underline{\tau} \in M_n(\mathbb{N})} \chi_{\underline{\tau}}\, \mathcal{M}_{\underline{\tau}}$ for some $\chi_{\underline{\tau}} \in \mathbb{Q}(q)$, almost all zero: our goal is to show that indeed they belong to $\mathbb{Z}\big[q,q^{-1}\big]$. Let $\tau_0 := \min\Big\{ \sum_{i \neq j} \tau'_{i,j} \,\Big|\, \underline{\tau} \in M_n(\mathbb{N})\,,\, \chi_{\underline{\tau}} \neq 0 \Big\}$. Then the previous analysis yields

$$\xi(f) \;=\; \textstyle\sum_{\underline{\tau} : \sum_{i\neq j} \tau_{i,j} = \tau_0} \chi_{\underline{\tau}}\, \varepsilon\, q^{\zeta} \displaystyle\prod_{i<j} E_{i,j}^{(\tau_{i,j})} \prod_{i<j} \Lambda_j^{\tau_{i,j}} \prod_{k=1}^{n} \binom{\Lambda_k\,;0}{\tau_k} \prod_{i<j} \Lambda_i^{\tau_{i,j}} \prod_{i<j} F_{i,j}^{(\tau_{i,j})} \;+\; h.d.t.$$

Pick now $\prod\limits_{i<j} \overline{E}_{i,j}^{\,\tau'_{i,j}} \prod\limits_{k=1}^{n} G_k^{\tau'_{k,k}} \prod\limits_{i<j} \overline{F}_{j,i}^{\,\tau'_{j,i}}$, a monomial in the PBW-like basis $\mathcal{B}^g$ of $\mathcal{U}_q(\mathfrak{gl}_n)$ (see §2.3). Then the above expansion of $\xi(f)$ along with formula (3.1) yields

$$\begin{aligned}
&\bigg\langle f\,,\, \prod_{i<j} \overline{E}_{i,j}^{\,\tau'_{i,j}} \prod_{k=1}^{n} G_k^{\tau'_{k,k}} \prod_{i<j} \overline{F}_{j,i}^{\,\tau'_{j,i}} \bigg\rangle \;=\; \bigg\langle \xi(f)\,,\, \prod_{i<j} \overline{E}_{i,j}^{\,\tau'_{i,j}} \prod_{k=1}^{n} G_k^{\tau'_{k,k}} \prod_{i<j} \overline{F}_{j,i}^{\,\tau'_{j,i}} \bigg\rangle \;= \\
&= \sum_{\substack{\tau_{i,j} = \tau'_{i,j} \\ \forall\, i \neq j}} \chi_{\underline{\tau}}\, \varepsilon\, q^{\zeta} \bigg\langle \prod_{i<j} E_{i,j}^{(\tau'_{i,j})} \prod_{i<j} \Lambda_j^{\tau'_{i,j}} \prod_{k=1}^{n} \binom{\Lambda_k\,;0}{\tau_{k,k}} \prod_{i<j} \Lambda_i^{\tau'_{i,j}} \prod_{i<j} F_{i,j}^{(\tau'_{i,j})}\,,\, \prod_{i<j} \overline{E}_{i,j}^{\,\tau'_{i,j}} \prod_{k=1}^{n} G_k^{\tau'_{k,k}} \prod_{i<j} \overline{F}_{j,i}^{\,\tau'_{j,i}} \bigg\rangle \;= \\
&\;= \sum_{\tau_{i,j} = \tau'_{i,j}\,,\,\forall\, i\neq j} \chi_{\underline{\tau}}\, \eta\, q^{\beta} \cdot \prod_{r,s=1}^{n} \bigg\langle \binom{\Lambda_r\,;0}{\tau_{r,r}}, G_k^{\tau'_{s,s}} \bigg\rangle \;=\; \textstyle\sum_{\tau_{i,j} = \tau'_{i,j}\,,\,\forall\, i\neq j} \chi_{\underline{\tau}}\, \eta\, q^{\beta} \cdot \displaystyle\prod_{k=1}^{n} \binom{\tau'_{k,k}}{\tau_{k,k}}
\end{aligned}$$

for some $\eta \in \{+1,-1\}$ and $\beta \in \mathbb{Z}$. Since $f \in F_q[M_n]$, the last term belongs to $\mathbb{Z}\big[q,q^{-1}\big]$.

Now choose any $\underline{\tau}'$ inside $\Big\{ \underline{\tau} \in M_n(\mathbb{N}) \,\Big|\, \sum_{i\neq j} \tau_{i,j} = \tau_0\,,\, \chi_{\underline{\tau}} \neq 0 \Big\}$ which is minimal (for the restriction of the product order in $M_n(\mathbb{N}) \cong \mathbb{N}^{n^2}$). Then $\prod_{k=1}^{n} \binom{\tau'_{k,k}}{\tau_{k,k}} = \delta_{\underline{\tau}',\underline{\tau}}$, and so $\chi_{\underline{\tau}'}\, \eta\, q^{\beta} = \Big\langle f\,,\, \prod_{i<j} \overline{E}_{i,j}^{\,\tau'_{i,j}} \prod_{k=1}^{n} G_k^{\tau'_{k,k}} \prod_{i<j} \overline{F}_{j,i}^{\,\tau'_{j,i}} \Big\rangle \in \mathbb{Z}\big[q,q^{-1}\big]$, hence $\chi_{\underline{\tau}'} \in \mathbb{Z}\big[q,q^{-1}\big]$.

Thus we get that $\chi_{\underline{T}'} \in \mathbb{Z}\big[q,q^{-1}\big]$ for the $\underline{T}' \in M_n(\mathbb{N})$ specified above. Since we already proved that $\mathcal{B}_{M_n} \subseteq \mathcal{F}_q[M_n]$, it follows that $f' := f - \chi_{\underline{T}'}\, \mathcal{M}_{\underline{T}'} \in \mathcal{F}_q[M_n]$. Now, by construction the expansion of f' as a $\mathbb{Q}(q)$–linear combination of elements of $\mathcal{B}_{M_n}$ has one non-trivial summand less than f: then we can apply again the same argument, and iterate till we find that all coefficients $\chi_{\underline{T}}$ in the original expansion of f do belong to $\mathbb{Z}\big[q,q^{-1}\big]$.

Finally, the last observation about other bases is clear.

(b) By claim *(a)*, every monomial of type $\mathcal{M}_{\underline{T}} := \prod_{i<j} \mathbf{t}_{i,j}^{(\tau_{i,j})} \prod_{k=1}^{n} \binom{t_{k,k}\,;\,0}{\tau_{k,k}} \prod_{i>j} \mathbf{t}_{i,j}^{(\tau_{i,j})}$ is $\mathbb{Z}\big[q,q^{-1}\big]$–valued on $\mathcal{U}_q(\mathfrak{gl}_n)$, i.e., it takes values in $\mathbb{Z}\big[q,q^{-1}\big]$ when paired with $\mathcal{U}_q(\mathfrak{gl}_n)$. On the other hand, the same is true for $D_q^{\,-\delta}$ ($\forall\, \delta \in \mathbb{N}$), because $D_q^{\,-\delta} \in \mathfrak{F}_q[GL_n]$ and $\mathfrak{F}_q[GL_n] \subseteq \mathcal{F}_q[GL_n]$ since $\mathfrak{U}_q(\mathfrak{gl}_n) \supseteq \mathcal{U}_q(\mathfrak{gl}_n)$. Finally,

$$\Big\langle \mathcal{M}_{\underline{T}}\, D_q^{\,-\delta}, \mathcal{U}_q(\mathfrak{gl}_n) \Big\rangle = \Big\langle \mathcal{M}_{\underline{T}} \otimes D_q^{\,-\delta},\, \Delta\big(\mathcal{U}_q(\mathfrak{gl}_n)\big) \Big\rangle \subseteq \big\langle \mathcal{M}_{\underline{T}}, \mathcal{U}_q(\mathfrak{gl}_n) \big\rangle \cdot \Big\langle D_q^{\,-\delta}, \mathcal{U}_q(\mathfrak{gl}_n) \Big\rangle$$

and the last product on right-hand side belongs to $\mathbb{Z}\big[q,q^{-1}\big]$. Thus $\mathcal{M}_{\underline{T}}\, D_q^{\,-\delta} \in \mathcal{F}_q[GL_n]$, for all $\underline{T} \in M_n(\mathbb{N})$, $\delta \in \mathbb{N}$, and so the $\mathbb{Z}\big[q,q^{-1}\big]$–span of $\mathcal{S}_{GL_n}$ is contained in $\mathcal{F}_q[GL_n]$.

Conversely, let $f \in \mathcal{F}_q[GL_n]$. Then there exists $\delta \in \mathbb{N}$ such that $f\, D_q^{\,\delta} \in F_q[M_n]$. In addition, $\Big\langle f\, D_q^{\,\delta}, \mathcal{U}_q(\mathfrak{gl}_n) \Big\rangle = \Big\langle f \otimes D_q^{\,\delta},\, \Delta\big(\mathcal{U}_q(\mathfrak{gl}_n)\big) \Big\rangle \subseteq \langle f, \mathcal{U}_q(\mathfrak{gl}_n) \rangle \cdot \Big\langle D_q^{\,\delta}, \mathcal{U}_q(\mathfrak{gl}_n) \Big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big]$ because $f, D_q^{\,\delta} \in \mathcal{F}_q[GL_n]$. Thus $f\, D_q^{\,\delta} \in \mathcal{F}_q[M_n]$; then, by claim *(a)*, $f\, D_q^{\,\delta}$ belongs to the $\mathbb{Z}\big[q,q^{-1}\big]$–span of $\mathcal{B}_{M_n}$, whence claim *(b)* follows at once.

Finally, the last observation about other spanning sets is self-evident.

(c) The projection epimorphism $F_q[M_n] \overset{\pi}{\twoheadrightarrow} F_q[M_n]\Big/\big(D_q - 1\big) \cong F_q[SL_n]$ maps $\mathcal{B}_{M_n}$ onto $\mathcal{S}_{SL_n}$. It follows directly from definitions that $\pi\big(\mathcal{F}_q[M_n]\big) \subseteq \mathcal{F}_q[SL_n]$, hence in particular (thanks to claim *(a)*) the $\mathbb{Z}\big[q,q^{-1}\big]$–span of $\mathcal{S}_{SL_n}$ is contained in $\mathcal{F}_q[SL_n]$.

Conversely, let $f \in \mathcal{F}_q[SL_n]$. Then $\Big\langle f\,,\, \prod_{i<j} \overline{E}_{i,j}^{\,\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} \overline{F}_{j,i}^{\,\varphi_{j,i}} \Big\rangle \in \mathbb{Z}\big[q,q^{-1}\big]$ for every monomial $\prod_{i<j} \overline{E}_{i,j}^{\,\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} \overline{F}_{j,i}^{\,\varphi_{j,i}}$ in the $\mathbb{Z}\big[q,q^{-1}\big]$–basis $\mathcal{B}^s$ of $\mathcal{U}_q(\mathfrak{sl}_n)$ (cf. §2.3). Moreover, $\sum_{(f)} f_{(1)} \otimes f_{(2)} = \Delta(f) \in \Delta\big(F_q[SL_n]\big) \subseteq F_q[SL_n] \otimes F_q[SL_n]$, hence

$$\Big\langle f\,,\, L_n^{\lambda} \cdot \prod_{i<j} \overline{E}_{i,j}^{\,\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} \overline{F}_{j,i}^{\,\varphi_{j,i}} \Big\rangle = \Big\langle \Delta(f)\,,\, L_n^{\lambda} \otimes \prod_{i<j} \overline{E}_{i,j}^{\,\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} \overline{F}_{j,i}^{\,\varphi_{j,i}} \Big\rangle =$$

$$= \sum_{(f)} \Big\langle f_{(1)}\,,\, L_n^{\lambda} \Big\rangle \cdot \Big\langle f_{(2)}\,,\, \prod_{i<j} \overline{E}_{i,j}^{\,\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} \overline{F}_{j,i}^{\,\varphi_{j,i}} \Big\rangle$$

On the other hand, the natural embedding $U_q(\mathfrak{sl}_n) \hookrightarrow U_q(\mathfrak{gl}_n)$ has a canonical section $U_q(\mathfrak{gl}_n) \overset{pr}{\twoheadrightarrow} U_q(\mathfrak{sl}_n)$ whose kernel is generated by $\big(L_n - 1\big)$. In fact, one has an algebra isomorphism $U_q(\mathfrak{gl}_n) \cong \big(\mathbb{Z}\big[q,q^{-1}\big]\,\big)\big[L_n, L_n^{-1}\big] \otimes_{\mathbb{Z}[q,q^{-1}]} U_q(\mathfrak{sl}_n)$. Dually, the natural epimorphism $F_q[GL_n] \twoheadrightarrow F_q[SL_n]$ (or $F_q[M_n] \twoheadrightarrow F_q[SL_n]$ as well) has a canonical section $F_q[SL_n] \hookrightarrow F_q[GL_n]$ (or $F_q[SL_n] \hookrightarrow F_q[M_n]$ as well) given by $\phi \mapsto \phi \circ pr$, for every $\phi \in F_q[SL_n]$. In particular this yields $\Big\langle \phi\,,\, L_n^{\lambda} \Big\rangle \equiv \Big\langle \phi \circ pr\,,\, L_n^{\lambda} \Big\rangle = \Big\langle \phi\,,\, pr\big(L_n^{\lambda}\big) \Big\rangle = \langle \phi\,, 1 \rangle = \epsilon(\phi)$ for all $\phi \in F_q[SL_n]$ and $\lambda \in \mathbb{Z}$. This and the previous analysis give

$$\Big\langle f\,,\, L_n^{\lambda} \cdot \prod_{i<j} \overline{E}_{i,j}^{\,\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} \overline{F}_{j,i}^{\,\varphi_{j,i}} \Big\rangle = \Big\langle \sum_{(f)} \epsilon\big(f_{(1)}\big) \cdot f_{(2)}\,,\, \prod_{i<j} \overline{E}_{i,j}^{\,\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} \overline{F}_{j,i}^{\,\varphi_{j,i}} \Big\rangle =$$

$$= \Big\langle f\,,\, \prod_{i<j} \overline{E}_{i,j}^{\,\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} \overline{F}_{j,i}^{\,\varphi_{j,i}} \Big\rangle \in \mathbb{Z}\big[q,q^{-1}\big]$$

Now observe that $L_n^\lambda \cdot \prod_{i<j} \overline{E}_{i,j}^{\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} \prod_{i<j} \overline{F}_{j,i}^{\varphi_{j,i}} = \prod_{i<j} \overline{E}_{i,j}^{\eta_{i,j}} \prod_{k=1}^{n-1} K_k^{\kappa_k} L_n^\lambda \prod_{i<j} \overline{F}_{j,i}^{\varphi_{j,i}}$, because L_n is central in $U_q(\mathfrak{gl}_n)$, and the latter monomials (for all $\eta_{i,j}, \varphi_{j,i} \in \mathbb{N}$ and all $\kappa_k, \lambda \in \mathbb{Z}$) clearly form a $\mathbb{Z}[q,q^{-1}]$–basis of $\mathcal{U}_q(\mathfrak{gl}_n)$, as $\mathcal{B}^g$ is a similar basis (see §2.2, and make an easy comparison, through definitions). But then we proved that any f in $\mathcal{F}_q[SL_n]$ takes values in $\mathbb{Z}[q,q^{-1}]$ onto elements of a $\mathbb{Z}[q,q^{-1}]$–basis of $\mathcal{U}_q(\mathfrak{gl}_n)$, hence also $f \in \mathcal{F}_q[GL_n]$ via the embedding $F_q[SL_n] \longrightarrow F_q[GL_n]$ and $f \in \mathcal{F}_q[M_n]$ via $F_q[SL_n] \longrightarrow F_q[M_n]$. By claim *(a)*, the last fact implies that f (in $F_q[M_n]$) is in the $\mathbb{Z}[q,q^{-1}]$–span of $\mathcal{B}_{M_n}$, hence also that $f = \pi(f)$ is in the $\mathbb{Z}[q,q^{-1}]$–span of $\pi(\mathcal{B}_{M_n}) = \mathcal{S}_{M_n}$.

Finally, the last remark about other bases is clear. □

4.4 Remarks. *(a) Exactly the same argument as above* — just using monomials in the generators $t_{i,j}$ instead of the $\mathbf{t}_{i,j}^{(m)}$'s and the $\binom{t_{k,k}\,;\,0}{\tau}$'s — *gives a new, independent proof of the identity* $\mathfrak{F}_q[M_n] = \Big\{ f \in F_q[M_n] \,\Big|\, \big\langle f, \mathfrak{U}_q(\mathfrak{gl}_n) \big\rangle \subseteq \mathbb{Z}[q,q^{-1}] \Big\}$ mentioned in §2.4, and, similarly, also the identities $\mathfrak{F}_q[GL_n] = \Big\{ f \in F_q[GL_n] \,\Big|\, \big\langle f, \mathfrak{U}_q(\mathfrak{gl}_n) \big\rangle \subseteq \mathbb{Z}[q,q^{-1}] \Big\}$ (see §2.4) and $\mathfrak{F}_q[SL_n] = \Big\{ f \in F_q[SL_n] \,\Big|\, \big\langle f, \mathfrak{U}_q(\mathfrak{sl}_n) \big\rangle \subseteq \mathbb{Z}[q,q^{-1}] \Big\}$ (see §2.4).

(b) Clearly the $\mathbb{Z}[q,q^{-1}]$–spanning set $\mathcal{S}_{GL_n}$ *is not* a $\mathbb{Z}[q,q^{-1}]$–basis of $\mathcal{F}_q[GL_n]$. For instance, expanding D_q as a $\mathbb{Z}[q,q^{-1}]$–linear combination of elements of the basis $\mathcal{B}_{M_n}$ (of $\mathcal{F}_q[M_n]$), the relation $D_q\, D_q^{-1} = 1$ (inside $\mathcal{F}_q[GL_n]$) yields a non-trivial $\mathbb{Z}[q,q^{-1}]$–linear dependence relation among elements of $\mathcal{S}_{GL_n}$.

(c) The $\mathbb{Z}[q,q^{-1}]$–spanning set $\mathcal{S}_{SL_n}$ *is definitely not* a $\mathbb{Z}[q,q^{-1}]$–basis of $\mathcal{F}_q[SL_n]$: e.g., for $n=2$ the relation $t_{1,1}\, t_{2,2} - q\, t_{1,2}\, t_{2,1} - 1 = 0$ yields the relation (in $\mathcal{F}_q[SL_2]$)

$$\binom{t_{1,1}\,;0}{1} + \binom{t_{2,2}\,;0}{1} + (q-1)\binom{t_{1,1}\,;0}{1}\binom{t_{2,2}\,;0}{1} - (1+q)\big(q-q^{-1}\big)\, \mathbf{t}_{1,2}^{(1)}\, \mathbf{t}_{2,1}^{(1)} = 0$$

(d) Theorem 4.3 has the following immediate consequence, whose proof is straightforward:

Corollary 4.5. *For every* $X \in \{M, GL\}$, *let* $\big(D_q - 1\big)$ *be the two-sided ideal of* $F_q[X_n]$ *generated by* $\big(D_q - 1\big)$, *and let* $\mathcal{D}(X_n) := \big(D_q - 1\big) \cap \mathcal{F}_q[X_n]$, *a two-sided ideal of* $\mathcal{F}_q[X_n]$.

(a) The epimorphism $F_q[M_n] \xrightarrow{\;\pi\;} F_q[M_n]\Big/\big(D_q - 1\big) \cong F_q[SL_n]$ *restricts to an epimorphism* $\mathcal{F}_q[M_n] \xrightarrow{\;\pi\;} \mathcal{F}_q[M_n]\Big/\mathcal{D}(M_n) \cong \mathcal{F}_q[SL_n]$ *of* $\mathbb{Z}[q,q^{-1}]$*–bialgebras.*

(b) The epimorphism $F_q[GL_n] \xrightarrow{\;\pi\;} F_q[GL_n]\Big/\big(D_q - 1\big) \cong F_q[SL_n]$ *restricts to an epimorphism* $\mathcal{F}_q[GL_n] \xrightarrow{\;\pi\;} \mathcal{F}_q[GL_n]\Big/\mathcal{D}(GL_n) \cong \mathcal{F}_q[SL_n]$ *of Hopf* $\mathbb{Z}[q,q^{-1}]$*–algebras.* □

Theorem 4.6.

(a) $\mathcal{F}_q[M_n]$ *is the unital associative* $\mathbb{Z}[q,q^{-1}]$*–algebra with generators* $\mathbf{t}_{i,j}^{(h)}$, $\binom{t_{\ell,\ell}\,;c}{k}$, *for all* $i, j, \ell \in \{1,\dots,n\}$, $i \neq j$, $h, k \in \mathbb{N}$, $c \in \mathbb{Z}$, *and relations*

[qDP] *relations* (1.1) *for* $X \in \big\{\, \mathbf{t}_{i,j} \,\big|\, i,j \in \{1,\dots,n\},\, i \neq j \,\big\}$

[qBC] *relations* (1.2) *for* $X \in \big\{\, \mathbf{t}_{\ell,\ell} \,\big|\, \ell = 1,\dots,n \,\big\}$

[H-V.1] $\mathbf{t}_{i,j}^{(h)}\, \mathbf{t}_{i,k}^{(f)} = q^{hf}\, \mathbf{t}_{i,k}^{(f)}\, \mathbf{t}_{i,j}^{(h)}$, $\mathbf{t}_{j,i}^{(h)}\, \mathbf{t}_{k,i}^{(f)} = q^{hf}\, \mathbf{t}_{k,i}^{(f)}\, \mathbf{t}_{j,i}^{(h)}$ $\forall\; i \neq j < k \neq i$

[H-V.2] $$\binom{t_{i,i}\,;c}{k}\mathbf{t}_{i,j}^{(h)} = \mathbf{t}_{i,j}^{(h)}\binom{t_{i,i}\,;c+h}{k}\ ,\quad \binom{t_{i,i}\,;c}{k}\mathbf{t}_{j,i}^{(h)} = \mathbf{t}_{j,i}^{(h)}\binom{t_{i,i}\,;c+h}{k}\qquad \forall\ i<j$$

[H-V.3] $$\binom{t_{i,i}\,;c}{k}\mathbf{t}_{i,j}^{(h)} = \mathbf{t}_{i,j}^{(h)}\binom{t_{i,i}\,;c-h}{k}\ ,\quad \binom{t_{i,i}\,;c}{k}\mathbf{t}_{j,i}^{(h)} = \mathbf{t}_{j,i}^{(h)}\binom{t_{i,i}\,;c-h}{k}\qquad \forall\ i>j$$

[CD.1] $$\binom{t_{\ell,\ell}\,;c}{k}\mathbf{t}_{i,j}^{(h)} = \mathbf{t}_{i,j}^{(h)}\binom{t_{\ell,\ell}\,;c}{k}\qquad \forall\ i<\ell<j\quad or\quad i>\ell>j$$

[CD.2] $$\mathbf{t}_{i,j}^{(h)}\,\mathbf{t}_{\ell,k}^{(f)} = \mathbf{t}_{\ell,k}^{(f)}\,\mathbf{t}_{i,j}^{(h)}\qquad \forall\ j\neq i<\ell\neq k<j$$

[D.1] *for all* $i<\ell\,,\ j<k\,,$ *with* $\big|\{i,j,\ell,k\}\big|=4\,,$

$$\mathbf{t}_{\ell,k}^{(f)}\,\mathbf{t}_{i,j}^{(h)} = \sum_{s=0}^{h\wedge f}(-1)^s\,q^{\binom{s+1}{2}-s\,(h+f-s)}\,\big(q-q^{-1}\big)^s\,[s]_q!\cdot\mathbf{t}_{i,j}^{(h-s)}\,\mathbf{t}_{i,k}^{(s)}\,\mathbf{t}_{\ell,j}^{(s)}\,\mathbf{t}_{\ell,k}^{(f-s)}$$

[D.2] *for all* $i<j\,,$ *with* $C(h,k,r,s,p):=p\,((h+k)-(r+s))-\binom{p}{2}\,,$

$$\binom{t_{j,j}\,;k}{r}\binom{t_{i,i}\,;h}{s} = \sum_{p=0}^{r\wedge s}(-1)^p\,q^{C(h,k,r,s,p)}\,\big(q-q^{-1}\big)^p\,[p]_q!\,\mathbf{t}_{i,j}^{(p)}\left\{\begin{matrix} t_{i,i}\,;\,h-p\\ s\,,\,p\end{matrix}\right\}\left\{\begin{matrix} t_{j,j}\,;\,k-p\\ r\,,\,p\end{matrix}\right\}\mathbf{t}_{j,i}^{(p)}$$

[D.3+] *for all* $i<j<k\,,$ *with* $A(h,f,r,s):=\binom{r+1}{2}+\binom{s}{2}-r\,(h+f-r)\,,$

$$\begin{aligned}\mathbf{t}_{j,k}^{(f)}\,\mathbf{t}_{i,j}^{(h)} &= \sum_{r=0}^{h\wedge f}\sum_{s=0}^{r}(-1)^r\,q^{A(h,k,r,s,p)}\,(q-1)^s\,(s)_q!\binom{r}{s}_q\mathbf{t}_{i,j}^{(h-r)}\,\mathbf{t}_{i,k}^{(r)}\,\mathbf{t}_{j,k}^{(f-r)}\binom{t_{j,j}\,;\,f-r}{s}\\ &=\sum_{r=0}^{h\wedge f}\sum_{s=0}^{r}(-1)^r\,q^{A(h,f,r,s)}\,(q-1)^s\,(s)_q!\binom{r}{s}_q\mathbf{t}_{i,j}^{(h-r)}\,\mathbf{t}_{i,k}^{(r)}\binom{t_{j,j}\,;\,0}{s}\mathbf{t}_{j,k}^{(f-r)}\end{aligned}$$

[D.3−] *for all* $i<j<\ell\,,$ *with* $A(h,f,r,s):=\binom{r+1}{2}+\binom{s}{2}-r\,(h+f-r)\,,$

$$\begin{aligned}\mathbf{t}_{\ell,j}^{(f)}\,\mathbf{t}_{j,i}^{(h)} &= \sum_{r=0}^{h\wedge f}\sum_{s=0}^{r}(-1)^r\,q^{A(h,f,r,s)}\,(q-1)^s\,(s)_q!\binom{r}{s}_q\binom{t_{j,j}\,;\,h-r}{s}\mathbf{t}_{j,i}^{(h-r)}\,\mathbf{t}_{\ell,i}^{(r)}\,\mathbf{t}_{\ell,j}^{(f-r)}\\ &=\sum_{r=0}^{h\wedge f}\sum_{s=0}^{r}(-1)^r\,q^{A(h,f,r,s)}\,(q-1)^s\,(s)_q!\binom{r}{s}_q\mathbf{t}_{j,i}^{(h-r)}\binom{t_{j,j}\,;\,0}{s}\mathbf{t}_{\ell,i}^{(r)}\,\mathbf{t}_{\ell,j}^{(f-r)}\end{aligned}$$

[D.4+] *for all* $i<\ell<j$ *or* $i<j<\ell$

$$\mathbf{t}_{\ell,j}^{(f)}\binom{t_{i,i}\,;c}{k} = \sum_{s=0}^{f\wedge k}q^{\binom{s+1}{2}-s\,(f-k+c)}\,\big(q-q^{-1}\big)^s\,[s]_q!\cdot\left\{\begin{matrix} t_{i,i}\,;\,c\\ k\,,\,s\end{matrix}\right\}\mathbf{t}_{i,j}^{(s)}\,\mathbf{t}_{\ell,i}^{(s)}\,\mathbf{t}_{\ell,j}^{(f-s)}$$

[D.4−] *for all* $i<h<\ell$ *or* $h<i<\ell$

$$\binom{t_{\ell,\ell}\,;c}{k}\mathbf{t}_{i,h}^{(f)} = \sum_{s=0}^{f\wedge k}q^{\binom{s+1}{2}-s\,(f-k+c)}\,\big(q-q^{-1}\big)^s\,[s]_q!\cdot\mathbf{t}_{i,h}^{(f-s)}\,\mathbf{t}_{i,\ell}^{(s)}\,\mathbf{t}_{\ell,h}^{(s)}\left\{\begin{matrix} t_{\ell,\ell}\,;\,c\\ k\,,\,s\end{matrix}\right\}$$

Moreover, $\mathcal{F}_q[M_n]$ *is a* $\mathbb{Z}\big[q,q^{-1}\big]$*–bialgebra, whose bialgebra structure is given by*

$$\begin{aligned}\Delta\left(\binom{t_{i,i}\,;c}{k}\right) &= \sum_{r=0}^{k}q^{r(c-k-1)}\big(q-q^{-1}\big)^r\sum_{s=0}^{r}q^{\binom{s+1}{2}}\binom{r}{s}_q\times\\ &\times\sum_{h=0}^{k-r}q^{h(c+2r+s-k-1)}\big(q-q^{-1}\big)^h\sum_{\sum_{j<i}e_j=h}\ \sum_{\sum_{j>i}e_j=r}\ \prod_{\ell=1}^{n}\,\widehat{{}^{i}}\,[e_\ell]_q!\,q^{\sum_{a=1}^{n}\widehat{{}^{i}}\binom{e_a+1}{2}-\binom{h+1}{2}-\binom{r+1}{2}}\times\\ &\times\prod_{\ell=1}^{i-1}\Big(\mathbf{t}_{i,\ell}^{(e_\ell)}\otimes\mathbf{t}_{\ell,i}^{(e_\ell)}\Big)\cdot\left\{\begin{matrix}\mathbf{t}_{i,i}\otimes\mathbf{t}_{i,i}\,;\,c+r+s-h\\ k-r\,,\,h\end{matrix}\right\}\cdot\prod_{\ell=i+1}^{n}\Big(\mathbf{t}_{i,\ell}^{(e_\ell)}\otimes\mathbf{t}_{\ell,i}^{(e_\ell)}\Big) =\end{aligned}$$

$$= \sum_{r+s\leq k} \sum_{h=0}^{r} \sum_{\sum_{j<i} e_j = s} \sum_{\sum_{j>i} e_j = r} \left(q-q^{-1}\right)^{r+s} \binom{r}{h}_q \prod_{\ell=1}^{n} {}^{\widehat{i}} \, [e_\ell]_q! \times$$
$$\times\, q^{\binom{h+1}{2} - \binom{r+1}{2} - \binom{s+1}{2} + (c-k-1)(r+s) + (2r+h)s + \sum_{a=1}^{n}{}^{\widehat{i}} \binom{e_a+1}{2}} \times$$
$$\times \prod_{\ell=1}^{i-1} \left(\mathbf{t}_{i,\ell}^{(e_\ell)} \otimes \mathbf{t}_{\ell,i}^{(e_\ell)} \right) \cdot \left\{ \begin{matrix} \mathbf{t}_{i,i} \otimes \mathbf{t}_{i,i} \,;\, c+r+h-s \\ k-r \,,\, s \end{matrix} \right\} \cdot \prod_{\ell=i+1}^{n} \left(\mathbf{t}_{i,\ell}^{(e_\ell)} \otimes \mathbf{t}_{\ell,i}^{(e_\ell)} \right) \quad \forall\ i$$

$$\Delta\left(\mathbf{t}_{i,j}^{(h)}\right) = \sum_{\sum_s e_s = h} q^{\sum_{r=1}^{n} \binom{e_r}{2} - \binom{h}{2}} \left(q-q^{-1}\right)^{h-e_i-e_j} \prod_{k=1}^{n} {}^{\widehat{i},\widehat{j}} \, [e_k]_q! \sum_{r=0}^{e_i} \sum_{s=0}^{e_j} q^{\binom{r}{2} + \binom{s}{2}} (r)_q! \, (s)_q! \times$$
$$\times \binom{e_i}{r}_q \binom{e_j}{s}_q (q-1)^{r+s} \cdot \prod_{k=1}^{i-1} \mathbf{t}_{i,k}^{(e_k)} \binom{t_{i,i}\,;\,0}{e_i} \prod_{k=i+1}^{n} \mathbf{t}_{i,k}^{(e_k)} \otimes \prod_{k=1}^{j-1} \mathbf{t}_{k,j}^{(e_k)} \binom{t_{j,j}\,;\,0}{e_j} \prod_{k=j+1}^{n} \mathbf{t}_{k,j}^{(e_k)} \quad \forall\ i \neq j$$

$$\epsilon\left(\binom{\mathbf{t}_{\ell,\ell}\,;\,c}{k}\right) = \binom{c}{k}_q \,, \quad \epsilon\left(\mathbf{t}_{i,j}^{(h)}\right) = 0 \quad \forall\ h,k \in \mathbb{N},\ c \in \mathbb{Z},\ \ell,i,j = 1,\dots,n,\ i \neq j$$

(notation of §1.3) where terms like $\binom{x\otimes x\,;\,\sigma}{t}$ *(with* $x \in \{a,d\}$, $\sigma \in \mathbb{Z}$ *and* $t \in \mathbb{N}_+$ *) read*

$$\binom{x\otimes x\,;\,2\tau}{t} = \sum_{r+s=\nu} q^{-sr} \binom{x\,;\,\tau}{r} \otimes \binom{x\,;\,\tau}{s} x^r = \sum_{r+s=\nu} q^{-rs} x^s \binom{x\,;\,\tau}{r} \otimes \binom{x\,;\,\tau}{s}$$
$$\binom{x\otimes x\,;\,2\tau+1}{t} = \sum_{r+s=\nu} q^{-(1-s)r} \binom{x\,;\,\tau}{r} \otimes \binom{x\,;\,\tau+1}{s} x^r = \sum_{r+s=\nu} q^{-(1-r)s} x^s \binom{x\,;\,\tau+1}{r} \otimes \binom{x\,;\,\tau}{s}$$

according to whether σ *is even* ($=2\tau$) *or odd* ($=2\tau+1$)*, and consequently for* $\left\{ \begin{matrix} x\otimes x\,;\,\sigma \\ t\,,\,\ell \end{matrix} \right\}$.

In particular, $\mathcal{F}_q[M_n]$ *is a* $\mathbb{Z}\big[q,q^{-1}\big]$*–integral form (as a bialgebra) of* $F_q[M_n]$.

(b) $\mathcal{F}_q[GL_n]$ *is the unital associative* $\mathbb{Z}\big[q,q^{-1}\big]$*–algebra with generators* $\mathbf{t}_{i,j}^{(h)}$, $\binom{t_{\ell,\ell}\,;\,c}{k}$, D_q^{-1}, *for all* $i,j,\ell \in \{1,\dots,n\}$, $i \neq j$, $h,k \in \mathbb{N}$, $c \in \mathbb{Z}$, *enjoying the same relations as in claim (a) plus the additional relations*

$$D_q^{-1} \binom{t_{\ell,\ell}\,;\,c}{k} = \binom{t_{\ell,\ell}\,;\,c}{k} D_q^{-1} \,, \quad D_q^{-1}\, \mathbf{t}_{i,j}^{(h)} = \mathbf{t}_{i,j}^{(h)}\, D_q^{-1} \qquad \forall\ i,j,\ell,h,k,c\ \ (i \neq j)$$
$$\sum_{\sigma \in \mathcal{S}_n} (-q)^{l(\sigma)} t_{1,\sigma(1)}\, t_{2,\sigma(2)} \cdots t_{n,\sigma(n)} \cdot D_q^{-1} = 1 = D_q^{-1} \cdot \sum_{\sigma \in \mathcal{S}_n} (-q)^{l(\sigma)} t_{1,\sigma(1)}\, t_{2,\sigma(2)} \cdots t_{n,\sigma(n)}$$

where any $t_{i,j}$ *reads* $t_{i,j} := \left(q-q^{-1}\right) \mathbf{t}_{i,j}^{(1)}$ *if* $i \neq j$ *and* $t_{i,j} := 1 + (q-1) \binom{t_{i,i}\,;\,0}{1}$ *if* $i=j$.

Moreover, $\mathcal{F}_q[GL_n]$ *is a Hopf* $\mathbb{Z}\big[q,q^{-1}\big]$*–algebra, with Hopf algebra structure given by the formulæ in claim (a) for* Δ *and* ϵ *plus the implicit formulæ for the antipode*

$$S\left(\binom{t_{\ell,\ell}\,;\,c}{k}\right) = \binom{\det_q\Big(\big(t_{i,j}\big)_{i,j=1,\dots,n}^{i,j\neq\ell}\Big)\, D_q^{-1}\,;\,c}{k}$$
$$S\left(\mathbf{t}_{i,j}^{(h)}\right) = (-1)^{(i+j)h} \left(q-q^{-1}\right)^{-h} \cdot \det_q\Big(\big(t_{r,s}\big)_{r,s=1,\dots,n}^{r\neq i;\, s\neq j}\Big)^h \cdot D_q^{-h} \Big/ [h]_q!$$

(for all $\ell,i,j = 1,\dots,n$, $h,k \in \mathbb{N}$, $c \in \mathbb{Z}$, *with* $i \neq j$ *), and*

$$\Delta\big(D_q^{-1}\big) = D_q^{-1} \otimes D_q^{-1} \,, \qquad \varepsilon\big(D_q^{-1}\big) = 1 \,, \qquad S\big(D_q^{-1}\big) = D_q \,.$$

In particular, $\mathcal{F}_q[GL_n]$ *is a* $\mathbb{Z}\big[q,q^{-1}\big]$*–integral form (as a Hopf algebra) of* $F_q[GL_n]$.

(c) $\mathcal{F}_q[SL_n]$ is generated, as a unital associative $\mathbb{Z}\big[q,q^{-1}\big]$–algebra, by generators as in claim (a). These generators enjoy all relations in claim (a), plus some additional relations (springing out of the relation $D_q = 1$ in $F_q[SL_n]$).

Moreover, $\mathcal{F}_q[SL_n]$ is a Hopf $\mathbb{Z}\big[q,q^{-1}\big]$–algebra, whose Hopf algebra structure is given by the formulæ in claim (a) for Δ and ϵ, and by the formulæ in (b), but taking into account $D_q = 1$ for the antipode.

In particular, $\mathcal{F}_q[SL_n]$ is a $\mathbb{Z}\big[q,q^{-1}\big]$–integral form (as a Hopf algebra) of $F_q[SL_n]$.

Proof. (a) By Theorem 4.3, the set of elements considered in the statement generates $\mathcal{F}_q[M_n]$ (and even more, as for binomial coefficients we can restrict to take only those with $r = 0 = s$).

As for relations, we set an alphanumerical key on their left, which refer to the type of relation, reminding what they arise from.

[qBC] is a reminder for "quantum binomial coefficients"; similarly **[qDP]** stand for "quantum divided powers". It is clear by definitions that all these relations do hold in $\mathcal{F}_q[M_n]$.

[H-V.n] (for $n = 1, 2, 3$) is a reminder for "horizontal-vertical": indeed, these are the relations induced — inside $\mathcal{F}_q[M_n]$ — by the "horizontal" or "vertical" relations holding true among matrix entries of the q–matrix $\big(t_{i,j}\big)_{i,j=1,\dots,n}$ of the generators of $F_q[M_n]$ which lie on the same row or the same column. Namely, $t_{i,j}\,t_{i,k} = q\,t_{i,k}\,t_{i,j}$ (for $j < k$) is a "horizontal" relation, and $t_{i,k}\,t_{h,k} = q\,t_{h,k}\,t_{i,k}$ (for $i < h$) is a "vertical" one. It is clear again by construction that both kind of relations induce the corresponding relations in $\mathcal{F}_q[M_n]$ as in the claim.

[CD.m] (for $m = 1, 2$) stands for "counter-diagonal". These are the relations induced — inside $\mathcal{F}_q[M_n]$ — by the relations in $F_q[M_n]$ involving a couple of generators which, as entries of the q–matrix of generators, are in "counter-diagonal" mutual position. Namely, we mean relations of type $t_{i,l}\,t_{j,k} = t_{j,k}\,t_{i,l}$ with $i < j$ and $k < l$, that are plain commutation relations. Then both **[CV.1]** and **[CV.2]** are trivially true in $\mathcal{F}_q[M_n]$.

Finally, **[D.m (±)]** (for $m = 1, \dots, 4$) are labels for "diagonal" relations, i.e. all those relations — among the generators in the claim — which are induced by relations in $F_q[M_n]$ involving a couple of entries in the q–matrix $\big(t_{i,j}\big)_{i,j=1,\dots,n}$ which stand in mutual diagonal position, that is relations of type $t_{i,k}\,t_{j,l} - t_{j,l}\,t_{i,k} = \big(q - q^{-1}\big)\,t_{i,l}\,t_{j,k}$ with $i < j$ and $k < l$. But now, this single type of relation in $F_q[M_n]$ gives rise to several types of relations in $\mathcal{F}_q[M_n]$. Indeed, once one fixes the ("mutually diagonal") positions, say (i,k) and (j,l), we can single out (within in the q–matrix) the unique rectangle R which has one diagonal with (i,k) and (j,l) as vertices. Then different types of relations occur depending on the position of the four corners of R w.r.t the main diagonal of the q–matrix. In the sequel, we call "diagonal" the corners of R which lie on the top-left to bottom-right diagonal of R, and "counter-diagonal" those on the top-right to bottom-left diagonal of R.

Relations of type **[D.1]** occur when no corner of R lies on the main diagonal.

Relations **[D.2]** show up when both diagonal corners of R are on the main diagonal.

Relations of type **[D.3±]** occur when one of the counter-diagonal corners of R lies on the main diagonal. If it is the bottom-left one, we have relation **[D.3+]**, with the "+" sign to remind us that the rectangle R lies above the main diagonal. If instead it is the top-right one, we have relation **[D.3−]**, where "−" reminds that R lies below the diagonal.

Similarly, we have relations of type **[D.4±]** when one of the diagonal corners of R lies on the main diagonal. If it is the top-left one, we have relation **[D.4+]**, while if it is the bottom-right one, we have relation **[D.4−]** instead.

In any case, these "diagonal relations" are the only ones which are not immediate from definitions. We shall now prove all of them by suitable induction arguments.

To simplify notations, we set the following terminology for the corners of R:

$$a := t_{i,j} \quad , \quad b := t_{i,k} \quad , \quad c := t_{\ell,j} \quad , \quad d := t_{\ell,k} \quad , \qquad \text{so that} \qquad R = \begin{matrix} a & \cdots & b \\ \vdots & & \vdots \\ c & \cdots & d \end{matrix} \tag{4.1}$$

where some of the indices may also coincide: they do iff a or d is on the main diagonal. These generators do generate a copy of $F_q[M_2]$ inside $F_q[M_n]$, hence the identities we are going to prove actually are identities in $F_q[M_2]$.

In terms of (4.1), formula **[D.1]** reads (for all $\big|\{i,j,\ell,k\}\big| = 4\,,\ i<\ell\,,\ j<k\,$)

$$d^{(f)}\,a^{(h)} \;=\; \sum_{s=0}^{h\wedge f} (-1)^s\, q^{\binom{s+1}{2} - s\,(h+f-s)} \big(q-q^{-1}\big)^s\, [s]_q! \,\cdot\, a^{(h-s)}\, b^{(s)}\, c^{(s)}\, d^{(f-s)} \quad ; \tag{4.2}$$

by the identities $x^{(m)} = x^m \big/ [m]_q!$ (for all m) this formula is equivalent, inside $F_q[M_n]$, to

$$d^f\, a^h \;=\; \sum_{s=0}^{h\wedge f} (-1)^s\, q^{\binom{s+1}{2} - s\,(h+f-s)} \big(q-q^{-1}\big)^s \begin{bmatrix} h \\ s \end{bmatrix}_q \begin{bmatrix} f \\ s \end{bmatrix}_q [s]_q! \,\cdot\, a^{h-s}\, b^s\, c^s\, d^{f-s} \tag{4.3}$$

so we shall prove the latter. The proof is inductive: first on h for $f=1$, and later on $f<h$. Then one can use the $\mathbb{Q}(q)$–algebra anti-automorphism of $F_q[M_2]$ given by $a \mapsto d$, $b \mapsto c$, $c \mapsto b$ and $d \mapsto a$, to handle the opposite case, that is for $f \geq h$.

For $h=f=1$ formula (4.3) is true: it is the relation $a\,d - d\,a = \big(q-q^{-1}\big)\, b\,c$. Then set $f=1$, and assume $d\,a^h = a^h\,d - q^{1-h}\big(q-q^{-1}\big)\begin{bmatrix} h \\ 1 \end{bmatrix}_q a^{h-1}\,b\,c$, by induction on h. Then

$$\begin{aligned} d\,a^{h+1} \;&=\; \big(d\,a^h\big)\,a \;=\; \Big(a^h\,d - q^{1-h}\big(q-q^{-1}\big)\begin{bmatrix} h \\ 1 \end{bmatrix}_q a^{h-1}\,b\,c\Big)\,a \;= \\ &=\; a^h\big(a\,d - \big(q-q^{-1}\big)\,b\,c\big) - q^{1-h-2}\big(q-q^{-1}\big)\begin{bmatrix} h \\ 1 \end{bmatrix}_q a^h\,b\,c \;= \\ &=\; a^{h+1}\,d - \big(q-q^{-1}\big)\Big(1 + q^{1-h-2}\begin{bmatrix} h \\ 1 \end{bmatrix}_q\Big)\,a^h\,b\,c \;=\; a^{h+1}\,d - \big(q-q^{-1}\big)\,q^{1-(h+1)}\begin{bmatrix} h+1 \\ 1 \end{bmatrix}_q a^h\,b\,c \end{aligned}$$

thanks to the identity $1 + q^{1-h-2}\begin{bmatrix} h \\ 1 \end{bmatrix}_q = q^{-h}\begin{bmatrix} h+1 \\ 1 \end{bmatrix}_q$. This ends the proof for $f=1$.

Now assuming that (4.3) hold for some f with $1<f<h$, we prove it for $f+1$. Using shorthand notation $\alpha^s_{f,h} := \binom{s+1}{2} - s\,(h+f-s)$, we have

$$\begin{aligned} d^{f+1}\,a^h \;&=\; d\big(d^f\,a^h\big) \;=\; d\left(\sum_{s=0}^{f} (-1)^s\, q^{\alpha^s_{f,h}} \big(q-q^{-1}\big)^s \begin{bmatrix} h \\ s \end{bmatrix}_q \begin{bmatrix} f \\ s \end{bmatrix}_q [s]_q! \,\cdot\, a^{h-s}\, b^s\, c^s\, d^{f-s}\right) \;= \\ &=\; \sum_{s=0}^{f} (-1)^s\, q^{\alpha^s_{f,h}} \big(q-q^{-1}\big)^s \begin{bmatrix} h \\ s \end{bmatrix}_q \begin{bmatrix} f \\ s \end{bmatrix}_q [s]_q! \,\cdot\, d\,a^{h-s}\, b^s\, c^s\, d^{f-s} \;= \\ &=\; \sum_{s=0}^{f} (-1)^s\, q^{\alpha^s_{f,h}-2\,s} \big(q-q^{-1}\big)^s \begin{bmatrix} h \\ s \end{bmatrix}_q \begin{bmatrix} f \\ s \end{bmatrix}_q [s]_q! \,\cdot\, a^{h-s}\, b^s\, c^s\, d^{f+1-s} \;+ \\ &\qquad +\; \sum_{s=0}^{f} (-1)^{s+1}\, q^{\alpha^s_{f,h}+1-h+s} \big(q-q^{-1}\big)^{s+1} \begin{bmatrix} h \\ s \end{bmatrix}_q \begin{bmatrix} f \\ s \end{bmatrix}_q \begin{bmatrix} h-s \\ 1 \end{bmatrix}_q [s]_q! \,\cdot\, a^{h-s-1}\, b^{s+1}\, c^{s+1}\, d^{f-s} \end{aligned}$$

Now, in the last sum change summation index from s to $s+1$. Then our last term reads

$$q^{\alpha^0_{f,h}}\, a^h\, d^{f+1} \;+\; (-1)^{f+1}\, q^{\alpha^f_{f,h}+f+1-h} \begin{bmatrix} h \\ f \end{bmatrix}_q \begin{bmatrix} f \\ f \end{bmatrix}_q \begin{bmatrix} h-f \\ 1 \end{bmatrix}_q [f]_q! \cdot a^{h-(f+1)}\, b^{f+1}\, c^{f+1} \;+$$
$$+ \sum_{s=1}^{f} (-1)^s \left(q-q^{-1}\right)^{s+1} \left(q^{\alpha^s_{f,h}-2s} \begin{bmatrix} h \\ s \end{bmatrix}_q \begin{bmatrix} f \\ s \end{bmatrix}_q [s]_q! \;+\; q^{\alpha^{s-1}_{f,h}+s-h} \times \right.$$
$$\left. \times \begin{bmatrix} h \\ s-1 \end{bmatrix}_q \begin{bmatrix} f \\ s-1 \end{bmatrix}_q \begin{bmatrix} h-s+1 \\ 1 \end{bmatrix}_q [s-1]_q! \right) \cdot a^{h-s}\, b^s\, c^s\, d^{f+1-s} \;=$$
$$= \sum_{s=0}^{f+1} (-1)^s\, q^{\alpha^s_{f+1,h}} \left(q-q^{-1}\right)^s \begin{bmatrix} h \\ s \end{bmatrix}_q \begin{bmatrix} f+1 \\ s \end{bmatrix}_q [s]_q! \cdot a^{h-s}\, b^s\, c^s\, d^{(f+1)-s}$$

thus ending the proof of **[D.1]**, thanks to the following identities:

$$\alpha^0_{f,h} \;=\; \alpha^0_{f+1,h} \;\;, \qquad\qquad \alpha^{s-1}_{f,h} + s - h \;=\; \binom{s+1}{2} - s\,(h+f+1-s) + f + 1 - s$$
$$\alpha^s_{f,h} - 2s \;=\; \binom{s+1}{2} - s\,(h+f+1-s) - s \;\;, \qquad\qquad \alpha^s_{f+1,h} \;=\; \binom{s+1}{2} - s\,(h+f+1-s)$$
$$q^{-s} \left([f-s+1]_q + q^{f+1}\, [s]_q \right) \;=\; [f+1]_q$$

As to **[D.2]**, in terms of (4.1) it reads, with $\mathbf{b} := b \big/ \left(q-q^{-1}\right)$ and $\mathbf{c} := c \big/ \left(q-q^{-1}\right)$,

$$\binom{d\,;k}{r}\binom{a\,;h}{s} = \sum_{p=0}^{r\wedge s} (-1)^p\, q^{p\,((h+k)-(r+s))-\binom{p}{2}} \left(q-q^{-1}\right)^p [p]_q!\, \mathbf{b}^{(p)} \left\{ \begin{matrix} a\,;\, h-p \\ s\,,\, p \end{matrix} \right\} \left\{ \begin{matrix} d\,;\, k-p \\ r\,,\, p \end{matrix} \right\} \mathbf{c}^{(p)}$$

a formula which is proved in [GR], §3.4.

For **[D.3+]**, set notation as in (4.1), now with $\ell = j$. Then (4.2) holds and, setting $\boldsymbol{x} := x \big/ \left(q-q^{-1}\right)$ for all $x \in \{a,b,c,d\}$, it yields

$$\mathbf{d}^{(f)}\, \mathbf{a}^{(h)} \;=\; \sum_{s=0}^{h\wedge f} (-1)^s\, q^{\binom{s+1}{2}-s\,(h+f-s)} \left(q-q^{-1}\right)^s [s]_q! \cdot \mathbf{a}^{(h-s)}\, \mathbf{b}^{(s)}\, \mathbf{c}^{(s)}\, \mathbf{d}^{(f-s)}$$

Now, for all $m \in \mathbb{N}$, the following formal identity hold (cf. [GR], Lemma 6.1):

$$x^m \;=\; \sum_{k=0}^{m} q^{\binom{k}{2}} \binom{m}{k}_q (q-1)^k\, (k)_q! \binom{x\,;\,0}{k} \quad .$$

Using this identity to expand $\mathbf{c}^{(s)} = \left(q-q^{-1}\right)^{-s} [s]_q^{-1} \cdot c^s$, our last formula turns into

$$\mathbf{d}^{(f)}\, \mathbf{a}^{(a)} \;=\; \sum_{s=0}^{h\wedge f} (-1)^s\, q^{\binom{s+1}{2}-s\,(h+f-s)}\, \mathbf{a}^{(h-s)}\, \mathbf{b}^{(s)}\, \mathbf{c}^{(s)}\, \mathbf{d}^{(f-s)} \;=$$
$$= \sum_{s=0}^{h\wedge f} \sum_{k=0}^{s} (-1)^s\, q^{\binom{s+1}{2}+\binom{k}{2}-s\,(h+f-s)} \binom{s}{k}_q (q-1)^k\, (k)_q! \cdot \mathbf{a}^{(h-s)}\, \mathbf{b}^{(s)} \binom{c\,;0}{k} \mathbf{d}^{(f-s)} \;=$$
$$= \sum_{s=0}^{h\wedge f} \sum_{k=0}^{s} (-1)^s\, q^{\binom{s+1}{2}+\binom{k}{2}-s\,(h+f-s)} \binom{s}{k}_q (q-1)^k\, (k)_q! \cdot \mathbf{a}^{(h-s)}\, \mathbf{b}^{(s)}\, \mathbf{d}^{(f-s)} \binom{c\,;\,f-r}{k}$$

where in the last step we used **[H-V.2]**. This proves **[D.3+]**, and **[D.3−]** is entirely similar.

Finally we prove [**D.4±**], starting with [**D.4+**]. We use notation of (4.1), but with $i = j$ and then using index j instead of k; more in general, we set $a := t_{i,i}$, $\mathbf{b}^{(r)} := \mathbf{t}_{i,j}^{(r)}$, $\mathbf{c}^{(r)} := \mathbf{t}_{i,j}^{(r)}$, $\mathbf{d}^{(r)} := \mathbf{t}_{\ell,j}^{(r)}$ (for $r \in \mathbb{N}$). Then formula [**D.4+**] reads (with index u instead of c)

$$\mathbf{d}^{(f)} \binom{a\,;\,u}{k} = \sum_{s=0}^{f \wedge k} (-1)^s \, q^{\binom{s+1}{2} - s\,(f+k-u)} \, [s]_q! \, \big(q - q^{-1}\big)^s \cdot \left\{ \begin{matrix} a\,;\,u-2s \\ k\,,s \end{matrix} \right\} \mathbf{b}^{(s)} \, \mathbf{c}^{(s)} \, \mathbf{d}^{(f-s)}$$

which is equivalent to the commutation relation in $F_q[M_2]$

$$d^f \binom{a\,;\,u}{k} = \sum_{s=0}^{f \wedge k} (-1)^s \, q^{\binom{s+1}{2} - s\,(f+k-u)} \begin{bmatrix} f \\ s \end{bmatrix}_q \cdot \left\{ \begin{matrix} a\,;\,u-2s \\ k\,,s \end{matrix} \right\} b^s \, c^s \, d^{f-s} \tag{4.4}$$

So we must prove this last identity: we do that by induction on k and f.

For $k = 1 = f$, formula (4.4) reads $d \binom{a\,;u}{1} = \binom{a\,;u}{1} d - q^{u-1}\,(q+1)\,b\,c$; this is equivalent to the identity $d\,\big(a\,q^u - 1\big) = \big(a\,q^u - 1\big)\,d - q^{u-1}\,\big(q^2 - 1\big)\,b\,c$, which is easy to prove.

For $k > 1 = f$, formula (4.4) reads

$$d \binom{a\,;u}{k} = \binom{a\,;u}{k} d - q^{u-k} \left(\binom{a\,;\,u-2}{k-1} + q \binom{a\,;\,u-1}{k-1} \right) b\,c \tag{4.5}$$

which, using compact notation $(a\,;k,u)_q := (q-1)^k\,(k)_q!\,\binom{a\,;u}{k}$, is equivalent to

$$d\,(a\,;k,u)_q = (a\,;k,u)_q\,d - q^{u-k}\,\big(q^k - 1\big)\,(a\,;k-2,u-2)_q \Big(\big(a\,q^{u-k} - 1\big) + q\,\big(a\,q^{u-1} - 1\big)\Big)\,b\,c$$

Making induction, the basis is proved above for $k=1$; the inductive step from k to $k+1$ is

$$\begin{aligned}
d\,(a\,;k+1,u)_q &= d\,\big(a\,q^{u-k} - 1\big)\,(a\,;k,u)_q = \\
&= \big(a\,q^{u-k}-1\big)\,d\,(a\,;k,u)_q - q^{u-(k+1)}\,\big(q^2-1\big)\,b\,c\,(a\,;k,u)_q = \big(a\,q^{u-k}-1\big)\,(a\,;k,u)_q\,d - \\
&\quad - q^{u-k}\,\big(q^k-1\big)\,\big(a\,q^{u-k}-1\big)\,(a\,;k-2,u-2)_q\,\Big(\big(a\,q^{u-k}-1\big) + q\,\big(a\,q^{u-1}-1\big)\Big)\,b\,c - \\
&\quad - q^{u-(k+1)}\,\big(q^2-1\big)\,(a\,;k,u-2)_q\,b\,c = \\
&= (a\,;k+1,u)_q\,d - q^{u-(k+1)}\,(a\,;k-1,u-2)_q \times \\
&\quad \times \Big(q\,\big(q^k-1\big)\,\Big(\big(a\,q^{u-k}-1\big) + q\,\big(a\,q^{u-1}-1\big)\Big) + \big(q^2-1\big)\,\big(a\,q^{u-k-1}-1\big)\Big)\,b\,c = \\
&= (a\,;k+1,u)_q\,d - q^{u-(k+1)}\,\big(q^{k+1}-1\big)\,(a\,;k-1,u-2)_q\,\Big(\big(a\,q^{u-(k+1)}-1\big) + q\,\big(a\,q^{u-1}-1\big)\Big)\,b\,c
\end{aligned}$$

where the very last step follows from the (straightforward) identity

$$\begin{aligned}
q\,\big(q^k-1\big)\,\Big(\big(a\,q^{u-k}-1\big) + q\,\big(a\,q^{u-1}-1\big)\Big) + \big(q^2-1\big)\,\big(a\,q^{u-k-1}-1\big)\Big) &= \\
&= \big(q^{k+1}-1\big)\,\Big(\big(a\,q^{u-(k+1)}-1\big) + q\,\big(a\,q^{u-1}-1\big)\Big)
\end{aligned}$$

To move next step, we need a more general formula than (4.5), namely

$$d \left\{ \begin{matrix} a\,;u \\ k\,,h \end{matrix} \right\} = \left\{ \begin{matrix} a\,;u \\ k\,,h \end{matrix} \right\} d - q^{u-k+h} \left\{ \begin{matrix} a\,;u-2 \\ k\,,h+1 \end{matrix} \right\} b\,c \tag{4.6}$$

We prove it by induction on h. The basis holds by (4.5), which also gives the inductive step:

$$d \left\{ \begin{matrix} a\,;u \\ k\,,h \end{matrix} \right\} = d \sum_{s=0}^{h} q^{\binom{s+1}{2}} \binom{h}{s}_q \binom{a;\,u+s}{k-h} = \sum_{s=0}^{h} q^{\binom{s+1}{2}} \binom{h}{s}_q d \binom{a;\,u+s}{k-h} =$$

$$= \sum_{s=0}^{h} q^{\binom{s+1}{2}} \binom{h}{s}_q \left(\binom{a;\,u+s}{k-h} d - q^{u+s-k+h} \left(\binom{a;\,u+s-2}{k-h-1} + q \binom{a;\,u+s-1}{k-h-1} \right) b\,c \right) =$$

$$= \left\{ \begin{matrix} a\,;u \\ k\,,h \end{matrix} \right\} d - q^{u-k+h} \left(\binom{a;\,u-2}{k-h-1} + \sum_{s=1}^{h} q^{\binom{s}{2}+s} \left(q^s \binom{h}{s}_q + \binom{h}{s-1}_q \right) \times \right.$$

$$\left. \times \binom{a;\,u+s-2}{k-h-1} + q^{\binom{h+1}{2}+h+1} \binom{a;\,u+h-1}{k-h-1} \right) b\,c =$$

$$= \left\{ \begin{matrix} a\,;u \\ k\,,h \end{matrix} \right\} d - q^{u-k+h} \sum_{s=0}^{h+1} q^{\binom{s+1}{2}} \binom{h+1}{s}_q \binom{a;\,u+s-2}{k-h-1} b\,c = \left\{ \begin{matrix} a\,;u \\ k,h \end{matrix} \right\} d - q^{u-k+h} \left\{ \begin{matrix} a\,;u-2 \\ k\,,h+1 \end{matrix} \right\} b\,c$$

Finally, we assume (4.4) holds for k and f, and we prove it for $f+1$ too. By (4.6) we have

$$d^{f+1} \binom{a;\,u}{k} = d\, d^f \binom{a;\,u}{k} = \sum_{s=0}^{f \wedge k} (-1)^s q^{\binom{s+1}{2} - s\,(f+k-u)} \begin{bmatrix} f \\ s \end{bmatrix}_q d \left\{ \begin{matrix} a;\,u-2\,s \\ k\,,s \end{matrix} \right\} b^s c^s d^{f-s} =$$

$$= \sum_{s=0}^{f \wedge k} (-1)^s q^{\binom{s+1}{2} - s\,(f+k-u) - 2\,s} \begin{bmatrix} f \\ s \end{bmatrix}_q \left\{ \begin{matrix} a;\,u-2\,s \\ k\,,s \end{matrix} \right\} b^s c^s d^{f+1-s} -$$

$$- \sum_{s=0}^{f \wedge k} (-1)^s q^{\binom{s+1}{2} - s\,(f+k-u) - u - s - k} \begin{bmatrix} f \\ s \end{bmatrix}_q \left\{ \begin{matrix} a;\,u-2\,(s+1) \\ k\,,s+1 \end{matrix} \right\} b^{s+1} c^{s+1} d^{f-s} =$$

$$= \sum_{s=0}^{(f+1) \wedge k} (-1)^s q^{\binom{s+1}{2} - s\,(f+1+k-u)} \left(q^{-s} \begin{bmatrix} f \\ s \end{bmatrix}_q + q^{f+1-s} \begin{bmatrix} f \\ s-1 \end{bmatrix}_q \right) \left\{ \begin{matrix} a;\,u-2\,s \\ k\,,s \end{matrix} \right\} b^s c^s d^{f+1-s} =$$

$$= \sum_{s=0}^{(f+1) \wedge k} (-1)^s q^{\binom{s+1}{2} - s\,(f+1+k-u)} \begin{bmatrix} f+1 \\ s \end{bmatrix}_q \left\{ \begin{matrix} a;\,u-2s \\ k\,,s \end{matrix} \right\} b^s c^s d^{f+1-s}$$

which eventually proves (4.4) for $f+1$ as well.

The analysis above proves the identity (4.4), which is equivalent to relation **[D.4+]**. The proof of relation **[D.4−]** is entirely similar, just along the same pattern.

So far we have proved that the given relations do hold in $\mathcal{F}_q[M_n]$; to prove the first part of the claim, we must show that these generate the ideal of *all* relations among generators. This amounts to show that the algebra enjoying only the given relations is isomorphic to $\mathcal{F}_q[M_n]$. In turn, this is equivalent to the following: if $\mathcal{B}'$ is any of the PBW–like bases of Theorem 4.3, then the given relations are enough to expand any product of generators as a $\mathbb{Z}\big[q, q^{-1}\big]$–linear combination of the monomials in $\mathcal{B}'$. But this is clear for the very relations themselves.

As to the bialgebra structure, everything is just a matter of computations, via a cute use of Lemma 1.5, and partially basing upon the case $n = 2$. We point out the main steps.

Assume $i < j$. By definitions we have

$$\Delta\left(\mathbf{t}_{i,j}^{(h)}\right) = \Delta\Big(\big(q-q^{-1}\big)^{-1} t_{i,j}\Big)^{(h)} = \big(q-q^{-1}\big)^{-h} \Big(\sum_{k=1}^{n} t_{i,k} \otimes t_{k,j}\Big)^{(h)} . \qquad (4.7)$$

But $k < k'$ implies $(t_{i,k} \otimes t_{k,j}) \cdot (t_{i,k'} \otimes t_{k',j}) = q^2 (t_{i,k'} \otimes t_{k',j}) \cdot (t_{i,k} \otimes t_{k,j})$, thus Lemma 1.5*(b)* and (4.7) yield

$$\Delta\left(\mathbf{t}_{i,j}^{(h)}\right) = \big(q-q^{-1}\big)^{-h} \sum_{e_1+\cdots+e_n=h} q^{\binom{e_1+1}{2}+\cdots+\binom{e_n+1}{2}-\binom{h+1}{2}} y_1^{(e_1)} y_2^{(e_2)} \cdots y_n^{(e_n)} \qquad (4.8)$$

for $y_k := t_{i,k} \otimes t_{k,j}$ $(k = 1, \dots, n)$. But now

$$y_k^{(e_k)} = \big(t_{i,k} \otimes t_{k,j}\big)^{(e_k)} = [e_k]_q! \cdot t_{i,k}^{(e_k)} \otimes t_{k,j}^{(e_k)} = \big(q-q^{-1}\big)^{2\,e_k} [e_k]_q! \cdot \mathbf{t}_{i,k}^{(e_k)} \otimes \mathbf{t}_{k,j}^{(e_k)} , \qquad \forall\, k \notin \{i,j\}$$

$$y_i^{(e_i)} = \big(t_{i,i} \otimes t_{i,j}\big)^{(e_i)} = t_{i,i}^{e_i} \otimes t_{i,j}^{(e_i)} = \big(q-q^{-1}\big)^{e_i} \cdot \sum_{r=0}^{e_i} q^{\binom{r}{2}} \binom{e_i}{r}_q (q-1)^r (r)_q! \cdot \binom{t_{i,i}\,;\,0}{r} \otimes \mathbf{t}_{i,j}^{(e_i)} ,$$

$$y_j^{(e_j)} = \big(t_{i,j} \otimes t_{j,j}\big)^{(e_j)} = t_{i,j}^{(e_j)} \otimes t_{j,j}^{e_j} = \big(q-q^{-1}\big)^{e_j} \cdot \sum_{s=0}^{e_j} q^{\binom{s}{2}} \binom{e_j}{s}_q (q-1)^s (s)_q! \cdot \mathbf{t}_{i,j}^{(e_j)} \otimes \binom{t_{j,j}\,;\,0}{s} .$$

Using these facts and the commutation relations between generators, (4.8) turns into the formula for $\Delta\left(\mathbf{t}_{i,j}^{(h)}\right)$ given in the claim. The case $i > j$ is entirely similar.

As to $\Delta\binom{t_{i,i}\,;\,c}{k}$, by definition of Δ we have $\Delta\left(\binom{t_{i,i}\,;\,c}{k}\right) = \binom{\sum_{s=1}^{n} t_{i,s} \otimes t_{s,i}\,;\,c}{k}$. To expand the latter term, we use twice Lemma 1.6 and once Lemma 1.5. First apply Lemma 1.6*(a-2)* to $x := \sum_{h=1}^{i} t_{i,h} \otimes t_{h,i}$, $w := \sum_{l=i+1}^{n} \mathbf{t}_{i,l} \otimes \mathbf{t}_{l,i}$, $t = c$, $m = k$; this gives an expansion formula in which some q–divided powers $w^{(r)} = \big(\sum_{l=i+1}^{n} \mathbf{t}_{i,l} \otimes \mathbf{t}_{l,i}\big)^{(r)}$ and some terms $\left\{ {x\,;\,c \atop m\,,\,r} \right\}$ occur. Expand the latter ones as a linear combination of some $\binom{x\,;\,a}{b}$'s, and then apply Lemma 1.6*(c-2)* to them with $t_{i,i} \otimes t_{i,i}$ in the rôle of x and $\sum_{h=1}^{i-1} \mathbf{t}_{i,h} \otimes \mathbf{t}_{h,i}$ acting as w. This eventually gives an expansion formula which is a linear combination of products of q–divided powers $\big(\sum_{h=1}^{i-1} \mathbf{t}_{i,h} \otimes \mathbf{t}_{h,i}\big)^{(s)}$, q–binomial coefficients $\binom{t_{i,i} \otimes t_{i,i}\,;\,a}{b}$ and q–divided powers $\big(\sum_{l=i+1}^{n} \mathbf{t}_{i,l} \otimes \mathbf{t}_{l,i}\big)^{(r)}$, in this order. Now, the $\binom{t_{i,i} \otimes t_{i,i}\,;\,a}{b}$'s can be patched together into some $\left\{ {t_{i,i} \otimes t_{i,i}\,;\,\alpha \atop \gamma\,,\,\delta} \right\}$'s, and the q–divided powers $\big(\sum_{h=1}^{i-1} \mathbf{t}_{i,h} \otimes \mathbf{t}_{h,i}\big)^{(s)}$ and $\big(\sum_{l=i+1}^{n} \mathbf{t}_{i,l} \otimes \mathbf{t}_{l,i}\big)^{(r)}$ can be expanded using Lemma 1.5*(b)*, and then also the obvious identities $\big(\mathbf{t}_{i,s} \otimes \mathbf{t}_{s\,i}\big)^{(e)} = [e]_q! \cdot \mathbf{t}_{i,s}^{(e)} \otimes \mathbf{t}_{s\,i}^{(e)}$. Using the commutation relations between generators, the final outcome can be written as claimed.

Eventually, $\epsilon\left(\binom{t_{\ell,\ell}\,;\,c}{k}\right) = \binom{c}{k}_q$ and $\epsilon\left(\mathbf{t}_{i,j}^{(h)}\right) = 0$ follow from the very definitions.

(b) The fact that $\mathcal{F}_q[GL_n]$ admits the presentation above is a direct consequence of Theorem 4.3*(b)* and of the presentation of $\mathcal{F}_q[M_n]$ given in claim *(a)*; in particular, the additional relations in first line simply mean that D_q^{-1} is central — because D_q itself is central — while the second line relation is a reformulation of the relation $D_q\, D_q^{-1} = 1$. The statement

about the Hopf structure also follows from claim *(a)* and Theorem 4.3*(b)* and from the formulæ for the antipode in $F_q[GL_n]$ (cf. §2.4), but for the formulæ for ${D_q}^{-1}$ which follow from $\Delta(D_q) = D_q \otimes D_q$, $\varepsilon(D_q) = 1$, $S(D_q) = {D_q}^{-1}$. Clearly, the formulæ for the antipode of generators $\binom{t_{\ell,\ell}\,;\,c}{k}$ and $\mathbf{t}_{i,j}^{(h)}$ are *implicit*, in that one should still expand the right-hand sides of them in terms of the generators themselves.

Now, the formulæ for Δ and ϵ do show that $\mathcal{F}_q[GL_n]$ is indeed a $\mathbb{Z}\big[q,q^{-1}\big]$–subbialgebra of $F_q[GL_n]$. For the antipode, $\Big\langle S\big(\mathcal{F}_q[GL_n]\big)\,,\,\mathcal{U}_q(\mathfrak{gl}_n)\Big\rangle = \Big\langle \mathcal{F}_q[GL_n]\,,\,S\big(\mathcal{U}_q(\mathfrak{gl}_n)\big)\Big\rangle \subseteq \mathbb{Z}\big[q,q^{-1}\big]$, which gives $S\big(\mathcal{F}_q[GL_n]\big) \subseteq \mathcal{F}_q[GL_n]$. The claim follows.

(c) This follows at once from claims *(a)* and *(b)*, along with Corollary 4.5. □

4.7 Remarks:

(a) The commutation relations in Theorem 4.6 are not the only possible ones, but several (equivalent) ones exist. The given ones are best suited to expand any product of the generators as a $\mathbb{Z}\big[q,q^{-1}\big]$–linear combination of the elements of a PBW basis as in Theorem 4.3.

(b) As they belong to $\mathcal{F}_q[GL_n]$, one could try to compute an explicit expression for the elements $S\Big(\binom{t_{\ell,\ell}\,;\,c}{k}\Big)$, as well as for $S\big(\mathbf{t}_{i,j}^{(h)}\big)$, *in terms of the generators given in Theorem 4.6(b)!* However, such a formula might be very complicated.

(c) In [GR], §5.6, some of the "additional relations" mentioned in Theorem 4.6*(c)* are found explicitly in case $n=2$.

(d) Theorem 4.6 also yields presentations at $q=1$, which implies the important corollary here below. As a consequence, since objects like $U_{\mathbb{Z}}\big({\mathfrak{gl}_n}^*\big)$ are usually called "hyperalgebras", the previous result allows us to call $\mathcal{F}_q[M_n]$, $\mathcal{F}_q[GL_n]$ and $\mathcal{F}_q[SL_n]$ *"quantum hyperalgebras"*.

Theorem 4.8.

(a) There exists a $\mathbb{Z}$–bialgebra isomorphism $\mathcal{F}_1[M_n] \cong U_{\mathbb{Z}}\big({\mathfrak{gl}_n}^*\big)$ given by

$$\mathbf{t}_{i,j}^{(h)}\Big|_{q=1} \mapsto (-1)^{h(j-i)}\,\mathrm{e}_{i,j}^{(h)} \; , \qquad \binom{t_{s,s}\,;\,0}{k}\bigg|_{q=1} \mapsto \binom{\mathrm{g}_s}{k} \; , \qquad \mathbf{t}_{j,i}^{(h)}\Big|_{q=1} \mapsto (-1)^{h(j-i-1)}\,\mathrm{f}_{j,i}^{(h)}$$

for all h, s and $i<j$. In particular $\mathcal{F}_1[M_n]$ is a Hopf $\mathbb{Z}$–algebra, isomorphic to $U_{\mathbb{Z}}\big({\mathfrak{gl}_n}^*\big)$.

(b) There exists a Hopf $\mathbb{Z}$–algebra isomorphism $\mathcal{F}_1[GL_n] \cong U_{\mathbb{Z}}\big({\mathfrak{gl}_n}^*\big)$, which is uniquely determined by the formulæ in claim (a).

(c) There exists a Hopf $\mathbb{Z}$–algebra isomorphism $\mathcal{F}_1[SL_n] \cong U_{\mathbb{Z}}\big({\mathfrak{sl}_n}^*\big)$ given by the same formulæ as in claim (a).

Proof. The presentation of $\mathcal{F}_q[M_n]$ given in Theorem 4.5 provides a similar presentation for $\mathcal{F}_1[M_n]$; a straightforward comparison then shows that the latter is the standard presentation of $U_{\mathbb{Z}}\big({\mathfrak{gl}_n}^*\big)$, following the correspondence given in the claim. To be precise in the first presentation one also has the specialization at $q=1$ of the $\binom{t_{\ell,\ell}\,;\,r}{k}$'s, but these are generated by the specializations of the $\binom{t_{\ell,\ell}\,;\,0}{\nu}$'s. This yields a $\mathbb{Z}$–algebra isomorphism: a moment's check shows that it is one of $\mathbb{Z}$–bialgebras too.

Up to minimal changes, the proof goes the same for claims *(b)* and *(c)* as well. □

The previous result regards specialization at $q = 1$. As to specializations at roots of 1, Theorem 3.7 yields the following important consequence:

Proposition 4.9. *Let ε be a root of unity, of odd order, and apply notation of §1.4.*

(a) The specialization $\mathcal{F}_\varepsilon[M_n]$ is a $\mathbb{Z}_\varepsilon$–bialgebra, isomorphic to $\mathfrak{H}_\varepsilon^g$ via the specialization of the embedding $\mathcal{F}_q[M_n] \longhookrightarrow \mathfrak{H}_q^g$.

(b) The embedding $\mathcal{F}_\varepsilon[M_n] \longhookrightarrow \mathcal{F}_\varepsilon[GL_n]$ of $\mathbb{Z}_\varepsilon$–bialgebras is an isomorphism. In particular, $\mathcal{F}_\varepsilon[M_n]$ and $\mathfrak{H}_\varepsilon^g$ both are Hopf $\mathbb{Z}_\varepsilon$–algebras isomorphic to $\mathcal{F}_\varepsilon[GL_n]$.

(c) The specialization $\mathcal{F}_\varepsilon[SL_n]$ is a Hopf $\mathbb{Z}_\varepsilon$–algebra, isomorphic to $\mathfrak{H}_\varepsilon^s$ via the specialization of the embedding $\mathcal{F}_q[SL_n] \hookrightarrow \mathfrak{H}_q^s$.

Proof. From Theorem 3.7*(a)* one argues $\mathcal{F}_\varepsilon[M_n]\big[\phi^{-1}\big] = \mathfrak{H}_\varepsilon^g$, which is an identity induced by the embedding $\widehat{\xi} : \mathcal{F}_\varepsilon[M_n] \longhookrightarrow \mathfrak{H}_\varepsilon^g$; note that we can see that the latter map is injective by looking at how the PBW basis of $\mathcal{F}_q[M_n]$ given in Theorem 4.3*(a)* expands w.r.t. to a suitable PBW basis of $\mathfrak{H}_q^g$, as we did exactly in the very proof of Theorem 4.3*(a)*. Now, for $\phi := \Lambda_1 \Lambda_2^2 \cdots \Lambda_n^n$ one has $\binom{\phi\,;\,c}{k} \in \mathfrak{H}_q^g$ for all $k \in \mathbb{N}$, $c \in \mathbb{Z}$. This is easily proved using Theorem 3.1 in [DL]; otherwise, it can also be proved by a brute force computation. Then

$$\binom{\phi\,;\,c}{k} \in \mathfrak{H}_q^g \bigcap F_q[M_n] = \mathcal{F}_q[M_n] \qquad\qquad \big(\forall\ k \in \mathbb{N}\,,\, c \in \mathbb{Z}\big)$$

Now we apply Lemma 1.8 to $x = \phi$, which gives $\phi^{\,\ell} \equiv 1 \mod (q-\varepsilon)\,\mathcal{F}_q[M_n]$. Thus $\phi^{\,\ell} = 1 \in \mathcal{F}_\varepsilon[M_n]$, so $\phi^{-1} = \phi^{\,\ell-1} \in \mathcal{F}_\varepsilon[M_n]$ and so $\mathcal{F}_\varepsilon[M_n] = \mathcal{F}_\varepsilon[M_n]\big[\phi^{-1}\big] = \mathfrak{H}_\varepsilon^g$.

The above proves claim *(a)*. Now, one can prove the identity $\mathcal{F}_\varepsilon[GL_n] = \mathfrak{H}_\varepsilon^g$ just like for claim *(a)*, or one can use the relations $\mathcal{F}_\varepsilon[M_n] \subseteq \mathcal{F}_\varepsilon[GL_n] \subseteq \mathfrak{H}_\varepsilon^g$ and $\mathfrak{H}_\varepsilon^g = \mathcal{F}_\varepsilon[M_n]$. As $\mathcal{F}_\varepsilon[GL_n]$ is clearly a Hopf $\mathbb{Z}_\varepsilon$–algebra, claim *(b)* is then proved. Similarly, *(c)* can be proved like *(a)*, or deduced as a corollary of *(a)* itself, of Corollary 4.5 and of Corollary 3.8. □

Theorem 4.6 and Proposition 3.3 also yield a description of quantum Frobenius morphisms:

Theorem 4.10. *Let ε be a root of unity, of odd order ℓ.*

(a) The quantum Frobenius morphism (2.1) for $\mathcal{F}_q[M_n]$ is defined over $\mathbb{Z}_\varepsilon$, i.e. it restricts to an epimorphism of $\mathbb{Z}_\varepsilon$–bialgebras

$$\mathcal{F}r_{M_n}^{\mathbb{Z}_\varepsilon} : \mathcal{F}_\varepsilon[M_n] \relbar\joinrel\twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{F}_1[M_n] \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{gl}_n^{\,*}\big)$$

coinciding, via Theorem 4.8 and Proposition 4.9, with (2.5), and given on generators by

$$\mathcal{F}r_{M_n}^{\mathbb{Z}_\varepsilon} : \begin{cases} (\,i<j\,)\ \mathbf{t}_{i,j}^{(k)}\Big|_{q=\varepsilon} \mapsto \mathbf{t}_{i,j}^{(k/\ell)}\Big|_{q=1} = (-1)^{(j-i)\,k/\ell}\, \mathrm{e}_{i,j}^{(k/\ell)} & \text{if } \ell \,\big|\, k\,, \quad \mathbf{t}_{i,j}^{(k)}\Big|_{q=\varepsilon} \mapsto 0 \quad \text{if } \ell \not\big|\, k \\ \binom{t_{i,i}\,;\,0}{k}\Big|_{q=\varepsilon} \mapsto \binom{t_{i,i}\,;\,0}{k/\ell}\Big|_{q=1} = \binom{\mathrm{g}_i}{k/\ell} & \text{if } \ell \,\big|\, k\,, \quad \binom{t_{i,i}\,;\,0}{n}\Big|_{q=\varepsilon} \mapsto 0 \quad \text{if } \ell \not\big|\, k \\ (\,i>j\,)\ \mathbf{t}_{i,j}^{(k)}\Big|_{q=\varepsilon} \mapsto \mathbf{t}_{i,j}^{(k/\ell)}\Big|_{q=1} = (-1)^{(j-i-1)\,k/\ell}\, \mathrm{f}_{i,j}^{(k/\ell)} & \text{if } \ell \,\big|\, k\,, \quad \mathbf{t}_{i,j}^{(k)}\Big|_{q=\varepsilon} \mapsto 0 \quad \text{if } \ell \not\big|\, k \end{cases}$$

(b) The quantum Frobenius morphism (2.2) for $\mathcal{F}_q[GL_n]$ is defined over $\mathbb{Z}_\varepsilon$, i.e. it restricts to a Hopf $\mathbb{Z}_\varepsilon$–algebra epimorphism

$$\mathcal{F}r_{GL_n}^{\mathbb{Z}_\varepsilon} : \mathcal{F}_\varepsilon[GL_n] \relbar\joinrel\twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{F}_1[GL_n] \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{gl}_n^{\,*}\big)$$

coinciding, via Theorem 4.8 and Proposition 4.9, with (2.5) and with $\mathcal{F}r_{M_n}^{\mathbb{Z}_\varepsilon}$ of claim (a), and described by the same formulæ.

(c) The quantum Frobenius morphism (2.3) for $\mathcal{F}_q[SL_n]$ is defined over $\mathbb{Z}_\varepsilon$, i.e. it restricts to a Hopf $\mathbb{Z}_\varepsilon$–algebra epimorphism

$$\mathcal{F}r_{SL_n}^{\mathbb{Z}_\varepsilon} : \mathcal{F}_\varepsilon[SL_n] \relbar\joinrel\twoheadrightarrow \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{F}_1[SL_n] \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{sl}_n{}^*\big)$$

coinciding with (2.4) via Theorem 4.8 and Proposition 4.9, described by the formulæ in (a).

Proof. By definition $\mathcal{F}r_{M_n}^{\mathbb{Q}_\varepsilon} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathcal{F}_\varepsilon[M_n] \relbar\joinrel\twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{gl}_n{}^*\big) = \mathbb{Q}_\varepsilon \otimes_{\mathbb{Q}} U\big(\mathfrak{gl}_n{}^*\big)$ is the restriction (via $\widehat{\xi} : \mathcal{F}_q[M_n] \hookrightarrow \mathfrak{H}_q^g$ at $q = \varepsilon$ and $q = 1$) of the similar epimorphism $\mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Q}_\varepsilon} : \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}_\varepsilon} \mathfrak{H}_\varepsilon^g \relbar\joinrel\twoheadrightarrow \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} \mathfrak{H}_1^g \cong \mathbb{Q}_\varepsilon \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\mathfrak{gl}_n{}^*\big)$ obtained by scalar extension from (2.5), see [Ga1]. Note also that all the generators $\binom{t_{i,i}\,;\,c}{k}$'s (for all i, c and k) are contained in the subalgebra generated by the $\binom{t_{j,j}\,;\,0}{h}$'s alone (for all j and k), thanks to relations (1.2). Therefore, the formulæ in the claim uniquely determine $\mathcal{F}r_{M_n}^{\mathbb{Z}_\varepsilon}$.

Now, the formulæ in Lemma 4.2 give, for $i < j$ (noting that $[\ell]_\varepsilon = 0$),

$$\mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Q}_\varepsilon}\Big(\widehat{\xi}\,\Big|_{q=\varepsilon}\Big(\mathbf{t}_{i,j}^{(k)}\Big|_{q=\varepsilon}\Big)\Big) = \sum_{\sum_{h=0}^{n-j} e_h = k} \varepsilon^{e_0 + \sum_{s=1}^{n-j}\binom{e_s}{2} + k(j-i-2) - \binom{k}{2}} (-1)^{k(j-i)} \big(\varepsilon^{-1} - \varepsilon\big)^{k-e_0} \times$$
$$\times \prod_{s=1}^{n-j} [e_s]_\varepsilon! \cdot \prod_{r=0}^{n-j} \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\Big(E_{i,j+r}^{(e_r)}\Big|_{q=\varepsilon}\Big) \cdot \prod_{r=0}^{n-j} \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\big(\Lambda_{j+r}\big|_{q=\varepsilon}\big)^{e_r} \cdot \prod_{r=1}^{n-j} \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\Big(F_{j+r,j}^{(e_r)}\Big|_{q=\varepsilon}\Big) =$$
$$= \varepsilon^{k(j-i-1)-\binom{k}{2}} (-1)^{k(j-i)} \cdot \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\Big(E_{i,j}^{(k)}\Big|_{q=\varepsilon}\Big)$$

The very last term is equal to $(-1)^{k(j-i)} E_{i,j}^{(k/\ell)}\Big|_{q=1} = (-1)^{(j-i)\,k/\ell}\, \mathrm{e}_{i,j}^{(k/\ell)} = \mathbf{t}_{i,j}^{(k/\ell)}\Big|_{q=1}$ if $\ell \big| n$, while it is zero if $\ell \not\big| n$ (note that in the present case k and k/ℓ have the same parity), via Theorem 4.8*(a)*. This proves the first formula in the claim.

Similarly, one proves the third formula. As for the second formula, we have

$$\mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Q}_\varepsilon}\left(\widehat{\xi}\,\Big|_{q=\varepsilon}\left(\binom{t_{i,i}\,;\,0}{k}\Big|_{q=\varepsilon}\right)\right) = \sum_{r=0}^{k} \varepsilon^{-r(k+2)-\binom{r}{2}} \big(\varepsilon - \varepsilon^{-1}\big)^r \sum_{\sum_{s=1}^{n-i} e_s = r} \prod_{s=1}^{n-i} \varepsilon^{\binom{e_s}{2}} [e_s]_\varepsilon! \times$$
$$\times \prod_{r=1}^{n-i} \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\Big(E_{i,i+r}^{(e_r)}\Big|_{q=\varepsilon}\Big)\, \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\left(\left\{\begin{matrix}\Lambda_i; -r \\ k\ ,\ r\end{matrix}\right\}\Big|_{q=\varepsilon}\right) \prod_{r=1}^{n-i} \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\big(\Lambda_{i+r}\big|_{q=\varepsilon}\big)^{e_r} \prod_{r=1}^{n-i} \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\Big(F_{i+r,i}^{(e_r)}\Big|_{q=\varepsilon}\Big) =$$
$$= \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\left(\left\{\begin{matrix}\Lambda_i\,;\,0 \\ k\,,0\end{matrix}\right\}\Big|_{q=\varepsilon}\right) = \mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}\left(\binom{\Lambda_i\,;\,0}{k}\Big|_{q=\varepsilon}\right) = \begin{cases} \binom{\Lambda_i\,;\,0}{k/\ell}\Big|_{q=1} = \binom{\mathrm{l}_i}{k/\ell} & \text{if } \ \ell \,|\, k \\ 0 & \text{if } \ \ell \not|\, k \end{cases}$$

by the very definition of $\mathfrak{Fr}_{\mathfrak{gl}_n^*}^{\mathbb{Z}_\varepsilon}$ again. On the other hand, if $\ell \big| k$ then $\widehat{\xi}\,\Big|_{q=1}\left(\binom{t_{i,i}\,;\,0}{k/\ell}\Big|_{q=1}\right) = \binom{\Lambda_i\,;\,0}{k/\ell}\Big|_{q=1} = \binom{\mathrm{g}_i}{k/\ell}$ thanks to Theorem 4.8 once more, whence our formula follows.

All this accounts for claim *(a)*. Claims *(b)* and *(c)* can be proved with the same arguments, or deduced from *(a)* in force of Proposition 4.9 and of Corollary 4.5. □

List of Symbols

$(n)_q$, $(n)_q!$, $\binom{n}{s}_q$, $\binom{n}{k_1,\dots,k_r}_q$, $[n]_q$ etc., $X^{(n)}$, $\binom{X\,;c}{n}$, $\left[\begin{smallmatrix} X\,;c \\ n \end{smallmatrix}\right]$, $\left\{\begin{smallmatrix} X\,;c \\ n\,,r \end{smallmatrix}\right\}$: see §1.3

$U_q(\mathfrak{gl}_n)$, $U_q(\mathfrak{sl}_n)$: see §2.1 — $\mathfrak{U}_q(\mathfrak{g})\,, \mathcal{U}_q(\mathfrak{g})$, $\mathfrak{Fr}_{\mathfrak{g}}^{\mathbb{Z}_\varepsilon}$, $\mathcal{F}r_{\mathfrak{g}}^{\mathbb{Z}_\varepsilon}$ (for $\mathfrak{g} \in \{\mathfrak{gl}_n, \mathfrak{sl}_n\}$): see §2.3

$F_q[G]$, $\mathfrak{F}_q[G]\,, \mathcal{F}_q[G]$, $\mathfrak{Fr}_G^{\mathbb{Z}}$, $\mathcal{F}r_G^{\mathbb{Q}_\varepsilon}$ (for $G \in \{M_n, GL_n, SL_n\}$): see §2.4

$U_q(\mathfrak{g}^*)\,, \mathfrak{U}_q(\mathfrak{g}^*)\,, \mathcal{U}_q(\mathfrak{g}^*)\,, \mathbf{H}_q^x\,, \mathfrak{H}_q^x\,, \mathcal{H}_q^x\,, \mathfrak{Fr}_{\mathfrak{g}^*}^{\mathbb{Z}_\varepsilon}\,, \mathcal{F}r_{\mathfrak{g}^*}^{\mathbb{Z}_\varepsilon}$ (for $x \in \{g, s\}$, $\mathfrak{g} \in \{\mathfrak{gl}_n, \mathfrak{sl}_n\}$): see §2.5

$\widetilde{\xi}\,,\ \xi\,,\ \widehat{\xi}$ *(embeddings of quantum function algebras into dual quantum groups)*: see §3.2

References

[APW] H. H. Andersen, P. Polo, K. Wen, *Representations of quantum algebras*, Inventiones Mathematicæ **104** (1991), 1–59.

[BGY] K. A. Brown, K. R. Goodearl, M. Yakimov, *Poisson structures on affine spaces and flag varieties. I. Matrix affine Poisson space*, Advances in Mathematics **206** (2006), 567–629.

[CP] V. Chari, A. Pressley, *A guide to Quantum Groups*, Cambridge Univ. Press, Cambridge, 1994.

[DL] C. De Concini, V. Lyubashenko, *Quantum function algebra at roots of 1*, Advances in Mathematics **108** (1994), 205–262.

[DP] C. De Concini, C. Procesi, *Quantum groups*, D-modules, Representation Theory, and Quantum Groups (L. Boutet de Monvel, C. De Concini, C. Procesi, P. Schapira, M. Vergne, eds.), Lecture Notes in Mathematics **1565**, Springer-Verlag, Berlin–Heidelberg–New York, 1993, pp. 798–820.

[Ga1] F. Gavarini, *Quantization of Poisson groups*, Pacific Journal of Mathematics **186** (1998), 217–266.

[Ga2] ———, *Quantum function algebras as quantum enveloping algebras*, Communications in Algebra **26** (1998), 1795–1818.

[Ga3] ———, *Presentation by Borel subalgebras and Chevalley generators for quantum enveloping algebras*, Proceedings of the Edinburgh Mathematical Society **49** (2006), 291–308.

[Ga4] ———, *PBW theorems and Frobenius structures for quantum matrices*, electronic preprint posted at `http://arxiv.org/abs/math.QA/0610691` (2006), 9 pages.

[GR] F. Gavarini, Z. Rakić, *$F_q[Mat_2]$, $F_q[GL_2]$ and $F_q[SL_2]$ as quantized hyperalgebras*, electronic preprint posted at `http://arxiv.org/abs/math.QA/0411440` (2004), 40 pages.

[Hu] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics **9**, Springer & Verlag, Berlin–Heidelberg–New York, 1972.

[Ji] M. Jimbo, *A q–analogue of $U(gl(N+1))$, Hecke Algebras and the Yang Baxter equation*, Letters in Mathematical Physics **10** (1985), 63–69.

[Lu] G. Lustzig, *Quantum groups at roots of 1*, GeometriæDedicata **35** (1990), 89–113.

[No] M. Noumi, *Macdonald's Symmetric Polynomials as Zonal Spherical Functions on Some Quantum Homogeneous Spaces*, Advances in Mathematics **123** (1996), 16–77.

Fabio Gavarini

Università degli Studi di Roma "Tor Vergata" — Dipartimento di Matematica

Via della Ricerca Scientifica 1, I-00133 Roma, ITALY — gavarini@mat.uniroma2.it

Zoran Rakić

Univerzitet u Beogradu — Matematički fakultet

Studentski trg 16, 11000 Beograd, SERBIA — zrakic@matf.bg.ac.yu