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THE CRYSTAL DUALITY PRINCIPLE: FROM HOPF ALGEBRAS TO GEOMETRICAL SYMMETRIES

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Abstract. We give functorial recipes to get, out of any Hopf algebra over a field, two pairs of Hopf algebras with some geometrical content. If the ground field has characteristic zero, the first pair is made by a function algebra $F[G_+]$ over a connected Poisson group and a universal enveloping algebra $U(\mathfrak{g}_-)$ over a Lie bialgebra $\mathfrak{g}_-$. In addition, the Poisson group as a variety is an affine space, and the Lie bialgebra as a Lie algebra is graded. Forgetting these last details, the second pair is of the same type, namely $\big(F[K_+], U(\mathfrak{k}_-)\big)$ for some Poisson group K_+ and some Lie bialgebra $\mathfrak{k}_-$. When the Hopf algebra H we start from is already of geometric type the result involves Poisson duality. The first Lie bialgebra associated to $H = F[G]$ is $\mathfrak{g}^*$ (with $\mathfrak{g} := \mathit{Lie}(G)$), and the first Poisson group associated to $H = U(\mathfrak{g})$ is of type G^*, i.e. it has $\mathfrak{g}$ as cotangent Lie bialgebra. If the ground field has positive characteristic, the same recipes give similar results, but the Poisson groups obtained have dimension 0 and height 1, and *restricted* universal enveloping algebras are obtained. We show how these geometrical Hopf algebras are linked to the initial one via 1-parameter deformations, and explain how these results follow from quantum group theory. We examine in detail the case of group algebras.

"Yet these crystals are to Hopf algebras
but as is the body to the Children of Rees:
the house of its inner fire, that is within it
and yet in all parts of it, and is its life"

N. Barbecue, "Scholia"

Introduction

Among all Hopf algebras over a field $\Bbbk$, there are two special families which are of relevant interest for their geometrical meaning. The function algebras $F[G]$ of algebraic groups G and the universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$, if $\mathit{Char}(\Bbbk) = 0$, or the restricted universal enveloping algebras $\mathbf{u}(\mathfrak{g})$ of restricted Lie algebras $\mathfrak{g}$, if $\mathit{Char}(\Bbbk) > 0$. For

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brevity, we call both $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$ "enveloping algebras" and denote them by $\mathcal{U}(\mathfrak{g})$. Similarly by "restricted Lie algebra", when $Char(\Bbbk) = 0$, we shall simply mean "Lie algebra". Function algebras are exactly those Hopf algebras which are commutative, and enveloping algebras those which are connected, cocommutative and generated by their primitives.

In this paper we give functorial recipes to get out from any Hopf algebra two pairs of Hopf algebras of geometrical type, say $\big(F[G_+],\mathcal{U}(\mathfrak{g}_-)\big)$ and $\big(F[K_+],\mathcal{U}(\mathfrak{k}_-)\big)$. In addition, the algebraic groups obtained in this way are connected *Poisson* groups, and the (restricted) Lie algebras are (restricted) *Lie bialgebras*. Therefore, to each Hopf algebra — which encodes a general notion of "symmetry" — we can associate in a functorial way some symmetries of geometrical type, where the geometry involved is in fact Poisson geometry. Moreover, if $Char(\Bbbk) > 0$ these Poisson groups have dimension 0 and height 1, which makes them very interesting for arithmetic geometry, hence for number theory too.

The construction of the pair $(G_+, \mathfrak{g}_-)$ uses pretty classical (as opposite to "quantum") methods: in fact, it might be part of the content of any basic textbook on Hopf algebras (and, surprisingly enough, it is not!). Instead, in order to obtain the pair $(K_+, \mathfrak{k}_-)$ one relies on the construction of the first pair, and uses the theory of quantum groups.

Let's describe our results in detail. Let $J := Ker(\epsilon_H)$ be the augmentation ideal of the Hopf algebra H (where ϵ_H is the counit of H), and let $\underline{J} := \big\{J^n\big\}_{n\in\mathbb{N}}$ be the associated J–adic filtration, $\widehat{H} := G_{\underline{J}}(H)$ the associated graded vector space and $H^\vee := H\Big/\bigcap_{n\in\mathbb{N}} J^n$. One proves that $\underline{J}$ is a Hopf algebra filtration, hence $\widehat{H}$ is a graded Hopf algebra: the latter happens to be connected, cocommutative and generated by its primitives, so $\widehat{H} \cong \mathcal{U}(\mathfrak{g}_-)$ for some restricted Lie algebra $\mathfrak{g}_-$. In addition, since $\widehat{H}$ is graded also $\mathfrak{g}_-$ itself is graded as a restricted Lie algebra. The fact that $\widehat{H}$ be cocommutative allows to define a Poisson cobracket on it (from the natural Poisson cobracket $\nabla := \Delta - \Delta^{\mathrm{op}}$ on H), which makes $\widehat{H}$ a graded *co-Poisson* Hopf algebra. Eventually, this implies that $\mathfrak{g}_-$ is a *Lie bialgebra*. So the right-hand side half of the first pair of "Poisson geometrical" Hopf algebras is just $\widehat{H}$.

On the other hand, one considers a second filtration — increasing, whereas $\underline{J}$ is decreasing — namely $\underline{D}$ which is defined in a dual manner to $\underline{J}$. For each $n \in \mathbb{N}$, let δ_n be the composition of the n–fold iterated coproduct followed by the projection onto $J^{\otimes n}$ (note that $H = \Bbbk \cdot 1_H \oplus J$); then $\underline{D} := \big\{D_n := Ker(\delta_{n+1})\big\}_{n\in\mathbb{N}}$. Let now $\widetilde{H} := G_{\underline{D}}(H)$ be the associated graded vector space and $H' := \bigcup_{n\in\mathbb{N}} D_n$. Again, one shows that $\underline{D}$ is a Hopf algebra filtration, hence $\widetilde{H}$ is a graded Hopf algebra: moreover, the latter is commutative, so $\widetilde{H} = F[G_+]$ for some algebraic group G_+. One proves also that $\widetilde{H} = F[G_+]$ has no non-trivial idempotents, thus G_+ is connected; in addition, since $\widetilde{H}$ is graded, G_+ as a variety is just an affine space. A deeper analysis shows that in the positive characteristic case G_+ has dimension 0 and height 1. The fact that $\widetilde{H}$ be commutative allows to define on it a Poisson bracket (from the natural Poisson bracket on H given by the commutator) which makes $\widetilde{H}$ a graded *Poisson* Hopf algebra. This means that G_+ is an algebraic *Poisson group*. So the left-hand side half of the first pair of "Poisson geometrical" Hopf algebras is just $\widetilde{H}$.

The relationship among H and the "geometrical" Hopf algebras $\widehat{H}$ and $\widetilde{H}$ can be expressed in terms of "reduction steps" and regular 1-parameter deformations, namely

$$\widetilde{H} \xleftrightarrow[\mathcal{R}^t_{\underline{D}}(H)]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftrightarrow[\mathcal{R}^t_{\underline{J}}(H^\vee)]{1 \leftarrow t \rightarrow 0} \widehat{H} \qquad (\bigstar)$$

Here the one-way arrows are Hopf algebra morphisms and the two-ways arrows are regular 1-parameter deformations of Hopf algebras, realized through the *Rees* Hopf algebras $\mathcal{R}^t_{\underline{D}}(H)$ and $\mathcal{R}^t_{\underline{J}}(H^\vee)$ associated to the filtration $\underline{D}$ of H and to the filtration $\underline{J}$ of $H^\vee$.

The construction of the pair $(K_+, \mathfrak{k}_-)$ uses quantum group theory, the basic ingredients being $\mathcal{R}^t_{\underline{D}}(H)$ and $\mathcal{R}^t_{\underline{J}}(H^\vee)$. In the present framework, by quantum group we mean, loosely speaking, a Hopf $\Bbbk[t]$–algebra (t an indeterminate) H_t such that either *(a)* $H_t \big/ t\, H_t \cong F[G]$ for some connected Poisson group G — then we say H_t is a QFA — or *(b)* $H_t \big/ t\, H_t \cong \mathcal{U}(\mathfrak{g})$, for some restricted Lie bialgebra $\mathfrak{g}$ — then we say H_t is a QrUEA. Formula $(\bigstar)$ says that $H'_t := \mathcal{R}^t_{\underline{D}}(H)$ is a QFA, with $H'_t \big/ t\, H'_t \cong \widetilde{H} = F[G_+]$, and that $H^\vee_t := \mathcal{R}^t_{\underline{J}}(H)$ is a QrUEA, with $H^\vee_t \big/ t\, H^\vee_t \cong \widehat{H} = \mathcal{U}(\mathfrak{g}_-)$. Now, a general result — the "Global Quantum Duality Principle", in short GQDP, see [Ga1–2] — teaches us how to construct from the QFA H'_t a QrUEA, call it $\big(H'_t\big)^\vee$, and how to build out of the QrUEA $H^\vee_t$ a QFA, say $\big(H^\vee_t\big)'$. Then $\big(H'_t\big)^\vee \big/ t\, \big(H'_t\big)^\vee = \mathcal{U}(\mathfrak{k}_-)$ for some restricted Lie bialgebra $\mathfrak{k}_-$, and $\big(H^\vee_t\big)' \big/ t\, \big(H^\vee_t\big)' = F[K_+]$ for some connected Poisson group K_+. This gives the pair $(K_+, \mathfrak{k}_-)$. The very construction implies that $\big(H'_t\big)^\vee$ and $\big(H^\vee_t\big)'$ yield another frame of regular 1-parameter deformations for H' and $H^\vee$, namely

$$\mathcal{U}(\mathfrak{k}_-) \xleftrightarrow[(H'_t)^\vee]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftrightarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0} F[K_+] \qquad (\maltese)$$

which is the analogue of $(\bigstar)$. In addition, when $\mathit{Char}(\Bbbk) = 0$ the GQDP also claims that the two pairs $(G_+, \mathfrak{g}_-)$ and $(K_+, \mathfrak{k}_-)$ are related by Poisson duality: namely, $\mathfrak{k}_-$ *is the cotangent Lie bialgebra of* G_+, and $\mathfrak{g}_-$ *is the cotangent Lie bialgebra* of K_+; in short, we write $\mathfrak{k}_- = \mathfrak{g}^\times$ and $K_+ = G^\star_-$. Therefore the four "Poisson symmetries" G_+, $\mathfrak{g}_-$, K_+ and $\mathfrak{k}_-$, attached to H are actually encoded simply by the pair (G_+, K_+).

In particular, when $H^\vee = H = H'$ from $(\bigstar)$ and $(\maltese)$ together we find

$$\begin{array}{ccccc}
\mathcal{U}(\mathfrak{g}_-) \xleftrightarrow[H^\vee_t]{0 \leftarrow t \rightarrow 1} & H^\vee & \xleftrightarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0} & F[K_+] & \Big(= F\big[G^\star_-\big] \text{ if } \mathit{Char}(\Bbbk) = 0 \Big) \\
 & \| & & & \\
 & H & & & \\
 & \| & & & \\
F[G_+] \xleftrightarrow[H'_t]{0 \leftarrow t \rightarrow 1} & H' & \xleftrightarrow[(H'_t)^\vee]{1 \leftarrow t \rightarrow 0} & \mathcal{U}(\mathfrak{k}_-) & \Big(= U\big(\mathfrak{g}^\times_+\big) \text{ if } \mathit{Char}(\Bbbk) = 0 \Big)
\end{array}$$

This gives *four* different regular 1-parameter deformations from H to Hopf algebras encoding geometrical objects of Poisson type, i.e. Lie bialgebras or Poisson algebraic groups.

When the Hopf algebra H we start from is already of geometric type, the result involves Poisson duality. Namely, if $\mathit{Char}(\Bbbk) = 0$ and $H = F[G]$, then $\mathfrak{g}_- = \mathfrak{g}^*$ (where $\mathfrak{g} := \mathit{Lie}(G)$), and if $H = \mathcal{U}(\mathfrak{g}) = U(\mathfrak{g})$, then $\mathit{Lie}(G_+) = \mathfrak{g}^*$, i.e. G_+ has $\mathfrak{g}$ as cotangent Lie bialgebra. If instead $\mathit{Char}(\Bbbk) > 0$, we have only a slight variation on this result.

The construction of $\widehat{H}$ and $\widetilde{H}$ needs only "half the notion" of a Hopf algebra. In fact, we construct $\widehat{A}$ for any augmented algebra A (roughly, an algebra with an augmentation, or counit, i.e. a character), and $\widetilde{C}$ for any coaugmented coalgebra C (a coalgebra with a coaugmentation, or unit, i.e. a coalgebra morphism from $\Bbbk$ to C). In particular this applies to bialgebras, for which both $\widehat{B}$ and $\widetilde{B}$ are (graded) Hopf algebras. We can also perform a second construction using $\big(B_t'\big)^\vee$ and $\big(B_t^\vee\big)'$ (via a stronger version of the GQDP), and get from these a second pair of bialgebras $\Big(\big(B_t'\big)^\vee\Big|_{t=0}, \big(B_t^\vee\big)'\Big|_{t=0} \Big)$. Then again $\big(B_t'\big)^\vee\Big|_{t=0} \cong \mathcal{U}(\mathfrak{k}_-)$ for some restricted Lie bialgebra $\mathfrak{k}_-$, while $\big(B_t^\vee\big)'\Big|_{t=0}$ is commutative with no non-trivial idempotents, but it's not, in general, a Hopf algebra. So the spectrum of $\big(B_t^\vee\big)'\Big|_{t=0}$ is a connected algebraic Poisson monoid, but not necessarily a Poisson group.

It is worthwhile pointing out that everything in fact follows from the GQDP, which — in the stronger formulation — deals with augmented algebras and coaugmented coalgebras over 1-dimensional domains. The content of this paper can in fact be obtained as a corollary of the GQDP as follows. Pick any augmented algebra or coaugmented coalgebra over $\Bbbk$, and take its scalar extension from $\Bbbk$ to $\Bbbk[t]$: the latter ring is a 1-dimensional domain, hence we can apply the GQDP, and (almost) every result in the present paper will follow.

In the last section we apply these results to the case of group algebras and their duals. Another interesting application — based on a non-commutative version of the function algebra of the group of formal diffeomorphism on the line, also called "Nottingham group" — is illustrated in detail in a separate paper (see [Ga3]).

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§ 1 Notation and terminology

1.1 Algebras, coalgebras, and further structures. Let $\Bbbk$ be a field, which will stand fixed throughout, with $p := \mathit{Char}(\Bbbk)$. In this paper we deal with (unital associative) $\Bbbk$–algebras and (counital coassociative) $\Bbbk$–coalgebras in the standard sense, cf. [Sw] or [Ab]. In particular we use notations as in [Ab]. For any (counital coassociative) $\Bbbk$–coalgebra C

we denote by $coRad(C)$ its coradical, and by $G(C) := \big\{ c \in C \,\big|\, \Delta(c) = c \otimes c \big\}$ its set of group-like elements. We say C is *monic* if $\big|G(C)\big| = 1$; we say it is *connected* if $coRad(C)$ is one-dimensional: of course "connected" implies "monic". Any unital associative $\Bbbk$–algebra A is said *idempotent-free* (in short, *i.p.-free*) iff it has no non-trivial idempotents.

We call *augmented algebra* any unital associative $\Bbbk$–algebra A *together with* a special unital algebra morphism $\underline{\epsilon} : A \longrightarrow \Bbbk$ (so the unit $u : \Bbbk \longrightarrow A$ is a section of $\underline{\epsilon}$): these form a category in the obvious way. We call *indecomposable elements* of an augmented algebra A the elements of the set $Q(A) := J_A \big/ J_A^{\,2}$ with $J_A := Ker\big(\underline{\epsilon} : A \longrightarrow \Bbbk\big)$. We denote by $\mathcal{A}^+$ the category of all augmented $\Bbbk$–algebras.

We call *coaugmented coalgebra* any counital coassociative $\Bbbk$–coalgebra C *together with* a special counital coalgebra morphism $\underline{u} : \Bbbk \longrightarrow C$ (so $\underline{u}$ is a section of $\epsilon : C \longrightarrow \Bbbk$), and let $\underline{1} := \underline{u}(1)$, a group-like element in C : these form a category in the obvious way. For such a C we said *primitive* the elements of the set $P(C) := \big\{ c \in C \mid \Delta(c) = c \otimes \underline{1} + \underline{1} \otimes c \big\}$. We denote by $\mathcal{C}^+$ the category of all coaugmented $\Bbbk$–coalgebras.

We denote by $\mathcal{B}$ the category of all $\Bbbk$–bialgebras. Clearly each bialgebra B can be seen both as an augmented algebra, w.r.t. $\underline{\epsilon} = \epsilon \equiv \epsilon_B$ (the counit of B) and as a coaugmented coalgebra, w.r.t. $\underline{u} = u \equiv u_B$ (the unit map of B), so that $\underline{1} = 1 = 1_B$: then $Q(B)$ is naturally a Lie coalgebra and $P(B)$ a Lie algebra over R. In the following we'll use such an interpretation throughout, looking at objects of $\mathcal{B}$ as objects of $\mathcal{A}^+$ and of $\mathcal{C}^+$. We call $\mathcal{HA}$ the category of all Hopf $\Bbbk$–algebras; this naturally identifies with a subcategory of $\mathcal{B}$.

We call *Poisson algebra* any (unital) commutative algebra A endowed with a Lie bracket $\{\ ,\ \} : A \otimes A \longrightarrow A$ (i.e., $\big(A, \{\ ,\ \}\big)$ is a Lie algebra) such that the *Leibnitz identities*

$$\{a\,b, c\} = \{a, c\}\, b + a\,\{b, c\}\,, \quad \{a, b\,c\} = \{a, b\}\, c + b\,\{a, c\}$$

hold (for all $a, b, c \in A$). We call *Poisson bialgebra*, or *Poisson Hopf algebra*, any bialgebra, or Hopf algebra, say H, which is also a Poisson algebra (w.r.t. the same product) enjoying

$$\Delta\big(\{a, b\}\big) = \big\{\Delta(a), \Delta(b)\big\}\,, \qquad \epsilon\big(\{a, b\}\big) = 0\,, \qquad S\big(\{a, b\}\big) = \big\{S(b), S(a)\big\}$$

(for all $a, b, c \in H$) — the condition on the antipode S being required in the Hopf algebra case — where the (Poisson) bracket on $H \otimes H$ is defined by $\{a \otimes b, c \otimes d\} := \{a, b\} \otimes c\,d + a\,b \otimes \{c, d\}$ (for all $a, b, c, d \in H$).

We call *co-Poisson coalgebra* any (counital) cocommutative coalgebra C with a Lie cobracket $\delta : C \longrightarrow C \otimes C$ (i.e. $\big(C, \delta\big)$ is a Lie coalgebra) such that the *co-Leibnitz identity*

$$\big(id \otimes \Delta\big) \circ \big(\delta(a)\big) \;=\; \textstyle\sum_{(a)} \Big(\delta\big(a_{(1)}\big) \otimes a_{(2)} + \sigma_{1,2}\Big(a_{(1)} \otimes \delta\big(a_{(2)}\big)\Big)\Big)$$

holds for all $a \in C$, where $\sigma_{1,2} : C^{\otimes 3} \longrightarrow C^{\otimes 3}$ is given by $x_1 \otimes x_2 \otimes x_3 \mapsto x_2 \otimes x_1 \otimes x_3$. We call *co-Poisson bialgebra*, or *co-Poisson Hopf algebra*, any bialgebra, or Hopf algebra, say H, which is also a co-Poisson algebra (w.r.t. the same co-product) enjoying

$$\delta(a\,b) = \delta(a)\,\Delta(b) + \Delta(a)\,\delta(b)\,, \qquad (\epsilon \otimes \epsilon)\big(\delta(a)\big) = 0\,, \qquad \delta\big(S(a)\big) = \tau\Big(\big(S \otimes S\big)\big(\delta(a)\big)\Big)$$

(where τ is the flip) for all $a, b \in H$, the condition on the antipode S being required in the Hopf algebra case. Finally, we call *bi-Poisson bialgebra*, or *bi-Poisson Hopf algebra*, any bialgebra, or Hopf algebra, say H, which is simultaneously a Poisson and co-Poisson bialgebra, or Hopf algebra, for some Poisson bracket and cobracket enjoying, for all $a, b \in H$,

$$\delta\big(\{a,b\}\big) = \big\{\delta(a), \Delta(b)\big\} + \big\{\Delta(a), \delta(b)\big\} \ .$$

See [CP] and [Tu1], [Tu2] for further details on the above notions.

A *graded algebra* is an algebra A which is $\mathbb{Z}$–graded as a vector space and whose structure maps m, u and S are morphisms of degree zero in the category of graded vector spaces, where $A \otimes A$ has the standard grading inherited from A and $\Bbbk$ has the trivial grading. Similarly we define the graded versions of coalgebras, bialgebras and Hopf algebras, and also the graded versions of Poisson algebras, co-Poisson coalgebras, Poisson/co-Poisson/bi-Poisson bialgebras, and Poisson/co-Poisson/bi-Poisson Hopf algebras, but for the fact that the Poisson bracket, resp. cobracket, must be a morphism (of graded spaces) of degree -1, resp. $+1$. We write $V = \oplus_{z \in \mathbb{Z}} V_z$ for the degree splitting of any graded vector space V.

1.2 Function algebras. According to standard theory, the category of commutative Hopf algebras is antiequivalent to the category of algebraic groups (over $\Bbbk$). Then we call $\mathit{Spec}\,(H)$ (*spectrum of* H) the image of a Hopf algebra H in this antiequivalence, and conversely we call *function algebra* or *algebra of regular functions* the preimage $F[G]$ of an algebraic group G. Note that we do *not* require algebraic groups to be reduced (i.e. $F[G]$ to have trivial nilradical) and we do *not* make any restrictions on dimensions: in particular we deal with *pro-affine* as well as *affine* algebraic groups. We say that G is *connected* if $F[G]$ is i.p.-free; this is equivalent to the classical topological notion when $\mathit{dim}(G)$ is finite.

Given an algebraic group G, let $J_G := \mathit{Ker}\,(\epsilon_{F[G]})$; the *cotangent space of* G (at its unity) is $\mathfrak{g}^\times := J_G \Big/ J_G^{\,2} = Q\big(F[G]\big)$, endowed with its weak topology. The *tangent space of* G (at its unity) is the topological dual $\mathfrak{g} := \big(\mathfrak{g}^\times\big)^\star$ of $\mathfrak{g}^\times$: this is a Lie algebra, the *tangent Lie algebra of* G. If $p > 0$, then $\mathfrak{g}$ is a *restricted* Lie algebra (also called "p–Lie algebra"). We say that G is an algebraic *Poisson* group if $F[G]$ is a Poisson Hopf algebra. Then the tangent Lie algebra $\mathfrak{g}$ of G is a *Lie bialgebra*, and the same holds for $\mathfrak{g}^\times$. If $p > 0$, then $\mathfrak{g}$ and $\mathfrak{g}^\times$ are *restricted Lie bialgebras*, the p–operation on $\mathfrak{g}^\times$ being trivial.

1.3 Enveloping algebras and symmetric algebras. Given a Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g})$ its *universal enveloping algebra*. If $p > 0$ and $\mathfrak{g}$ is a *restricted* Lie algebra, we denote $\mathbf{u}(\mathfrak{g}) = U(\mathfrak{g}) \Big/ \big(\big\{\, x^p - x^{[p]} \,\big|\, x \in \mathfrak{g} \,\big\}\big)$ its *restricted universal enveloping algebra*. If $p = 0$, then $P\big(U(\mathfrak{g})\big) = \mathfrak{g}$; if instead $p > 0$, then $P\big(U(\mathfrak{g})\big) = \mathfrak{g}^\infty := \mathit{Span}\Big(\big\{x^{p^n}\big\}_{n \in \mathbb{N}}\Big)$, the latter carrying a natural structure of restricted Lie algebra with $X^{[p]} := X^p$.

Note that $U(\mathfrak{g}) = \mathbf{u}(\mathfrak{g}^\infty)$ for any Lie algebra $\mathfrak{g}$, so any universal enveloping algebra can be thought of as a restricted universal enveloping algebra. Both $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$ are cocommutative connected Hopf algebras, generated by $\mathfrak{g}$ itself. Conversely, if $p = 0$ then each cocommutative connected Hopf algebra is the universal enveloping algebra of some Lie algebra, and if $p > 0$ then each cocommutative connected Hopf algebra H which is generated by $P(H)$ is the restricted universal enveloping algebra of some restricted Lie algebra (cf. [Mo], Theorem 5.6.5, and references therein). Thus, in order to unify terminology and notations, we call both universal enveloping algebras (when $p = 0$) and restricted universal enveloping algebras "enveloping algebras", and denote them by $\mathcal{U}(\mathfrak{g})$; similarly, we talk of "restricted Lie algebra" even when $p = 0$ simply meaning "Lie algebra".

If a cocommutative connected Hopf algebra generated by its primitive elements is also *co-Poisson*, then the restricted Lie algebra $\mathfrak{g}$ such that $H = \mathcal{U}(\mathfrak{g})$ is indeed a (restricted) Lie bialgebra. Conversely, if a (restricted) Lie algebra $\mathfrak{g}$ is also a Lie bialgebra then $\mathcal{U}(\mathfrak{g})$ is a cocommutative connected *co-Poisson* Hopf algebra (cf. [CP]).

Let V be a vector space: then the symmetric algebra $S(V)$ has a natural structure of Hopf algebra, given by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$ for all $x \in V$. If $\mathfrak{g}$ is a Lie algebra, then $S(\mathfrak{g})$ is also a Poisson Hopf algebra w.r.t. the Poisson bracket given by $\{x, y\}_{S(\mathfrak{g})} = [x, y]_{\mathfrak{g}}$ for all $x,\, y \in \mathfrak{g}$. If $\mathfrak{g}$ is a Lie coalgebra, then $S(\mathfrak{g})$ is also a co-Poisson Hopf algebra w.r.t. the Poisson cobracket determined by $\delta_{S(\mathfrak{g})}(x) = \delta_{\mathfrak{g}}(x)$ for all $x \in \mathfrak{g}$. Finally, if $\mathfrak{g}$ is a Lie bialgebra, then $S(\mathfrak{g})$ is a bi-Poisson Hopf algebra with respect to the previous Poisson bracket and cobracket (cf. [Tu1] and [Tu2] for details).

1.4 Filtrations. Let $\big\{F_z\big\}_{z \in \mathbb{Z}} =: \underline{F} : \big(\{0\} \subseteq \big) \cdots \subseteq F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \big(\subseteq V \big)$ be a filtration of a vector space V. We denote by $G_{\underline{F}}(V) := \bigoplus_{z \in \mathbb{Z}} F_z \big/ F_{z-1}$ the associated graded vector space. We say that $\underline{F}$ is *exhaustive* if $V^{\underline{F}} := \bigcup_{z \in \mathbb{Z}} F_z = V$; we say it is *separating* if $V_{\downarrow} := \bigcap_{z \in \mathbb{Z}} F_z = \{0\}$. We say that a filtered vector space is *exhausted* if the filtration is exhaustive; we say that it is *separated* if the filtration is separating.

A filtration $\underline{F} = \big\{F_z\big\}_{z \in \mathbb{Z}}$ in an algebra A is said to be an *algebra filtration* iff $m\big(F_\ell \otimes F_m\big) \subseteq F_{\ell+m}$ for all $\ell,\, m,\, n \in \mathbb{Z}$. Similarly, a filtration $\underline{F} = \big\{F_z\big\}_{z \in \mathbb{Z}}$ in a coalgebra C is said to be a *coalgebra filtration* iff $\Delta\big(F_z\big) \subseteq \sum_{r+s=z} F_r \otimes F_s$ for all $z \in \mathbb{Z}$. Finally, a filtration $\underline{F} = \big\{F_z\big\}_{z \in \mathbb{Z}}$ in a bialgebra, or in a Hopf algebra, H is said to be a *bialgebra filtration*, or a *Hopf (algebra) filtration*, iff it is both an algebra and a coalgebra filtration and — in the Hopf case — in addition $S\big(F_z\big) \subseteq F_z$ for all $z \in \mathbb{Z}$. The notions of *exhausted* and *separated* for filtered algebras, coalgebras, bialgebras and Hopf algebras are defined like for vector spaces, with respect to the proper type of filtrations.

Lemma 1.5. *Let $\underline{F}$ be an algebra filtration of an algebra A. Then $G_{\underline{F}}(A)$ is a graded algebra; if, in addition, it is commutative, then it is a commutative graded Poisson algebra. If E is another algebra with algebra filtration $\underline{\varPhi}$ and $\phi : A \longrightarrow E$ is a morphism of algebras such that $\phi(F_z) \subseteq \varPhi_z$ for all $z \in \mathbb{Z}$, then the morphism $G(\phi) : G_{\underline{F}}(A) \longrightarrow G_{\underline{\varPhi}}(E)$ associated to ϕ is a morphism of graded algebras. In addition, if $G_{\underline{F}}(A)$ and $G_{\underline{\varPhi}}(E)$ are commutative, then $G(\phi)$ is a morphism of graded commutative Poisson algebras.*

The analogous statement holds replacing "algebra" with "coalgebra", "commutative" with "cocommutative" and "Poisson" with "co-Poisson". In addition, if we start from bialgebras, or Hopf algebras, with bialgebra filtrations, or Hopf filtrations, then we end up with graded commutative Poisson bialgebras, or Poisson Hopf algebras, and graded cocommutative co-Poisson bialgebras, or co-Poisson Hopf algebras, respectively.

Proof. The only non-trivial part concerns the Poisson structure on $G_{\underline{F}}(A)$ and the co-Poisson structure on $G_{\underline{F}}(C)$ (for a coalgebra C). Indeed, let $\underline{F} := \big\{F_z\big\}_{z\in\mathbb{Z}}$ be an algebra filtration of A. The Poisson bracket on $G_{\underline{F}}(A)$ is defined as follows. For any $\overline{x} \in F_z \big/ F_{z-1}$, $\overline{y} \in F_\zeta \big/ F_{\zeta-1}$ $(z, \zeta \in \mathbb{Z})$, let $x \in F_z$, resp. $y \in F_\zeta$, be a lift of $\overline{x}$, resp. of $\overline{y}$. Then $[x,y] := (xy - yx) \in F_{z+\zeta-1}$ because $G_{\underline{F}}(A)$ is commutative; thus we set $\big\{\overline{x},\overline{y}\big\} := \overline{[x,y]} \equiv [x,y] \mod F_{z+\zeta-2} \in F_{z+\zeta-1}\big/F_{z+\zeta-2}$, which is easily seen to define a Poisson bracket on $G_{\underline{F}}(A)$ which makes it into a graded commutative *Poisson* algebra. Similarly, if $\underline{F} := \big\{F_z\big\}_{z\in\mathbb{Z}}$ is a coalgebra filtration of C, we define a co-Poisson bracket on $G_{\underline{F}}(C)$ as follows. For any $\overline{x} \in F_z \big/ F_{z-1}$ $(z \in \mathbb{Z})$, let $x \in F_z$ be a lift of $\overline{x}$. Then $\nabla(x) := \big(\Delta(x) - \Delta^{\mathrm{op}}(x)\big) \in \sum_{r+s=z-1} F_r \otimes F_s$ because $G_{\underline{F}}(C)$ is cocommutative; thus $\delta(\overline{x}) := \overline{\nabla(x)} \equiv \nabla(x) \mod \sum\limits_{r+s=z-2} F_r \otimes F_s \in \sum\limits_{r+s=z-1} \big(F_r\big/F_{r-1}\big) \otimes \big(F_s\big/F_{s-1}\big)$ defines a Poisson cobracket which makes $G_{\underline{F}}(C)$ into a graded cocommutative *co-Poisson* algebra. $\square$

Lemma 1.6. *Let C be a coalgebra. If $\underline{F} := \big\{F_z\big\}_{z\in\mathbb{Z}}$ is a coalgebra filtration, then $C^{\underline{F}} := \bigcup\limits_{z\in\mathbb{Z}} F_z$ is a coalgebra, which injects into C, and $C_{\underline{F}} := C \Big/ \bigcap\limits_{z\in\mathbb{Z}} F_z$ is a coalgebra, which C surjects onto. The same holds for algebras, bialgebras, Hopf algebras, with algebra, bialgebra, Hopf algebra filtrations respectively.*

Proof. The claim for $C^{\underline{F}}$ is trivial, while for $C_{\underline{F}}$ we must prove $F_\downarrow := \bigcap\limits_{z\in\mathbb{Z}} F_z$ is a coideal.

Fix a basis $B_\downarrow$ of $F_\downarrow$ and a basis B_+ of any chosen complement of $F_\downarrow$, so that $B := B_\downarrow \cup B_+$ be a basis of C. In addition, we can choose B_+ to have the following property: the span of $B_+ \cap \big(F_z \setminus F_{z-1}\big)$ has trivial intersection with F_{z-1}, for all $z \in \mathbb{Z}$. Then $C \otimes F_\downarrow + F_\downarrow \otimes C$ has basis $\big(B_\downarrow \otimes B_+\big) \cup \big(B_+ \otimes B_\downarrow\big) \cup \big(B_\downarrow \otimes B_\downarrow\big)$. Moreover, for each $b_+ \in B_+$ there is a unique $z(b_+) \in \mathbb{Z}$ such that $b_+ \in F_{z(b_+)} \setminus F_{z(b_+)-1}$. Now pick $f \in F_\downarrow$, and let $\Delta(f) = \sum_{b,b' \in B} c_{b,b'} \cdot b \otimes b'$ be the expansion of $\Delta(f)$ w.r.t. the basis $B \otimes B$ of $C \otimes C$: then $\Delta(f) \subseteq C \otimes F_\downarrow + F_\downarrow \otimes C$ if and only if $\big\{(b,b') \in B_+ \times B_+ \,\big|\, c_{b,b'} \neq 0\big\} = \emptyset$. So assume the latter set is non empty, and let $\nu := \min\big\{ z(b) + z(b') \,\big|\, b, b' \in B_+, \, c_{b,b'} \neq 0 \big\}$. Then $\Delta(f) \notin \sum\limits_{r+s=\nu-1} F_r \otimes F_s$, which contradicts $\Delta(f) \in \Delta\big(F_\downarrow\big) \subseteq \Delta\big(F_{\nu-1}\big) \subseteq \sum\limits_{r+s=\nu-1} F_r \otimes F_s$.

As for algebras, the definition of algebra filtration implies that all terms of $\underline{F}$ are ideals, so $F_\downarrow$ is an ideal too, and we conclude. Finally, the bialgebra case follows from the previous two cases, and the Hopf algebra case follows too once we note that in addition each term of $\underline{F}$ is S–stable (by assumption), hence the same is true for $F_\downarrow$ as well. $\square$

§ 2 Connecting functors on (co)augmented (co)algebras

2.1 The $\underline{\epsilon}$– filtration $\underline{J}$ on augmented algebras. Let A be an augmented algebra (cf. §1.1). Let $J := Ker(\underline{\epsilon})$: then $\underline{J} := \big\{ J_{-n} := J^n, \, J_n := A \big\}_{n \in \mathbb{N}}$ is clearly an algebra filtration of A, which we call *the $\underline{\epsilon}$–filtration* of A. To simplify notation we shall usually forget its (fixed) positive part. We say that A is *$\underline{\epsilon}$–separated* if $\underline{J}$ is separating, that is $J^\infty := \bigcap_{n \in \mathbb{N}} J^n = \{0\}$. Next lemma points out some properties of the $\underline{\epsilon}$–filtration $\underline{J}$:

Lemma 2.2.

(a) $\underline{J}$ is an algebra filtration of A, which contains the radical filtration of A, that is $J^n \supseteq Rad(A)^n$ for all $n \in \mathbb{N}$ where $Rad(A)$ is the (Jacobson) radical of A.

(b) If A is $\underline{\epsilon}$–separated, then it is i.p.-free.

(c) $A^\vee := A \Big/ \bigcap_{n \in \mathbb{N}} J^n$ is a quotient augmented algebra of A, which is $\underline{\epsilon}$–separated.

Proof. *(a)* By definition $Rad(A)$ is the intersection of all maximal left (or right) ideals of A, and J is one of them: so $J \supseteq Rad(A)$, whence the claim follows at once.

(b) Let $e \in A$ be idempotent, and let $e_0 := \underline{\epsilon}(e)$, $e_+ := e - e_0 \cdot 1$: then $e_+ \in J$, and ${e_0}^2 = \underline{\epsilon}\big(e^2\big) = \underline{\epsilon}\,(e) = e_0$, i.e. $e_0 \in \Bbbk$ is idempotent, whence $e_0 \in \{0,1\}$. If $e_0 = 0$ then $e_+ = e = e^n = {e_+}^n \in J^n$ for all $n \in \mathbb{N}$ so $e_+ \in J^\infty := \bigcap_{n \in \mathbb{N}}$. If $e_0 = 1$ then ${e_+}^2 = e^2 - 2\,e + 1 = -(e-1) = -e_+$ whence $e_+ = (-1)^{n+1} {e_+}^n \in J^n$ for all $n \in \mathbb{N}$, so $e_+ \in J^\infty$. Therefore, if A is $\underline{\epsilon}$–separated it is also i.p.-free.

(c) Lemma 1.6 proves that $A_{\underline{J}} = A^\vee$ is a quotient algebra of A. The augmentation of A induces an augmentation on $A^\vee$ too, and the latter is $\underline{\epsilon}$–separated by construction. $\square$

Proposition 2.3. *Mapping $A \mapsto A^\vee := A \big/ \cap_{n \in \mathbb{N}} J^n$ gives a well-defined functor from the category $\mathcal{A}^+$ to the subcategory of all $\underline{\epsilon}$–separated augmented algebras. Moreover, the augmented algebras A of the latter subcategory are characterized by $A^\vee = A$.*

Proof. By Lemma 2.2*(c)*, the functor is well-defined on objects, and we are left with defining the functor on morphisms. The last part of the claim will be immediate.

Let $\varphi : A \longrightarrow E$ be a morphism of augmented algebras, i.e. such that $\underline{\epsilon}_E \circ \varphi = \underline{\epsilon}_A$. Then $J_A = {\underline{\epsilon}_A}^{-1}(0) = \varphi^{-1}\big({\underline{\epsilon}_E}^{-1}(0)\big) = \varphi^{-1}\big(J_E\big)$, so $\varphi\big({J_A}^n\big) \subseteq {J_E}^n$ for all $n \in \mathbb{N}$: thus $\varphi\big({J_A}^\infty\big) \subseteq {J_E}^\infty$, whence φ induces a morphism of $\underline{\epsilon}$–separated augmented algebras. $\square$

Remark 2.4: It is worthwhile mentioning a special example of $\underline{\epsilon}$–separated augmented algebras, namely the *graded* ones (i.e. those augmented algebras A with an algebra grading such that $\underline{\epsilon}_A$ is a morphism of graded algebras w.r.t. the trivial grading on the ground field $\Bbbk$) which are also *connected*, i.e. their zero-degree subspace is the $\Bbbk$–span of 1_A. Then

Every graded connected augmented algebra A is ϵ–separated, or equivalently $A = A^\vee$. Indeed, by definition, each non-zero homogeneous element in $J := J_A$ has positive degree: thus any non-zero homogeneous element of J^n has degree at least n, so any non-zero homogeneous element of J^∞ should have degree at least n *for any* $n \in \mathbb{N}$. Then $J^\infty = \{0\}$.

2.5 Drinfeld's $\delta_\bullet$– maps. Let C be a coaugmented coalgebra (cf. §1.1). For every $n \in \mathbb{N}$, define $\Delta^n \colon H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := \mathrm{id}_C$, and $\Delta^n := \big(\Delta \otimes \mathrm{id}_C^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ if $n > 2$. For any ordered subset $\Phi = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, define the linear map $j_\Phi : H^{\otimes k} \longrightarrow H^{\otimes n}$ by $j_\Phi(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := \underline{1}$ if $i \notin \Phi$ and $b_{i_m} := a_m$ for $1 \leq m \leq k$. Then set $\Delta_\Phi := j_\Phi \circ \Delta^k$, $\Delta_\emptyset := \Delta^0$, and $\delta_\Phi := \sum_{\Psi \subset \Phi} (-1)^{n - |\Psi|} \Delta_\Psi$, $\delta_\emptyset := \epsilon$. By the inclusion-exclusion principle, the inverse formula $\Delta_\Phi = \sum_{\Psi \subseteq \Phi} \delta_\Psi$ holds. To be short we'll write $\delta_0 := \delta_\emptyset$ and $\delta_n := \delta_{\{1,2,\dots,n\}}$ too.

Lemma 2.6.

(a) $\quad \delta_n = \big(\mathrm{id}_C - \underline{u} \circ \epsilon\big)^{\otimes n} \circ \Delta^n \qquad$ *for all* $n \in \mathbb{N}_+$ *;*

(b) The maps δ_n *(and similarly the* δ_Φ*'s, for all finite* $\Phi \subseteq \mathbb{N}$*) are coassociative, i.e.*

$$\big(\mathrm{id}_C^{\otimes s} \otimes \delta_\ell \otimes \mathrm{id}_C^{\otimes (n-1-s)}\big) \circ \delta_n = \delta_{n+\ell-1} \qquad \textit{for all} \quad n, \ell, s \in \mathbb{N}, \ 0 \leq s \leq n-1 .$$

Proof. Claim *(a)* is proved by an easy induction, and claim *(b)* follows from *(a)*. $\square$

2.7 The $\delta_\bullet$– filtration $\underline{D}$ on coaugmented coalgebras. Let C be as above, and take notations of §2.5. For all $n \in \mathbb{N}$, let $D_n := Ker(\delta_{n+1})$: then $\underline{D} := \big\{ D_{-n} := \{0\}, D_n \big\}_{n \in \mathbb{N}}$ is clearly a filtration of C, which we call *the $\delta_\bullet$–filtration* of C; to simplify notation we shall usually forget its (fixed) negative part. We say that C is *$\delta_\bullet$–exhausted* if $\underline{D}$ is exhaustive, i.e. $\bigcup_{n \in \mathbb{N}} D_n = C$.

The $\delta_\bullet$–filtration has several remarkable properties: next lemma highlights some of them, in particular shows that $\underline{D}$ is sort of a refinement of the coradical filtration of C. We make use of the notion of "wedge" product, namely $X \bigwedge Y := \Delta^{-1}\big(C \otimes Y + X \otimes C\big)$ for all subspaces X, Y of C, with $\bigwedge^1 X := X$ and $\bigwedge^{n+1} X := \big(\bigwedge^n X\big) \bigwedge X$ for all $n \in \mathbb{N}_+$.

Lemma 2.8.

(a) $D_0 = \Bbbk \cdot \underline{1}$, $\quad D_n = \Delta^{-1}\big(C \otimes D_{n-1} + D_0 \otimes C\big) = \bigwedge^{n+1} D_0 \quad$ *for all* $n \in \mathbb{N}$.

(b) $\underline{D}$ *is a coalgebra filtration of* C*, which is contained in the coradical filtration of* C*, that is* $D_n \subseteq C_n$ *if* $\underline{C} := \big\{C_n\big\}_{n \in \mathbb{N}}$ *is the coradical filtration of* C.

(c) C *is* $\delta_\bullet$*–exhausted* $\iff$ C *is connected* $\iff$ $\underline{D} = \underline{C}$.

(d) $C' := \bigcup_{n \in \mathbb{N}} D_n$ *is a subcoalgebra of* C*: more precisely, it is the irreducible (hence connected) component of* C *containing* $\underline{1}$.

Proof. (a) Definitions give $D_0 := Ker(\delta_1) = \Bbbk \cdot 1$ and, using induction and coassociativity, $D_n := Ker(\delta_{n+1}) = Ker\big((\delta_1 \otimes \delta_n) \circ \Delta\big) = \Delta^{-1}\big(C \otimes Ker(\delta_n) + Ker(\delta_1) \otimes C\big) = \Delta^{-1}\big(C \otimes D_{n-1} + D_0 \otimes C\big)$ for all $n \in \mathbb{N}$. Since $D_0 := \bigwedge^1 D_0$ we have by induction $D_n = \Delta^{-1}\big(C \otimes D_{n-1} + D_0 \otimes C\big) = D_{n-1} \bigwedge D_0 = \big(\bigwedge^n D_0\big) \bigwedge D_0 = \bigwedge^{n+1} D_0$ for all $n \in \mathbb{N}$.

(b) (cf. [Ab], Theorem 2.4.1) From *(a)* we have that D_0 is a subcoalgebra, and then the D_n are subcoalgebras too, because the wedge of two subcoalgebras is again a subcoalgebra. In addition, $D_n = \bigwedge^{n+1} D_0 = \big(\bigwedge^i D_0\big) \bigwedge \big(\bigwedge^{n+1-i} D_0\big) = D_{i-1} \bigwedge D_{n-i}$ so $\Delta(D_n) \subseteq$

$C \otimes D_{n-i} + D_{i-1} \otimes C$ for $1 \leq i \leq n$, and also for $i \in \{0, n+1\}$ because D_n is a subcoalgebra. Then $\Delta(D_n) \subseteq \sum_{r+s=n} D_r \otimes D_s$ (for all n) follows from the fact (cf. [Sw], §9.1.5) that for any vector space V and any filtration $\big\{V_{-\ell} := 0, V_\ell\big\}_{\ell \in \mathbb{N}}$ of V one has $\bigcap_{i=0}^{\ell+1} \big(V \otimes V_{\ell+1-i} + V_i \otimes V\big) = \sum_{i=1}^{\ell+1} V_i \otimes V_{\ell+2-1}$ (set $V = C$ and $V_\ell = D_{\ell-1}$ for all ℓ).

Second we prove that $\underline{D}$ is contained in $\underline{C}$. Recall that $\underline{C}$ may be defined inductively by $C_0 := \mathit{coRad}\,(C)$, $C_n := \Delta^{-1}\big(C \otimes C_{n-1} + C_0 \otimes C\big) = C_0 \bigwedge C_{n-1} = \bigwedge^{n+1} C_0$; then we get $D_n \subseteq C_n$ for all $n \in \mathbb{N}$ by induction using $C_0 \supseteq \Bbbk \cdot 1 = D_0$, by *(a)*.

(c) C is connected iff $C_0 = \Bbbk \cdot 1$. But in this case $C_0 = D_0$, hence $C_n = \bigwedge^{n+1} C_0 = \bigwedge^{n+1} D_0 = D_n$ for all n (by *(a)*), that is $\underline{D} = \underline{C}$. Conversely, if $\underline{D} = \underline{C}$ then $\mathit{coRad}\,(C) = C_0 = D_0 = \Bbbk \cdot 1$ thus C is connected. Further, if $\underline{D} = \underline{C}$ then C is $\delta_\bullet$–exhausted, because the coradical filtration $\underline{C}$ is always exhaustive. Conversely, assume C is $\delta_\bullet$–exhausted: then by a standard result (cf. [Ab], Theorem 2.3.9(ii), or [Mo], Lemma 5.3.4) we have $D_0 \supseteq C_0$, so $D_0 = C_0$ by *(a)*, and we conclude by induction as above that $\underline{D} = \underline{C}$.

(d) $C' := \bigcup_{n \in \mathbb{N}} D_n$ is a subcoalgebra of C because $\underline{D}$ is a coalgebra filtration, so Lemma 1.6 applies. Moreover, by construction $\underline{1} \in D_0 \subseteq C'$, and C' is $\delta_\bullet$–exhausted (w.r.t. the same group-like element $\underline{1}$), hence it is connected by *(c)*. Thus C' is the irreducible (connected) component of C containing $\underline{1}$. $\square$

Proposition 2.9. *Mapping $C \mapsto C' := \bigcup_{n \in \mathbb{N}} D_n$ gives a well-defined functor from $\mathcal{C}^+$ to the subcategory of all $\delta_\bullet$–exhausted (=connected) coaugmented coalgebras. Moreover, the coaugmented coalgebras C of the latter subcategory are characterized by $C' = C$.*

Proof. By Lemma 2.8*(d)*, if C is a coaugmented coalgebra, then C' with its $\delta_\bullet$–filtration (w.r.t. the same group-like element $\underline{1}$) is an exhausted coaugmented coalgebra: in other words it is $\delta_\bullet$–exhausted, so by Lemma 2.8*(c)* it is connected too. The last part of the claim being immediate, we are left to define the functor on morphisms.

Let $\varphi : C \longrightarrow K$ be a morphism of coaugmented coalgebras (mapping $\underline{1} \in G(C)$ to $\underline{1} \in G(K)$). Then $\delta_m \circ \varphi = \big(\mathrm{id} - \underline{u} \circ \epsilon\big)^{\otimes m} \circ \Delta^m \circ \varphi = \varphi^{\otimes m} \circ \big(\mathrm{id} - \underline{u} \circ \epsilon\big)^{\otimes m} \circ \Delta^m = \varphi^{\otimes m} \circ \delta_m$ for all $m \in \mathbb{N}$, which yields $\varphi\big(D_n(C)\big) = \varphi\big(\mathit{Ker}\big(\delta_{n+1}\big|_C\big)\big) \subseteq \mathit{Ker}\big(\delta_{n+1}\big|_K\big) = D_n(K)$. Thus φ maps $\underline{D}(C)$ to $\underline{D}(K)$ and so C' to K', so the restriction $\varphi\big|_{C'}$ of φ to C' is a morphism of exhausted coaugmented coalgebras which we take as $\varphi' \in \mathit{Mor}\big(C', K'\big)$. $\square$

Remark 2.10: Lemma 2.8*(c)* yields alternative proofs of two well-known facts.

(a) Every graded connected (in the graded sense, i.e. $\dim(C_0) = 1$*) coaugmented coalgebra C is connected (in the mere coalgebra sense).* Indeed, given $c \in C_{\partial(c)}$ homogeneous of degree $\partial(c)$, one has $\delta_n(c) \in C_+^{\otimes n} \bigcap \big(\sum_{r+s=\partial(c)} C_r \otimes C_s\big)$ for all n. Thus $c \in \mathit{Ker}\,(\delta_{\partial(c)+1}) =: D_{\partial(c)}$, hence $C = \bigoplus_{n \in \mathbb{N}} C_n \subseteq \bigcup_{n \in \mathbb{N}} D_n = C' \subseteq C$ so $C = C'$.

(b) Every connected coaugmented coalgebra is monic. By Lemma 2.8*(c)–(d)* it is enough to prove $G(C') = \{\underline{1}\}$. As $G(C') = G(C) \cap C'$, let $g \in G(C)$: then $\delta_n(g) = (g - \underline{1})^{\otimes n}$ for all $n \in \mathbb{N}$, so $g \in C' := \bigcup_{n \in \mathbb{N}} D_n = \bigcup_{n \in \mathbb{N}} \mathit{Ker}\,(\delta_{n+1})$ if and only if $g = \underline{1}$.

2.11 Connecting functors and Hopf duality. Let's start from an augmented algebra A, with $J := Ker(\underline{\epsilon})$. Its $\underline{\epsilon}$–filtration has $J^n := Im\Big(J^{\otimes n} \hookrightarrow H^{\otimes n} \overset{\mu^n}{\twoheadrightarrow} H\Big)$ as n–th term, the left hand side arrow being the natural embedding induced by $J \hookrightarrow H$ and μ^n being the n–fold iterated multiplication of H. Similarly, let C be a coaugmented coalgebra. The s–th term of its $\delta_\bullet$–filtration is $D_s := Ker\Big(H \xhookrightarrow{\Delta^{s+1}} H^{\otimes(s+1)} \twoheadrightarrow J^{\otimes(s+1)}\Big)$ where the right hand side arrow is the natural projection induced by $\big(\mathrm{id}_C - \underline{u} \circ \epsilon\big) : H \twoheadrightarrow J$ and Δ^{s+1} is the $(s+1)$–fold iterated comultiplication of C. This clearly means that

(a) The notions of $\underline{\epsilon}$–filtration and of $\delta_\bullet$–filtration are dual to each other.

Similarly, as taking intersection and taking unions are mutually dual operations, and taking submodules and quotient modules are mutually dual too, we have the following two facts:

(b) The notions of $A^\vee := A \Big/ \bigcap_{n \in \mathbb{N}} J^n$ and of $C' := \bigcup_{n \in \mathbb{N}} D_n$ are dual to each other;

(c) The notions of $\underline{\epsilon}$–separated (for an augmented algebra) and $\delta_\bullet$–exhausted (for a coaugmented coalgebra) are dual to each other.

Remark 2.12: *(a) $(\ )^\vee$ and $(\ )'$ as "connecting functors".* We now explain in what sense both $A^\vee$ and C' are "connected" objects. Indeed, C' is truly connected, in the sense of coalgebra theory (cf. Lemma 2.8*(d)*). By duality, we might expect that $A^\vee$ be (or correspond to) a "connected" object too. In fact, when A is commutative, it is the algebra of regular functions $F[\mathcal{V}]$ of an algebraic variety $\mathcal{V}$; the augmentation on A is a character, hence corresponds to the choice of a point $P_0 \in \mathcal{V}$: thus A does correspond to the *pointed variety* $(\mathcal{V}, P_0)$. Then $A^\vee = F[\mathcal{V}_0]$ where $\mathcal{V}_0$ is the connected component of $\mathcal{V}$ containing P_0. This follows at once because $A^\vee$ is i.p.-free (cf. Lemma 2.2). More in general, for any $A \in \mathcal{A}^+$, if $A^\vee$ is commutative then its spectrum is a *connected* algebraic variety. Therefore we shall call "connecting functors" both functors $A \mapsto A^\vee$ and $C \mapsto C'$.

(b) Asymmetry of connecting functors on bialgebras. Let B be a bialgebra. As the notion of $B^\vee$ is dual to that of B', and since $B = B'$ implies that B is a *Hopf* algebra (cf. Corollary 3.4*(b)*), one might dually conjecture that $B = B^\vee$ imply that B is a *Hopf* algebra. Actually, *this is false*, the bialgebra $B := F\big[Mat(n, \Bbbk)\big]$ yielding a counter-example: $F\big[Mat(n, \Bbbk)\big] = F\big[Mat(n, \Bbbk)\big]^\vee$, and yet $F\big[Mat(n, \Bbbk)\big]$ is *not* a Hopf algebra.

(c) Hopf duality and augmented pairings. The most precise description of the relationship between connecting functors of the two types uses the notion of "augmented pairing":

Definition 2.13. *Let $A \in \mathcal{A}^+$, $C \in \mathcal{C}^+$. We call augmented pairing between A and C any bilinear mapping $\langle\ ,\ \rangle : A \times C \longrightarrow \Bbbk$ such that, for all $x, x_1, x_2 \in A$ and $y \in C$, $\big\langle x_1 \cdot x_2 , y \big\rangle = \big\langle x_1 \otimes x_2 , \Delta(y) \big\rangle := \sum_{(y)} \big\langle x_1 , y_{(1)} \big\rangle \cdot \big\langle x_2 , y_{(2)} \big\rangle$, $\langle 1 , y \rangle = \epsilon(y)$, $\langle x , \underline{1} \rangle = \underline{\epsilon}(x)$. For any $B, P \in \mathcal{B}$ we call bialgebra pairing between B and P any augmented pairing between the latters (see §1.1) such that we have also, symmetrically, $\big\langle x , y_1 \cdot y_2 \big\rangle = \big\langle \Delta(x) , y_1 \otimes y_2 \big\rangle := \sum_{(x)} \big\langle x_{(1)} , y_1 \big\rangle \cdot \big\langle x_{(2)} , y_2 \big\rangle$, for all $x \in B$, $y_1, y_2 \in P$.*

For any $H, K \in \mathcal{HA}$ *we call Hopf algebra pairing (or Hopf pairing) between* H *and* K *any bialgebra pairing such that, in addition,* $\big\langle S(x), y \big\rangle = \big\langle x, S(y) \big\rangle$ *for all* $x \in H$, $y \in K$.

We say that a pairing as above is perfect on the left (right) if its left (right) kernel is trivial; we say it is perfect if it is both left and right perfect.

Theorem 2.14. *Let* $A \in \mathcal{A}^+$, $C \in \mathcal{C}^+$ *and let* $\pi \colon A \times C \longrightarrow \Bbbk$ *be an augmented pairing. Then* π *induces a filtered augmented pairing* $\pi_f : A^\vee \times C' \longrightarrow \Bbbk$ *and a graded augmented pairing* $\pi_{{}_G} : G_{\underline{J}}(A) \times G_{\underline{D}}(C) \longrightarrow \Bbbk$ *(notation of §1.4), both perfect on the right. If in addition* π *is perfect then* π_f *and* $\pi_{{}_G}$ *are perfect as well.*

Proof. Consider the filtrations $\underline{J} = \big\{ {J_{{}_A}}^n \big\}_{n \in \mathbb{N}}$ and $\underline{D} = \big\{ D_n^{{}_C} \big\}_{n \in \mathbb{N}}$. The key fact is that

$$D_n^{{}_C} = \big({J_{{}_A}}^{n+1} \big)^\perp \qquad \text{and} \qquad {J_{{}_A}}^{n+1} \subseteq \big(D_n^{{}_C} \big)^\perp \qquad \text{for all } n \in \mathbb{N} \, . \tag{2.1}$$

Indeed, let E be an algebra, K a coalgebra, and assume there is a bilinear pairing $E \times K \longrightarrow \Bbbk$ enjoying $\big\langle x_1 \cdot x_2, y \big\rangle = \big\langle x_1 \otimes x_2, \Delta(y) \big\rangle$ (for all $x_1, x_2 \in E$ and $y \in K$). If X is a subspace of K, then $\bigwedge^n X = \big((X^\perp)^n \big)^\perp$ (cf. §2.7) for all $n \in \mathbb{N}$, where the superscript $\perp$ means "orthogonal subspace" (either in E or in K) w.r.t. the pairing under exam (cf. [Ab] or [Mo]). Now, Lemma 2.8*(a)* gives $D_n^{{}_C} = \bigwedge^{n+1}(\Bbbk \cdot \underline{1}_{{}_C})$, thus $D_n^{{}_C} = \bigwedge^{n+1}(\Bbbk \cdot \underline{1}_{{}_C}) = \Big(\big((\Bbbk \cdot \underline{1}_{{}_C})^\perp \big)^{n+1} \Big)^\perp = \big({J_{{}_A}}^{n+1} \big)^\perp$ because $(\Bbbk \cdot \underline{1}_{{}_C})^\perp = J_{{}_A}$ (w.r.t. the pairing π above). Therefore $D_n^{{}_C} = \big({J_{{}_A}}^{n+1} \big)^\perp$, and this also implies ${J_{{}_A}}^{n+1} \subseteq \big(D_n^{{}_C} \big)^\perp$.

Now $C' := \bigcup_{n \in \mathbb{N}} D_n^{{}_C} = \bigcup_{n \in \mathbb{N}} \big({J_{{}_A}}^{n+1} \big)^\perp = \big(\bigcap_{n \in \mathbb{N}} {J_{{}_A}}^{n+1} \big)^\perp = \big({J_{{}_A}}^\infty \big)^\perp$. So π induces a Hopf pairing $\pi_f : A^\vee \times C' \longrightarrow \Bbbk$ as required, and by (2.1) this respects the filtrations on either side. Then by general theory π_f induces a graded Hopf pairing $\pi_{{}_G}$ as required: in particular $\pi_{{}_G}$ is well-defined because $D_n^{{}_C} \subseteq \big({J_{{}_A}}^{n+1} \big)^\perp$ and ${J_{{}_A}}^{n+1} \subseteq \big(D_n^{{}_C} \big)^\perp$ ($n \in \mathbb{N}_+$) by (2.1). Moreover, both π_f and $\pi_{{}_G}$ are perfect on the right because all the inclusions $D_n^{{}_C} \subseteq \big({J_{{}_A}}^{n+1} \big)^\perp$ happen to be identities. Finally if π_f is perfect it is easy to see that $\pi_{{}_G}$ is perfect as well; note that this improves (2.1), for then ${J_{{}_A}}^{n+1} = \big(D_n^{{}_C} \big)^\perp$ for all $n \in \mathbb{N}$. $\square$

2.15 The crystal functors. It is clear from the very construction that mapping $A \mapsto \widehat{A} := G_{\underline{J}}(A)$ (for all $A \in \mathcal{A}^+$) defines a functor from $\mathcal{A}^+$ to the category of graded, augmented $\Bbbk$–algebras, which factors through the functor $A \mapsto A^\vee$. Similarly, mapping $C \mapsto \widetilde{C} := G_{\underline{D}}(C)$ (for all $C \in \mathcal{C}^+$) defines a functor from $\mathcal{C}^+$ to the category of graded, coaugmented $\Bbbk$–coalgebras, which factors through the functor $C \mapsto C'$.

In §4 we'll show that each of these functors can be seen as a "crystallization process", in the sense, loosely speaking, of Kashiwara's terminology of "crystal basis" for quantum groups: we move from one fiber to another, very special one, within a 1-parameter deformation. Therefore, we call them *"crystal functors"*.

§ 3 Connecting and crystal functors on bialgebras and Hopf algebras

3.1 The program. We apply now the connecting functors to bialgebras and Hopf algebras. Then we look at the graded objects associated to the filtrations $\underline{J}$ and $\underline{D}$ in a bialgebra: this leads to the crystal functors on $\mathcal{B}$ and $\mathcal{HA}$, the main topic of this section.

From now on, every bialgebra B will be considered as a coaugmented coalgebra w.r.t. its unit map, hence w.r.t the group-like element 1 (the unit of B), and the corresponding maps δ_n ($n \in \mathbb{N}$) and $\delta_\bullet$–filtration $\underline{D}$ will be taken into account. Similarly, B will be considered as an augmented algebra w.r.t. the special algebra morphism $\underline{\epsilon} = \epsilon$ (the counit of B), and the corresponding $\underline{\epsilon}$–filtration (also called ϵ–filtration) $\underline{J}$ will be considered.

We begin with a technical result about the "multiplicative" properties of the maps δ_n.

Lemma 3.2. *([KT], Lemma 3.2) Let $B \in \mathcal{B}$, $a, b \in B$, and $\Phi \subseteq \mathbb{N}$, with Φ finite. Then*

(a) $\quad \delta_\Phi(ab) \, = \, \sum_{\Lambda \cup Y = \Phi} \delta_\Lambda(a) \, \delta_Y(b)$;

(b) *if* $\Phi \neq \emptyset$, *then* $\quad \delta_\Phi(ab - ba) \, = \, \sum_{\substack{\Lambda \cup Y = \Phi \\ \Lambda \cap Y \neq \emptyset}} \big(\delta_\Lambda(a) \, \delta_Y(b) - \delta_Y(b) \, \delta_\Lambda(a) \big)$. □

Lemma 3.3. *Let B be a bialgebra. Then $\underline{J}$ and $\underline{D}$ are bialgebra filtrations. If B is also a Hopf algebra, then $\underline{J}$ and $\underline{D}$ are Hopf algebra filtrations.*

Proof. By Lemma 2.2*(a)*, $\underline{J}$ is an algebra filtration. In addition, since $J := Ker(\epsilon)$ is a biideal we have $\Delta\big(J^n\big) = \Delta(J)^n \subseteq \big(B \otimes J + J \otimes B\big)^n \subseteq \sum_{r+s=n} J^r \otimes J^s$ for all $n \in \mathbb{N}$, so that $\underline{J}$ is a bialgebra filtration as well. Moreover, in the Hopf algebra case J is a Hopf ideal, hence $S\big(J^n\big) \subseteq J^n$ for all $n \in \mathbb{N}$, which means $\underline{J}$ is a Hopf algebra filtration.

As for $\underline{D}$, by Lemma 2.8*(b)* it is a coalgebra filtration. The fact that it is also a bialgebra filtration is a direct consequence of Lemma 3.2*(a)*. The claim in the Hopf case follows noting that $\delta_n \circ S = S^{\otimes n} \circ \delta_n$ so $S(D_n) = S\big(Ker(\delta_{n+1})\big) \subseteq Ker(\delta_{n+1}) = D_n$ for all $n \in \mathbb{N}$. Otherwise, the claim for $\underline{D}$ follows from [Ab], Theorem 2.4.1, or [Mo], 5.2.8. □

Corollary 3.4. *Let B be a bialgebra. Then*

(a) $B^\vee := B \Big/ \bigcap_{n \in \mathbb{N}} J^n$ *is an ϵ–separated (i.p.-free) bialgebra, which B surjects onto.*

(b) $B' := \bigcup_{n \in \mathbb{N}} D_n$ *is a $\delta_\bullet$–exhausted (connected) Hopf algebra, which injects into B: more precisely, it is the irreducible (actually, connected) component of B containing* 1.

(c) If in addition $B = H$ is a Hopf algebra, then $H^\vee$ is a Hopf algebra quotient of H and H' is a Hopf subalgebra of H.

Proof. Claim *(a)* follows from the results in §§2–3. As for *(b)*, the results of §§2–3 imply that B' is a $\delta_\bullet$–exhausted (connected) bialgebra, embedding into B, namely the irreducible component of B containing 1. In addition, a general result ensures that any connected bialgebra is a Hopf algebra (cf. [Ab], Theorem 2.4.24), whence this holds true for B'. Finally, claim *(c)* follows from *(a)*, *(b)*, and Lemma 3.3. □

Theorem 3.5. *Let B be a bialgebra, $\underline{J}$, $\underline{D}$ its ϵ–filtration and $\delta_\bullet$–filtration respectively.*

(a) $\widehat{B} := G_{\underline{J}}(B)$ *is a graded cocommutative co-Poisson Hopf algebra generated by* $P\big(G_{\underline{J}}(B)\big)$, *the set of its primitive elements. Therefore* $\widehat{B} \cong \mathcal{U}(\mathfrak{g}_-)$ *as graded co-Poisson Hopf algebras, for some restricted Lie bialgebra* $\mathfrak{g}_-$ *which is graded as a Lie algebra. In particular, if* $p=0$ *and* $\dim(B)\in\mathbb{N}$ *then* $\widehat{B}=\Bbbk\cdot 1$ *and* $\mathfrak{g}_-=\{0\}$.

(b) $\widetilde{B} := G_{\underline{D}}(B)$ *is a graded commutative Poisson Hopf algebra. Therefore,* $\widetilde{B} \cong F[G_+]$ *for some connected algebraic Poisson group* G_+ *which, as a variety, is a (pro)affine space. If* $p=0$ *then* $\widetilde{B} \cong F[G_+]$ *is a polynomial algebra, i.e.* $F[G_+]=\Bbbk\big[\{x_i\}_{i\in\mathcal{I}}\big]$ *(for some set $\mathcal{I}$); in particular, if* $\dim(B)\in\mathbb{N}$ *then* $\widetilde{B}=\Bbbk\cdot 1$ *and* $G_+=\{1\}$. *If* $p>0$ *then* G_+ *has dimension 0 and height 1, and if* $\Bbbk$ *is perfect then* $\widetilde{B}\cong F[G_+]$ *is a truncated polynomial algebra, i.e.* $F[G_+]=\Bbbk\big[\{x_i\}_{i\in\mathcal{I}}\big]\Big/\big(\{x_i^p\}_{i\in\mathcal{I}}\big)$ *(for some set $\mathcal{I}$).*

Proof. *(a)* Thanks to Corollary 3.4*(a)*, I must only prove that $G_{\underline{J}}(B)$ is cocommutative, and then apply Lemma 1.5 to have the co-Poisson structure. In fact, even for this, it is enough to show that $G_{\underline{J}}(B)$ is generated by $P\big(G_{\underline{J}}(B)\big)$. Moreover, I must show that $G_{\underline{J}}(B)$ is in fact a (graded ϵ–separated cocommutative co-Poisson) *Hopf algebra*, not only a bialgebra! The last part of the statement then will follow from §1.3.

Since $G_{\underline{J}}(B)$ is generated by J/J^2, it is enough to show that $J/J^2 \subseteq P\big(G_{\underline{J}}(B)\big)$. Let $0\neq\overline{\eta}\in J/J^2$, and let $\eta\in J$ be a lift of $\overline{\eta}$: then $\Delta(\eta)=\epsilon(\eta)\cdot 1\otimes 1+\delta_1(\eta)\otimes 1+1\otimes\delta_1(\eta)+\delta_2(\eta)=\eta\otimes 1+1\otimes\eta+\delta_2(\eta)$ (by the very definitions and by $\eta\in J$). Now, $\delta_2(\eta)\in J\otimes J$, hence $\overline{\delta_2(\eta)}=\overline{0}\in\big(G_{\underline{J}}(B)\otimes G_{\underline{J}}(B)\big)_1=\big(B/J\big)\otimes J/J^2+J/J^2\otimes\big(B/J\big)$. So $\Delta(\overline{\eta}):=\overline{\Delta(\eta)}=\overline{\eta\otimes 1+1\otimes\eta}=\overline{\eta}\otimes 1+1\otimes\overline{\eta}$, which gives $J/J^2\subseteq P\big(G_{\underline{J}}(B)\big)$.

Finally, $G_{\underline{J}}(B)$ is connected by Remark 2.10*(a)*, so is monic. As it's also cocommutative, by a standard result (cf. [Ab], Corollary 2.4.29) we deduce that it is a Hopf algebra.

(b) Thanks to Corollary 3.4*(b)*, the sole non-trivial thing to prove is that $G_{\underline{D}}(B)$ is commutative, for then Lemma 1.5 applies. Indeed, let $0\neq\overline{a}\in D_r/D_{r-1}$, $0\neq\overline{b}\in D_s/D_{s-1}$ (for $r,s\in\mathbb{N}$, with $D_{-1}:=\{0\}$), and let a and b be a lift of $\overline{a}$ and $\overline{b}$ in D_r and in D_s respectively: then $\delta_{r+1}(a)=0$, $\delta_r(a)\neq 0$, and $\delta_{s+1}(b)=0$, $\delta_s(b)\neq 0$. Now $\big(\overline{a}\,\overline{b}-\overline{b}\,\overline{a}\big)\in D_{r+s}/D_{r+s-1}$ by construction, but

$$\delta_{r+s}\big(a\,b-b\,a\big) \;=\; \sum_{\substack{\Lambda\cap Y\neq\emptyset \\ \Lambda\cup Y=\{1,\dots,r+s\}}} \big(\delta_\Lambda(a)\,\delta_Y(b)-\delta_Y(b)\,\delta_\Lambda(a)\big)$$

by Lemma 3.2*(b)*. In the formula above one has $\Lambda\cup Y=\{1,\dots,r+s\}$ *and* $\Lambda\cap Y\neq\emptyset$ if and only if either $|\Lambda|>r$, whence $\delta_\Lambda(a)=0$, or $|Y|>s$, whence $\delta_Y(b)=0$, thus in any case $\delta_\Lambda(a)\,\delta_Y(b)=0$, and similarly $\delta_Y(b)\,\delta_\Lambda(a)=0$. The outset is $\delta_{r+s}\big(a\,b-b\,a\big)=0$, whence $(a\,b-b\,a)\in D_{r+s-1}$ so $\big(\overline{a}\,\overline{b}-\overline{b}\,\overline{a}\big)=\overline{0}\in D_{r+s}/D_{r+s-1}$, which implies commutativity.

The "geometrical part" of the statement is clear by §1.2. In particular, note that the Poisson group G_+ is connected because $F[G_+]\cong\widetilde{B}$ is i.p.-free because it is ϵ–separated, by Remark 2.4, and then Lemma 2.2*(b)* applies. In addition, since $\widetilde{B}\cong F[G_+]$ is graded, when $p=0$ the (pro)affine variety G_+ is a *cone*: but it is also smooth, since it is an algebraic group, hence it has no vertex. So it is a (pro)affine space, say $G_+ \cong \mathbb{A}_{\Bbbk}^{\times\mathcal{I}} = \Bbbk^{\mathcal{I}}$ for some index set $\mathcal{I}$. Then $\widetilde{B} \cong F[G_+] \cong \Bbbk\big[\{x_i\}_{i\in\mathcal{I}}\big]$ is a polynomial algebra, as claimed.

Finally, when $p>0$ the group G_+ has dimension 0 and height 1. Indeed, we must show that $\bar{\eta}^p = 0$ for each $\eta \in \widetilde{B} \setminus (\Bbbk \cdot 1)$, which may be taken homogeneous. Letting $\eta \in B'$ be any lift of $\bar{\eta}$, we have $\eta \in D_\ell \setminus D_{\ell-1}$ for a unique $\ell \in \mathbb{N}$, hence $\delta_{\ell+1}(x) = 0$. From $\Delta^{\ell+1}(\eta) = \sum_{\Lambda\subseteq\{1,\dots,\ell+1\}} \delta_\Lambda(\eta)$ (cf. §2.5) and the multiplicativity of $\Delta^{\ell+1}$ we have

$$\Delta^{\ell+1}(\eta^p) \;=\; \big(\Delta^{\ell+1}(\eta)\big)^p \;=\; \Big(\textstyle\sum_{\Lambda\subseteq\{1,\dots,\ell+1\}} \delta_\Lambda(\eta)\Big)^p \;\in$$

$$\in\; \textstyle\sum_{\Lambda\subseteq\{1,\dots,\ell+1\}} \delta_\Lambda(\eta)^p \;+\; \displaystyle\sum_{\substack{e_1,\dots,e_p<p\\ e_1+\cdots+e_p=p}} \binom{p}{e_1,\dots,e_p} \sum_{\Lambda_1,\dots,\Lambda_p\subseteq\{1,\dots,\ell+1\}} \prod_{k=1}^{p} \delta_{\Lambda_k}(\eta)^{e_k} \;+$$

$$+\; \textstyle\sum_{k=0}^{\ell} \displaystyle\sum_{\substack{\Psi\subseteq\{1,\dots,\ell+1\}\\ |\Psi|=k}} j_\Psi\big(J_{B'}^{\,\otimes k}\big) \;+\; \big(\mathrm{ad}_{[\;,\;]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big)$$

(since $\delta_\Lambda(\eta) \in j_\Lambda\Big(J_{B'}^{\,\otimes|\Lambda|}\Big)$ for all $\Lambda \subseteq \{1,\dots,\ell+1\}$) with $D_{(\ell)} := \sum\limits_{\sum_k s_k=\ell} \otimes_{k=1}^{\ell+1} D_{s_k}$ and $\big(\mathrm{ad}_{[\;,\;]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big) := \big[D_{(\ell)},\big[D_{(\ell)},\dots,\big[D_{(\ell)},\big[D_{(\ell)},D_{(\ell)}\big]\big]\cdots\big]\big]$ ($(p-1)$ brackets). Then

$$\delta^{\ell+1}(\eta^p) \;=\; \big(\mathrm{id}_H - \epsilon\big)^{\otimes(\ell+1)}\big(\Delta^{\ell+1}(\eta^p)\big) \;\in\; \delta_{\ell+1}(\eta)^p \;+$$

$$+ \sum_{\substack{e_1,\dots,e_p<p\\ e_1+\cdots e_p=p}} \binom{p}{e_1,\dots,e_p} \sum_{\cup_k\Lambda_k=\{1,\dots,\ell+1\}} \prod_{k=1}^{p} \delta_{\Lambda_k}(\eta)^{e_k} + \big(\mathrm{id}_H - \epsilon\big)^{\otimes(\ell+1)}\Big(\big(\mathrm{ad}_{[\;,\;]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big)\Big).$$

Now, $\delta^{\ell+1}(\eta)^p = 0$ by construction and $\binom{p}{e_1,\dots,e_p}$ is a multiple of p, hence it is zero because $p = \mathit{Char}(\Bbbk)$; therefore $\delta_{\ell+1}(\eta) \in \big(\mathrm{id}_H - \epsilon\big)^{\otimes(\ell+1)}\Big(\big(\mathrm{ad}_{[\;,\;]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big)\Big)$. By Lemma 3.2–3, $D_{s_i}\cdot D_{s_j} \subseteq D_{s_i+s_j}$ and $\big[D_{s_i},D_{s_j}\big] \subseteq D_{(s_i+s_j)-1}$ by the commutativity of $\widetilde{B}$. This together with Leibniz' rule implies $\big(\mathrm{ad}_{[\;,\;]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big) \subseteq \sum\limits_{\sum_t r_t = p\,\ell+1-p} \otimes_{t=1}^{\ell+1} D_{r_t}$; moreover $\big(\mathrm{id}_H - \epsilon\big)^{\otimes(\ell+1)}\Big(\big(\mathrm{ad}_{[\;,\;]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big)\Big) \subseteq \sum_{\substack{\sum_t r_t = p\,\ell+1-p\\ r_1,\dots,r_{\ell+1}>0}} \otimes_{t=1}^{\ell+1} D_{r_t}$ because $D_0 = \mathit{Ker}(\delta_1) = \mathit{Ker}\big(\mathrm{id}_H - \epsilon\big)$. In particular, in the last term above we have $D_{r_1} \subseteq D_{(p-1)\ell+1-p} := \mathit{Ker}\big(\delta_{(p-1)\ell+2-p}\big) \subseteq \mathit{Ker}\big(\delta_{(p-1)\ell}\big)$: thus by coassociativity of the δ_n's

$$\delta_{p\,\ell}(\eta) = \big(\big(\delta_{(p-1)\ell} \otimes \mathrm{id}^\ell\big) \circ \delta_{\ell+1}\big)(\eta) \;\subseteq \sum_{\substack{\sum_t r_t = p\,\ell-1\\ r_1,\dots,r_{\ell+1}>0}} \delta_{(p-1)\ell}(D_{r_1}) \otimes D_{r_2} \otimes \cdots \otimes D_{r_{\ell+1}} \;=\; 0$$

i.e. $\delta_{p\,\ell}(\eta) = 0$. This means $\eta \in D_{p\,\ell-1}$, whereas, on the other hand, $\eta^p \in D_\ell^{\,p} \subseteq D_{p\,\ell}$: then $\bar{\eta}^p := \overline{\eta^p} = \bar{0} \in D_{p\,\ell}\Big/D_{p\,\ell-1} \subseteq \widetilde{B}$, by the definition of the product in $\widetilde{B}$. Finally, if $\Bbbk$ is perfect by general theory since G_+ has dimension 0 and height 1, then $F[G_+] \cong \widetilde{B}$ is *truncated polynomial*, namely $F[G_+] \cong \Bbbk\big[\{x_i\}_{i\in\mathcal{I}}\big]\Big/\big(\{x_i^{\,p}\}_{i\in\mathcal{I}}\big)$ for some index set $\mathcal{I}$. $\square$

3.6 Crystal functors on bialgebras and Hopf algebras. The analysis in the present section shows that, when restricted to $\Bbbk$–bialgebras, the output of the previous functors are objects of Poisson-geometric type: Lie bialgebras and Poisson groups. Therefore $B \mapsto \widehat{B}$ and $B \mapsto \widetilde{B}$ (for $B \in \mathcal{B}$) are "geometrification functors", in that they associate to B some *geometrical* symmetries.

It is worthwhile pointing out that, by construction, either crystal functor forgets some information about the initial object, yet still saves something. So $\widehat{B}$ tells nothing about the coalgebra structure of $B^\vee$ (for all enveloping algebras — like $\widehat{B}$ — roughly look the same from the coalgebra point of view), yet it grasps some information on its algebra structure. Conversely, $\widetilde{B}$ gives no information about the algebra structure of B' (in that the latter is simply a polynomial algebra), but tells something non-trivial about its coalgebra structure.

We finish this section with the Hopf duality relationship between these functors:

Theorem 3.7. *Let $B, P \in \mathcal{B}$ and let $\pi : B \times P \longrightarrow \Bbbk$ be a bialgebra pairing. Then π induces filtered bialgebra pairings $\pi_f : B^\vee \times P' \longrightarrow \Bbbk$, $\pi^f : B' \times P^\vee \longrightarrow \Bbbk$, and graded bialgebra pairings $\pi_{{}_G} : \widehat{B} \times \widetilde{P} \longrightarrow \Bbbk$, $\pi^{{}^G} : \widetilde{B} \times \widehat{P} \longrightarrow \Bbbk$; π_f and $\pi_{{}_G}$ are perfect on the right, π^f and $\pi^{{}^G}$ on the left. If in addition π is perfect then all these induced pairings are perfect as well. If in particular $B, P \in \mathcal{HA}$ are Hopf algebras and π is a Hopf algebra pairing, then all the induced pairings are (filtered or graded) Hopf algebra pairings.*

Proof. The notion of bialgebra pairing is the "left-right symmetrization" of the notion of augmented pairing when both the augmented and the coaugmented coalgebra involved are bialgebras. Thus the claim above for bialgebras follows simply by a twofold application of Theorem 2.14. As for Hopf algebras, it is enough to remark in addition that the antipode preserves on both sides of the pairing the filtrations involved (by Lemma 3.3). Hence, if the initial pairing is a Hopf one, the induced pairings will clearly be Hopf pairings as well. □

Remark: the $\delta_\bullet$–filtration has been independently introduced and studied by L. Foissy in the case of graded Hopf algebras (see [Fo1–2]).

§ 4 Deformations I — Rees algebras, Rees coalgebras, etc.

4.1 Filtrations and "Rees objects". Let V be a vector space over $\Bbbk$, and let $\{F_z\}_{z \in \mathbb{Z}} := \underline{F} : \big(\{0\} \subseteq \big) \cdots \subseteq F_{-m} \subseteq \cdots \subseteq F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots \big(\subseteq V \big)$ be a filtration of V by vector subspaces F_z $(z \in \mathbb{Z})$.

First we define the associated *blowing space* as the $\Bbbk$–subspace $\mathcal{B}_{\underline{F}}(V)$ of $V\big[t, t^{-1}\big]$ (with t an indeterminate) given by $\mathcal{B}_{\underline{F}}(V) := \sum_{z \in \mathbb{Z}} t^z F_z$; this is isomorphic to $\bigoplus_{z \in \mathbb{Z}} F_z$. Second, we define the associated *Rees module* as the $\Bbbk[t]$–submodule $\mathcal{R}_{\underline{F}}^t(V)$ of $V\big[t, t^{-1}\big]$ generated by $\mathcal{B}_{\underline{F}}(V)$. Straightforward computations give $\Bbbk$–vector space isomorphisms

$$\mathcal{R}_{\underline{F}}^t(V) \Big/ (t-1)\, \mathcal{R}_{\underline{F}}^t(V) \;\cong\; \bigcup\nolimits_{z \in \mathbb{Z}} F_z =: V^{\underline{F}} \;, \quad \mathcal{R}_{\underline{F}}^t(V) \Big/ t\, \mathcal{R}_{\underline{F}}^t(V) \;\cong\; \bigoplus_{z \in Z} F_z \big/ F_{z-1} =: G_{\underline{F}}(V)$$

In other words, $\mathcal{R}^t_{\underline{F}}(V)$ is a $\Bbbk[t]$–module which specializes to $V^{\underline{F}}$ for $t=1$ and specializes to $G_{\underline{F}}(V)$ for $t=0$. Therefore the $\Bbbk$–vector spaces $V^{\underline{F}}$ and $G_{\underline{F}}(V)$ can be seen as 1-parameter (polynomial) deformations of each other via the 1-parameter family of $\Bbbk$–vector spaces given by $\mathcal{R}^t_{\underline{F}}(V)$, in short $V^{\underline{F}} \xleftrightarrow[\mathcal{R}^t_{\underline{F}}(V)]{1 \leftarrow t \rightarrow 0} G_{\underline{F}}(V)$.

We can repeat this construction within the category of algebras, coalgebras, bialgebras or Hopf algebras over $\Bbbk$ with a filtration in the proper sense: then we'll end up with corresponding objects $\mathcal{B}_{\underline{F}}(V)$, $\mathcal{R}^t_{\underline{F}}(V)$, etc. of the same type (algebras, coalgebras, etc.).

4.2 Connecting functors and Rees modules. Let $A \in \mathcal{A}^+$ be an augmented algebra. By Lemma 2.2 the ϵ–filtration $\underline{J}$ of A is an algebra filtration, hence we have the Rees *algebra* $\mathcal{R}^t_{\underline{J}}(A)$. By the previous analysis, this yields a 1-parameter deformation $A \cong \mathcal{R}^t_{\underline{J}}(A)\Big|_{t=1} \xleftrightarrow[\mathcal{R}^t_{\underline{J}}(A)]{1 \leftarrow t \rightarrow 0} \mathcal{R}^t_{\underline{J}}(A)\Big|_{t=0} \cong G_{\underline{J}}(A) =: \widehat{A}$, where $M\Big|_{t=c} := M\big/(t-c)\,M$ for any $\Bbbk[t]$–module M and any $c \in \Bbbk$. All fibers in this deformation are pairwise isomorphic as vector spaces *but* perhaps for the fiber at $t=0$, i.e. exactly $\widehat{A}$, for there the subspace J^∞ is "shrunk to zero". This is settled passing from A to $A^\vee$, for which the same deformation is *regular*, i.e. *all* fibers in it are pairwise isomorphic (as vector spaces): the scheme is

$$A^\vee := A\big/J^\infty \cong \mathcal{R}^t_{\underline{J}}(A^\vee)\Big|_{t=1} \xleftrightarrow[\mathcal{R}^t_{\underline{J}}(A^\vee)]{1 \leftarrow t \rightarrow 0} \mathcal{R}^t_{\underline{J}}(A)\Big|_{t=0} \cong G_{\underline{J}}(A) =: \widehat{A} \tag{4.1}$$

where we implicitly used the identities $\mathcal{R}^t_{\underline{J}}(A^\vee)\Big|_{t=0} = \widehat{A^\vee} = \widehat{A} = \mathcal{R}^t_{\underline{J}}(A)\Big|_{t=0}$.

The situation is (dually!) similar for the connecting functor on coaugmented coalgebras. Indeed, let $C \in \mathcal{C}^+$: by Lemma 2.8 the $\delta_\bullet$–filtration $\underline{D}$ of C is a coalgebra filtration, so we have the Rees *coalgebra* $\mathcal{R}^t_{\underline{D}}(C)$. The analysis above yields a 1-parameter deformation

$$C' := \bigcup\nolimits_{n\in\mathbb{N}} D_n \cong \mathcal{R}^t_{\underline{D}}(C)\Big|_{t=1} \xleftrightarrow[\mathcal{R}^t_{\underline{D}}(C')]{1 \leftarrow t \rightarrow 0} \mathcal{R}^t_{\underline{D}}(C)\Big|_{t=0} \cong G_{\underline{D}}(C) =: \widetilde{C} \tag{4.2}$$

which is also *regular*, i.e. all its fibers are pairwise isomorphic as vector spaces.

4.3 The bialgebra and Hopf algebra case. We now consider a bialgebra $B \in \mathcal{B}$. In this case, the results of §3 ensure that $B^\vee$ is a bialgebra, $\widehat{B}$ is a (graded, etc.) Hopf algebra, and $\mathcal{R}^t_{\underline{J}}(B^\vee)$ is a $\Bbbk[t]$–bialgebra, because $\underline{J}$ is a bialgebra filtration. Using Theorem 3.5, formula (4.1) becomes $B^\vee \xleftrightarrow[\mathcal{R}^t_{\underline{J}}(B^\vee)]{1 \leftarrow t \rightarrow 0} \widehat{B} \cong \mathcal{U}(\mathfrak{g}_-)$ for some restricted Lie bialgebra $\mathfrak{g}_-$ as in Theorem 3.5*(a)*. Similarly, by §3 we know that B' is a Hopf algebra, $\widetilde{B}$ is a (graded, etc.) Hopf algebra and $\mathcal{R}^t_{\underline{D}}(B')$ is a Hopf $\Bbbk[t]$–algebra, because $\underline{D}(B') = \underline{D}(B)$ is a Hopf algebra filtration of B. Thus, again by Theorem 3.5, formula (4.2) becomes $B' \xleftrightarrow[\mathcal{R}^t_{\underline{D}}(B')]{1 \leftarrow t \rightarrow 0} \widetilde{B} \cong F[G_+]$ for some Poisson algebraic group G_+, as in Theorem 3.5*(b)*. The overall outcome is

$$F[G_+] \cong \widetilde{B} \xleftrightarrow[\mathcal{R}^t_{\underline{D}}(B')]{0 \leftarrow t \rightarrow 1} B' \hookrightarrow B \twoheadrightarrow B^\vee \xleftrightarrow[\mathcal{R}^t_{\underline{J}}(B^\vee)]{1 \leftarrow t \rightarrow 0} \widehat{B} \cong \mathcal{U}(\mathfrak{g}_-) \tag{4.3}$$

This drawing shows how the bialgebra B gives rise to two Hopf algebras of Poisson geometrical type, namely $F[G_+]$ on the left-hand side and $\mathcal{U}(\mathfrak{g}_-)$ on the right-hand side, through bialgebra morphisms and regular bialgebra deformations. Indeed, in both cases one has first a "reduction step", i.e. $B \mapsto B'$ or $B \mapsto B^\vee$, (yielding "connected" objects, cf. Remark 2.12*(a)*), then a regular 1-parameter deformation via Rees bialgebras.

Finally, if $H \in \mathcal{HA}$ is a Hopf algebra then all objects in (4.3) are Hopf algebras too, i.e. also $H^\vee$ (over $\Bbbk$) and $\mathcal{R}_{\underline{J}}^t(B^\vee)$ (over $\Bbbk[t]$). Therefore (4.3) reads

$$F[G_+] \cong \widetilde{H} \xleftrightarrow[\mathcal{R}_{\underline{D}}^t(H')]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftrightarrow[\mathcal{R}_{\underline{J}}^t(H^\vee)]{1 \leftarrow t \rightarrow 0} \widehat{H} \cong \mathcal{U}(\mathfrak{g}_-) \tag{4.4}$$

with the one-way arrows being morphisms *of Hopf algebras* and the two-ways arrows being 1-parameter regular deformations *of Hopf algebras*. When in addition H is connected, that is $H = H'$, and "coconnected", that is $H = H^\vee$, the scheme (4.4) takes the simpler form

$$F[G_+] \cong \widetilde{H} \xleftrightarrow[\mathcal{R}_{\underline{D}}^t(H)]{0 \leftarrow t \rightarrow 1} H \xleftrightarrow[\mathcal{R}_{\underline{J}}^t(H)]{1 \leftarrow t \rightarrow 0} \widehat{H} \cong \mathcal{U}(\mathfrak{g}_-) \tag{4.5}$$

which means we can (regularly) deform H itself to "Poisson geometrical" Hopf algebras.

<u>Remarks</u>: *(a)* There is no simple relationship, *a priori*, between the Poisson group G_+ and the Lie bialgebra $\mathfrak{g}_-$ in (4.3) or (4.4), or even (4.5): examples do show that. In particular, either G_+ or $\mathfrak{g}_-$ may be trivial while the other is not (see §8).

(b) The Hopf duality relationship between connecting functors of the two types explained in §§2.11–14 extends to deformations via Rees modules. Indeed, by construction there is a neat category-theoretical duality between the definition of $\mathcal{R}_{\underline{J}}^t(A)$ and of $\mathcal{R}_{\underline{D}}^t(C)$ (for $A \in \mathcal{A}^+$ and $C \in \mathcal{C}^+$). Even more, Theorem 2.14 extends to the following:

Theorem 4.4. *Let $A \in \mathcal{A}^+$, $C \in \mathcal{C}^+$ and let $\pi\colon A \times C \longrightarrow \Bbbk$ be an augmented pairing. Then π induces an augmented pairing $\pi_{\mathcal{R}} : \mathcal{R}_{\underline{J}}^t(A) \times \mathcal{R}_{\underline{D}}^t(C) \longrightarrow \Bbbk[t]$ which is perfect on the right. If in addition π is perfect then $\pi_{\mathcal{R}}$ is perfect as well, and*

$$\begin{aligned} \mathcal{R}_{\underline{J}}^t(A) &= \Big\{ \eta \in A(t) \,\Big|\, \pi_t\big(\eta,\kappa\big) \in \Bbbk[t]\,,\ \forall\, \kappa \in \mathcal{R}_{\underline{D}}^t(C) \Big\} =: \big(\mathcal{R}_{\underline{D}}^t(C)\big)^\bullet \\ \mathcal{R}_{\underline{D}}^t(C) &= \Big\{ \kappa \in C(t) \,\Big|\, \pi_t\big(\eta,\kappa\big) \in \Bbbk[t]\,,\ \forall\, \eta \in \mathcal{R}_{\underline{J}}^t(A) \Big\} =: \big(\mathcal{R}_{\underline{J}}^t(A)\big)^\bullet \end{aligned} \tag{4.6}$$

where $S(t) := \Bbbk(t) \otimes_\Bbbk S$ for $S \in \{A, C\}$ and $\pi_t : A(t) \times C(t) \longrightarrow \Bbbk(t)$ is the obvious $\Bbbk(t)$–linear pairing induced by π. If A and C are bialgebras, resp. Hopf algebras, then everything holds with bialgebra, resp. Hopf algebra, pairings instead of augmented pairings.

Proof. The obvious augmented $\Bbbk(t)$–linear pairing $\pi_t : A(t) \times C(t) \longrightarrow \Bbbk(t)$ induced by π restricts to an augmented pairing $\pi_{\mathcal{R}} : \mathcal{R}_{\underline{J}}^t(A) \times \mathcal{R}_{\underline{D}}^t(C) \longrightarrow \Bbbk(t)$. By (2.1) both these pairings are perfect on the right, and $\pi_{\mathcal{R}}$ takes value into $\Bbbk[t]$. In addition, in the proof of Theorem 2.14 we showed that if π is perfect formula (2.1) improves, in that $J_A^{\,n+1} = \big(D_n^C\big)^\perp$ for all $n \in \mathbb{N}$. This implies that both π_t and $\pi_{\mathcal{R}}$ are perfect as well, and also that the identities (4.6) do hold, q.e.d. The final part about bialgebras or Hopf algebras is clear. $\square$

§ 5 Deformations II — from Rees bialgebras to quantum groups

5.1 From Rees bialgebras to quantum groups via the GQDP. In this section we show how, for any $\Bbbk$–bialgebra B, we can get another deformation scheme like (4.3). In fact, this will be built upon the latter, applying part of the "Global Quantum Duality Principle" explained in [Ga1–2] in its stronger form about bialgebras.

Indeed, the deformations in (4.3) were realized through Rees bialgebras, namely $\mathcal{R}^t_{\underline{J}}\big(B^\vee\big)$ and $\mathcal{R}^t_{\underline{D}}\big(B'\big)$. These are torsion-free (actually, free) as $\Bbbk[t]$–modules, hence one can apply the construction made in [*loc. cit.*] via the so-called Drinfeld's functors, to get some new torsion-free $\Bbbk[t]$–bialgebras. The latters (just like the Rees bialgebras we start from) again specialize to special bialgebras at $t=0$. In particular, if B is a Hopf algebra the new bialgebras are Hopf algebras too, and precisely "quantum groups" in the sense of [*loc. cit.*].

To begin with, set $B_t^\vee := \mathcal{R}^t_{\underline{J}}\big(B^\vee\big)$: this is a torsion-free, $\Bbbk[t]$–bialgebra. We define

$$\big(B_t^\vee\big)' := \Big\{\, b \in B_t^\vee \,\Big|\, \delta_n(b) \in t^n \big(B_t^\vee\big)^{\otimes n} \, , \ \forall \ n \in \mathbb{N} \,\Big\} \, .$$

On the other hand, let $B'_t := \mathcal{R}^t_{\underline{D}}(B) = \mathcal{R}^t_{\underline{D}}\big(B'\big)$: this is a torsion-free $\Bbbk[t]$–bialgebra as well. Letting $J' := Ker\big(\epsilon \colon B'_t \longrightarrow \Bbbk[t]\big)$ and $B'(t) := \Bbbk(t) \otimes_{\Bbbk[t]} B'_t = \Bbbk(t) \otimes_{\Bbbk} B'$, set

$$\big(B'_t\big)^\vee := \textstyle\sum_{n \geq 0} t^{-n} \big(J'\big)^n = \sum_{n \geq 0} \big(t^{-1} J'\big)^n \qquad \big(\subseteq B'(t)\big) \, .$$

The first important point is the following:

Proposition 5.2. *Both $\big(B_t^\vee\big)'$ and $\big(B'_t\big)^\vee$ are free (hence torsion-free) $\Bbbk[t]$–bialgebras; moreover, the mappings $B \mapsto \big(B_t^\vee\big)'$ and $B \mapsto \big(B'_t\big)^\vee$ are functorial. The analogous results hold for Hopf $\Bbbk$–algebras, just replacing "bialgebra(s)" with "Hopf algebra(s)" throughout.*

Proof. This is a special instance of Theorem 3.6 in [Ga1] (or in [Ga2]), but for the fact that the final objects are free, not only torsion-free. This is easy to see: taking a $\Bbbk$–basis of $B^\vee$ or B' respectively which is "compatible with $\underline{J}$", resp. "with $\underline{D}$", in the obvious sense, one gets at once from that a $\Bbbk[t]$–basis of $B_t^\vee$ or of B'_t, which then are free. Now $\big(B_t^\vee\big)'$ is a $\Bbbk[t]$–submodule of the free $\Bbbk[t]$–module $B_t^\vee$, hence it is free as well. On the other hand, one can rearrange a basis of B'_t so to get another one "compatible" with the filtration $\Big\{\big(J'\big)^n\Big\}_{n \in \mathbb{N}}$. Then from this new basis one immediately obtains a $\Bbbk[t]$–basis for $\big(B'_t\big)^\vee$, via suitable rescaling by negative powers of t.

The Hopf algebra case goes through in the same way, by similar arguments. $\square$

Next we show that $\big(B_t^\vee\big)'$ and $\big(B'_t\big)^\vee$ are regular deformations of $B^\vee$ and B' respectively:

Proposition 5.3. *Let B be a $\Bbbk$–bialgebra (or a Hopf $\Bbbk$–algebra). Then:*

(a) $\big(B_t^\vee\big)'\Big|_{t=1} := \big(B_t^\vee\big)' \Big/ (t-1)\big(B_t^\vee\big)' \cong B^\vee$ *as $\Bbbk$–bialgebras (or as Hopf $\Bbbk$–algebras);*

(b) $\big(B'_t\big)^\vee\Big|_{t=1} := \big(B'_t\big)^\vee \Big/ (t-1)\big(B'_t\big)^\vee \cong B'$ *as $\Bbbk$–bialgebras (or as Hopf $\Bbbk$–algebras).*

Proof. Claim *(b)* follows from the chain of isomorphisms $\left(B_t'\right)^\vee\Big|_{t=1} \cong B_t'\Big|_{t=1} \cong B'$ of $\Bbbk$–bialgebras, which follow directly from definitions. As for *(a)*, we can define another $\Bbbk$–bialgebra $\left(\left(B_t^\vee\right)'\right)^\vee$ as we did for $\left(B_t'\right)^\vee$: the very definition then gives $\left(\left(B_t^\vee\right)'\right)^\vee\Big|_{t=1} \cong \left(B_t^\vee\right)'\Big|_{t=1}$. The claim then follows from $\left(\left(B_t^\vee\right)'\right)^\vee = B_t^\vee$, by Theorem 3.6*(c)* of [Ga1–2]. $\square$

The key result is then the following

Theorem 5.4.

(a) $\left(B_t^\vee\right)'\Big|_{t=0} := \left(B_t^\vee\right)' \Big/ t \left(B_t^\vee\right)'$ *is a commutative, i.p.-free Poisson* $\Bbbk$*–bialgebra. Moreover, if* $p > 0$ *each non-zero element of* $\mathrm{Ker}\left(\epsilon \colon \left(B_t^\vee\right)'\Big|_{t=0} \longrightarrow \Bbbk\right)$ *has nilpotency order* p. *Therefore* $\left(B_t^\vee\right)'\Big|_{t=0}$ *is the function algebra* $F[M]$ *of some connected Poisson algebraic monoid* M, *and if* $p > 0$ *then* M *has dimension 0 and height 1.*

If in addition $B = H \in \mathcal{HA}$ *is a Hopf* $\Bbbk$*–algebra, then* $\left(H_t^\vee\right)'\Big|_{t=0}$ *is a Poisson Hopf* $\Bbbk$*–algebra, and* $K_+ := \mathit{Spec}\left(\left(H_t^\vee\right)'\Big|_{t=0}\right) = M$ *is a (connected) algebraic Poisson group.*

(b) $\left(B_t'\right)^\vee\Big|_{t=0} := \left(B_t'\right)^\vee \Big/ t \left(B_t'\right)^\vee$ *is a connected cocommutative Hopf* $\Bbbk$*–algebra generated by* $P\left(\left(B_t'\right)^\vee\Big|_{t=0}\right)$. *Therefore* $\left(B_t'\right)^\vee\Big|_{t=0} = \mathcal{U}(\mathfrak{k}_-)$ *for some Lie bialgebra* $\mathfrak{k}_-$.

(c) If $p = 0$ *and* $B = H \in \mathcal{HA}$ *is a Hopf* $\Bbbk$*–algebra, let* $\widehat{H} = U(\mathfrak{g}_-)$ *and* $\widetilde{H} = F[G_+]$ *as in Theorem 3.5. Then (notation of (a) and (b))* $K_+ = G_-^\star$ *and* $\mathfrak{k}_- = \mathfrak{g}_+^\times$, *that is* $\mathit{coLie}(K_+) = \mathfrak{g}_-$ *and* $\mathfrak{k}_- = \mathit{coLie}(G_+)$ *as Lie bialgebras.*

Proof. *(a)* This follows from Theorem 3.8 in [Ga1–2], applied to the $\Bbbk[t]$–bialgebra $B_t^\vee$.

(b) This follows from Theorem 3.7 in [*loc. cit.*], applied to the $\Bbbk[t]$–bialgebra B_t'.

(c) This follows from Theorem 4.8 in [*loc. cit.*] applied to the Hopf $\Bbbk[t]$–algebra $H_t^\vee$, because by construction the latter is a QrUEA (in the terminology therein), and from Theorem 4.7 in [*loc. cit.*] applied to the Hopf $\Bbbk[t]$–algebra H_t', because by construction the latter is a QFA (in the terminology therein, noting that the condition $\bigcap_{n\in\mathbb{N}} \left(\mathit{Ker}\left(H \xrightarrow{\;\epsilon\;} \Bbbk[t] \xrightarrow{t \mapsto 0} \Bbbk\right)\right)^n = \bigcap_{n\in\mathbb{N}} t^n H$ required in [*loc. cit.*] for a QFA is satisfied). $\square$

5.5 Deformations through Drinfeld's functors. The outcome of the previous analysis is that, for each $\Bbbk$–bialgebra $B \in \mathcal{B}$, there is a second scheme — besides (4.3) — which yields regular 1-parameter deformations, namely

$$\mathcal{U}(\mathfrak{k}_-) \xleftarrow[\left(B_t'\right)^\vee]{0 \leftarrow t \rightarrow 1} B' \hookrightarrow B \twoheadrightarrow B^\vee \xleftarrow[\left(B_t^\vee\right)']{1 \leftarrow t \rightarrow 0} F[M] \tag{5.1}$$

This gives another recipe, besides (4.3), to make two other bialgebras of Poisson geometrical type, namely $F[M]$ and $\mathcal{U}(\mathfrak{k}_-)$, out of the bialgebra B, through bialgebra morphisms and regular bialgebra deformations. Like for (4.3), in both cases there is first the "reduction step" $B \mapsto B'$ or $B \mapsto B^\vee$ and then regular 1-parameter deformations via $\Bbbk[t]$–bialgebras. However, this time on right-hand side we have in general only a bialgebra, not a Hopf algebra. When $B = H \in \mathcal{HA}$ is a Hopf $\Bbbk$–algebra, then (5.1) improves: all objects therein are Hopf algebras too, and morphisms and deformations are ones of Hopf algebras. In particular $M = K_+$ is a Poisson *group*, not only a monoid: at a glance

$$\mathcal{U}(\mathfrak{k}_-) \xleftarrow[(H'_t)^\vee]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0} F[K_+] \tag{5.2}$$

This yields another recipe, besides (4.4), to make two Hopf algebras of Poisson geometrical type — i.e. $F[K_+]$ and $\mathcal{U}(\mathfrak{k}_-)$ — out of H, through Hopf algebra morphisms and regular Hopf algebra deformations. Again we have first a "reduction step" $H \mapsto H'$ or $H \mapsto H^\vee$, then regular 1-parameter deformations via Hopf $\Bbbk[t]$–algebras.

In the special case when H is connected, that is $H = H'$, and "coconnected", that is $H = H^\vee$, formula (5.2) takes the simpler form, the analogue of (4.5),

$$\mathcal{U}(\mathfrak{k}_-) \xleftarrow[(H'_t)^\vee]{0 \leftarrow t \rightarrow 1} H \xleftarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0} F[K_+] \tag{5.3}$$

so we can again (regularly) deform H into Poisson geometrical Hopf algebras.

In particular, when $H' = H = H^\vee$ patching together (4.5) and (5.3) we find

$$\begin{array}{ccccc} F[G_+] & \xleftarrow[H'_t]{0 \leftarrow t \rightarrow 1} & H' & \xleftarrow[(H'_t)^\vee]{1 \leftarrow t \rightarrow 0} & \mathcal{U}(\mathfrak{k}_-) \quad \Big(= U\big(\mathfrak{g}_+^\times\big) \text{ if } \mathit{Char}(\Bbbk) = 0\Big) \\ & & \Big\Vert & & \\ & & H & & \\ & & \Big\Vert & & \\ \mathcal{U}(\mathfrak{g}_-) & \xleftarrow[H^\vee_t]{0 \leftarrow t \rightarrow 1} & H^\vee & \xleftarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0} & F[K_+] \quad \Big(= F\big[G_-^\star\big] \text{ if } \mathit{Char}(\Bbbk) = 0\Big) \end{array}$$

This gives *four* different regular 1-parameter deformations from H to Hopf algebras encoding geometrical objects of Poisson type (i.e. Lie bialgebras or Poisson algebraic groups).

5.6 Drinfeld-like functors. The constructions in the present section show that mapping $B \mapsto \big(B'_t\big)^\vee\big|_{t=0}$ and $B \mapsto \big(B^\vee_t\big)'\big|_{t=0}$ (for all $B \in \mathcal{B}$) define two endofunctors of $\mathcal{B}$. Their output describes objects of Poisson-geometric type, namely Lie bialgebras and connected Poisson algebraic monoids. Therefore, both $B \mapsto \big(B'_t\big)^\vee\big|_{t=0}$ and $B \mapsto \big(B^\vee_t\big)'\big|_{t=0}$ (for $B \in \mathcal{B}$) are *geometrification functors* on $\Bbbk$–bialgebras, just like the ones in §3.6: we call them "Drinfeld-like functors". Thus we have four functorial recipes (our *geometrification functors*) to sort out of the generalized symmetry B some *geometrical* symmetries. Now we explain the duality relation between Drinfeld-like functors and associated deformations:

Theorem 5.7. *Let* $B, P \in \mathcal{B}$, *and let* $\pi : B \times P \longrightarrow \Bbbk$ *be a* $\Bbbk$*–bialgebra pairing. Then* π *induces a* $\Bbbk[t]$*–bialgebra pairing* $\pi'_\vee : \big(B_t^\vee\big)' \times \big(P_t'\big)^\vee \longrightarrow \Bbbk[t]$ *and a* $\Bbbk$*–bialgebra pairing* $\pi'_\vee\big|_{t=0} : \big(B_t^\vee\big)'\Big|_{t=0} \times \big(P_t'\big)^\vee\Big|_{t=0} \longrightarrow \Bbbk$. *If in addition* π *is perfect and* $Char(\Bbbk) = 0$ *then both the induced pairings are perfect as well, and*

$$\begin{aligned} \big(B_t^\vee\big)' \; &= \; \Big\{ \eta \in B(t) \,\Big|\, \pi_t(\eta,\kappa) \in \Bbbk[t] \,,\; \forall\, \kappa \in \big(P_t'\big)^\vee \Big\} \;\; =: \; \Big(\big(P_t'\big)^\vee\Big)^\bullet \\ \big(P_t'\big)^\vee \; &= \; \Big\{ \kappa \in P(t) \,\Big|\, \pi_t(\eta,\kappa) \in \Bbbk[t] \,,\; \forall\, \eta \in \big(B_t^\vee\big)' \Big\} \;\; =: \; \Big(\big(B_t^\vee\big)'\Big)^\bullet \end{aligned} \tag{5.4}$$

where $S(t) := \Bbbk(t) \otimes_\Bbbk S$ *for* $S \in \{B, P\}$ *and* $\pi_t : B(t) \times P(t) \longrightarrow \Bbbk(t)$ *is the obvious* $\Bbbk(t)$*–linear pairing induced by* π. *If* $B, P \in \mathcal{HA}$ *are Hopf* $\Bbbk$*–algebras then everything holds with Hopf* $\Bbbk$*–algebra pairings instead of bialgebra pairings.*

Proof. By Theorem 4.4, π induces a pairing $\pi_{\mathcal{R}} : B_t^\vee \times P_t' \longrightarrow \Bbbk[t]$ of $\Bbbk[t]$–bialgebras. Applying to it Proposition 4.4 in [Ga1–2] we get a pairing $\pi'_\vee : \big(B_t^\vee\big)' \times \big(P_t'\big)^\vee \longrightarrow \Bbbk[t]$ as claimed, and a pairing $\pi'_\vee\big|_{t=0} : \big(B_t^\vee\big)'\Big|_{t=0} \times \big(P_t'\big)^\vee\Big|_{t=0} \longrightarrow \Bbbk$ as required, by specialization.

When π is perfect, by Theorem 4.4 again $\pi_{\mathcal{R}}$ is perfect as well, and formulas (4.6) hold. Then if $Char(\Bbbk) = 0$ we can apply Theorem 4.11 of [Ga1–2] to $\pi_{\mathcal{R}}$ (switching the positions of $F_\hbar$ and $U_\hbar$ therein), since P_t' is a QFA and $B_t^\vee$ is a QrUEA (notation of [*loc. cit.*]). The outcome is that formulas (5.4) hold, which implies that both $\pi'_\vee$ and $\pi'_\vee\big|_{t=0}$ are perfect.

The final claim about the Hopf algebra case is clear. $\square$

§ 6 The case of group-related Hopf algebras

6.1 The function algebra case. Let G be any algebraic group over the field $\Bbbk$. Let $F[G]_t := \Bbbk[t] \otimes_\Bbbk F[G]$: this is trivially a QFA at t (in the sense of [Ga1–2]), for $F[G]_t \big/ t\, F[G]_t = F[G]$, which induces on G the trivial Poisson structure. Then the cotangent Lie bialgebra of G is simply $\mathfrak{g}^\times := J \big/ J^2$, where $J := J_{F[G]} \equiv Ker\big(\epsilon_{F[G]}\big)$, with trivial Lie bracket and Lie cobracket dual to the Lie bracket of $\mathfrak{g}$.

We begin by computing $\widehat{F[G]} := G_{\underline{J}}\big(F[G]\big)$ and its deformation $F[G]_t^\vee := \mathcal{R}_{\underline{J}}^t\big(F[G]\big)$.

Let $\{j_b\}_{b \in \mathcal{S}} \, (\subseteq J)$ be a system of parameters of $F[G]$, i.e. $\big\{ y_b := j_b \mod J^2 \big\}_{b \in \mathcal{S}}$ is a $\Bbbk$–basis of $\mathfrak{g}^\times := J \Big/ J^2$. Then, for all $n \in \mathbb{N}$, the set $\big\{ j^{\,\underline{e}} \mod J^{n+1} \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}} \,,\; |\underline{e}| = n \big\}$ spans $J^n \big/ J^{n+1}$ over $\Bbbk$, where $\mathbb{N}_f^{\mathcal{S}} := \big\{ \underline{e} \in \mathbb{N}^{\mathcal{S}} \,\big|\, \underline{e}(s) = 0 \text{ for almost all } s \in \mathcal{S} \big\}$, $|\underline{e}| := \sum_{b \in \mathcal{S}} \underline{e}(b)$, and $j^{\,\underline{e}} := \prod_{b \in \mathcal{S}} j_b^{\underline{e}(b)}$ (w.r.t. some fixed total order on $\mathcal{S}$). This implies that

$$F[G]_t^\vee \; = \; \textstyle\sum_{\underline{e} \in \mathbb{N}_f^{\mathcal{S}}} \Bbbk[t] \cdot t^{-|\underline{e}|} j^{\,\underline{e}} \bigoplus \Bbbk\big[t, t^{-1}\big]\, J^\infty \; = \; \sum_{\underline{e} \in \mathbb{N}_f^{\mathcal{S}}} \Bbbk[t] \cdot (j^\vee)^{\,\underline{e}} \bigoplus \Bbbk\big[t, t^{-1}\big]\, J^\infty$$

where $J^\infty := \bigcap_{n \in \mathbb{N}} J^n$ and $j_s^\vee := t^{-1} j_s$ for all $s \in \mathcal{S}$, and also that $\widehat{F[G]} = G_{\underline{J}}\big(F[G]\big)$ is $\Bbbk$–spanned by $\big\{ j^{\,\underline{e}} \mod J^{|\underline{e}|+1} \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}} \big\}$, so $\widehat{F[G]} = G_{\underline{J}}\big(F[G]\big)$ is a quotient of $S(\mathfrak{g}^\times)$.

Assume for simplicity that $\Bbbk$ be perfect. Let $F[[G]]$ be the J–adic completion of $H = F[G]$. By standard results on algebraic groups (cf. [DG]) one can choose the above system of parameters so that $F[[G]] \cong \Bbbk\big[\big[\{x_b\}_{b\in\mathcal{S}}\big]\big]\Big/\Big(\Big\{x_b^{\,p^{n(b)}}\Big\}_{b\in\mathcal{S}}\Big)$ (the algebra of truncated formal power series), as $\Bbbk$–algebras via $y_b \mapsto x_b$, where $p^{n(b)}$ is the nilpotency order of y_b, for all $b \in \mathcal{S}$. Since $\widehat{F[G]} := G_{\underline{J}}\big(F[G]\big) = G_{\underline{J}}\big(F[[G]]\big) \cong \Bbbk\big[\{\overline{x}_b\}_{b\in\mathcal{S}}\big] \cong S(\mathfrak{g}^\times)$, we deduce that $\widehat{F[G]} \cong S(\mathfrak{g}^\times)\Big/\Big(\Big\{\,\overline{x}^{\,p^{n(x)}}\Big\}_{x\in\mathcal{N}(F[G])}\Big) =: S(\mathfrak{g}^\times)_{red}$ as algebras, where $\mathcal{N}\big(F[G]\big)$ is the nilradical of $F[G]$ and $p^{n(x)}$ is the order of nilpotency of $x \in \mathcal{N}\big(F[G]\big)$. In addition, tracking the very definition of the co-Poisson Hopf structure onto $\widehat{F[G]}$ (after our construction) we see that $\widehat{F[G]} \cong S(\mathfrak{g}^\times)_{red}$ *as co-Poisson Hopf algebras.* Here the Hopf structure on $S(\mathfrak{g}^\times)_{red}$ is induced from the standard one on $S(\mathfrak{g}^\times)$ (the ideal to be modded out being a *Hopf* ideal), and the co-Poisson structure is the one induced from the Lie cobracket of $\mathfrak{g}^\times$. Indeed, for all $y \in J$ and $\overline{y} := y \mod J^2 \in J\big/J^2 \subseteq \widehat{F[G]}$ we have $\Delta\big(\overline{y}\big) \equiv \overline{y} \otimes 1 + 1 \otimes \overline{y} \in \widehat{F[G]}^{\otimes 2}$ (cf. the proof of Theorem 3.5*(a)*), so $\widehat{F[G]} \cong S(\mathfrak{g}^\times)_{red}$ as *Hopf* algebras too. The co-Poisson structure on both sides is uniquely determined by the Lie cobracket of $\mathfrak{g}^\times$, hence $\widehat{F[G]} \cong S(\mathfrak{g}^\times)_{red}$ as *co-Poisson* Hopf algebras too. Finally, if G is a *Poisson* group then both $\widehat{F[G]}$ and $S(\mathfrak{g}^\times)$ are canonically *bi-Poisson* Hopf algebras, with $\{\ ,\ \}\Big|_{\mathfrak{g}^\times} = [\ ,\ ]_{\mathfrak{g}^\times}$, isomorphic to each other through the above isomorphism.

Now, for any Lie algebra $\mathfrak{h}$ we can consider $\mathfrak{h}^{[p]^\infty} := \Big\{\, x^{[p]^n} := x^{p^n} \,\Big|\, x \in \mathfrak{h}\,, n \in \mathbb{N} \,\Big\}$, the *restricted Lie algebra generated by* $\mathfrak{h}$ inside $U(\mathfrak{h})$, with the p–operation given by $x^{[p]} := x^p$; thus one always has $U(\mathfrak{h}) = \mathbf{u}\big(\mathfrak{h}^{[p]^\infty}\big)$. Then the previous analysis gives $S(\mathfrak{g}^\times)_{red} = \mathbf{u}\left(\mathfrak{g}^\times_{res}\right)$ (for $\mathfrak{g}^\times$ is Abelian), hence finally $\widehat{F[G]} \cong \mathbf{u}\left(\mathfrak{g}^\times_{res}\right)$ as co-Poisson Hopf algebras, and even as bi-Poisson Hopf algebras whenever G is a Poisson group.

If $\Bbbk$ is not perfect the same analysis applies, but modifying a bit the previous arguments.

As for $F[G]^\vee$, it is known (by [Ab], Lemma 4.6.4) that $F[G]^\vee = F[G]$ whenever G is finite dimensional and there exists no $f \in F[G] \setminus \Bbbk$ which is separable algebraic over $\Bbbk$.

It is also interesting to consider $\big(F[G]_t^\vee\big)'$. If $Char(\Bbbk) = 0$, Proposition 4.3 in [Ga1–2] gives $\big(F[G]_t^\vee\big)' = F[G]_t$. If instead $Char(\Bbbk) = p > 0$, then the situation might change drastically. Indeed, if G has eight 1 — i.e., $F[G] = \Bbbk\big[\{x_i\}_{i\in\mathcal{I}}\big]\big]\Big/\big(\big\{x_i^p \,\big|\, i \in \mathcal{I}\,\big\}\big)$ as a $\Bbbk$–algebra — then the same analysis as in the characteristic zero case applies, with a few minor changes, whence one gets again $\big(F[G]_t^\vee\big)' = F[G]_t$. Otherwise, let $y \in J \setminus \{0\}$ be primitive and such that $y^p \neq 0$ (for instance, this occurs for $G \cong \mathbb{G}_a$). Then y^p is primitive as well, hence $\delta_n(y^p) = 0$ for each $n > 1$. It follows that $0 \neq t\,(y^\vee)^p \in \big(F[G]_t^\vee\big)'$, whereas $t\,(y^\vee)^p \notin F[G]_t$, as follows from our previous description of $F[G]_t^\vee$. Thus $\big(F[G]_t^\vee\big)' \supsetneqq F[G]_t^\vee$, a *counterexample to Proposition 4.3 of [Ga1–2].*

What for $\widetilde{F[G]}$, $F[G]'$ and $F[G]_t'$? This heavily depends on the group G under consideration. We provide two simple examples, both "extreme", and opposite to each other.

First, let $G := \mathbb{G}_a = Spec\big(\Bbbk[x]\big)$, so $F[G] = F[\mathbb{G}_a] = \Bbbk[x]$ and $F[\mathbb{G}_a]_t := \Bbbk[t] \otimes_{\Bbbk} \Bbbk[x] = \big(\Bbbk[t]\big)[x]$. Then since $\Delta(x) := x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$ we find that $F[\mathbb{G}_a]_t' = \big(\Bbbk[t]\big)[tx]$

(like in §6.3 below), which at $t=1$ gives $F[\mathbb{G}_a]' = F[\mathbb{G}_a]$. Second, let $G := \mathbb{G}_m = \mathit{Spec}\Big(\Bbbk\big[z^{+1}, z^{-1}\big]\Big)$, that is $F[G] = F[\mathbb{G}_m] = \Bbbk\big[z^{+1}, z^{-1}\big]$ so that $F[\mathbb{G}_m]_t := \Bbbk[t] \otimes_{\Bbbk} \Bbbk\big[z^{+1}, z^{-1}\big] = \Bbbk[t]\big[z^{+1}, z^{-1}\big]$. Then since $\Delta\big(z^{\pm 1}\big) := z^{\pm 1} \otimes z^{\pm 1}$ and $\epsilon\big(z^{\pm 1}\big) = 1$ we find $\delta_n\big(z^{\pm 1}\big) = \big(z^{\pm 1} - 1\big)^{\otimes n}$ for all $n \in \mathbb{N}$, whence $F[\mathbb{G}_m]_t' = \Bbbk[t] \cdot 1$ and $F[\mathbb{G}_m]' = \Bbbk \cdot 1$.

We resume the main facts above in the following statement:

Theorem 6.2. *Let $H = F[G]$ be the function algebra of an algebraic Poisson group. Then $F[G]^\vee = F[G]$ whenever G is finite dimensional and there exists no $f \in F[G] \setminus \Bbbk$ which is separable algebraic over $\Bbbk$. In any case, $\widehat{F[G]}$ is a bi-Poisson Hopf algebra, namely*

$$\widehat{F[G]} := G_{\underline{J}}\big(F[G]\big) \;\cong\; S(\mathfrak{g}^\times)\Big/\Big(\Big\{\, \overline{x}^{\,p^{n(x)}} \,\Big|\, x \in \mathcal{N}\big(F[G]\big) \Big\}\Big) \;\cong\; \mathbf{u}\big(\mathfrak{g}^\times_{res}\big)$$

where $\mathcal{N}\big(F[G]\big)$ is the nilradical of $F[G]$, $p^{n(x)}$ is the order of nilpotency of $x \in \mathcal{N}\big(F[G]\big)$ and the bi-Poisson Hopf structure of $S(\mathfrak{g}^\times)\Big/\Big(\Big\{\, \overline{x}^{\,p^{n(x)}} \,\Big|\, x \in \mathcal{N}\big(F[G]\big) \Big\}\Big)$ is the quotient one from $S(\mathfrak{g}^\times)$. In particular, if G is smooth then $\widehat{F[G]} \cong S(\mathfrak{g}^\times) = U(\mathfrak{g}^\times)$. $\square$

6.3 The enveloping algebra case. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. Assume $p=0$, and set $U(\mathfrak{g})_t := \Bbbk[t] \otimes_{\Bbbk} U(\mathfrak{g})$. Then $U(\mathfrak{g})_t$ is trivially a QrUEA at t (cf. [Ga1–2]), for $U(\mathfrak{g})_t \big/ t\, U(\mathfrak{g})_t = U(\mathfrak{g})$, inducing on $\mathfrak{g}$ the trivial Lie cobracket. The dual Poisson group is $\mathfrak{g}^\star$, the topological dual of $\mathfrak{g}$ w.r.t. the weak topology, w.r.t. addition, with $\mathfrak{g}$ as cotangent Lie bialgebra and $F[\mathfrak{g}^\star] = S(\mathfrak{g})$. The Hopf structure is the standard one, and the Poisson structure is the Kostant-Kirillov one, induced by $\{x,y\} := [x,y]$ for $x,\, y \in \mathfrak{g}$, as in §1.3.

Similarly, if $p>0$ and $\mathfrak{g}$ is restricted, the Hopf $\Bbbk[t]$–algebra $\mathbf{u}(\mathfrak{g})_t := \Bbbk[t] \otimes_{\Bbbk} \mathbf{u}(\mathfrak{g})$ is a QrUEA at t, because $\mathbf{u}(\mathfrak{g})_t \big/ t\, \mathbf{u}(\mathfrak{g})_t = \mathbf{u}(\mathfrak{g})$, inducing on $\mathfrak{g}$ the trivial Lie cobracket. Then the dual Poisson group is again $\mathfrak{g}^\star$, with cotangent Lie bialgebra $\mathfrak{g}$ and with $F[\mathfrak{g}^\star] = S(\mathfrak{g})$, the Poisson Hopf structure being as above. Recall also that $U(\mathfrak{g}) = \mathbf{u}\big(\mathfrak{g}^{[p]^\infty}\big)$ (cf. §1.3).

First we describe $\mathbf{u}(\mathfrak{g})_t' := \mathcal{R}_{\underline{D}}^t$, $\mathbf{u}(\mathfrak{g})'$ and $\widetilde{\mathbf{u}(\mathfrak{g})}$, computing the $\delta_\bullet$–filtration $\underline{D}$ of $\mathbf{u}(\mathfrak{g})$.

By the PBW theorem, once an ordered basis B of $\mathfrak{g}$ is fixed $\mathbf{u}(\mathfrak{g})$ admits as basis the set of ordered monomials in the elements of B whose degree, w.r.t. each element of B, is less than p. This yields a Hopf algebra filtration of $\mathbf{u}(\mathfrak{g})$ by the total degree, the so-called *standard filtration*. A straightforward calculation shows that $\underline{D}$ coincides with the standard filtration. This implies $\mathbf{u}(\mathfrak{g})' = \mathbf{u}(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})_t' = \langle \tilde{\mathfrak{g}} \rangle = \langle t\, \mathfrak{g} \rangle$, where hereafter $\tilde{\mathfrak{g}} := t\, \mathfrak{g}$, and similarly $\tilde{x} := t\, x$ for all $x \in \mathfrak{g}$. Therefore the presentation† $\mathbf{u}(\mathfrak{g})_t = T_{\Bbbk[t]}(\mathfrak{g}) \Big/ \big(\big\{\, x\, y - y\, x - [x,y]\,,\; z^p - z^{[p]} \;\big|\; x,y,z \in \mathfrak{g} \,\big\}\big)$ implies

$$\mathbf{u}(\mathfrak{g})_t' \;=\; T_R\big(\tilde{\mathfrak{g}}\big) \Big/ \big(\big\{\, \tilde{x}\, \tilde{y} - \tilde{y}\, \tilde{x} - t\, \widetilde{[x,y]}\,,\; \tilde{z}^{\,p} - t^{\,p-1}\, \widetilde{z^{[p]}} \;\big|\; x,y,z \in \mathfrak{g} \,\big\}\big)$$

$$\text{and} \qquad \widetilde{\mathbf{u}(\mathfrak{g})} \;=\; T_{\Bbbk}\big(\tilde{\mathfrak{g}}\big) \Big/ \big(\big\{\, \tilde{x}\, \tilde{y} - \tilde{y}\, \tilde{x}\,,\; \tilde{z}^{\,p} \;\big|\; \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{\mathfrak{g}} \,\big\}\big) \;=\; F[\mathfrak{g}^\star] \Big/ \big(\big\{\, z^p \;\big|\; z \in \mathfrak{g} \,\big\}\big)$$

as *Hopf* algebras and as *Poisson Hopf* algebras respectively.

† Hereafter, $T_R(M)$, resp. $S_R(M)$, is the tensor, resp. symmetric algebra of an R–module M.

Therefore $\widetilde{\mathbf{u}(\mathfrak{g})}$ *is the function algebra of, and* $\mathbf{u}(\mathfrak{g})'_t$ *is a QFA for, a connected algebraic Poisson group of dimension 0 and height 1 with cotangent Lie bialgebra* $\mathfrak{g}$ *, hence dual to* $\mathfrak{g}$ *.*

Since $U(\mathfrak{g}) = \mathbf{u}\big(\mathfrak{g}^{[p]^\infty}\big)$ the previous analysis yields $U(\mathfrak{g})' = U(\mathfrak{g})$ and a description of $U(\mathfrak{g})'_t$ and $\widetilde{U(\mathfrak{g})}$, in particular $\widetilde{U(\mathfrak{g})} \,=\, F\Big[\big(\mathfrak{g}^{[p]^\infty}\big)^\star\Big]\Big/\Big(\big\{\, z^p \,\big\}_{z \in \mathfrak{g}^{[p]^\infty}}\Big)$.

Furthermore, $\mathbf{u}(\mathfrak{g})'_t \,=\, \langle \tilde{\mathfrak{g}} \rangle$ implies that $J_{\mathbf{u}(\mathfrak{g})'_t}$ is generated (as an ideal) by $\tilde{\mathfrak{g}}$, so $t^{-1} J_{\mathbf{u}(\mathfrak{g})'_t}$ is generated by $\mathfrak{g}$, thus $\big(\mathbf{u}(\mathfrak{g})'_t\big)^\vee := \sum_{n \geq 0} \big(t^{-1} J_{\mathbf{u}(\mathfrak{g})'_t}\big)^n = \sum_{n \geq 0} \mathfrak{g}^n = \mathbf{u}(\mathfrak{g})_t$.

When *Char*$(\Bbbk) \,=\, 0$ and we look at $U(\mathfrak{g})$, the like argument applies: $\underline{D}$ coincides with the standard filtration of $U(\mathfrak{g})$ given by total degree, via the PBW theorem. This and definitions imply $U(\mathfrak{g})' \,=\, U(\mathfrak{g})$ and $U(\mathfrak{g})'_t \,=\, \langle \tilde{\mathfrak{g}} \rangle \,=\, \langle t\, \mathfrak{g} \rangle$, so that the presentation $U(\mathfrak{g})_t = T_{\Bbbk[t]}(\mathfrak{g})\Big/\big(\big\{\, x\,y - y\,x - [x,y] \,\big\}_{x,y,z \in \mathfrak{g}}\big)$ yields

$$U_t(\mathfrak{g})' \;=\; T_{\Bbbk[t]}(\tilde{\mathfrak{g}})\Big/\big(\big\{\tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} - t\,\widetilde{[x,y]} \,\big|\, \tilde{x}, \tilde{y} \in \tilde{\mathfrak{g}} \;\big\}\big)$$

$$\text{and} \qquad \widetilde{U(\mathfrak{g})} \;=\; T_{\Bbbk}(\tilde{\mathfrak{g}})\Big/\big(\big\{\, \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} \,\big|\, \tilde{x}, \tilde{y} \in \tilde{\mathfrak{g}} \,\big\}\big) \;=\; S(\mathfrak{g}) \;=\; F[\mathfrak{g}^\star]$$

as *Hopf* and as *Poisson Hopf* algebras (as predicted by Theorem 2.2*(c)* in [Ga1-2]). Furthermore, $J_{U(\mathfrak{g})'_t}$ is generated by $\tilde{\mathfrak{g}}$: therefore $t^{-1} J_{U(\mathfrak{g})'_t}$ is generated by $\mathfrak{g}$, whence eventually $\big(U(\mathfrak{g})'_t\big)^\vee := \sum_{n \geq 0} \big(\, t^{-1} J_{U(\mathfrak{g})'_t}\,\big)^n = \sum_{n \geq 0} \mathfrak{g}^n = U(\mathfrak{g})_t$.

What for the functor $(\;)^\vee$? This heavily depends on the $\mathfrak{g}$ we start from.

First assume *Char*$(\Bbbk) = 0$. Let $\mathfrak{g}_{(1)} := \mathfrak{g}$, $\mathfrak{g}_{(k)} := \big[\mathfrak{g}, \mathfrak{g}_{(k-1)}\big]$ $(k \in \mathbb{N}_+)$, be the *lower central series* of $\mathfrak{g}$. Set $\mathfrak{g}_{\langle k \rangle} := \mathfrak{g}_{(k)}\big/\mathfrak{g}_{(k+1)}$ for all $k \in \mathbb{N}$, and $\mathfrak{g}_{gr} := \oplus_{k \in \mathbb{N}} \mathfrak{g}_{\langle k \rangle}$, the latter being a *graded* Lie algebra in a natural way; set also $\mathfrak{g}_{(\infty)} := \cap_{k \in \mathbb{N}} \mathfrak{g}_{(k)}$. Recall that $U(\mathfrak{g})_t^\vee := \sum_{n \geq 0} t^{-n} J^n$ for $J := J_{U_t(\mathfrak{g})}$. Pick subsets $B_k \,\big(\subset \mathfrak{g}_{(k)}\big)$ such that B_k mod $\mathfrak{g}_{(k+1)}$ be a $\Bbbk$–basis of $\mathfrak{g}_{\langle k \rangle}$ (for all $k \in \mathbb{N}$), give any total order to $B := \cup_{k \in \mathbb{N}} B_k$ and set $\partial(b) := k$ iff $b \in B_k$. Applying the PBW theorem to this ordered basis of $\mathfrak{g}$, we get that $J^n\big/J^{n+1}$ has basis $\big\{\, b_1^{e_1} b_2^{e_2} \cdots b_s^{e_s} \mod J^{n+1} \,\big|\, s \in \mathbb{N} \,, b_r \in B \,, \sum_{r=1}^s b_r\, \partial(b_r) = n \,\big\}$. Then one finds that $U(\mathfrak{g})_t^\vee$ is generated by $\big\{\, t^{-1} b_r \,\big|\, b_r \in B_1 \,\big\} \cup \big(\, \cup_{n \in \mathbb{N}}\, t^{-n} \mathfrak{g}_{(\infty)}\big)$, while $U(\mathfrak{g})^\vee \,=\, U(\mathfrak{g})\Big/\mathfrak{g}_{(\infty)}\, U(\mathfrak{g})$ and $\widehat{U(\mathfrak{g})} \,\cong\, U(\mathfrak{g}_{gr})$ as *graded* Hopf algebras.

Now assume *Char*$(\Bbbk) = p > 0$. Let $\mathfrak{g}_n := \Big\langle \bigcup_{m\, p^k \geq n} \big(\mathfrak{g}_{(m)}\big)^{[p^k]} \Big\rangle$ for all $n \in \mathbb{N}_+$ (where $\langle X \rangle$ denotes the Lie subalgebra of $\mathfrak{g}$ generated by X): then $\big\{\mathfrak{g}_n\big\}_{n \in \mathbb{N}_+}$ is *the p–lower central series of* $\mathfrak{g}$, which is a *strongly central series* of $\mathfrak{g}$. Set $\mathfrak{g}_{[k]} := \mathfrak{g}_k\big/\mathfrak{g}_{k+1}$ for all $k \in \mathbb{N}$, and $\mathfrak{g}_{p\text{-}gr} := \oplus_{k \in \mathbb{N}}\, \mathfrak{g}_{[k]}$, the latter being a *graded* restricted Lie algebra in a natural way; set also $\mathfrak{g}_\infty := \cap_{k \in \mathbb{N}}\, \mathfrak{g}_k$. Now definitions give $\mathfrak{g}_n \subseteq J^n$ for all $n \in \mathbb{N}$, where $J := J_{\mathbf{u}(\mathfrak{g})}$. More precisely, we can proceed as above, taking suitable lifts B_k of bases of each $\mathfrak{g}_{[k]}$. Then the (restricted) PBW theorem for $\mathbf{u}(\mathfrak{g})$ implies that $J^n\big/J^{n+1}$ has $\Bbbk$–basis the set of ordered monomials $x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_s}^{e_s}$ (with the x_i's in the union of the B_k's) such that $\sum_{r=1}^s e_r \partial(x_{i_r}) = n$, where $\partial(x_{i_r}) \in \mathbb{N}$ is uniquely determined by the condition $x_{i_r} \in \mathfrak{g}_{\partial(x_{i_r})} \setminus \mathfrak{g}_{\partial(x_{i_r})+1}$. This yields a description of $\underline{J}$, hence of $\mathbf{u}(\mathfrak{g})^\vee$, $\mathbf{u}_t(\mathfrak{g})^\vee$, and $\widehat{\mathbf{u}(\mathfrak{g})}$: in particular, $\mathbf{u}(\mathfrak{g})^\vee \,=\, \mathbf{u}(\mathfrak{g})\Big/\mathfrak{g}_\infty\, \mathbf{u}(\mathfrak{g})$ and $\widehat{\mathbf{u}(\mathfrak{g})} \,\cong\, \mathbf{u}(\mathfrak{g}_{p\text{-}gr})$ as *graded* Hopf algebras.

We collect the main results above in the following statement:

Theorem 6.4.

(a) Let $Char(\Bbbk) = 0$, *and* $\mathfrak{g}$ *be a Lie bialgebra. Then* $U(\mathfrak{g})' = U(\mathfrak{g})$, *and* $\widetilde{U(\mathfrak{g})} := G_{\underline{D}}\big(U(\mathfrak{g})\big)$ *is a bi-Poisson Hopf algebra, namely* $\widetilde{U(\mathfrak{g})} \cong S(\mathfrak{g}) = F[\mathfrak{g}^\star]$ *(with notation of §1), where the bi-Poisson Hopf structure on* $S(\mathfrak{g}^\star)$ *is the canonical one (see §1.3). On the other hand,* $U(\mathfrak{g})^\vee = U(\mathfrak{g}) \big/ \mathfrak{g}_{(\infty)} U(\mathfrak{g})$ *and* $\widehat{U(\mathfrak{g})} \cong U(\mathfrak{g}_{gr})$ *as graded Hopf algebras.*

(b) Let $Char(\Bbbk) = p > 0$, *and* $\mathfrak{g}$ *be a restricted Lie bialgebra. Then* $\mathbf{u}(\mathfrak{g})' = \mathbf{u}(\mathfrak{g})$, *and* $\widetilde{\mathbf{u}(\mathfrak{g})} := G_{\underline{D}}\big(\mathbf{u}(\mathfrak{g})\big)$ *is a bi-Poisson Hopf algebra, namely* $\widetilde{\mathbf{u}(\mathfrak{g})} \cong S(\mathfrak{g}) \Big/ \big(\{x^p \,|\, x \in \mathfrak{g}\}\big) = F[G^\star]$ *(with notation of §1). Here the bi-Poisson Hopf structure on* $S(\mathfrak{g}) \Big/ \big(\{x^p \,|\, x \in \mathfrak{g}\}\big)$ *is induced by the canonical one on* $S(\mathfrak{g})$ *(see §1.3) and* $G^\star$ *is a connected algebraic Poisson group of dimension 0 and height 1 whose cotangent Lie bialgebra is* $\mathfrak{g}$. *On the other hand,* $\mathbf{u}(\mathfrak{g})^\vee = \mathbf{u}(\mathfrak{g}) \Big/ \mathfrak{g}_\infty \mathbf{u}(\mathfrak{g})$ *and* $\widehat{\mathbf{u}(\mathfrak{g})} \cong \mathbf{u}(\mathfrak{g}_{p\text{-}gr})$ *as graded Hopf algebras.* $\square$

Remark: For any given Lie algebra $\mathfrak{g}$, the group-scheme theoretic version of Lie's Third Theorem claims the existence of a connected algebraic group-scheme *of height 1* having $\mathfrak{g}$ as tangent Lie algebra. Part *(b)* of Theorem 6.4 gives a Poisson-dual result: $G^\star = \mathfrak{g}^\star$ is an algebraic Poisson group-scheme of height 1 having $\mathfrak{g}$ as *cotangent* Lie algebra.

6.5 The hyperalgebra case. Let G be an algebraic group, which for simplicity we assume to be finite-dimensional. The hyperalgebra associated to G is $Hyp\,(G) := \big(F[G]^\circ\big)_\epsilon = \big\{\, \phi \in F[G]^\circ \,\big|\, \phi(\mathfrak{m}_e{}^n) = 0\,,\ \forall\, n \gg 0 \,\big\}$, that is the irreducible component of the *dual* Hopf algebra $F[G]^\circ$ containing $\epsilon = \epsilon_{F[G]}$. This is a Hopf subalgebra of $F[G]^\circ$, connected and cocommutative. There is a Hopf algebra morphism $\Phi : U(\mathfrak{g}) \longrightarrow Hyp\,(G)$; if $p = 0$ then Φ is an isomorphism, so $Hyp\,(G)$ identifies to $U(\mathfrak{g})$; if $p > 0$ then Φ factors through $\mathbf{u}(\mathfrak{g})$ and the induced morphism $\overline{\Phi} : \mathbf{u}(\mathfrak{g}) \longrightarrow Hyp\,(G)$ is injective, so that $\mathbf{u}(\mathfrak{g})$ identifies with a Hopf subalgebra of $Hyp\,(G)$. Now we study $Hyp\,(G)'$, $Hyp\,(G)^\vee$, $\widetilde{Hyp\,(G)}$, $\widehat{Hyp\,(G)}$.

As $Hyp\,(G)$ is connected, we have $Hyp\,(G) = Hyp\,(G)'$. Now, Theorem 3.5*(b)* gives $\widetilde{Hyp\,(G)} = F[\varGamma\,]$ for some connected algebraic Poisson group $\varGamma$. Theorem 6.2 yields $\widehat{F[G]} \cong S(\mathfrak{g}^*) \bigg/ \Big(\Big\{ \overline{x}^{\,p^{n(x)}} \Big\}_{x \in \mathcal{N}_{F[G]}} \Big) = \mathbf{u}\bigg(P\bigg(S(\mathfrak{g}^*) \bigg/ \Big(\Big\{ \overline{x}^{\,p^{n(x)}} \Big\}_{x \in \mathcal{N}_{F[G]}} \Big) \bigg)\bigg) = \mathbf{u}\Big(\big(\mathfrak{g}^*\big)^{p^\infty} \Big)$, with $\big(\mathfrak{g}^*\big)^{p^\infty} := Span\Big(\Big\{ x^{p^n} \,\Big|\, x \in \mathfrak{g}^*, n \in \mathbb{N} \Big\} \Big) \subseteq \widehat{F[G]}$, and noting that $\mathfrak{g}^\times = \mathfrak{g}^*$. On the other hand, exactly like for $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$, respectively in case $Char(\Bbbk) = 0$ and $Char(\Bbbk) > 0$, the filtration $\underline{D}$ of $Hyp\,(G)$ is the natural filtration given by the order of differential operators. This implies $Hyp\,(G)'_t := \mathcal{R}_t\big(Hyp\,(G)\big) = \big\langle \big\{\, t^n x^{(n)} \,\big|\, x \in \mathfrak{g}\,, n \in \mathbb{N} \,\big\} \big\rangle$, where hereafter $x^{(n)}$ is the n–th divided power or the n–th binomial coefficient of $x \in \mathfrak{g}$ (such $x^{(n)}$'s generate $Hyp\,(G)$ as an algebra). Then one easily checks that the graded Hopf pairing between $Hyp\,(G)'_t \Big/ t\, Hyp\,(G)'_t = \widetilde{Hyp\,(G)} = F[\varGamma]$ and $\widehat{F[G]}$ in Theorem 3.7 is perfect, from which we deduce that $\varGamma$ has cotangent Lie bialgebra isomorphic to $\Big(\big(\mathfrak{g}^*\big)^{p^\infty} \Big)^*$.

§ 7 The Crystal duality Principle

7.1 The Crystal Duality Principle. We can finally motivate the expression "Crystal Duality Principle". In short, we gave functorial recipes to get, out from *any* Hopf algebra H, four Hopf algebras of Poisson-geometrical type arranged in couples, namely $\big(\widehat{H}, \widetilde{H}\,\big) = \big(\mathcal{U}(\mathfrak{g}_-), F[G_+]\big)$ and $\Big(\big(H_t^\vee\big)'\big|_{t=0}, \big(H_t'\big)^\vee\big|_{t=0}\Big) = \big(F[K_+], \mathcal{U}(\mathfrak{k}_-)\big)$, hence four Poisson-geometrical symmetries G_+, $\mathfrak{g}_-$, K_+ and $\mathfrak{k}_-$: this is the "Principle". The word "Crystal" reminds the fact that the first couple, namely $\big(\widehat{H}, \widetilde{H}\,\big)$, is obtained via a *crystallization process* (cf. §§2.15 and 3.6). Finally, "Duality" witnesses that if $\mathit{Char}(\Bbbk) = 0$ then the link between the two couples of special Hopf algebras is Poisson duality (see Theorem 5.4*(c)*), in that $K_+ = G_-^\star$ and $\mathfrak{k}_- = \mathfrak{g}_+^\times$. Moreover, in *any* characteristic, when H is a Hopf algebra of (Poisson-)geometrical type applying the crystal functor leading to Hopf algebras of dual type the result is ruled by Poisson duality (see Theorems 6.2 and 6.4).

7.2 The CDP as corollary of the GQDP. The construction of Drinfeld-like functors passes through the application of the Global Quantum Duality Principle (=GQDP in the sequel). In this section we briefly outline how *the whole CDP can be obtained as a corollary of the GQDP* (but for some minor details); see also [Ga1–2], §5.

For any $H \in \mathcal{HA}$, let $H_t := H[t] \equiv \Bbbk[t] \otimes_{\Bbbk} H$. Then H_t is a torsionless Hopf algebra over $\Bbbk[t]$, hence one of those to which the constructions in [Ga1–2] can be applied. In particular, we can act on it with Drinfeld's functors considered therein: these give quantum groups, namely a quantized (restricted) universal enveloping algebra (=QrUEA) and a quantized function algebra (=QFA). Straightforward computations show that the QrUEA is just $H_t^\vee := \mathcal{R}_{\underline{J}}^t(H)$, and the QFA is $H_t' := \mathcal{R}_{\underline{D}}^t(H)$, with $\widehat{H} \cong H_t^\vee\big|_{t=0}$ and $\widetilde{H} \cong H_t'\big|_{t=0}$. It follows that all properties of $\widehat{H}$ and $\widetilde{H}$ spring out as special cases of the results proved in [Ga1–2] for Drinfeld's functors, but for their being graded. Similarly, the fact that H' be a Hopf subalgebra of H follows from the fact that H_t' itself is a Hopf algebra and $H' = H_t'\big|_{t=1}$. Instead, $H^\vee$ is a quotient Hopf algebra of H because $H_t^\vee$ is a Hopf algebra, hence $\overline{H_t^\vee} := H_t^\vee \Big/ \bigcap_{n \in \mathbb{N}} t^n H_t^\vee$ is a Hopf algebra, and finally $H^\vee = \overline{H_t^\vee}\big|_{t=1}$. The fact that H_t' and $H_t^\vee$ be regular 1-parameter deformations of H' and $H^\vee$ is then clear by construction. Finally, the parts of the CDP dealing with Poisson duality are direct consequences of the like items in the GQDP applied to H_t' and to $H_t^\vee$ (but for Theorem 6.4*(b)*). The cases of (co)augmented (co)algebras or bialgebras can be easily treated the same.

§ 8 The Crystal Duality Principle on group algebras

8.1 Group-related algebras. In this section, G is any abstract group. For any commutative unital ring $\mathbb{A}$, by $\mathbb{A}[G]$ we mean the group algebra of G over $\mathbb{A}$ and, when

G is *finite*, we denote by $A_{\mathbb{A}}(G) := \mathbb{A}[G]^*$ (the linear dual of $\mathbb{A}[G]$) the function algebra of G over $\mathbb{A}$. Hereafter $\Bbbk$ will be a field with $p := \mathit{Char}(\Bbbk)$, and $R := \Bbbk[t]$.

Recall that $H := \mathbb{A}[G]$ admits G itself as a special basis, with Hopf algebra structure given by $g \cdot_H \gamma := g \cdot_G \gamma$, $1_H := 1_G$, $\Delta(g) := g \otimes g$, $\epsilon(g) := 1$, $S(g) := g^{-1}$, for all $g, \gamma \in G$. Dually, $H := A_{\mathbb{A}}(G)$ has basis $\big\{\varphi_g \,\big|\, g \in G\big\}$ dual to the basis G of $\mathbb{A}[G]$, with $\varphi_g(\gamma) := \delta_{g,\gamma}$ for all $g, \gamma \in G$. Its Hopf algebra structure is given by $\varphi_g \cdot \varphi_\gamma := \delta_{g,\gamma}\varphi_g$, $1_H := \sum_{g \in G} \varphi_g$, $\Delta(\varphi_g) := \sum_{\gamma \cdot \ell = g} \varphi_\gamma \otimes \varphi_\ell$, $\epsilon(\varphi_g) := \delta_{g,1_G}$, $S(\varphi_g) := \varphi_{g^{-1}}$, for all g, $\gamma \in G$. Thus $\Bbbk[G]_t = R \otimes_{\Bbbk} \Bbbk[G] = R[G]$, $A_{\Bbbk}[G]_t = R \otimes_{\Bbbk} A_{\Bbbk}[G] = A_R[G]$. First we have

Theorem 8.2. $\Bbbk[G]_t^{\,\prime} = R \cdot 1$, $\Bbbk[G]' = \Bbbk \cdot 1$ *and* $\widetilde{\Bbbk[G]} = \Bbbk \cdot 1 = F\big[\{*\}\big]$.

Proof. The claim follows easily from the formula $\delta_n(g) = (g-1)^{\otimes n}$, for $g \in G$, $n \in \mathbb{N}$. $\square$

8.3 $\Bbbk[G]_t^{\vee}$, $\Bbbk[G]^{\vee}$ and $\widehat{\Bbbk[G]}$ and the dimension subgroup problem. In contrast with the triviality result in Theorem 8.2, things are more interesting for $R[G]^{\vee} = \Bbbk[G]_t^{\vee}$, $\Bbbk[G]^{\vee}$ and $\widehat{\Bbbk[G]}$. Note, however, that since $\Bbbk[G]$ is cocommutative the induced Poisson cobracket on $\widehat{\Bbbk[G]}$ is trivial, and the Lie cobracket of $\mathfrak{k}_G := P\Big(\widehat{\Bbbk[G]}\Big)$ is trivial as well.

Studying $\Bbbk[G]^{\vee}$ and $\widehat{\Bbbk[G]}$ amounts to study the filtration $\big\{J^n\big\}_{n \in \mathbb{N}}$, with $J := \mathit{Ker}\,(\epsilon_{\Bbbk[G]})$, which is a classical topic. Indeed, for $n \in \mathbb{N}$ let $\mathcal{D}_n(G) := \big\{\, g \in G \,\big|\, (g-1) \in J^n \,\big\}$: this is a characteristic subgroup of G, called *the* n^{th} *dimension subgroup of* G. All these form a filtration inside G: characterizing it in terms of G is the *dimension subgroup problem*. For group algebras over fields, it is completely solved (see [Pa], Ch. 11, §1, and [HB], and references therein); this also gives a description of $\big\{J^n\big\}_{n \in \mathbb{N}_+}$. Thus we find ourselves within the domain of classical group theory. Now we use the results which solve the dimension subgroup problem to deduce a description of $\Bbbk[G]^{\vee}$, $\widehat{\Bbbk[G]}$ and $\Bbbk[G]_t^{\vee}$. Later on we'll get from this a description of $\big(\Bbbk[G]_t^{\vee}\big)'$ and its semiclassical limit too.

By construction, J has $\Bbbk$–basis $\big\{\eta_g \,\big|\, g \in G \setminus \{1_G\}\big\}$, where $\eta_g := (g-1)$. Then $\Bbbk[G]^{\vee}$ is generated by $\big\{\, \eta_g \mod J^\infty \,\big|\, g \in G \setminus \{1_G\}\big\}$, and $\widehat{\Bbbk[G]}$ by $\big\{\, \overline{\eta_g} \,\big|\, g \in G \setminus \{1_G\}\big\}$. Hereafter $\overline{x} := x \mod J^{n+1}$ for all $x \in J^n$, that is $\overline{x}$ is the element in $\widehat{\Bbbk[G]}$ which corresponds to $x \in \Bbbk[G]$. Moreover, $\overline{g} = \overline{1 + \eta_g} = \overline{1}$ for all $g \in G$; also, $\Delta\big(\overline{\eta_g}\big) = \overline{\eta_g} \otimes \overline{g} + 1 \otimes \overline{\eta_g} = \overline{\eta_g} \otimes 1 + 1 \otimes \overline{\eta_g}$: thus $\overline{\eta_g}$ is primitive, so $\big\{\, \overline{\eta_g} \,\big|\, g \in G \setminus \{1_G\}\big\}$ generates $\mathfrak{k}_G := P\Big(\widehat{\Bbbk[G]}\Big)$.

8.4 The Jennings-Hall theorem. The description of $\mathcal{D}_n(G)$ is given by the Jennings-Hall theorem, which we now recall. The involved construction strongly depends on whether $p := \mathit{Char}(\Bbbk)$ is zero or not, so we shall distinguish these two cases.

First assume $p = 0$. Let $G_{(1)} := G$, $G_{(k)} := (G, G_{(k-1)})$ $(k \in \mathbb{N}_+)$, form the *lower central series* of G; hereafter (X, Y) is the commutator subgroup of G generated by the set of commutators $\big\{(x,y) := x\,y\,x^{-1}y^{-1} \,\big|\, x \in X, y \in Y\big\}$. Then let $\sqrt{G_{(n)}} := \big\{x \in G \,\big|\, \exists\, s \in \mathbb{N}_+ : x^s \in G_{(n)}\big\}$ for all $n \in \mathbb{N}_+$: these form a descending series of characteristic subgroups in G, such that each composition factor $A_{(n)}^G := \sqrt{G_{(n)}}\Big/\sqrt{G_{(n+1)}}$ is torsion-free Abelian.

Therefore $\mathcal{L}_0(G) := \bigoplus_{n\in\mathbb{N}_+} A^G_{(n)}$ is a graded Lie ring, with Lie bracket $\big[\,\overline{g}, \overline{\ell}\,\big] := \overline{(g,\ell\,)}$ for all *homogeneous* $\overline{g}$, $\overline{\ell} \in \mathcal{L}_0(G)$, with obvious notation. It is easy to see that the map $\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_0(G) \longrightarrow \mathfrak{k}_G$, $\overline{g} \mapsto \overline{\eta_g}$, is an epimorphism *of graded Lie rings*: therefore *the Lie algebra* $\mathfrak{k}_G$ *is a quotient of* $\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_0(G)$; in fact, the above is an isomorphism (see below). We shall use notation $\partial(g) := n$ for all $g \in \sqrt{G_{(n)}} \setminus \sqrt{G_{(n+1)}}$.

For each $k \in \mathbb{N}_+$ pick in $A^G_{(k)}$ a subset $\overline{B}_k$ which is a $\mathbb{Q}$–basis of $\mathbb{Q} \otimes_{\mathbb{Z}} A^G_{(k)}$. For each $\overline{b} \in \overline{B}_k$, choose a fixed $b \in \sqrt{G_{(k)}}$ such that its coset in $A^G_{(k)}$ be $\overline{b}$, and denote by B_k the set of all such elements b. Let $B := \bigcup_{k\in\mathbb{N}_+} B_k$: we call such a set *t.f.l.c.s.-net* (= "torsion-free-lower-central-series-net") on G. Clearly $B_k = \Big(B \cap \sqrt{G_{(k)}}\,\Big) \setminus \Big(B \cap \sqrt{G_{(k+1)}}\,\Big)$ for all k. By an *ordered t.f.l.c.s.-net* we mean a t.f.l.c.s.-net B which is totally ordered in such a way that: *(i)* if $a \in B_m$, $b \in B_n$, $m < n$, then $a \preceq b$; *(ii)* for each k, every non-empty subset of B_k has a greatest element. An ordered t.f.l.c.s.-net always exists.

Now assume instead $p > 0$. Starting from the lower central series $\big\{ G_{(k)} \big\}_{k\in\mathbb{N}_+}$, define $G_{[n]} := \prod_{kp^\ell \geq n} \big(G_{(k)}\big)^{p^\ell}$ for all $n \in \mathbb{N}_+$ (hereafter, for any group $\varGamma$ we denote $\varGamma^{p^e}$ the subgroup generated by $\big\{ \gamma^{p^e} \,\big|\, \gamma \in \varGamma \,\big\}$). This gives another strongly central series $\big\{ G_{[n]} \big\}_{n\in\mathbb{N}_+}$ in G, with the additional property that $\big(G_{[n]}\big)^p \leq G_{[n+1]}$ for all n, called *the p–lower central series of G*. Then $\mathcal{L}_p(G) := \oplus_{n\in\mathbb{N}_+} G_{[n]} \big/ G_{[n+1]}$ is a graded restricted Lie algebra over $\mathbb{Z}_p := \mathbb{Z} \big/ p\,\mathbb{Z}$, with operations $\overline{g} + \overline{\ell} := \overline{g \cdot \ell}$, $\big[\,\overline{g}, \overline{\ell}\,\big] := \overline{(g,\ell\,)}$, $\overline{g}^{\,[p\,]} := \overline{g^p}$, for all $g, \ell \in G$ (cf. [HB], Ch. VIII, §9). Like before, we consider the map $\Bbbk \otimes_{\mathbb{Z}_p} \mathcal{L}_p(G) \longrightarrow \mathfrak{k}_G$, $\overline{g} \mapsto \overline{\eta_g}$, which now is an epimorphism *of graded restricted Lie $\mathbb{Z}_p$–algebras*, whose image spans $\mathfrak{k}_G$ over $\Bbbk$. Thus $\mathfrak{k}_G$ *is a quotient of* $\Bbbk \otimes_{\mathbb{Z}_p} \mathcal{L}_p(G)$; in fact, the above is an isomorphism (see below). Finally, we introduce notation $d(g) := n$ for all $g \in G_{[n]} \setminus G_{[n+1]}$.

For each $k \in \mathbb{N}_+$, choose a $\mathbb{Z}_p$–basis $\overline{B}_k$ of the $\mathbb{Z}_p$–vector space $G_{[k]} \big/ G_{[k+1]}$. For each $\overline{b} \in \overline{B}_k$, fix $b \in G_{[k]}$ such that $\overline{b} = b\, G_{[k+1]}$, and let B_k be the set of all such elements b. Let $B := \bigcup_{k\in\mathbb{N}_+} B_k$: such a set will be called a *p-l.c.s.-net* (= "p-lower-central-series-net"; the terminology in [HB] is "κ-net") on G. Of course $B_k = \big(B \cap G_{[k]} \big) \setminus \big(B \cap G_{[k+1]} \big)$ for all k. By an *ordered p-l.c.s.-net* we mean a p-l.c.s.-net B which is totally ordered in such a way that: *(i)* if $a \in B_m$, $b \in B_n$, $m < n$, then $a \preceq b$; *(ii)* for each k, every non-empty subset of B_k has a greatest element (like for $p = 0$). Again, p-l.c.s.-nets do exist.

We can now describe each $\mathcal{D}_n(G)$, hence also each graded summand $J^n \big/ J^{n+1}$ of $\widehat{\Bbbk[G]}$, in terms of a fixed ordered t.f.l.c.s.-net or p-l.c.s.-net. To unify notations, set $G_n := G_{(n)}$, $\theta(g) := \partial(g)$ if $p = 0$, and $G_n := G_{[n]}$, $\theta(g) := d(g)$ if $p > 0$, set $G_\infty := \bigcap_{n\in N_+} G_n$, let $B := \bigcup_{k\in\mathbb{N}_+} B_k$ be an ordered t.f.l.c.s.-net or p-l.c.s.-net according to whether $p = 0$ or $p > 0$, and set $\ell(0) := +\infty$ and $\ell(p) := p$ for $p > 0$. The key result we need is

<u>Jennings-Hall theorem</u> *(cf. [HB], [Pa] and references therein). Let $p := Char(\Bbbk)$.*

(a) For all $g \in G$, $\eta_g \in J^n \iff g \in G_n$. Therefore $\mathcal{D}_n(G) = G_n$ for all $n \in \mathbb{N}_+$.

(b) For any $n \in \mathbb{N}_+$, the set of ordered monomials

$$\mathbb{B}_n := \left\{ \overline{\eta_{b_1}}^{\,e_1} \cdots \overline{\eta_{b_r}}^{\,e_r} \;\middle|\; b_i \in B_{d_i},\ e_i \in \mathbb{N}_+,\ e_i < \ell(p),\ b_1 \precneqq \cdots \precneqq b_r,\ \textstyle\sum_{i=1}^r e_i\, d_i = n \right\}$$

is a $\Bbbk$*–basis of* $J^n \big/ J^{n+1}$*, and* $\mathbb{B} := \{1\} \cup \bigcup_{n \in \mathbb{N}} \mathbb{B}_n$ *is a* $\Bbbk$*–basis of* $\widehat{\Bbbk[G]}$*.*

(c) $\big\{\, \overline{\eta_b} \;\big|\; b \in B_n \big\}$ *is a* $\Bbbk$*–basis of the* n*–th graded summand* $\mathfrak{k}_G \cap \big(J^n \big/ J^{n+1}\big)$ *of the graded restricted Lie algebra* $\mathfrak{k}_G$*, and* $\big\{\, \overline{\eta_b} \;\big|\; b \in B \,\big\}$ *is a* $\Bbbk$*–basis of* $\mathfrak{k}_G$*.*

(d) $\big\{\, \overline{\eta_b} \;\big|\; b \in B_1 \big\}$ *is a minimal set of generators of the (restricted) Lie algebra* $\mathfrak{k}_G$*.*

(e) The map $\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_p(G) \longrightarrow \mathfrak{k}_G$*,* $\overline{g} \mapsto \overline{\eta_g}$*, is an isomorphism of graded restricted Lie algebras. Therefore* $\widehat{\Bbbk[G]} \cong \mathcal{U}\big(\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_p(G)\big)$ *as Hopf algebras (notation of §1.3).*

(f) $J^\infty = \mathit{Span}\big(\{\, \eta_g \,|\, g \in G_\infty \,\}\big)$*, whence* $\Bbbk[G]^\vee \cong \bigoplus_{\overline{g} \in G/G_\infty} \Bbbk \cdot \overline{g} \;\cong\; \Bbbk\big[G/G_\infty\big]$*.* □

Recall that $A\big[x, x^{-1}\big]$, for any A, has A–basis $\big\{ (x-1)^n x^{-[n/2]} \,\big|\, n \in \mathbb{N} \big\}$, where $[q]$ is the integer part of $q \in \mathbb{Q}$. Then from Jennings-Hall theorem and definitions we deduce

Proposition 8.5. *Let* $\chi_g := t^{-\theta(g)} \eta_g$*, for all* $g \in \{G\} \setminus \{1\}$*. Then*

$$\begin{aligned} \Bbbk[G]_t^\vee &= \Big(\bigoplus_{\substack{b_i \in B,\, 0 < e_i < \ell(p) \\ r \in \mathbb{N},\, b_1 \gneqq \cdots \gneqq b_r}} R \cdot \chi_{b_1}^{e_1} \, b_1^{-[e_1/2]} \cdots \chi_{b_r}^{e_r} \, b_r^{-[e_r/2]} \Big) \bigoplus R\big[t^{-1}\big] \cdot J^\infty = \\ &= \Big(\bigoplus_{\substack{b_i \in B,\, 0 < e_i < \ell(p) \\ r \in \mathbb{N},\, b_1 \gneqq \cdots \gneqq b_r}} R \cdot \chi_{b_1}^{e_1} \, b_1^{-[e_1/2]} \cdots \chi_{b_r}^{e_r} \, b_r^{-[e_r/2]} \Big) \bigoplus \Big(\sum\nolimits_{\gamma \in G_\infty} R\big[t^{-1}\big] \cdot \eta_\gamma \Big) \, ; \end{aligned}$$

If $J^\infty = J^n$ *for some* $n \in \mathbb{N}$ *(iff* $G_\infty = G_n$*) we can drop the factors* $b_1^{-[e_1/2]}, \dots, b_r^{-[e_r/2]}$*.* □

8.6 Poisson groups from $\Bbbk[G]$. The previous discussion attached to G and $\Bbbk$ the (maybe restricted) Lie algebra $\mathfrak{k}_G$. By Jennings-Hall theorem, this is just the scalar extension of the Lie ring $\mathcal{L}_p$ associated to G via the central series of the G_n's. In particular, the functor $G \mapsto \mathfrak{k}_G$ is one considered since a long time in group theory.

By Theorem 5.4*(a)* we know that $\big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0} = F\big[\mathit{\Gamma}_G\big]$ for some connected Poisson group $\mathit{\Gamma}_G$, of dimension zero and height 1 if $p > 0$. This defines a functor $G \mapsto \mathit{\Gamma}_G$: in particular, $\mathit{\Gamma}_G$ *is a new invariant for abstract groups*.

The description of $\Bbbk[G]_t^\vee$ in Proposition 8.5 leads us to an explicit description of $\big(\Bbbk[G]_t^\vee\big)'$, hence of $\big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0} = F\big[\mathit{\Gamma}_G\big]$ and $\mathit{\Gamma}_G$ too. Indeed, direct inspection gives $\delta_n\big(\chi_g\big) = t^{(n-1)\,\theta(g)} \chi_g^{\otimes n}$, so $\psi_g := t\,\chi_g = t^{1-\theta(g)} \eta_g \in \big(\Bbbk[G]_t^\vee\big)' \setminus t\,\big(\Bbbk[G]_t^\vee\big)'$ for each $g \in G \setminus G_\infty$. Instead for $\gamma \in G_\infty$ we have $\eta_\gamma \in J^\infty$, which implies also $\eta_\gamma \in \big(\Bbbk[G]_t^\vee\big)'$, and even $\eta_\gamma \in \bigcap_{n \in \mathbb{N}} t^n \big(\Bbbk[G]_t^\vee\big)'$. Therefore $\big(\Bbbk[G]_t^\vee\big)'$ is generated by $\big\{\, \psi_g \;\big|\; g \in G \setminus \{1\} \big\} \cup \big\{ \eta_\gamma \,\big|\, \gamma \in G_\infty \big\}$. Moreover, $g = 1 + t^{\theta(g)-1} \psi_g \in \big(\Bbbk[G]_t^\vee\big)'$ for every $g \in G \setminus G_\infty$, and $\gamma = 1 + (\gamma - 1) \in 1 + J^\infty \subseteq \big(\Bbbk[G]_t^\vee\big)'$ for $\gamma \in G_\infty$. This and the previous analysis, along with Proposition 8.5, prove next result, which in turn is the basis for Theorem 8.8 below.

Proposition 8.7.

$$\begin{aligned} \big(\Bbbk[G]_t^\vee\big)' &= \Big(\bigoplus_{\substack{b_i \in B,\, 0 < e_i < \ell(p) \\ r \in \mathbb{N},\, b_1 \gneqq \cdots \gneqq b_r}} R \cdot \psi_{b_1}^{e_1} \, b_1^{-[e_1/2]} \cdots \psi_{b_r}^{e_r} \, b_r^{-[e_r/2]} \Big) \bigoplus R\big[t^{-1}\big] \cdot J^\infty = \\ &= \Big(\bigoplus_{\substack{b_i \in B,\, 0 < e_i < \ell(p) \\ r \in \mathbb{N},\, b_1 \gneqq \cdots \gneqq b_r}} R \cdot \psi_{b_1}^{e_1} \, b_1^{-[e_1/2]} \cdots \psi_{b_r}^{e_r} \, b_r^{-[e_r/2]} \Big) \bigoplus \Big(\sum\nolimits_{\gamma \in G_\infty} R\big[t^{-1}\big] \cdot \eta_\gamma \Big) \, . \end{aligned}$$

In particular, $\big(\Bbbk[G]_t^\vee\big)' = \Bbbk[G]_t$ *if and only if* $G_2 = \{1\} = G_\infty$. *If in addition* $J^\infty = J^n$ *for some* $n \in \mathbb{N}$ *(iff* $G_\infty = G_n$*), then we can drop the factors* $b_1^{-[e_1/2]}, \dots, b_r^{-[e_r/2]}$. $\square$

Theorem 8.8. *Let* $x_g := \psi_g \mod t\, \big(\Bbbk[G]_t^\vee\big)'$, $z_g := g \mod t\, \big(\Bbbk[G]_t^\vee\big)'$ *for all* $g \neq 1$, *and* $B_1 := \big\{\, b \in B \,\big|\, \theta(b) = 1 \big\}$, $B_> := \big\{\, b \in B \,\big|\, \theta(b) > 1 \big\}$.

(a) If $p = 0$, *then* $F\big[\varGamma_G\big] = \big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0}$ *is a polynomial/Laurent polynomial algebra, namely* $F\big[\varGamma_G\big] = \Bbbk\big[\big\{z_b^{\pm 1}\big\}_{b \in B_1} \cup \{x_b\}_{b \in B_>}\big]$, *the* z_b*'s being group-like and the* x_b*'s being primitive. In particular* $\varGamma_G \cong \big(\mathbb{G}_m^{\times B_1}\big) \times \big(\mathbb{G}_a^{\times B_>}\big)$ *as algebraic groups.*

(b) If $p > 0$, *then* $F\big[\varGamma_G\big] = \big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0}$ *is a truncated polynomial/Laurent polynomial algebra, namely* $F\big[\varGamma_G\big] = \Bbbk\big[\big\{z_b^{\pm 1}\big\}_{b \in B_1} \cup \{x_b\}_{b \in B_>}\big] \Big/ \big(\{z_b^p - 1\}_{b \in B_1} \cup \{x_b^p\}_{b \in B_>}\big)$, *the* z_b*'s being group-like and the* x_b*'s being primitive. In particular* $\varGamma_G \cong \big(\boldsymbol{\mu}_p{}^{\times B_1}\big) \times \big(\boldsymbol{\alpha}_p{}^{\times B_>}\big)$ *as algebraic groups of dimension zero and height 1.*

(c) The Poisson group $\varGamma_G$ *has cotangent Lie bialgebra* $\mathfrak{k}_G$, *that is* $\mathrm{coLie}\,(\varGamma_G) = \mathfrak{k}_G$.

Proof. (a) Definitions give $\partial(g\,\ell) \geq \partial(g) + \partial(\ell)$ for all $g, \ell \in G$, so that $[\psi_g, \psi_\ell] = t^{1-\partial(g)-\partial(\ell)+\partial((g,\ell))}\, \psi_{(g,\ell)}\, g\,\ell \in t \cdot \big(\Bbbk[G]_t^\vee\big)'$, which proves (directly) that $\big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0}$ is commutative. Moreover, the relation $1 = g^{-1}\, g = g^{-1}\big(1 + t^{\partial(g)-1}\psi_g\big)$ (for any $g \in G$) yields $z_{g^{-1}} = z_g^{\,-1}$, and $z_{g^{-1}} = 1$ iff $\partial(g) > 1$. Note also that $J^\infty \equiv 0 \mod t\, \big(\Bbbk[G]_t^\vee\big)'$ and $g = 1 + t^{\partial(g)-1}\psi_g \equiv 1 \mod t\, \big(\Bbbk[G]_t^\vee\big)'$ for $g \in G \setminus G_\infty$, and also $\gamma = 1 + (\gamma - 1) \in 1 + J^\infty \equiv 1 \mod t\, \big(\Bbbk[G]_t^\vee\big)'$ for $\gamma \in G_\infty$. Then Proposition 8.7 gives

$$F\big[\varGamma_G\big] \;=\; \big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0} = \Big(\bigoplus\nolimits_{\substack{b_i \in B_1,\ a_i \in \mathbb{Z} \\ s \in \mathbb{N},\ b_1 \precneqq \cdots \precneqq b_s}} \Bbbk \cdot z_{b_1}^{a_1} \cdots z_{b_s}^{a_s}\Big) \bigotimes \Big(\bigoplus\nolimits_{\substack{b_i \in B_>,\ e_i \in \mathbb{N}_+ \\ r \in \mathbb{N},\ b_1 \precneqq \cdots \precneqq b_r}} \Bbbk \cdot x_{b_1}^{e_1} \cdots x_{b_r}^{e_r}\Big)$$

which means that $F\big[\varGamma_G\big]$ is a polynomial/Laurent polynomial algebra, as claimed.

Again definitions imply $\Delta(z_g) = z_g \otimes z_g$ for all $g \in G$ and $\Delta(x_g) = x_g \otimes 1 + 1 \otimes x_g$ iff $\partial(g) > 1$. Thus the z_b's are group-like and the x_b's are primitive, as claimed.

(b) The definition of d implies $d(g\,\ell) \geq d(g) + d(\ell)$ $(g, \ell \in G)$, whence we get $[\psi_g, \psi_\ell] = t^{1-d(g)-d(\ell)+d((g,\ell))}\, \psi_{(g,\ell)}\, g\,\ell \in t \cdot \big(\Bbbk[G]_t^\vee\big)'$, proving that $\big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0}$ is commutative. In addition $d(g^p) \geq p\, d(g)$, so $\psi_g^{\,p} = t^{\,p\,(1-d(g))}\, \eta_g^{\,p} = t^{\,p-1+d(g^p)-p\,d(g)}\, \psi_{g^p} \in t \cdot \big(\Bbbk[G]_t^\vee\big)'$, whence $\big(\psi_g^{\,p}\big|_{t=0}\big)^p = 0$ inside $\big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0} = F\big[\varGamma_G\big]$, which proves that $\varGamma_G$ has dimension 0 and height 1. Finally, $b^p = (1 + \psi_b)^p = 1 + \psi_b^{\,p} \equiv 1 \mod t\, \big(\Bbbk[G]_t^\vee\big)'$ for all $b \in B_1$, so $b^{-1} \equiv b^{p-1} \mod t\, \big(\Bbbk[G]_t^\vee\big)'$. Thus, letting $x_g := \psi_g \mod t\, \big(\Bbbk[G]_t^\vee\big)'$ (for $g \neq 1$), we get

$$F\big[\varGamma_G\big] = \big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0} = \Big(\bigoplus\nolimits_{\substack{b_i \in B_1,\ 0 < e_i < p \\ s \in \mathbb{N},\ b_1 \precneqq \cdots \precneqq b_s}} \Bbbk \cdot z_{b_1}^{e_1} \cdots z_{b_s}^{e_s}\Big) \bigotimes \Big(\bigoplus\nolimits_{\substack{b_i \in B_>,\ 0 < e_i < p \\ r \in \mathbb{N},\ b_1 \precneqq \cdots \precneqq b_r}} \Bbbk \cdot x_{b_1}^{e_1} \cdots x_{b_r}^{e_r}\Big)$$

just like for *(a)*, and also taking care that $z_b = x_b + 1$ and $z_b^{\,p} = 1$ for $b \in B_1$. Therefore $\big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0}$ is a truncated polynomial/Laurent polynomial algebra as claimed. The properties of the x_b's and the z_b's w.r.t. the Hopf structure are then proved like for *(a)* again.

(c) The augmentation ideal $\mathfrak{m}_e$ of $\left(\Bbbk[G]_t^\vee\right)'\Big|_{t=0} = F\big[\varGamma_G\big]$ is generated by $\{x_b\}_{b \in B}$. Then $t^{-1}\big[\psi_g, \psi_\ell\big] = t^{\,\theta((g,\ell))-\theta(g)-\theta(\ell)}\,\psi_{(g,\ell)}\,\big(1+t^{\,\theta(g)-1}\psi_g\big)\,\big(1+t^{\,\theta(\ell)-1}\psi_\ell\big)$ by the previous computation, whence at $t=0$ one has $\big\{x_g\,, x_\ell\big\} \equiv x_{(g,\ell)} \mod \mathfrak{m}_e^2$ if $\theta\big((g,\ell)\big) = \theta(g) + \theta(\ell)$, and $\big\{x_g\,, x_\ell\big\} \equiv 0 \mod \mathfrak{m}_e^2$ if $\theta\big((g,\ell)\big) > \theta(g) + \theta(\ell)$. This means that the cotangent Lie bialgebra $\mathfrak{m}_e \Big/ \mathfrak{m}_e^2$ of $\varGamma_G$ is isomorphic to $\mathfrak{k}_G$, as claimed. $\square$

Remark 8.9: Theorem 8.8 yields functorial recipes to attach to each abstract group G and each field $\Bbbk$ a connected Abelian algebraic Poisson group over $\Bbbk$, namely $G \mapsto \varGamma_G \equiv K_G^\star$, with $coLie\,(\varGamma_G) = \mathfrak{k}_G$. *Every such $\varGamma_G$ (for given $\Bbbk$) is then an invariant of G*, a new one to the author's knowledge. Yet this invariant is perfectly *equivalent* to the well-known invariant $\mathfrak{k}_G$ (over the same $\Bbbk$). In fact, $\varGamma_{G_1} \cong \varGamma_{G_2}$ implies $\mathfrak{k}_{G_1} \cong \mathfrak{k}_{G_2}$, whereas $\mathfrak{k}_{G_1} \cong \mathfrak{k}_{G_2}$ implies that $\varGamma_{G_1}$ and $\varGamma_{G_2}$ are isomorphic as algebraic groups, by Theorem 8.8*(a–b)*, and bear isomorphic Poisson structures, by Theorem 8.8*(c)*, so $\varGamma_{G_1} \cong \varGamma_{G_2}$ as Poisson groups.

8.10 The case of $A_\Bbbk(G)$. Let's now G be a *finite* group, $\mathbb{A}$ any commutative unital ring, and $\Bbbk$, $R := \Bbbk[t]$ be as before. By definition $A_\mathbb{A}(G) = \mathbb{A}[G]^*$, hence $\mathbb{A}[G] = A_\mathbb{A}(G)^*$, and we have a natural perfect Hopf pairing $A_\mathbb{A}(G) \times \mathbb{A}[G] \longrightarrow \mathbb{A}$. Our first result is

Theorem 8.11. $A_\Bbbk(G)_t^\vee = R \cdot 1 \oplus R\big[t^{-1}\big]\,J = \big(A_\Bbbk(G)_t^\vee\big)'$, $A_\Bbbk(G)^\vee = \Bbbk \cdot 1$, $\widehat{A_\Bbbk(G)} = A_\Bbbk(G)_t^\vee\Big|_{t=0} = \Bbbk \cdot 1 = \mathcal{U}(\mathbf{0})$ *and* $\big(A_\Bbbk(G)_t^\vee\big)'\Big|_{t=0} = \Bbbk \cdot 1 = F\big[\{*\}\big]$.

Proof. By construction $J := Ker\,(\epsilon_{A_\Bbbk(G)})$ has $\Bbbk$–basis $\big\{\varphi_g\big\}_{g \in G \setminus \{1_G\}} \cup \big\{\varphi_{1_G} - 1_{A_\Bbbk(G)}\big\}$, and since $\varphi_g = {\varphi_g}^2$ for all g and $(\varphi_{1_G} - 1)^2 = -(\varphi_{1_G} - 1)$ we have $J = J^\infty$, so $A_\Bbbk(G)^\vee = \Bbbk{\cdot}1$ and $\widehat{A_\Bbbk(G)} = \Bbbk{\cdot}1$. Similarly, the set $\big\{t^{-1}\varphi_g\big\}_{g \in G \setminus \{1_G\}} \cup \big\{t^{-1}(\varphi_{1_G} - 1_{A_R(G)})\big\}$ generates $A_\Bbbk(G)_t^\vee$, since $A_\Bbbk(G)_t = A_R(G)$. Moreover, $J = J^\infty$ implies $t^{-n}J \subseteq A_\Bbbk(G)_t^\vee$ for all n, so $A_\Bbbk(G)_t^\vee = R\,1 \oplus R[t^{-1}]J$. Then $J_{A_\Bbbk(G)_t^\vee} = R\big[t^{-1}\big]J \subseteq t\,A_\Bbbk(G)_t^\vee$, which implies $\big(A_\Bbbk(G)_t^\vee\big)' = A_\Bbbk(G)_t^\vee$: in particular $\big(A_\Bbbk(G)_t^\vee\big)'\Big|_{t=0} = A_\Bbbk(G)_t^\vee\Big|_{t=0} = \Bbbk \cdot 1$, as claimed. $\square$

8.12 Poisson groups from $A_\Bbbk(G)$. Now we look at $A_\Bbbk(G)_t'$, $A_\Bbbk(G)'$ and $\widetilde{A_\Bbbk(G)}$. By construction $A_\Bbbk(G)_t$ and $\Bbbk[G]_t$ are in perfect Hopf pairing, and are free R–modules of *finite rank*. In this case Theorem 4.4 yields $A_\Bbbk(G)_t' = \big(\Bbbk[G]_t^\vee\big)^\bullet$ (see (4.6)), hence we have $A_\Bbbk(G)_t' = \big(\Bbbk[G]_t^\vee\big)^\bullet = \big(\Bbbk[G]_t^\vee\big)^*$: thus $A_\Bbbk(G)_t'$ is the *dual* Hopf algebra to $\Bbbk[G]_t^\vee$. Then from Proposition 8.5 we can deduce an explicit description of $A_\Bbbk(G)_t'$, whence also of $\big(A_\Bbbk(G)_t'\big)^\vee$. By Theorem 3.7 there are a perfect filtered Hopf pairing $\Bbbk[G]^\vee \times A_\Bbbk(G)' \longrightarrow \Bbbk$ and a perfect graded Hopf pairing $\widehat{\Bbbk[G]} \times \widetilde{A_\Bbbk(G)} \longrightarrow \Bbbk$: thus $A_\Bbbk(G)' \cong \big(\Bbbk[G]^\vee\big)^*$ as filtered Hopf algebras and $\widetilde{A_\Bbbk(G)} \cong \big(\widehat{\Bbbk[G]}\big)^*$ as graded Hopf algebras.

If $p = 0$ then $J = J^\infty$, as each $g \in G$ has finite order and $g^n = 1$ implies $g \in G_\infty$. Then $\Bbbk[G]^\vee = \Bbbk \cdot 1 = \widehat{\Bbbk[G]}$, so $A_\Bbbk(G)' = \Bbbk \cdot 1 = \widetilde{A_\Bbbk(G)}$. If $p > 0$ instead, this analysis gives $\widetilde{A_\Bbbk(G)} = \big(\widehat{\Bbbk[G]}\big)^* = \big(\mathbf{u}(\mathfrak{k}_G)\big)^* = F[K_G]$, where K_G is a connected Poisson group of dimension 0, height 1 and tangent Lie bialgebra $\mathfrak{k}_G$. Thus we get:

Theorem 8.13.

(a) There is a second functorial recipe to attach to each finite abstract group a connected algebraic Poisson group of dimension zero and height 1 over any field $\Bbbk$ *with* $Char(\Bbbk) > 0$, *namely* $G \mapsto K_G := Spec\left(\widetilde{A_{\Bbbk}(G)}\right)$, *with* $Lie(K_G) = \mathfrak{k}_G = coLie(\Gamma_G)$.

(b) If $p := Char(\Bbbk) > 0$, *then* $\left(A_{\Bbbk}(G)_t'\right)^{\vee}\Big|_{t=0} = \mathbf{u}\big(\mathfrak{k}_G^{\times}\big) = S\big(\mathfrak{k}_G^{\times}\big)\Big/\big(\big\{x^p \,\big|\, x \in \mathfrak{k}_G^{\times}\big\}\big)$.

Proof. Claim *(a)* is the outcome of the discussion above. Part *(b)* instead requires an explicit description of $\left(A_{\Bbbk}(G)_t'\right)^{\vee}$. Since $A_{\Bbbk}(G)_t' \cong \left(\Bbbk[G]_t^{\vee}\right)^*$, from Proposition 8.5 we get $A_{\Bbbk}(G)_t' = \left(\bigoplus_{\substack{b_i \in B,\, 0<e_i<p \\ r\in\mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot \rho_{b_1,\dots,b_r}^{e_1,\dots,e_r}\right)$ where each $\rho_{b_1,\dots,b_r}^{e_1,\dots,e_r}$ is defined by

$$\left\langle \rho_{b_1,\dots,b_r}^{e_1,\dots,e_r}\,,\, \chi_{\beta_1}^{\varepsilon_1}\,\beta_1^{-[\varepsilon_1/2]}\cdots\chi_{\beta_s}^{\varepsilon_s}\,\beta_s^{-[\varepsilon_s/2]} \right\rangle \;=\; \delta_{r,s}\prod\nolimits_{i=1}^{r}\delta_{b_i,\beta_i}\delta_{e_i,\varepsilon_i}$$

(for all $b_i, \beta_j \in B$ and $0 < e_i, \varepsilon_j < p$). Now, let $K := \Bbbk[G]_t^{\vee}$, $H := A_{\Bbbk}(G)_t'$; by the previous description of H and K, the natural pairing $H \times K \longrightarrow R$, which is perfect on the left, has $K_\infty := R\big[t^{-1}\big]\,J^\infty$ as right kernel: so it induces a *perfect* Hopf pairing $H \times \overline{K} \longrightarrow R$, where $\overline{K} := K\big/K_\infty$. By construction the latter specializes at $t = 0$ to the natural pairing $F[K_G] \times \mathbf{u}(\mathfrak{k}_G) \longrightarrow \Bbbk$, which is perfect too. Then we can apply Proposition 4.4*(c)* in [Ga1–2] (with $\overline{K}$ playing the rôle of K therein) which yields $\overline{K}\,' = \left(H^{\vee}\right)^{\bullet} = \left(\left(A_{\Bbbk}(G)_t'\right)^{\vee}\right)^{\bullet}$. By definitions one sees that $R\big[t^{-1}\big]\,J^\infty \subseteq \left(\Bbbk[G]_t^{\vee}\right)'$, whence $\overline{K}\,' = \left(\Bbbk[G]_t^{\vee}\right)'\Big/\left(R\big[t^{-1}\big]\,J^\infty\right)$ follows at once; Proposition 8.7 describes the latter space as $\overline{K}\,' = \left(\bigoplus_{\substack{b_i \in B,\, 0<e_i<p \\ r\in\mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot \overline{\psi}_{b_1}^{\,e_1}\cdots\overline{\psi}_{b_r}^{\,e_r}\right)$, where $\overline{\psi}_{b_i} := \psi_{b_i} \mod R\big[t^{-1}\big]\,J^\infty$ for all i.

Since we saw that $\overline{K}\,' = \left(\left(A_{\Bbbk}(G)_t'\right)^{\vee}\right)^{\bullet}$, and $\psi_g = t^{+1}\chi_g$, this analysis yields

$$\left(A_{\Bbbk}(G)_t'\right)^{\vee} = \left(\bigoplus_{\substack{b_i \in B,\, 0<e_i<p \\ r\in\mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot t^{-\sum_i e_i}\rho_{b_1,\dots,b_r}^{e_1,\dots,e_r}\right) \;\cong\; \left(\overline{K}\,'\right)^*$$

whence $\left(A_{\Bbbk}(G)_t'\right)^{\vee}\Big|_{t=0} \cong \left(\overline{K}\,'\right)^*\Big|_{t=0} = \left(K_t'\big|_{t=0}\right)^* = \left(\left(\Bbbk[G]_t^{\vee}\right)'\big|_{t=0}\right)^* \cong F\big[\Gamma_G\big]^* = \mathbf{u}\big(\mathfrak{k}_G^{\times}\big) = S\big(\mathfrak{k}_G^{\times}\big)\Big/\big(\big\{x^p \,\big|\, x \in \mathfrak{k}_G^{\times}\big\}\big)$ as claimed, the latter identity being trivial (as $\mathfrak{k}_G^{\times}$ is Abelian). □

8.14 Remark: for each field $\Bbbk$ of positive characteristic, *the functor* $G \mapsto K_G$ *is an invariant for* G, a new one's to the author knowledge, but again equivalent to $\mathfrak{k}_G$.

8.15 Examples. *(1) Finite Abelian* p*–groups.* Let p be a prime number, $\Bbbk$ a field with $Char(\Bbbk) = p$, and $G := \mathbb{Z}_{p^{e_1}} \times \cdots \times \mathbb{Z}_{p^{e_k}}$ $(k, e_1, \dots, e_k \in \mathbb{N})$, with $e_1 \geq \cdots \geq e_k$.

First, $\mathfrak{k}_G$ is Abelian, because G is. Let g_i be a generator of $\mathbb{Z}_{p^{e_i}}$ (for all i), identified with its image in G. Since G is Abelian we have $G_{[n]} = G^{p^n}$ (for all n), and an ordered p-l.c.s.-net is $B := \bigcup_{r\in\mathbb{N}_+} B_r$ with $B_r := \left\{ g_1^{p^r}, g_2^{p^r}, \dots, g_{j_r}^{p^r} \right\}$ where j_r is uniquely defined by $e_{j_r} > r$, $e_{j_r+1} \leq r$. Then $\mathfrak{k}_G$ has $\Bbbk$–basis $\left\{ \overline{\eta_{g_i^{p^{s_i}}}} \right\}_{1\leq i\leq k;\; 0\leq s_i<e_i}$, and minimal set of

generators (as a restricted Lie algebra) $\big\{ \overline{\eta_{g_1}}, \overline{\eta_{g_2}}, \dots, \overline{\eta_{g_k}} \big\}$. In fact, the p–operation of $\mathfrak{k}_G$ is $\big(\overline{\eta_{g_i^{p^s}}}\big)^{[p]} = \overline{\eta_{g_i^{p^{s+1}}}}$, and the order of nilpotency of each $\overline{\eta_{g_i}}$ is exactly p^{e_i}, i.e. the order of g_i. In addition $J^\infty = \{0\}$ so $\Bbbk[G]^\vee = \Bbbk[G]$. The outcome is $\Bbbk[G]^\vee = \Bbbk[G]$ and

$$\widehat{\Bbbk[G]} \; = \; \mathbf{u}(\mathfrak{k}_G) \; = \; U(\mathfrak{k}_G) \Big/ \Big(\Big\{ \big(\overline{\eta_{g_i^{p^s}}}\big)^p - \overline{\eta_{g_i^{p^{s+1}}}} \Big\}_{1 \leq i \leq k}^{0 \leq s < e_i} \bigcup \Big\{ \big(\overline{\eta_{g_i^{p^{e_i - 1}}}}\big)^p \Big\}_{1 \leq i \leq k} \Big)$$

whence $\widehat{\Bbbk[G]} \cong \Bbbk[x_1, \dots, x_k] \Big/ \Big(\Big\{ x_i^{p^{e_i}} \;\Big|\; 1 \leq i \leq k \Big\} \Big)$, via $\overline{\eta_{g_i^{p^s}}} \mapsto x_i^{p^s}$ (for all i, s).

As for $\Bbbk[G]_t^\vee$, for all $r < e_i$ we have $d\big(g_i^{p^r}\big) = p^r$ and so $\chi_{g_i^{p^r}} = t^{-p^r}\big(g_i^{p^r} - 1\big)$ and $\psi_{g_i^{p^r}} = t^{1-p^r}\big(g_i^{p^r} - 1\big)$. Now $G_{[\infty]} = \{1\}$ (or, equivalently, $J^\infty = \{0\}$) and everything is Abelian, so from the general theory we conclude that both $\Bbbk[G]_t^\vee$ and $\big(\Bbbk[G]_t^\vee\big)'$ are truncated-polynomial algebras, in the $\chi_{g_i^{p^r}}$'s and in the $\psi_{g_i^{p^r}}$'s respectively, namely

$$\Bbbk[G]_t^\vee = \Bbbk[t]\Big[\big\{ \chi_{g_i^{p^s}} \big\}_{1 \leq i \leq k \,;\, 0 \leq s < e_i}\Big] \cong \big(\Bbbk[t]\big)\,[y_1, \dots, y_k] \Big/ \Big(\Big\{ y_i^{p^{e_i}} \;\Big|\; 1 \leq i \leq k \Big\} \Big)$$

$$\big(\Bbbk[G]_t^\vee\big)' = \Bbbk[t]\Big[\big\{ \psi_{g_i^{p^s}} \big\}_{1 \leq i \leq k \,;\, 0 \leq s < e_i}\Big] \cong \big(\Bbbk[t]\big) \Big[\big\{ z_{i,s} \big\}_{1 \leq i \leq k \,;\, 0 \leq s < e_i}\Big] \Big/ \Big(\Big\{ {z_{i,s}}^p \Big\}_{1 \leq i \leq k} \Big)$$

via the isomorphisms given by $\overline{\chi_{g_i^{p^s}}} \mapsto y_i^{p^s}$, $\overline{\psi_{g_i^{p^s}}} \mapsto z_{i,s}$. When $e_1 > 1$ this implies $\big(\Bbbk[G]_t^\vee\big)' \supsetneqq \Bbbk[G]_t$. Setting $\overline{\psi_{g_i^{p^s}}} := \psi_{g_i^{p^s}} \mod t\,\big(\Bbbk[G]_t^\vee\big)'$ $(1 \leq i \leq k,\ 0 \leq s < e_i)$ we have

$$F\big[\varGamma_G\big] = \big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0} = \Bbbk\Big[\big\{\overline{\psi_{g_i^{p^s}}}\big\}_{1 \leq i \leq k}^{0 \leq s < e_i}\Big] \cong \Bbbk\Big[\big\{ w_{i,s} \big\}_{1 \leq i \leq k}^{0 \leq s < e_i}\Big] \Big/ \Big(\Big\{ w_{i,s}^{\,p} \Big|\, 1 \leq i \leq k \Big\} \Big)$$

(via $\overline{\psi_{g_i^{p^s}}} \mapsto w_{i,s}$) as a $\Bbbk$–algebra. The Poisson bracket is trivial, and the $w_{i,s}$'s are primitive for $s > 1$ and $\Delta(w_{i,1}) = w_{i,1} \otimes 1 + 1 \otimes w_{i,1} + w_{i,1} \otimes w_{i,1}$ for all $1 \leq i \leq k$. If instead $e_1 = \cdots = e_k = 1$, then $\big(\Bbbk[G]_t^\vee\big)' = \Bbbk[G]_t$. This is an analogue of Theorem 2.2*(b)* in [Ga1–2], although now *Char*$(\Bbbk) > 0$, in that in this case $\Bbbk[G]_t$ is a QFA, with $\Bbbk[G]_t\Big|_{t=0} = \Bbbk[G] = F\big[\widehat{G}\,\big]$ where $\widehat{G}$ is the *group of characters* of G. But then $F\big[\widehat{G}\,\big] = \Bbbk[G] = \Bbbk[G]_t\Big|_{t=0} = \big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0} = F\big[\varGamma_G\big]$ (by the general analysis), which means that $\widehat{G}$ is a finite connected dimension 0 height 1 group-scheme dual to $\mathfrak{k}_G$, namely $K_G^\star = \varGamma_G$.

Finally, a direct easy calculation shows that — letting $\chi_g^* := t^{\,d(g)}\,(\varphi_g - \varphi_1) \in A_\Bbbk(G)_t'$ and $\psi_g^* := t^{\,d(g)-1}\,(\varphi_g - \varphi_1) \in \big(A_\Bbbk(G)_t'\big)^\vee$, for all $g \in G \setminus \{1\}$ — we have also

$$A_\Bbbk(G)_t' = \Bbbk[t]\Big[\big\{ \chi_{g_i^{p^s}}^* \big\}_{1 \leq i \leq k}^{0 \leq s < e_i}\Big] \cong \Bbbk[t]\Big[\big\{ Y_{i,j} \big\}_{1 \leq i \leq k}^{0 \leq s < e_i}\Big] \Big/ \Big(\big\{ Y_{i,j}^{\,p} \big\}_{1 \leq i \leq k}^{0 \leq s < e_i} \Big)$$

$$\big(A_\Bbbk(G)_t'\big)^\vee = \Bbbk[t]\Big[\big\{ \psi_{g_i^{p^s}}^* \big\}_{1 \leq i \leq k}^{0 \leq s < e_i}\Big] \cong \Bbbk[t]\Big[\big\{ Z_{i,s} \big\}_{1 \leq i \leq k}^{0 \leq s < e_i}\Big] \Big/ \Big(\big\{ Z_{i,s}^{\,p} - Z_{i,s} \big\}_{1 \leq i \leq k}^{0 \leq s < e_i} \Big)$$

via the isomorphisms given by $\chi_{g_i^{p^s}}^* \mapsto Y_{i,s}$ and $\psi_{g_i^{p^s}}^* \mapsto Z_{i,s}$, from which one also gets the analogous descriptions of $A_\Bbbk(G)_t'\Big|_{t=0} = \widetilde{A_\Bbbk(G)} = F[K_G]$ and of $\big(A_\Bbbk(G)_t'\big)^\vee\Big|_{t=0} = \mathbf{u}(\mathfrak{k}_G^\times)$.

(2) A non-Abelian p–group. Let p be a prime number, and let $Char(\Bbbk) = p > 0$.

Let $G := \mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$, that is the group with generators ν, τ and relations $\nu^p = 1$, $\tau^{p^2} = 1$, $\nu \, \tau \, \nu^{-1} = \tau^{1+p}$. In this case, $G_{[2]} = \cdots = G_{[p]} = \big\{1, \tau^p\big\}$, $G_{[p+1]} = \{1\}$, so we can take $B_1 = \{\nu \, , \tau\}$ and $B_p = \big\{\tau^p\big\}$ to form an ordered p-l.c.s.-net $B := B_1 \cup B_p$ w.r.t. the ordering $\nu \preceq \tau \preceq \tau^p$. Noting also that $J^\infty = \{0\}$ (for $G_{[\infty]} = \{1\}$), we have

$$\Bbbk[G]_t^\vee \; = \; \bigoplus\nolimits_{a,b,c=0}^{p-1} \Bbbk[t] \cdot \chi_\nu^{\,a} \, \chi_\tau^{\,b} \, \chi_{\tau^p}^{\,c} \; = \; \bigoplus\nolimits_{a,b,c=0}^{p-1} \Bbbk[t] \, t^{-a-b-c\,p} \cdot (\nu-1)^a \, (\tau-1)^b \, \big(\tau^p - 1\big)^c$$

as $\Bbbk[t]$–modules, since $d(\nu) = 1 = d(\tau)$ and $d\big(\tau^p)\big) = p$, with $\Delta(\chi_g) = \chi_g \otimes 1 + 1 \otimes \chi_g + t^{d(g)} \, \chi_g \otimes \chi_g$ for all $g \in B$. As a direct consequence we have also

$$\bigoplus\nolimits_{a,b,c=0}^{p-1} \Bbbk \cdot \overline{\chi_\nu}^{\;a} \, \overline{\chi_\tau}^{\;b} \, \overline{\chi_{\tau^p}}^{\;c} \; = \; \Bbbk[G]_t^\vee\Big|_{t=0} \; \cong \; \widehat{\Bbbk[G]} \; = \; \bigoplus\nolimits_{a,b,c=0}^{p-1} \Bbbk \cdot \overline{\eta_\nu}^{\;a} \, \overline{\eta_\tau}^{\;b} \, \overline{\eta_{\tau^p}}^{\;c} \, .$$

The relations $\nu^p = 1$ and $\tau^{p^2} = 1$ in G yield trivial relations in $\Bbbk[G]$ and $\big(\Bbbk[t]\big)[G]$. Instead, the relation $\nu \, \tau \, \nu^{-1} = \tau^{1+p}$ turns into $[\eta_\nu, \eta_\tau] = \eta_{\tau^p} \cdot \tau \, \nu$, whence $[\chi_\nu, \chi_\tau] = t^{p-2} \, \chi_{\tau^p} \cdot \tau \, \nu$ in $\Bbbk[G]_t^\vee$; thus $[\overline{\chi_\nu}\, , \overline{\chi_\tau}\,] = \delta_{p,2} \, \overline{\chi_{\tau^p}}$. Since $[\overline{\chi_\tau}\, , \overline{\chi_{\tau^p}}\,] = 0 = [\overline{\chi_\nu}\, , \overline{\chi_{\tau^p}}\,]$ (for $\nu \, \tau^p \, \nu^{-1} = \big(\tau^{1+p}\big)^p = \tau^{p+p^2} = \tau^p$) and $\{\overline{\chi_\nu}\, , \overline{\chi_\tau}\, , \overline{\chi_{\tau^p}}\,\}$ is a $\Bbbk$–basis of $\mathfrak{k}_G = \mathcal{L}_p(G)$, the latter has trivial or non-trivial Lie bracket, according to whether $p \neq 2$ or $p = 2$. In addition, $\chi_\nu^{\,p} = 0$, $\chi_{\tau^p}^{\,p} = 0$ and $\chi_\tau^{\,p} = \chi_{\tau^p}$: these give analogous relations in $\Bbbk[G]_t^\vee\Big|_{t=0} = \widehat{\Bbbk[G]}$, which read as formulas for the p–operation, namely $\overline{\chi_\nu}^{\;[p]} = 0$, $\overline{\chi_{\tau^p}}^{\;[p]} = 0$, $\overline{\chi_\tau}^{\;[p]} = \chi_{\tau^p}$.

Reassuming, we have a complete presentation for $\Bbbk[G]_t^\vee$ by generators and relations, i.e.

$$\Bbbk[G]_t^\vee \;\; \cong \;\; \Bbbk[t] \, \big\langle x_1, x_2, x_3 \big\rangle \Bigg/ \! \left(\begin{matrix} x_1 \, x_2 - x_2 \, x_1 - t^{p-2} \, x_3 \, (1 + t \, x_\tau) \, (1 + t \, x_\nu) \\ x_1 \, x_3 - x_3 \, x_1 \, , \quad x_1^{\,p} \, , \quad x_2^{\,p} - x_3 \, , \quad x_3^{\,p} \, , \quad x_2 \, x_3 - x_3 \, x_2 \end{matrix} \right)$$

via $\chi_\nu \mapsto x_1$, $\chi_\tau \mapsto x_2$, $\chi_{\tau^p} \mapsto x_3$. Similarly (as a consequence) we have the presentation

$$\widehat{\Bbbk[G]} \; = \; \Bbbk[G]_t^\vee\Big|_{t=0} \;\; \cong \;\; \Bbbk \, \big\langle y_1, y_2, y_3 \big\rangle \Bigg/ \! \left(\begin{matrix} y_1 \, y_2 - y_2 \, y_1 - \delta_{p,2} \, y_3 \, , \quad y_2^{\,p} - y_3 \\ y_1 \, y_3 - y_3 \, y_1 \, , \quad y_1^{\,p} \, , \quad y_3^{\,p} \, , \quad y_2 \, y_3 - y_3 \, y_2 \end{matrix} \right)$$

via $\overline{\chi_\nu} \mapsto y_1$, $\overline{\chi_\tau} \mapsto y_2$, $\overline{\chi_{\tau^p}} \mapsto y_3$, with p–operation as above and the y_i's being primitive

__Remark:__ if $p \neq 2$ exactly the same result holds for $G = \mathbb{Z}_p \times \mathbb{Z}_{p^2}$, i.e. $\mathfrak{k}_{\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}} = \mathfrak{k}_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$: this shows that the restricted Lie bialgebra $\mathfrak{k}_G$ may be not enough to recover the group G.

As for $\big(\Bbbk[G]_t^\vee\big)'$, it is generated by $\psi_\nu = \nu - 1$, $\psi_\tau = \tau - 1$, $\psi_{\tau^p} = t^{1-p}\big(\tau^p - 1\big)$, with relations $\psi_\nu^{\,p} = 0$, $\psi_\tau^{\,p} = t^{p-1} \psi_{\tau^p}$, $\psi_{\tau^p}^{\,p} = 0$, $\psi_\nu \, \psi_\tau - \psi_\tau \, \psi_\nu = t^{\,p-1} \psi_{\tau^p} \, (1 + \psi_\tau) \, (1 + \psi_\nu)$, $\psi_\tau \, \psi_{\tau^p} - \psi_{\tau^p} \, \psi_\tau = 0$, and $\psi_\nu \, \psi_{\tau^p} - \psi_{\tau^p} \, \psi_\nu = 0$. In particular $\big(\Bbbk[G]_t^\vee\big)' \supsetneqq \big(\Bbbk[t]\big)[G]$, and

$$\big(\Bbbk[G]_t^\vee\big)' \;\; \cong \;\; \Bbbk[t] \, \big\langle y_1, y_2, y_3 \big\rangle \Bigg/ \! \left(\begin{matrix} y_\nu \, y_{\tau^p} - y_{\tau^p} \, y_\nu \, , \quad y_\tau^{\,p} - t^{p-1} y_{\tau^p} \, , \quad y_\tau \, y_{\tau^p} - y_{\tau^p} \, y_\tau \\ y_\nu^{\,p} \, , \quad y_\nu \, y_\tau - y_\tau \, y_\nu - t^{\,p-1} y_{\tau^p} \, (1 + y_\tau) \, (1 + y_\nu) \, , \quad y_{\tau^p}^{\,p} \end{matrix} \right)$$

via $\psi_\nu \mapsto y_1$, $\psi_\tau \mapsto y_2$, $\psi_{\tau^p} \mapsto y_3$. Letting $z_1 := \psi_\nu\big|_{t=0} + 1$, $z_2 := \psi_\tau\big|_{t=0} + 1$ and $x_3 := \psi_{\tau^p}\big|_{t=0}$ this gives $\big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0} = \Bbbk\big[z_1, z_2, x_3\big] \Big/ \big(z_1^{\,p} - 1, z_2^{\,p} - 1, x_3^{\,p}\,\big)$ as a $\Bbbk$–algebra, with the z_i's group-like, x_3 primitive (cf. Theorem 8.8*(b)*), and Poisson bracket given by $\big\{z_1, z_2\big\} = \delta_{p,2} \, z_1 \, z_2 \, x_3$, $\big\{z_2, x_3\big\} = 0$ and $\big\{z_1, x_3\big\} = 0$. Thus $\big(\Bbbk[G]_t^\vee\big)'\Big|_{t=0} = F[\varGamma_G]$ with

$\Gamma_G \cong \boldsymbol{\mu}_p \times \boldsymbol{\mu}_p \times \boldsymbol{\alpha}_p$ as algebraic groups, with Poisson structure such that $coLie\,(\Gamma_G) \cong \mathfrak{k}_G$.

Since $G_\infty = \{1\}$ the general theory ensures that $A_{\Bbbk}(G)' = A_{\Bbbk}(G)$. We leave to the interested reader the task of computing the filtration $\underline{D}$ of $A_{\Bbbk}(G)$, and consequently describe $A_R(G)'$, $\big(A_R(G)'\big)^\vee$, $\widetilde{A_{\Bbbk}(G)}$ and the connected Poisson group $K_G := Spec\big(\widetilde{A_{\Bbbk}(G)}\big)$.

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