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ABSTRACT

Let R be a 1-dimensional integral domain, let $\hbar \in R \setminus \{0\}$ be prime, and let $\mathcal{HA}$ be the category of torsionless Hopf algebras over R. We call $H \in \mathcal{HA}$ a *"quantized function algebra" (=QFA)*, resp. *"quantized restricted universal enveloping algebras" (=QrUEA)*, at $\hbar$ if $H \big/ \hbar H$ is the function algebra of a connected Poisson group, resp. the (restricted, if $R \big/ \hbar R$ has positive characteristic) universal enveloping algebra of a (restricted) Lie bialgebra.

An "inner" Galois correspondence on $\mathcal{HA}$ is established via the definition of two endofunctors, $(\ )^\vee$ and $(\ )'$, of $\mathcal{HA}$ such that: *(a)* the image of $(\ )^\vee$, resp. of $(\ )'$, is the full subcategory of all QrUEAs, resp. QFAs, at $\hbar$; *(b)* if $p := Char\big(R \big/ \hbar R\big) = 0$, the restrictions $(\ )^\vee\big|_{\text{QFAs}}$ and $(\ )'\big|_{\text{QrUEAs}}$ yield equivalences inverse to each other; *(c)* if $p = 0$, starting from a QFA over a Poisson group G, resp. from a QrUEA over a Lie bialgebra $\mathfrak{g}$, the functor $(\ )^\vee$, resp. $(\ )'$, gives a QrUEA, resp. a QFA, over the dual Lie bialgebra, resp. a dual Poisson group. In particular, *(a)* provides a machine to produce quantum groups of both types (either QFAs or QrUEAs), *(b)* gives a characterization of them among objects of $\mathcal{HA}$, and *(c)* gives a "global" version of the so-called "quantum duality principle" (after Drinfeld's, cf. [Dr]).

This result applies in particular to Hopf algebras of the form $\Bbbk[\hbar] \otimes_{\Bbbk} H$ where H is a Hopf algebra over the field $\Bbbk$: this yields quantum groups, hence "classical" geometrical symmetries of Poisson type (Poisson groups or Lie bialgebras, via specialization) associated to the "generalized" symmetry encoded by H. Both our main result and the above mentioned application are illustrated by means of several examples, which are studied in some detail.

These notes draw a sketch of the theoretical construction leading to the "global quantum duality principle". Besides, the principle itself, and in particular the above mentioned application, is illustrated by means of several examples: group algebras, the standard quantization of the Kostant-Kirillov structure on any Lie algebra, the quantum semisimple groups, the quantum Euclidean group and the quantum Heisenberg group.

Introduction

The most general notion of "symmetry" in mathematics is encoded in the definition of Hopf algebra. Among Hopf algebras H over a field, the commutative and the cocommutative ones encode "geometrical" symmetries, in that they correspond, under some technical conditions, to algebraic groups and to (restricted, if the ground field has positive characteristic) Lie algebras respectively: in the first case H is the algebra $F[G]$ of regular functions over an algebraic group G, whereas in the second case it is the (restricted) universal enveloping algebra $U(\mathfrak{g})$ ($\mathbf{u}(\mathfrak{g})$ in the restricted case) of a (restricted) Lie algebra $\mathfrak{g}$. A popular generalization of these two types of "geometrical symmetry" is given by quantum groups: roughly, these are Hopf algebras H depending on a parameter $\hbar$ such that setting $\hbar = 0$ the Hopf algebra one gets is either of the type $F[G]$ — hence H is a *quantized function algebra*, in short QFA — or of the type $U(\mathfrak{g})$ or $\mathbf{u}(\mathfrak{g})$ (according to the characteristic of the ground field) — hence H is a *quantized (restricted) universal enveloping algebra*, in short QrUEA. When a QFA exists whose specialization (i.e. its "value" at $\hbar = 0$) is $F[G]$, the group G inherits from this "quantization" a Poisson bracket, which makes it a Poisson (algebraic) group; similarly, if a QrUEA exists whose specialization is $U(\mathfrak{g})$ or $\mathbf{u}(\mathfrak{g})$, the (restricted) Lie algebra $\mathfrak{g}$ inherits a Lie cobracket which makes it a Lie bialgebra. Then by Poisson group theory one has Poisson groups G^* dual to G and a Lie bialgebra $\mathfrak{g}^*$ dual to $\mathfrak{g}$, so other geometrical symmetries are related to the initial ones.

The dependence of a Hopf algebra on $\hbar$ can be described as saying that it is defined over a ring R and $\hbar \in R$: so one is lead to dwell upon the category $\mathcal{HA}$ of Hopf R–algebras (maybe with some further conditions), and then raises three basic questions:

— (1) *How can we produce quantum groups?*

— (2) *How can we characterize quantum groups (of either kind) within $\mathcal{HA}$?*

— (3) *What kind of relationship, if any, does exist between quantum groups over mutually dual Poisson groups, or mutually dual Lie bialgebras?*

A first answer to question **(1)** and **(3)** together is given, in characteristic zero, by the so-called "quantum duality principle", known in literature with at least two formulations. One claims that quantum function algebras associated to dual Poisson groups can be taken to be dual — in the Hopf sense — to each other; and similarly for quantum enveloping algebras (cf. [FRT1] and [Se]). The second one, formulated by Drinfeld in local terms (i.e., using formal groups, rather than algebraic groups, and complete topological Hopf algebras; cf. [Dr], §7, and see [Ga4] for a proof), provides a recipe to get, out of a QFA over G, a QrUEA over $\mathfrak{g}^*$, and, conversely, to get a QFA over G^* out of a QrUEA over $\mathfrak{g}$. More precisely, Drinfeld defines two functors, inverse to each other, from the category of quantized universal enveloping algebras (in his sense) to the category of quantum formal series Hopf algebras (his analogue of QFAs) and viceversa, such that $U_\hbar(\mathfrak{g}) \mapsto F_\hbar[[G^*]]$ and $F_\hbar[[G]] \mapsto U_\hbar(\mathfrak{g}^*)$ (in his notation, where the subscript $\hbar$ stands as a reminder for "quantized" and the double square brackets stand for "formal series Hopf algebra").

In this paper we present a *global* version of the quantum duality principle which gives a complete answer to questions **(1)** through **(3)**. The idea is to push as far as possible Drinfeld's original method so to apply it to the category $\mathcal{HA}$ of all Hopf algebras which are torsion-free modules over some 1-dimensional domain (in short, 1dD), say R, and to do it for each non-zero prime element $\hbar$ in R.

To be precise, we extend Drinfeld's recipe so to define functors from $\mathcal{HA}$ to itself. We show that the image of these "generalized" Drinfeld's functors is contained in a category of quantum groups — one gives QFAs, the other QrUEAs — so we answer question **(1)**. Then, in the zero characteristic case, we prove that when restricted to quantum groups these functors yield equivalences inverse to each other. Moreover, we show that these equivalences exchange the types of quantum group (switching QFA with QrUEA) and the underlying Poisson symmetries (interchanging G or $\mathfrak{g}$ with G^* or $\mathfrak{g}^*$), thus solving **(3)**. Other details enter the picture to show that these functors endow $\mathcal{HA}$ with sort of a (inner) "Galois correspondence", in which QFAs on one side and QrUEAs on the other side are the subcategories (in $\mathcal{HA}$) of "fixed points" for the composition of both Drinfeld's functors (in the suitable order): in particular, this answers question **(2)**. It is worth stressing that, since our "Drinfeld's functors" are defined for each non-trivial point $(\hbar)$ of *Spec* (R), for any such $(\hbar)$ and for any H in $\mathcal{HA}$ they yield two quantum groups, namely a QFA and a QrUEA, w.r.t. $\hbar$ itself. Thus we have a method to get, out of any single $H \in \mathcal{HA}$, several quantum groups.

Therefore the "global" in the title is meant in several respects: geometrically, we consider global objects (namely Poisson groups rather than Poisson *formal* groups, as in Drinfeld's approach); algebraically we consider quantum groups over any 1dD R, so there may be several different "semi-classical limits" (=specialization) to consider, one for each non-trivial point in the spectrum of R (whereas in Drinfeld's context $R = \Bbbk[[\hbar]]$ so one can specialise only at $\hbar = 0$); more generally, our recipe applies to *any* Hopf algebra, i.e. not only to quantum groups; finally, most of our results are characteristic-free, i.e. they hold not only in zero characteristic (as in Drinfeld's case) but also in positive characteristic. As a further outcome, this "global version" of the quantum duality principle leads to formulate a "quantum duality principle for subgroups and homogeneous spaces", see [CG].

A key, long-ranging application of our *global quantum duality principle* (GQDP) is the following. Take as R the polynomial ring $R = \Bbbk[\hbar]$, where $\Bbbk$ is a field: then for any Hopf algebra over $\Bbbk$ we have that $H[\hbar] := R \otimes_{\Bbbk} H$ is a torsion-free Hopf R–algebra, hence we can apply Drinfeld's functors to it. The outcome of this procedure is the *crystal duality principle* (CDP), whose statement strictly resembles that of the GQDP: now Hopf $\Bbbk$–algebras are looked at instead of torsionless Hopf R–algebras, and quantum groups are replaced by Hopf algebras with canonical filtrations such that the associated graded Hopf algebra is either commutative or cocommutative. Correspondingly, we have a method to associate to H a Poisson group G and a Lie bialgebra $\mathfrak{k}$ such that G is an affine space (as an algebraic variety) and $\mathfrak{k}$ is graded (as a Lie algebra); in both cases, the "geometrical" Hopf algebra can be attained — roughly — through a continuous 1-parameter deformation process. This result can also be formulated in purely classical — i.e. "non-quantum" — terms and proved by purely classical means. However, the approach via the GQDP also yields further possibilities to deform H into other Hopf algebras of geometrical type, which is out of reach of any classical approach.

The purpose of these notes is to illustrate the global quantum duality principle in some detail through some relevant examples, namely the application to the "Crystal Duality Principle" (§3) and some quantum groups: the standard quantization of the Kostant-Kirillov structure on a Lie algebra (§4), the quantum semisimple groups (§5), the three dimensional quantum Euclidean group (§6), the quantum Heisenberg group. All details and technicalities which are skipped in the present paper can be found in [Ga5], together with another relevant example (see also [Ga6] and [Ga7]).

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§ 1 Notation and terminology

1.1 The classical setting. Let $\Bbbk$ be a fixed field of any characteristic. We call "algebraic group" the maximal spectrum G associated to any commutative Hopf $\Bbbk$–algebra H (in particular, we deal with *proaffine* as well as *affine* algebraic groups); then H is called the algebra of regular function on G, denoted with $F[G]$. We say that G is connected if $F[G]$ has no non-trivial idempotents; this is equivalent to the classical topological notion when $\dim(G)$ is finite. If G is an algebraic group, we denote by $\mathfrak{m}_e$ the defining ideal of the unit element $e \in G$ (in fact $\mathfrak{m}_e$ is the augmentation ideal of $F[G]$). The cotangent space of G at e is $\mathfrak{g}^\times := \mathfrak{m}_e \Big/ {\mathfrak{m}_e}^2$, endowed with its weak topology, which is naturally a Lie coalgebra. By $\mathfrak{g}$ we mean the tangent space of G at e, realized as the topological dual $\mathfrak{g} := \big(\mathfrak{g}^\times\big)^\star$ of $\mathfrak{g}^\times$: this is the tangent Lie algebra of G. By $U(\mathfrak{g})$ we mean the universal enveloping algebra of $\mathfrak{g}$: this is a connected cocommutative Hopf algebra, and there is a natural Hopf pairing (see §1.2*(a)*) between $F[G]$ and $U(\mathfrak{g})$. If $\mathit{Char}\,(\Bbbk) = p > 0$, then $\mathfrak{g}$ is a restricted Lie algebra, and $\mathbf{u}(\mathfrak{g}) := U(\mathfrak{g}) \Big/ \big(\big\{\, x^p - x^{[p]} \,\big|\, x \in \mathfrak{g} \,\big\}\big)$ is the restricted universal enveloping algebra of $\mathfrak{g}$. In the sequel, in order to unify notation and terminology, when $\mathit{Char}\,(\Bbbk) = 0$ we call any Lie algebra $\mathfrak{g}$ "restricted", and its "restricted universal enveloping algebra" will be $U(\mathfrak{g})$, and we write $\mathcal{U}(\mathfrak{g}) := U(\mathfrak{g})$ if $\mathit{Char}\,(\Bbbk) = 0$ and $\mathcal{U}(\mathfrak{g}) := \mathbf{u}(\mathfrak{g})$ if $\mathit{Char}\,(\Bbbk) > 0$.

We shall also consider $\mathit{Hyp}\,(G) := \big(F[G]^\circ\big)_\varepsilon = \big\{\, f \in F[G]^\circ \,\big|\, f({\mathfrak{m}_e}^n) = 0 \ \forall\, n \geqq 0 \,\big\}$, i.e. the connected component of the Hopf algebra $F[G]^\circ$ dual to $F[G]$; this is called the *hyperalgebra* of G. By construction $\mathit{Hyp}\,(G)$ is a connected Hopf algebra, containing $\mathfrak{g} = \mathit{Lie}(G)$; if $\mathit{Char}\,(\Bbbk) = 0$ one has $\mathit{Hyp}\,(G) = U(\mathfrak{g})$, whereas if $\mathit{Char}\,(\Bbbk) > 0$ one has a sequence of Hopf algebra morphisms $U(\mathfrak{g}) \relbar\joinrel\twoheadrightarrow \mathbf{u}(\mathfrak{g}) \lhook\joinrel\longrightarrow \mathit{Hyp}\,(G)$. In any case, there exists a natural perfect (= non-degenerate) Hopf pairing between $F[G]$ and $\mathit{Hyp}\,(G)$.

Now assume G is a Poisson group (for this and other notions hereafter see, e.g., [CP], but within an *algebraic geometry* setting): then $F[G]$ is a Poisson Hopf algebra, and its Poisson bracket induces on $\mathfrak{g}^\times$ a Lie bracket which makes it into a Lie bialgebra;so $U(\mathfrak{g}^\times)$ and $\mathcal{U}(\mathfrak{g}^\times)$ are co-Poisson Hopf algebras too. On the other hand, $\mathfrak{g}$ turns into a Lie bialgebra — maybe in topological sense, if G is infinite dimensional — and $U(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$ are (maybe topological) co-Poisson Hopf algebras. The Hopf pairing above between $F[G]$ and $\mathcal{U}(\mathfrak{g})$ then is compatible with these additional co-Poisson and Poisson structures. Similarly, $\mathit{Hyp}\,(G)$ is a co-Poisson Hopf algebra as well and the Hopf pairing between $F[G]$ and $\mathit{Hyp}\,(G)$ is compatible with the additional structures. Moreover, the perfect pairing $\mathfrak{g} \times \mathfrak{g}^\times \longrightarrow \Bbbk$ given by evaluation is compatible with the Lie bialgebra structure on either side (see §1.2*(b)*): so $\mathfrak{g}$ and $\mathfrak{g}^\times$ are Lie bialgebras *dual to each other*. In the sequel, we denote by $G^\star$ any connected algebraic Poisson group with $\mathfrak{g}$ as cotangent Lie bialgebra, and say it is *(Poisson) dual* to G.

For the Hopf operations in any Hopf algebra we shall use standard notation, as in [Ab].

Definition 1.2.

(a) Let H, K be Hopf algebras (in any category). A pairing $\langle\ ,\ \rangle : H \times K \longrightarrow R$ (where R is the ground ring) is a Hopf (algebra) pairing if $\big\langle x, y_1 \cdot y_2 \big\rangle = \big\langle \Delta(x), y_1 \otimes y_2 \big\rangle := \sum_{(x)} \big\langle x_{(1)}, y_1 \big\rangle \cdot \big\langle x_{(2)}, y_2 \big\rangle$, $\big\langle x_1 \cdot x_2, y \big\rangle = \big\langle x_1 \otimes x_2, \Delta(y) \big\rangle := \sum_{(y)} \big\langle x_1, y_{(1)} \big\rangle \cdot \big\langle x_2, y_{(2)} \big\rangle$, $\langle x, 1 \rangle = \epsilon(x)$, $\langle 1, y \rangle = \epsilon(y)$, $\big\langle S(x), y \big\rangle = \big\langle x, S(y) \big\rangle$, for all $x, x_1, x_2 \in H$, $y, y_1, y_2 \in K$.

(b) Let $\mathfrak{g}$, $\mathfrak{h}$ be Lie bialgebras (in any category). A pairing $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{h} \longrightarrow \Bbbk$ (where $\Bbbk$ is the ground ring) is called a Lie bialgebra pairing if $\big\langle x, [y_1, y_2] \big\rangle = \big\langle \delta(x), y_1 \otimes y_2 \big\rangle := \sum_{[x]} \big\langle x_{[1]}, y_1 \big\rangle \cdot$

$\langle x_{[2]}, y_2 \rangle$, $\langle [x_1,x_2], y \rangle = \langle x_1 \otimes x_2, \delta(y) \rangle := \sum_{[y]} \langle x_1, y_{[1]} \rangle \cdot \langle x_2, y_{[2]} \rangle$, *for all* $x, x_1, x_2 \in \mathfrak{g}$ *and* $y, y_1, y_2 \in \mathfrak{h}$, *with* $\delta(x) = \sum_{[x]} x_{[1]} \otimes x_{[2]}$ *and* $\delta(y) = \sum_{[x]} y_{[1]} \otimes y_{[2]}$.

1.3 The quantum setting. Let R be a 1-dimensional (integral) domain (=1dD), and let $F = F(R)$ be its quotient field. Denote by $\mathcal{M}$ the category of torsion-free R–modules, and by $\mathcal{HA}$ the category of all Hopf algebras in $\mathcal{M}$. Let $\mathcal{M}_F$ be the category of F–vector spaces, and $\mathcal{HA}_F$ be the category of all Hopf algebras in $\mathcal{M}_F$. For any $M \in \mathcal{M}$, set $M_F := F(R) \otimes_R M$. Scalar extension gives a functor $\mathcal{M} \longrightarrow \mathcal{M}_F$, $M \mapsto M_F$, which restricts to a functor $\mathcal{HA} \longrightarrow \mathcal{HA}_F$.

Let $\hbar \in R$ be a non-zero prime element (which will be fixed throughout), and $\Bbbk := R/(\hbar) = R/\hbar R$ the corresponding quotient field. For any R–module M, we set $M_\hbar\big|_{\hbar=0} := M/\hbar M = \Bbbk \otimes_R M$: this is a $\Bbbk$–module (via scalar restriction $R \to R/\hbar R =: \Bbbk$), which we call the *specialization* of M at $\hbar = 0$; we use also notation $M \xrightarrow{\hbar \to 0} \overline{N}$ to mean that $M_\hbar\big|_{\hbar=0} \cong \overline{N}$. Moreover, set $M_\infty := \bigcap_{n=0}^{+\infty} \hbar^n M$ (this is the closure of $\{0\}$ in the $\hbar$–adic topology of M). In addition, for any $H \in \mathcal{HA}$, let $I_H := Ker\Big(H \stackrel{\epsilon}{\twoheadrightarrow} R \xrightarrow{\hbar \to 0} \Bbbk\Big)$ and set $I_H^{\,\infty} := \bigcap_{n=0}^{+\infty} I_H^{\,n}$.

Finally, given $\mathbb{H}$ in $\mathcal{HA}_F$, a subset $\overline{H}$ of $\mathbb{H}$ is called *an R–integer form* (or simply *an R–form*) *of* $\mathbb{H}$ if $\overline{H}$ is a Hopf R–subalgebra of $\mathbb{H}$ (hence $\overline{H} \in \mathcal{HA}$) and $H_F := F(R) \otimes_R \overline{H} = \mathbb{H}$.

Definition 1.4. *(“Global quantum groups” [or “algebras”]) Let $\hbar \in R \setminus \{0\}$ be a prime.*

(a) We call quantized restricted universal enveloping algebra (at $\hbar$) (in short, QrUEA) any $\mathcal{U}_\hbar \in \mathcal{HA}$ such that $\mathcal{U}_\hbar\big|_{\hbar=0} := \mathcal{U}_\hbar / \hbar\, \mathcal{U}_\hbar$ is (isomorphic to) the restricted universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of some restricted Lie algebra $\mathfrak{g}$.

We call $\mathcal{QrUEA}$ the full subcategory of $\mathcal{HA}$ whose objects are all the QrUEAs (at $\hbar$).

(b) We call quantized function algebra (at $\hbar$) (in short, QFA) any $F_\hbar \in \mathcal{HA}$ such that $(F_\hbar)_\infty = I_{F_\hbar}^{\,\infty}$ (notation of §1.3) and $F_\hbar\big|_{\hbar=0} := F_\hbar / \hbar\, F_\hbar$ is (isomorphic to) the algebra of regular functions $F[G]$ of some connected algebraic group G.

We call $\mathcal{QFA}$ the full subcategory of $\mathcal{HA}$ whose objects are all the QFAs (at $\hbar$).

Remark 1.5. If $\mathcal{U}_\hbar$ is a QrUEA (at $\hbar$, that is w.r.t. to $\hbar$) then $\mathcal{U}_\hbar\big|_{\hbar=0}$ is a co-Poisson Hopf algebra, w.r.t. the Poisson cobracket δ defined as follows: if $x \in \mathcal{U}_\hbar\big|_{\hbar=0}$ and $x' \in \mathcal{U}_\hbar$ gives $x = x' \mod \hbar\, \mathcal{U}_\hbar$, then $\delta(x) := \big(\hbar^{-1}\left(\Delta(x') - \Delta^{\mathrm{op}}(x')\right)\big) \mod \hbar \left(\mathcal{U}_\hbar \otimes \mathcal{U}_\hbar\right)$. So $\mathcal{U}_\hbar\big|_{\hbar=0} \cong \mathcal{U}(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$, and by [Dr], §3, the restriction of δ makes $\mathfrak{g}$ into a *Lie bialgebra* (the isomorphism $\mathcal{U}_\hbar\big|_{\hbar=0} \cong \mathcal{U}(\mathfrak{g})$ being one of *co-Poisson* Hopf algebras); in this case we write $\mathcal{U}_\hbar = \mathcal{U}_\hbar(\mathfrak{g})$. Similarly, if $F_\hbar$ is a QFA at $\hbar$, then $F_\hbar\big|_{\hbar=0}$ is a *Poisson* Hopf algebra, w.r.t. the Poisson bracket $\{\ ,\ \}$ defined as follows: if $x, y \in F_\hbar\big|_{\hbar=0}$ and $x', y' \in F_\hbar$ give $x = x' \mod \hbar\, F_\hbar$, $y = y' \mod \hbar\, F_\hbar$, then $\{x, y\} := \big(\hbar^{-1}(x'\, y' - y'\, x')\big) \mod \hbar\, F_\hbar$. So $F_\hbar\big|_{\hbar=0} \cong F[G]$ for some connected *Poisson* algebraic group G (the isomorphism being one of *Poisson* Hopf algebras): in this case we write $F_\hbar = F_\hbar[G]$.

Definition 1.6.

(a) Let R be any (integral) domain, and let F be its field of fractions. Given two F–modules $\mathbb{A}$, $\mathbb{B}$, and an F–bilinear pairing $\mathbb{A} \times \mathbb{B} \longrightarrow F$, for any R–submodule $A \subseteq \mathbb{A}$ and $B \subseteq \mathbb{B}$ we set

$A^\bullet := \Big\{ b \in \mathbb{B} \,\Big|\, \big\langle A, b \big\rangle \subseteq R \Big\}$ *and* $B^\bullet := \Big\{ a \in \mathbb{A} \,\Big|\, \big\langle a, B \big\rangle \subseteq R \Big\}$.

(b) Let R be a 1dD. Given $H, K \in \mathcal{HA}$, we say that H and K are dual to each other if there exists a perfect Hopf pairing between them for which $H = K^\bullet$ and $K = H^\bullet$.

§ 2 The global quantum duality principle

2.1 Drinfeld's functors. (Cf. [Dr], §7) Let R, $\mathcal{HA}$ and $\hbar \in R$ be as in §1.3. For any $H \in \mathcal{HA}$, let $I = I_H := Ker\Big(H \overset{\epsilon}{\twoheadrightarrow} R \overset{\hbar \mapsto 0}{\twoheadrightarrow} R\big/\hbar R = \Bbbk\Big) = Ker\Big(H \overset{\hbar \mapsto 0}{\twoheadrightarrow} H\big/\hbar H \overset{\bar{\epsilon}}{\twoheadrightarrow} \Bbbk\Big)$ (as in §1.3), a maximal Hopf ideal of H (where $\bar{\epsilon}$ is the counit of $H\big|_{\hbar=0}$, and the two composed maps clearly coincide): we define

$$H^\vee := \sum_{n \geq 0} \hbar^{-n} I^n = \sum_{n \geq 0} \big(\hbar^{-1} I\big)^n = \bigcup_{n \geq 0} \big(\hbar^{-1} I\big)^n \quad \big(\subseteq H_F\big) .$$

If $J = J_H := Ker(\epsilon_H)$ then $I = J + \hbar \cdot 1_H$, thus $H^\vee = \sum_{n \geq 0} \hbar^{-n} J^n = \sum_{n \geq 0} \big(\hbar^{-1} J\big)^n$ too.

Given any Hopf algebra H, for every $n \in \mathbb{N}$ define $\Delta^n \colon H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := \mathrm{id}_H$, and $\Delta^n := \big(\Delta \otimes \mathrm{id}_H^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ if $n > 2$. For any ordered subset $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, define the morphism $j_\Sigma : H^{\otimes k} \longrightarrow H^{\otimes n}$ by $j_\Sigma(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_{i_m} := a_m$ for $1 \leq m \leq k$; then set $\Delta_\Sigma := j_\Sigma \circ \Delta^k$, $\Delta_\emptyset := \Delta^0$, and $\delta_\Sigma := \sum_{\Sigma' \subset \Sigma} (-1)^{n - |\Sigma'|} \Delta_{\Sigma'}$, $\delta_\emptyset := \epsilon$. By the inclusion-exclusion principle, this definition admits the inverse formula $\Delta_\Sigma = \sum_{\Psi \subseteq \Sigma} \delta_\Psi$. We shall also use the notation $\delta_0 := \delta_\emptyset$, $\delta_n := \delta_{\{1,2,\dots,n\}}$, and the useful formula $\delta_n = (\mathrm{id}_H - \epsilon)^{\otimes n} \circ \Delta^n$, for all $n \in \mathbb{N}_+$.

Now consider any $H \in \mathcal{HA}$ and $\hbar \in R$ as in §1.3: we define

$$H' := \big\{ a \in H \,\big|\, \delta_n(a) \in \hbar^n H^{\otimes n}, \ \forall\, n \in \mathbb{N} \big\} \quad \big(\subseteq H\big) .$$

Theorem 2.2 *("The Global Quantum Duality Principle")*

(a) The assignment $H \mapsto H^\vee$, resp. $H \mapsto H'$, defines a functor $(\ )^\vee \colon \mathcal{HA} \longrightarrow \mathcal{HA}$, resp. $(\ )' \colon \mathcal{HA} \longrightarrow \mathcal{HA}$, whose image lies in $\mathcal{QrUEA}$, resp. in $\mathcal{QFA}$. In particular, when $Char(\Bbbk) > 0$ the algebraic Poisson group G such that $H'\big|_{\hbar=0} = F[G]$ is zero-dimensional of height 1. Moreover, for all $H \in \mathcal{HA}$ we have $H \subseteq \big(H^\vee\big)'$ and $H \supseteq \big(H'\big)^\vee$, hence also $H^\vee = \Big(\big(H^\vee\big)'\Big)^\vee$ and $H' = \Big(\big(H'\big)^\vee\Big)'$.

(b) Let $Char(\Bbbk) = 0$. Then for any $H \in \mathcal{HA}$ one has

$$H = \big(H^\vee\big)' \iff H \in \mathcal{QFA} \qquad \text{and} \qquad H = \big(H'\big)^\vee \iff H \in \mathcal{QrUEA} ,$$

thus we have two induced equivalences, namely $(\ )^\vee \colon \mathcal{QFA} \longrightarrow \mathcal{QrUEA}$, $H \mapsto H^\vee$, and $(\ )' \colon \mathcal{QrUEA} \longrightarrow \mathcal{QFA}$, $H \mapsto H'$, which are inverse to each other.

(c) ("Quantum Duality Principle") Let $Char(\Bbbk) = 0$. Then

$$F_\hbar[G]^\vee\Big|_{\hbar=0} := F_\hbar[G]^\vee \Big/ \hbar\, F_\hbar[G]^\vee = U(\mathfrak{g}^\times) , \quad U_\hbar(\mathfrak{g})'\Big|_{\hbar=0} := U_\hbar(\mathfrak{g})' \Big/ \hbar\, U_\hbar(\mathfrak{g})' = F\big[G^\star\big]$$

(with G, $\mathfrak{g}$, $\mathfrak{g}^\times$, $\mathfrak{g}^\star$ and $G^\star$ as in §1.1, and $U_\hbar(\mathfrak{g})$ has the obvious meaning, cf. §1.5) where the choice of the group $G^\star$ (among all the connected Poisson algebraic groups with tangent Lie bialgebra $\mathfrak{g}^\star$) depends on the choice of the QrUEA $U_\hbar(\mathfrak{g})$. In other words, $F_\hbar[G]^\vee$ is a QrUEA for the Lie bialgebra $\mathfrak{g}^\times$, and $U_\hbar(\mathfrak{g})'$ is a QFA for the Poisson group $G^\star$.

(d) Let Char $(\Bbbk) = 0$. Let $F_\hbar \in \mathcal{QFA}$, $U_\hbar \in \mathcal{QrUEA}$ be dual to each other (with respect to some pairing). Then ${F_\hbar}^\vee$ and ${U_\hbar}'$ are dual to each other (w.r.t. the same pairing).

(e) Let Char $(\Bbbk) = 0$. Then for all $\mathbb{H} \in \mathcal{HA}_F$ the following are equivalent:

$\mathbb{H}$ has an R–integer form $H_{(f)}$ which is a QFA at $\hbar$;

$\mathbb{H}$ has an R–integer form $H_{(u)}$ which is a QrUEA at $\hbar$.

Remarks 2.3. After stating our main theorem, some comments are in order.

(a) The Global Quantum Duality Principle as a "Galois correspondence" type theorem. Let $L \subseteq E$ be a Galois (not necessarily finite) field extension, and let $G := \mathit{Gal}\big(E/L\big)$ be its Galois group. Let $\mathcal{F}$ be the set of intermediate extensions (i.e. all fields F such that $L \subseteq F \subseteq E$), let $\mathcal{S}$ be the set of all subgroups of G and let $\mathcal{S}^c$ be the set of all subgroups of G which are *closed* w.r.t. the Krull topology of G. Note that $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{S}^c$ can all be seen as lattices w.r.t. set-theoretical inclusion — $\mathcal{S}^c$ being a sublattice of $\mathcal{S}$ — hence as categories too. The celebrated Galois Theorem provides two maps, namely $\Phi : \mathcal{F} \longrightarrow \mathcal{S}$, $F \mapsto \mathit{Gal}\big(E/F\big) := \big\{ \gamma \in G \bigm| \gamma\big|_F = \mathrm{id}_F \big\}$, and $\Psi : \mathcal{S} \longrightarrow \mathcal{F}$, $H \mapsto E^H := \big\{ e \in E \bigm| \eta(e) = e \;\; \forall\, \eta \in H \big\}$, such that:

— 1) Φ and Ψ are *contravariant* functors (that is, they are order-reversing maps of lattices, i.e. lattice antimorphisms); moreover, the image of Φ lies in the subcategory $\mathcal{S}^c$;

— 2) for $H \in \mathcal{S}$ one has $\Phi\big(\Psi(H)\big) = \overline{H}$, the *closure* of H w.r.t. the Krull topology: thus $H \subseteq \Phi\big(\Psi(H)\big)$, and $\Phi \circ \Psi$ is a *closure operator*, so that $H \in \mathcal{S}^c$ iff $H = \Phi\big(\Psi(H)\big)$;

— 3) for $F \in \mathcal{F}$ one has $\Psi\big(\Phi(F)\big) = F$;

— 4) Φ and Ψ restrict to antiequivalences $\Phi : \mathcal{F} \to \mathcal{S}^c$ and $\Psi : \mathcal{S}^c \to \mathcal{F}$ which are inverse to each other.

Then one can see that Theorem 2.2 establishes a strikingly similar result, which in addition is much more symmetric: $\mathcal{HA}$ plays the role of both $\mathcal{F}$ and $\mathcal{S}$, whereas $(\ )'$ stands for Ψ and $(\ )^\vee$ stands for Φ. $\mathcal{QFA}$ plays the role of the distinguished subcategory $\mathcal{S}^c$, and symmetrically we have the distinguished subcategory $\mathcal{QrUEA}$. The composed operator $\big((\ )^\vee\big)' = (\ )' \circ (\ )^\vee$ plays the role of a "closure operator", and symmetrically $\big((\ )'\big)^\vee = (\ )^\vee \circ (\ )'$ plays the role of a "taking-the-interior operator": in other words, QFAs may be thought of as "closed sets" and QrUEAs as "open sets" in $\mathcal{HA}$. Yet note also that now all involved functors are *covariant.*

(b) Duality between Drinfeld's functors. For any $n \in \mathbb{N}$ let $\mu_n : {J_H}^{\otimes n} \lhook\joinrel\longrightarrow H^{\otimes n} \xrightarrow{m^n} H$ be the composition of the natural embedding of ${J_H}^{\otimes n}$ into $H^{\otimes n}$ with the n–fold multiplication (in H): then μ_n is the "Hopf dual" to δ_n. By construction we have $H^\vee = \sum_{n \in \mathbb{N}} \mu_n\big(\hbar^{-n} {J_H}^{\otimes n}\big)$ and $H' = \bigcap_{n \in \mathbb{N}} {\delta_n}^{-1}\big(\hbar^{+n} {J_H}^{\otimes n}\big)$: this shows that the two functors are built up as "dual" to each other (see also part *(d)* of Theorem 2.2).

(c) Ambivalence QrUEA $\leftrightarrow$ QFA in $\mathcal{HA}_F$ *.* Part *(e)* of Theorem 2.2 means that some Hopf algebras *over* $F(R)$ might be thought of *both* as "quantum function algebras" *and* as "quantum enveloping algebras": examples are U_F and F_F for $U \in \mathcal{Q}r\mathcal{UEA}$ and $F \in \mathcal{QFA}$.

(d) Drinfeld's functors for algebras, coalgebras and bialgebras. The definition of either of Drinfeld functors requires only "half of" the notion of Hopf algebra. In fact, one can define $(\ )^\vee$ for all "augmented algebras" (that is, roughly speaking, "algebras with a counit") and $(\ )'$ for all "coaugmented coalgebras" (roughly, "coalgebras with a unit"), and in particular for bialgebras: this yields again nice functors, and neat results extending the global quantum duality principle, cf. [Ga5], §§3–4.

(e) Relaxing the assumptions. We chose to work over $\mathcal{HA}$ for simplicity: in fact, this ensures that the specialization functor $H \mapsto H\big|_{\hbar=0}$ yields Hopf algebras over a field, so that we can use the more elementary geometric language of algebraic groups and Lie algebras in the easiest sense. Nevertheless, what is really necessary to let the machine work is to consider any (commutative, unital) ring R, any $\hbar \in R$ and then define Drinfeld's functors over Hopf R–algebras *which are $\hbar$–torsionless.* For instance, this is — essentially — what is done in [KT], where the ground ring is $R = \Bbbk[[u,v]]$, and the role of $\hbar$ is played by either u or v . In general, working in such a more general setting amounts to consider, at the semiclassical level (i.e. after specialization), *Poisson group schemes over* $R\big/\hbar\,R$ (i.e. over *Spec*$\big(R/\hbar\,R\big)$) and *Lie* $R\big/\hbar\,R$*–bialgebras*, where $R/\hbar\,R$ might not be a field.

Similar considerations — about R and $\hbar$ — hold w.r.t. remark *(d)* above.

§ 3 Application to trivial deformations: the Crystal Duality Principle

3.1 Drinfeld's functors on trivial deformations. Let $\mathcal{HA}_\Bbbk$ be the category of all Hopf algebras over the field $\Bbbk$. For all $n \in \mathbb{N}$, let $J^n := \big(\mathit{Ker}(\epsilon\colon\ H \longrightarrow \Bbbk)\big)^n$ and $D_n := \mathit{Ker}\big(\delta_{n+1}\colon\ H \longrightarrow H^{\otimes n}\big)$, and set $\underline{J} := \big\{J^n\big\}_{n\in\mathbb{N}}$, $\underline{D} := \big\{D_n\big\}_{n\in\mathbb{N}}$. Of course $\underline{J}$ is a decreasing filtration of H (maybe with $\bigcap_{n\geq 0} J^n \supsetneqq \{0\}$), and $\underline{D}$ is an increasing filtration of H (maybe with $\bigcup_{n\geq 0} D_n \subsetneqq H$), by coassociativity of the δ_n's.

Let $R := \Bbbk[\hbar]$ be the polynomial ring in the indeterminate $\hbar$: then R is a PID (= principal ideal domain), hence a 1dD, and $\hbar$ is a non-zero prime in R . Let $H_\hbar := H[\hbar] = R \otimes_\Bbbk H$, the scalar extension of H : this is a torsion free Hopf algebra over R, hence one can apply Drinfeld's functors to $H_\hbar$; in this section we do that with respect to the prime $\hbar$ itself. We shall see that the outcome is quite neat, and can be expressed purely in terms of Hopf algebras in $\mathcal{HA}_\Bbbk$: because of the special relation between some features of H — namely, the filtrations $\underline{J}$ and $\underline{D}$ — and some properties of Drinfeld's functors, we call this result "Crystal Duality Principle", in that it is obtained through sort of a "crystallization" process (bearing in mind, in a sense, Kashiwara's motivation for the terminology "crystal bases" in the context of quantum groups: see [CP], §14.1, and references therein). Indeed, this theorem can also be proved almost entirely by using only classical Hopf algebraic methods within $\mathcal{HA}_\Bbbk$, i.e. without resorting to deformations: this is accomplished in [Ga6]. We first discuss the general situation (§§3.2–5), second we look at the case of function algebras and enveloping algebras (§§3.6–7), then we state and prove the theorem of Crystal Duality Principle (§3.9). Eventually (§§3.11–12) we dwell upon two other interesting applications: hyperalgebras, and group algebras and their dual.

Note that the same analysis and results (with only a few more details to take care of) still hold if we take as R any 1dD and as $\hbar$ any prime element in R such that $R\big/\hbar\, R = \Bbbk$ and R carries a structure of $\Bbbk$–algebra; for instance, one can take $R = \Bbbk[[h]]$ and $\hbar = h$, or $R = \Bbbk\big[q, q^{-1}\big]$ and $\hbar = q - 1$. Finally, in the sequel to be short we perform our analysis for Hopf algebras only: however, as Drinfeld's functors are defined not only for Hopf algebras but for augmented algebras and coaugmented coalgebras too, we might do the same study for them as well. In particular, the Crystal Duality Principle has a stronger version which concerns these more general objects too (cf. [Ga6]).

Lemma 3.2

$$H_\hbar^\vee \;=\; \sum_{n\geq 0} R\cdot\hbar^{-n}J^n \;=\; R\cdot J^0 + R\cdot\hbar^{-1}J^1 + \cdots + R\cdot\hbar^{-n}J^n + \cdots \qquad (3.1)$$

$$H_\hbar{}' \;=\; \sum_{n\geq 0} R\cdot\hbar^{+n}D_n \;=\; R\cdot D_0 + R\cdot\hbar^{+1}D_1 + \cdots + R\cdot\hbar^{+n}D_n + \cdots \qquad (3.2)$$

Sketch of proof. (3.1) follows directly from definitions, while (3.2) is an easy exercise. □

3.3 Rees Hopf algebras and their specializations. Let M be a module over a commutative unitary ring R, and let $\underline{M} := \{M_z\}_{z\in\mathbb{Z}} = \Big(\cdots \subseteq M_{-m} \subseteq \cdots \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots\Big)$ be a bi-infinite filtration of M by submodules M_z ($z\in\mathbb{Z}$). In particular, we consider increasing filtrations (i.e., those with $M_z = \{0\}$ for $z < 0$) and decreasing filtrations (those with $M_z = \{0\}$ for all $z > 0$) as special cases of bi-infinite filtrations. First we define the associated *blowing module* to be the R–submodule $\mathcal{B}_{\underline{M}}(M)$ of $M\big[t,t^{-1}\big]$ (where t is any indeterminate) given by $\mathcal{B}_{\underline{M}}(M) := \sum_{z\in\mathbb{Z}} t^z M_z$; this is isomorphic to the *first graded module*[1] *associated to* M, namely $\bigoplus_{z\in\mathbb{Z}} M_z$. Second, we define the associated *Rees module* to be the $R[t]$–submodule $\mathcal{R}^t_{\underline{M}}(M)$ of $M\big[t,t^{-1}\big]$ generated by $\mathcal{B}_{\underline{M}}(M)$; straightforward computations then give R–module isomorphisms

$$\mathcal{R}^t_{\underline{M}}(M)\Big/(t-1)\,\mathcal{R}^t_{\underline{M}}(M) \;\cong\; \bigcup_{z\in\mathbb{Z}} M_z\,, \qquad \mathcal{R}^t_{\underline{M}}(M)\Big/t\,\mathcal{R}^t_{\underline{M}}(M) \;\cong\; G_{\underline{M}}(M)$$

where $G_{\underline{M}}(M) := \bigoplus_{z\in Z} M_z\big/M_{z-1}$ is the *second graded module*[1] *associated to* M. In other words, $\mathcal{R}^t_{\underline{M}}(M)$ is an $R[t]$–module which specializes to $\bigcup_{z\in\mathbb{Z}} M_z$ for $t=1$ and specializes to $G_{\underline{M}}(M)$ for $t=0$; therefore the R–modules $\bigcup_{z\in\mathbb{Z}} M_z$ and $G_{\underline{M}}(M)$ can be seen as 1-parameter (polynomial) deformations of each other via the 1-parameter family of R–modules given by $\mathcal{R}^t_{\underline{M}}(M)$. We can repeat this construction within the category of algebras, coalgebras, bialgebras or Hopf algebras over R with a filtration in the proper sense: then we'll end up with corresponding objects $\mathcal{B}_{\underline{M}}(M)$, $\mathcal{R}^t_{\underline{M}}(M)$, etc. of the like type (algebras, coalgebras, etc.). In particular we'll deal with Rees Hopf algebras.

[1] I pick this terminology from Serge Lang's textbook *"Algebra"*.

3.4 Drinfeld's functors on $H_\hbar$ and filtrations on H. Lemma 3.2 sets a link between properties of ${H_\hbar}'$, resp. of ${H_\hbar}^\vee$, and properties of the filtration $\underline{D}$, resp. $\underline{J}$, of H.

First, (3.1) together with ${H_\hbar}^\vee \in \mathcal{HA}$ implies that $\underline{J}$ is a Hopf algebra filtration of H; conversely, if one proves that $\underline{J}$ is a Hopf algebra filtration of H (which is straightforward) then from (3.1) we get a one-line proof that ${H_\hbar}^\vee \in \mathcal{HA}$. Second, we can look at $\underline{J}$ as a bi-infinite filtration, reversing index notation and extending trivially on positive indices, $\underline{J} = \Big(\cdots \subseteq J^n \subseteq \cdots J^2 \subseteq J \subseteq J^0 \big(= H \big) \subseteq H \subseteq \cdots \subseteq H \subseteq \cdots \Big)$; then the Rees Hopf algebra $\mathcal{R}_{\underline{J}}^\hbar(H)$ is defined (see §3.3). Now (3.1) give ${H_\hbar}^\vee = \mathcal{R}_{\underline{J}}^\hbar(H)$, so ${H_\hbar}^\vee \big/ \hbar\, {H_\hbar}^\vee \cong \mathcal{R}_{\underline{J}}^\hbar(H) \big/ \hbar\, \mathcal{R}_{\underline{J}}^\hbar(H) \cong G_{\underline{J}}(H)$. Thus $G_{\underline{J}}(H)$ is cocommutative because ${H_\hbar}^\vee \big/ \hbar\, {H_\hbar}^\vee$ is; conversely, we get an easy proof of the cocommutativity of ${H_\hbar}^\vee \big/ \hbar\, {H_\hbar}^\vee$ once we prove that $G_{\underline{J}}(H)$ is cocommutative, which is straightforward. Finally, $G_{\underline{J}}(H)$ is generated by $Q(H) = J \big/ J^2$ whose elements are primitive, so *a fortiori* $G_{\underline{J}}(H)$ is generated by its primitive elements; then the latter holds for ${H_\hbar}^\vee \big/ \hbar\, {H_\hbar}^\vee$ as well. To sum up, as ${H_\hbar}^\vee \in \mathcal{Q}r\mathcal{UEA}$ we argue that $G_{\underline{J}}(H) = \mathcal{U}(\mathfrak{g})$ for some restricted Lie bialgebra $\mathfrak{g}$; conversely, we can get ${H_\hbar}^\vee \in \mathcal{Q}r\mathcal{UEA}$ directly from the properties of the filtration $\underline{J}$ of H. Moreover, since $G_{\underline{J}}(H) = \mathcal{U}(\mathfrak{g})$ is graded, $\mathfrak{g}$ as a restricted Lie algebra is *graded* too.

On the other hand, it is easy to see that (3.2) and ${H_\hbar}' \in \mathcal{HA}$ imply that $\underline{D}$ is a Hopf algebra filtration of H; conversely, if one shows that $\underline{D}$ is a Hopf algebra filtration of H (which can be done) then (3.2) yields a direct proof that ${H_\hbar}' \in \mathcal{HA}$. Second, we can look at $\underline{D}$ as a bi-infinite filtration, extending it trivially on negative indices, namely $\underline{D} = \Big(\cdots \subseteq \{0\} \subseteq \cdots \{0\} \subseteq \big(\{0\} = \big) D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n \subseteq \cdots \Big)$; then the Rees Hopf algebra $\mathcal{R}_{\underline{D}}^\hbar(H)$ is defined (see §3.3). Now (3.2) gives ${H_\hbar}' = \mathcal{R}_{\underline{D}}^\hbar(H)$; but then ${H_\hbar}' \big/ \hbar\, {H_\hbar}' \cong \mathcal{R}_{\underline{D}}^\hbar(H) \big/ \hbar\, \mathcal{R}_{\underline{D}}^\hbar(H) \cong G_{\underline{D}}(H)$. Thus $G_{\underline{D}}(H)$ is commutative because ${H_\hbar}' \big/ \hbar\, {H_\hbar}'$ is; viceversa, we get an easy proof of the commutativity of ${H_\hbar}' \big/ \hbar\, {H_\hbar}'$ once we prove that $G_{\underline{D}}(H)$ is commutative (which can be done too). Finally, $G_{\underline{D}}(H)$ is graded with 1-dimensional degree 0 component (by construction) hence it has no non-trivial idempotents; so the latter is true for ${H_\hbar}' \big/ \hbar\, {H_\hbar}'$ too. Note also that $I_{{H_\hbar}'}^\infty = \{0\}$ by construction (because $H_\hbar$ is free over R). To sum up, since ${H_\hbar}' \in \mathcal{QFA}$ we get $G_{\underline{D}}(H) = F[G]$ for some connected algebraic Poisson group G; conversely, we can argue that ${H_\hbar}' \in \mathcal{QFA}$ directly from the properties of the filtration $\underline{D}$.

In addition, since $G_{\underline{D}}(H) = F[G]$ is graded, when *Char* $(\Bbbk) = 0$ the (pro)affine variety $G_{(cl)}$ of closed points of G is a (pro)affine space [2], that is $G_{(cl)} \cong \mathbb{A}_\Bbbk^{\times \mathcal{I}} = \Bbbk^{\mathcal{I}}$ for some index set $\mathcal{I}$, and so $F[G] = \Bbbk\big[\{x_i\}_{i \in \mathcal{I}}\big]$ is a polynomial algebra.

Finally, when $p :=$ *Char* $(\Bbbk) > 0$ the group G has dimension 0 and height 1: indeed, we can see this as a consequence of part of Theorem 2.2*(a)* via the identity $H_\hbar' \big/ \hbar\, H_\hbar' = G_{\underline{D}}(H)$, or conversely we can prove the relevant part of Theorem 2.2*(a)* via this identity by observing that G has those properties (cf. [Ga5], §5.4). At last, by general theory since G has dimension 0 and height 1

[2] For it is a *cone* — since H is graded — without vertex — since $G_{(cl)}$, being a group, is smooth.

the function algebra $F[G] = G_{\underline{D}}(H) = H'_\hbar \Big/ \hbar\, H'_\hbar$ is a *truncated polynomial algebra*, namely of type $F[G] = \Bbbk\big[\{x_i\}_{i \in \mathcal{I}}\big] \Big/ \big(\{x_i^p\}_{i \in \mathcal{I}}\big)$ for some index set $\mathcal{I}$.

3.5 Special fibers of ${H_\hbar}'$ and $H_\hbar^\vee$ and deformations. Given $H \in \mathcal{HA}_\Bbbk$, consider $H_\hbar$: our goal is to study $H_\hbar^\vee$ and ${H_\hbar}'$.

As for $H_\hbar^\vee$, the natural map from H to $\widehat{H} := G_{\underline{J}}(H) = H_\hbar^\vee \Big/ \hbar\, H_\hbar^\vee =: H_\hbar^\vee \Big|_{\hbar=0}$ sends $J^{\,\infty} := \bigcap_{n \geq 0} J^{\,n}$ to zero, by definition; also, letting $H^\vee := H \Big/ J^{\,\infty}$ (a Hopf algebra quotient of H, for $\underline{J}$ is a Hopf algebra filtration), we have $\widehat{H} = \widehat{H^\vee}$. Thus $\left(H^\vee\right)_\hbar^\vee \Big|_{\hbar=0} = \widehat{H^\vee} = \widehat{H} = \mathcal{U}(\mathfrak{g}_-)$ for some graded restricted Lie bialgebra $\mathfrak{g}_-$; also, $\left(H^\vee\right)_\hbar^\vee \Big|_{\hbar=1} := \left(H^\vee\right)_\hbar^\vee \Big/ (\hbar - 1) \left(H^\vee\right)_\hbar^\vee = \sum_{n \geq 0} \overline{J}^{\,n} = H^\vee$ (see §3.3). Thus we can see $\left(H^\vee\right)_\hbar^\vee = \mathcal{R}_{\underline{J}}^\hbar(H^\vee)$ as a 1-parameter family inside $\mathcal{HA}_\Bbbk$ with *regular* fibers (that is, they are isomorphic to each other as $\Bbbk$–vector spaces; indeed, we switch from H to $H^\vee$ just to achieve this regularity) which links $\widehat{H^\vee}$ and $H^\vee$ as (polynomial) deformations of each other, namely

$$\mathcal{U}(\mathfrak{g}_-) = \widehat{H^\vee} = \left(H^\vee\right)_\hbar^\vee \Big|_{\hbar=0} \xleftarrow[\left(H^\vee\right)_\hbar^\vee]{0 \leftarrow \hbar \rightarrow 1} \left(H^\vee\right)_\hbar^\vee \Big|_{\hbar=1} = H^\vee .$$

Now look at $\left(\left(H^\vee\right)_\hbar^\vee\right)'$. By construction, $\left(\left(H^\vee\right)_\hbar^\vee\right)' \Big|_{\hbar=1} = \left(H^\vee\right)_\hbar^\vee \Big|_{\hbar=1} = H^\vee$, whereas $\left(\left(H^\vee\right)_\hbar^\vee\right)' \Big|_{\hbar=1} = F[K_-]$ for some connected algebraic Poisson group K_-: in addition, if *Char* $(\Bbbk) = 0$ then $K_- = G_-^\star$ by Theorem 2.2*(c)*. So $\left(\left(H^\vee\right)_\hbar^\vee\right)'$ can be thought of as a 1-parameter family inside $\mathcal{HA}_\Bbbk$, with regular fibers, linking $H^\vee$ and $F[G_-^\star]$ as (polynomial) deformations of each other, namely

$$H^\vee = \left(\left(H^\vee\right)_\hbar^\vee\right)' \Big|_{\hbar=1} \xleftarrow[\left(\left(H^\vee\right)_\hbar^\vee\right)']{1 \leftarrow \hbar \rightarrow 0} \left(\left(H^\vee\right)_\hbar^\vee\right)' \Big|_{\hbar=0} = F[K_-] \quad \Big(= F[G_-^\star] \text{ if } \textit{Char}\,(\Bbbk) = 0 \Big) .$$

Therefore $H^\vee$ is *both* a deformation of an enveloping algebra *and* a deformation of a function algebra, via two different 1-parameter families (with regular fibers) in $\mathcal{HA}_\Bbbk$ which match at the value $\hbar = 1$, corresponding to the common element $H^\vee$. At a glance,

$$\mathcal{U}(\mathfrak{g}_-) \xleftarrow[\left(H^\vee\right)_\hbar^\vee]{0 \leftarrow \hbar \rightarrow 1} H^\vee \xleftarrow[\left(\left(H^\vee\right)_\hbar^\vee\right)']{1 \leftarrow \hbar \rightarrow 0} F[K_-] \quad \Big(= F[G_-^\star] \text{ if } \textit{Char}\,(\Bbbk) = 0 \Big) . \tag{3.3}$$

Now consider ${H_\hbar}'$. We have ${H_\hbar}' \Big|_{\hbar=0} := {H_\hbar}' \Big/ \hbar\, {H_\hbar}' = G_{\underline{D}}(H) =: \widetilde{H}$, and $\widetilde{H} = F[G_+]$ for some connected algebraic Poisson group G_+. On the other hand, we have also ${H_\hbar}' \Big|_{\hbar=1} := {H_\hbar}' \Big/ (\hbar - 1)\, {H_\hbar}' = \sum_{n \geq 0} D_n =: H'$; note that the latter is a Hopf subalgebra of H, because $\underline{D}$ is a Hopf algebra filtration; moreover we have $\widetilde{H} = \widetilde{H'}$, by the very definitions. Therefore we can think of ${H_\hbar}' = \mathcal{R}_{\underline{D}}^\hbar(H')$ as a 1-parameter family inside $\mathcal{HA}_\Bbbk$ with regular fibers which links $\widetilde{H}$ and H' as (polynomial) deformations of each other, namely

$$F[G_+] = \widetilde{H} = {H_\hbar}' \Big|_{\hbar=0} \xleftarrow[{H_\hbar}']{0 \leftarrow \hbar \rightarrow 1} {H_\hbar}' \Big|_{\hbar=1} = H' .$$

Consider also $\left({H_\hbar}'\right)^\vee$: by construction $\left({H_\hbar}'\right)^\vee \Big|_{\hbar=1} = {H_\hbar}' \Big|_{\hbar=1} = H'$, whereas $\left({H_\hbar}'\right)^\vee \Big|_{\hbar=0} = \mathcal{U}(\mathfrak{k}_+)$ for some restricted Lie bialgebra $\mathfrak{k}_+$: in addition, if *Char* $(\Bbbk) = 0$ then $\mathfrak{k}_+ = \mathfrak{g}_+^\times$ by Theorem

2.2*(c)*. Thus $\big(H_\hbar{}'\big)^\vee$ can be seen as a 1-parameter family with regular fibers, inside $\mathcal{HA}_\Bbbk$, which links $\mathcal{U}(\mathfrak{k}_+)$ and H' as (polynomial) deformations of each other, namely

$$H' = \big(H_\hbar{}'\big)^\vee\Big|_{\hbar=1} \xleftarrow[\ (H_\hbar{}')^\vee\ ]{1 \leftarrow \hbar \rightarrow 0} \big(H_\hbar{}'\big)^\vee\Big|_{\hbar=0} = \mathcal{U}(\mathfrak{k}_+) \quad \Big(= U(\mathfrak{g}_+^\times) \text{ if } \mathit{Char}\,(\Bbbk) = 0 \Big) .$$

Therefore, H' is *at the same time* a deformation of a function algebra *and* a deformation of an enveloping algebra, via two different 1-parameter families inside $\mathcal{HA}_\Bbbk$ (with regular fibers) which match at the value $\hbar = 1$, corresponding (in both families) to H'. In short,

$$F[G_+] \xleftrightarrow[\ H_\hbar{}'\ ]{0 \leftarrow \hbar \rightarrow 1} H' \xleftrightarrow[\ (H_\hbar{}')^\vee\ ]{1 \leftarrow \hbar \rightarrow 0} \mathcal{U}(\mathfrak{k}_+) \quad \Big(= U(\mathfrak{g}_+^\times) \text{ if } \mathit{Char}\,(\Bbbk) = 0 \Big) . \qquad (3.4)$$

Finally, it is worth noticing that when $H' = H = H^\vee$ formulas (3.3–4) give

$$\begin{array}{ccccc} F[G_+] & \xleftrightarrow[\ H_\hbar{}'\ ]{0 \leftarrow \hbar \rightarrow 1} & H' & \xleftrightarrow[\ (H_\hbar{}')^\vee\ ]{1 \leftarrow \hbar \rightarrow 0} & \mathcal{U}(\mathfrak{k}_+) \quad \Big(= U(\mathfrak{g}_+^\times) \text{ if } \mathit{Char}\,(\Bbbk) = 0 \Big) \\ & & \| & & \\ & & H & & \\ & & \| & & \\ \mathcal{U}(\mathfrak{g}_-) & \xleftrightarrow[\ (H^\vee)_\hbar^\vee\ ]{0 \leftarrow \hbar \rightarrow 1} & H^\vee & \xleftrightarrow[\ ((H^\vee)_\hbar^\vee)'\ ]{1 \leftarrow \hbar \rightarrow 0} & F[K_-] \quad \Big(= F[G_-^\star] \text{ if } \mathit{Char}\,(\Bbbk) = 0 \Big) \end{array} \qquad (3.5)$$

which provides *four* different regular 1-parameter (polynomial) deformations from H to Hopf algebras encoding geometrical objects of Poisson type, i.e. Lie bialgebras or Poisson algebraic groups.

3.6 The function algebra case. Let G be any algebraic group over the field $\Bbbk$. Let $R := \Bbbk[\hbar]$ be as in §3.1, and set $F_\hbar[G] := \big(F[G]\big)_\hbar = R \otimes_\Bbbk F[G]$: this is trivially a QFA at $\hbar$, because $F_\hbar[G] \big/ \hbar\, F_\hbar[G] = F[G]$, inducing on G the trivial Poisson structure, so that its cotangent Lie bialgebra is simply $\mathfrak{g}^\times$ with trivial Lie bracket and Lie cobracket dual to the Lie bracket of $\mathfrak{g}$. In the sequel we identify $F[G]$ with $1 \otimes F[G] \subset F_\hbar[G]$.

We begin by computing $F_\hbar[G]^\vee$ (w.r.t. $\hbar$) and $F_\hbar[G]^\vee\Big|_{\hbar=0} = \widehat{F[G]} = G_{\underline{J}}\big(F[G]\big)$.

Let $J := J_{F[G]} \equiv \mathit{Ker}\big(\epsilon_{F[G]}\big)$, let $\{j_b\}_{b \in \mathcal{S}}$ $(\subseteq J)$ be a system of parameters of $F[G]$, i.e. $\{y_b := j_b \mod J^2\}_{b \in \mathcal{S}}$ is a $\Bbbk$–basis of $Q\big(F[G]\big) = J \big/ J^2 = \mathfrak{g}^\times$. Then $J^n \big/ J^{n+1}$ is $\Bbbk$–spanned by $\big\{ j^{\,\underline{e}} \mod J^{n+1} \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}},\ |\underline{e}| = n \big\}$ for all n, where $\mathbb{N}_f^{\mathcal{S}} := \big\{ \sigma \in \mathbb{N}^{\mathcal{S}} \,\big|\, \sigma(b) = 0 \ \textit{for almost all}\ b \in \mathcal{S} \big\}$ (hereafter, monomials like the previous ones are *ordered* w.r.t. some fixed order of the index set $\mathcal{S}$) and $|\underline{e}| := \sum_{b \in \mathcal{S}} \underline{e}(b)$. This implies that

$$F[G]^\vee = \textstyle\sum_{\underline{e} \in \mathbb{N}_f^{\mathcal{S}}} \Bbbk[\hbar] \cdot \hbar^{-|\underline{e}|} j^{\,\underline{e}} \bigoplus \Bbbk[\hbar]\big[\hbar^{-1}\big]\, J^\infty = \sum_{\underline{e} \in \mathbb{N}_f^{\mathcal{S}}} \Bbbk[\hbar] \cdot (j^\vee)^{\underline{e}} \bigoplus \Bbbk[\hbar]\big[\hbar^{-1}\big]\, J^\infty$$

where $J^\infty := \bigcap_{n \in \mathbb{N}} J^n$ and $j_s^\vee := \hbar^{-1} j_s$ for all $s \in \mathcal{S}$. We also get that $\widehat{F[G]} = G_{\underline{J}}\big(F[G]\big)$ is $\Bbbk$–spanned by $\big\{ j^{\,\underline{e}} \mod J^{n+1} \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}} \big\}$, so $\widehat{F[G]} = G_{\underline{J}}\big(F[G]\big)$ is a quotient of $S(\mathfrak{g}^\times)$.

Now we distinguish various cases. First assume G is *smooth*, i.e. $\Bbbk^a \otimes_\Bbbk F[G]$ is *reduced* (where $\Bbbk^a$ is the algebraic closure of $\Bbbk$), which is always the case if $\mathit{Char}\,(\Bbbk) = 0$. Then (by standard results on algebraic groups) the above set spanning $\widehat{F[G]}$ is a $\Bbbk$–basis: thus $F_\hbar[G]^\vee\Big|_{\hbar=0} = \widehat{F[G]} = G_{\underline{J}}\big(F[G]\big) \cong S(\mathfrak{g}^\times)$ as $\Bbbk$–algebras. In addition, tracking the construction of the co-Poisson Hopf structure onto $\widehat{F[G]}$ we see at once that $\widehat{F[G]} \cong S(\mathfrak{g}^\times)$ *as co-Poisson Hopf algebras too*, where the Hopf structure on $S(\mathfrak{g}^\times)$ is the standard one and the co-Poisson structure is the one induced from the Lie cobracket of $\mathfrak{g}^\times$ (cf. [Ga5] for details). Note also that $S(\mathfrak{g}^\times) = U(\mathfrak{g}^\times)$ because $\mathfrak{g}^\times$ is Abelian.

Another "extreme" case is when G is a *finite connected group scheme*: then, assuming for simplicity that $\Bbbk$ be perfect, we have $F[G] = \Bbbk[x_1, \dots, x_n] \Big/ \big(x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}}\big)$ for some $n, e_1, \dots, e_n \in \mathbb{N}$. Modifying a bit the analysis of the smooth case one gets

$$F[G]^\vee = \textstyle\sum_{\underline{e} \in \mathbb{N}^n} \Bbbk[\hbar] \cdot \hbar^{-|\underline{e}|} x^{\underline{e}} = \sum_{\underline{e} \in \mathbb{N}^n} \Bbbk[\hbar] \cdot (x^\vee)^{\underline{e}}$$

(now $J^\infty = \{0\}$), and $F_\hbar[G]^\vee\Big|_{\hbar=0} = \widehat{F[G]} = G_{\underline{J}}\big(F[G]\big) \cong S(\mathfrak{g}^\times) \Big/ \big(\bar{x}_1^{p^{e_1}}, \dots, \bar{x}_n^{p^{e_n}}\big)$, where $\bar{x}_i := x_i \mod J^2 \in \mathfrak{g}^\times$. Now, recall that for any Lie algebra $\mathfrak{h}$ there is $\mathfrak{h}^{[p]^\infty} := \Big\{ x^{[p]^n} := x^{p^n} \,\Big|\, x \in \mathfrak{h}\,, n \in \mathbb{N} \Big\}$, the *restricted Lie algebra generated by* $\mathfrak{h}$ inside $U(\mathfrak{h})$, with p–operation given by $x^{[p]} := x^p$; then one always has $U(\mathfrak{h}) = \mathbf{u}\big(\mathfrak{h}^{[p]^\infty}\big)$. In our case $\Big\{\bar{x}_1^{p^{e_1}}, \dots, \bar{x}_n^{p^{e_n}}\Big\}$ generates a p–ideal $\mathcal{I}$ of $(\mathfrak{g}^\times)^{[p]^\infty}$, hence $\mathfrak{g}^\times_{res} := \mathfrak{g}^{[p]^\infty} \Big/ \mathcal{I}$ is a restricted Lie algebra too, with $\Bbbk$–basis $\Big\{\bar{x}_1^{p^{a_1}}, \dots, \bar{x}_n^{p^{a_n}} \,\Big|\, a_1 < e_1, \dots, a_n < e_n\Big\}$. Then the previous analysis gives $F_\hbar[G]^\vee\Big|_{\hbar=0} = \mathbf{u}\,(\mathfrak{g}^\times_{res}) \equiv S(\mathfrak{g}^\times) \Big/ \Big(\Big\{\bar{x}_1^{p^{e_1}}, \dots, \bar{x}_n^{p^{e_n}}\Big\}\Big)$ as co-Poisson Hopf algebras.

The general case is intermediate. Assume again for simplicity that $\Bbbk$ be perfect. Let $F[[G]]$ be the J–adic completion of $H = F[G]$. By standard results on algebraic groups (cf. [DG]) there is a subset $\{x_i\}_{i \in \mathcal{I}}$ of J such that $\big\{\overline{x}_i := x_i \mod J^2\big\}_{i \in \mathcal{I}}$ is a basis of $\mathfrak{g}^\times = J \big/ J^2$ and $F[[G]] \cong \Bbbk\big[\big[\{x_i\}_{i \in \mathcal{I}}\big]\big] \Big/ \Big(\Big\{x_i^{p^{n(x_i)}}\Big\}_{i \in \mathcal{I}_0}\Big)$ (the algebra of truncated formal power series), for some $\mathcal{I}_0 \subset \mathcal{I}$ and $\big(n(x_i)\big)_{i \in \mathcal{I}_0} \in \mathbb{N}^{\mathcal{I}_0}$. Since $G_{\underline{J}}\big(F[G]\big) = G_{\underline{J}}\big(F[[G]]\big)$, we argue that $G_{\underline{J}}\big(F[G]\big) \cong \Bbbk\big[\{\overline{x}_i\}_{i \in \mathcal{I}}\big] \Big/ \Big(\Big\{\overline{x}_i^{p^{n(x_i)}}\Big\}_{i \in \mathcal{I}_0}\Big)$; finally, since $\Bbbk\big[\{\overline{x}_i\}_{i \in \mathcal{I}}\big] \cong S(\mathfrak{g}^\times)$ we get

$$G_{\underline{J}}\big(F[G]\big) \cong S(\mathfrak{g}^\times) \Big/ \Big(\Big\{\, \overline{x}^{\,p^{n(x)}}\Big\}_{x \in \mathcal{N}(F[G])}\Big)$$

as algebras, $\mathcal{N}\big(F[G]\big)$ being the nilradical of $F[G]$ and $p^{n(x)}$ is the nilpotency order of $x \in \mathcal{N}\big(F[G]\big)$.

Finally, noting that $\Big(\Big\{\, \overline{x}^{\,p^{n(x)}}\Big\}_{x \in \mathcal{N}(F[G])}\Big)$ is a co-Poisson Hopf ideal of $S(\mathfrak{g}^\times)$, like in the smooth case we argue that the above isomorphism is one of *co-Poisson Hopf* algebras.

If $\Bbbk$ is not perfect the same analysis applies, but modifying a bit the previous arguments.

As for $F[G]^\vee := F[G] \big/ J^\infty$, one has (cf. [Ab], Lemma 4.6.4) $F[G]^\vee = F[G]$ whenever G is finite dimensional and there exists no $f \in F[G] \setminus \Bbbk$ which is separable algebraic over $\Bbbk$.

It is also interesting to consider $\big(F_\hbar[G]^\vee\big)'$. If *Char* $(\Bbbk) = 0$ Theorem 2.2*(c)* gives $\big(F_\hbar[G]^\vee\big)' = F_\hbar[G]$. If instead *Char* $(\Bbbk) = p > 0$, then the situation might change dramatically. Indeed, if G has dimension 0 and eight 1 then — i.e., if $F[G] = \Bbbk\big[\{x_i\}_{i \in \mathcal{I}}\big]\big] \Big/ \big(\big\{x_i^p \,\big|\, i \in \mathcal{I}\big\}\big)$ as a $\Bbbk$–algebra — the same analysis as in the zero characteristic case applies, with a few minor changes, whence one gets again $\big(F_\hbar[G]^\vee\big)' = F_\hbar[G]$. Otherwise, let $y \in J \setminus \{0\}$ be primitive and such that $y^p \neq 0$ (for instance, this occurs for $G \cong \mathbb{G}_a$). Then y^p is primitive as well, hence $\delta_n(y^p) = 0$ for each $n > 1$.

It follows that $0 \neq \hbar\,(y^\vee)^p \in \big(F_\hbar[G]^\vee\big)'$, whereas $\hbar\,(y^\vee)^p \notin F_\hbar[G]$, due to our previous description of $F_\hbar[G]^\vee$. Thus $\big(F_\hbar[G]^\vee\big)' \supsetneqq F_\hbar[G]^\vee$, a *counterexample to the first part of Theorem 2.2(c)*.

What for $F[G]'$ and $\widetilde{F[G]}$? Again, this depends on the group G under consideration. We provide two simple examples, both "extreme", in a sense, and opposite to each other.

Let $G := \mathbb{G}_a = \mathit{Spec}\big(\Bbbk[x]\big)$, so $F[G] = F[\mathbb{G}_a] = \Bbbk[x]$ and $F_\hbar[\mathbb{G}_a] := R \otimes_\Bbbk \Bbbk[x] = R[x]$. Then since $\Delta(x) := x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$ we find $F_\hbar[\mathbb{G}_a]' = R[\hbar x]$ (like in §3.7 below: indeed, this is just a special instance, for $F[\mathbb{G}_a] = U(\mathfrak{g})$ where $\mathfrak{g}$ is the 1-dimensional Lie algebra). Moreover, iterating one gets easily $\big(F_\hbar[\mathbb{G}_a]\big)' = R\big[\hbar^2 x\big]$, $\Big(\big(F_\hbar[\mathbb{G}_a]'\big)'\Big)' = R\big[\hbar^3 x\big]$, and in general $\bigg(\Big(\big(F_\hbar[\mathbb{G}_a\underbrace{]'\big)'\Big)'\cdots\bigg)'}_n = R\big[\hbar^n x\big] \cong R[x] = F_\hbar[\mathbb{G}_a]$ for all $n \in \mathbb{N}$.

Second, let $G := \mathbb{G}_m = \mathit{Spec}\Big(\Bbbk\big[z^{+1}, z^{-1}\big]\Big)$, that is $F[G] = F[\mathbb{G}_m] = \Bbbk\big[z^{+1}, z^{-1}\big]$ so that $F_\hbar[\mathbb{G}_m] := R \otimes_\Bbbk \Bbbk\big[z^{+1}, z^{-1}\big] = R\big[z^{+1}, z^{-1}\big]$. Then since $\Delta\big(z^{\pm 1}\big) := z^{\pm 1} \otimes z^{\pm 1}$ and $\epsilon\big(z^{\pm 1}\big) = 1$ we find $\Delta^n\big(z^{\pm 1}\big) = \big(z^{\pm 1}\big)^{\otimes n}$ and $\delta_n\big(z^{\pm 1}\big) = \big(z^{\pm 1} - 1\big)^{\otimes n}$ for all $n \in \mathbb{N}$. It follows easily from that $F_\hbar[\mathbb{G}_m]' = R \cdot 1$, the trivial possibility (see also §3.12 later on).

3.7 The enveloping algebra case. Let $\mathfrak{g}$ be any Lie algebra over the field $\Bbbk$, and $U(\mathfrak{g})$ its universal enveloping algebra with its standard Hopf structure. Assume $\mathit{Char}\,(\Bbbk) = 0$, and let $R = \Bbbk[\hbar]$, as in §3.1, and set $U_\hbar(\mathfrak{g}) := R \otimes_\Bbbk U(\mathfrak{g}) = \big(U(\mathfrak{g})\big)_\hbar$. Then $U_\hbar(\mathfrak{g})$ is trivially a QrUEA at $\hbar$, for $U_\hbar(\mathfrak{g}) \big/ \hbar\, U_\hbar(\mathfrak{g}) = U(\mathfrak{g})$, inducing on $\mathfrak{g}$ the trivial Lie cobracket. Thus the dual Poisson group is just $\mathfrak{g}^\star$ (the topological dual of $\mathfrak{g}$ w.r.t. the weak topology) w.r.t. addition, with $\mathfrak{g}$ as cotangent Lie bialgebra and function algebra $F[\mathfrak{g}^\star] = S(\mathfrak{g})$: the Hopf structure is the standard one, and the Poisson structure is the one induced by $\{x, y\} := [x, y]$ for all $x, y \in \mathfrak{g}$ (it is the *Kostant-Kirillov structure* on $\mathfrak{g}^\star$).

Similarly, if $\mathit{Char}\,(\Bbbk) = p > 0$ and $\mathfrak{g}$ is any restricted Lie algebra over $\Bbbk$, let $\mathbf{u}(\mathfrak{g})$ be its restricted universal enveloping algebra, with its standard Hopf structure. Then if $R = \Bbbk[\hbar]$ the Hopf R–algebra $U_\hbar(\mathfrak{g}) := R \otimes_\Bbbk \mathbf{u}(\mathfrak{g}) = \big(\mathbf{u}(\mathfrak{g})\big)_\hbar$ is a QrUEA at $\hbar$, because $\mathbf{u}_\hbar(\mathfrak{g}) \big/ \hbar\, \mathbf{u}_\hbar(\mathfrak{g}) = \mathbf{u}(\mathfrak{g})$, inducing on $\mathfrak{g}$ the trivial Lie cobracket: then the dual Poisson group is again $\mathfrak{g}^\star$, with cotangent Lie bialgebra $\mathfrak{g}$ and function algebra $F[\mathfrak{g}^\star] = S(\mathfrak{g})$ (the Poisson Hopf structure being as above). Recall also that $U(\mathfrak{g}) = \mathbf{u}\big(\mathfrak{g}^{[p]^\infty}\big)$ (cf. §3.6).

First we compute $\mathbf{u}_\hbar(\mathfrak{g})'$ (w.r.t. the prime $\hbar$) using (3.2), i.e. computing the filtration $\underline{D}$.

By the PBW theorem, once an ordered basis B of $\mathfrak{g}$ is fixed $\mathbf{u}(\mathfrak{g})$ admits as basis the set of ordered monomials in the elements of B whose degree (w.r.t. each element of B) is less than p; this yields a Hopf algebra filtration of $\mathbf{u}(\mathfrak{g})$ by the total degree, which we refer to as *the standard filtration*. Then a straightforward calculation shows that $\underline{D}$ coincides with the standard filtration. This and (3.2) imply $\mathbf{u}_\hbar(\mathfrak{g})' = \langle\tilde{\mathfrak{g}}\rangle = \langle\hbar\,\mathfrak{g}\rangle$: hereafter $\tilde{\mathfrak{g}} := \hbar\,\mathfrak{g}$, and similarly $\tilde{x} := \hbar\, x$ for all $x \in \mathfrak{g}$. Then the relations $x\,y - y\,x = [x, y]$ and $z^p = z^{[p]}$ in $\mathbf{u}(\mathfrak{g})$ yield $\tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} = \hbar\,\widetilde{[x, y]} \equiv 0 \mod \hbar\,\mathbf{u}_\hbar(\mathfrak{g})'$ and also $\tilde{z}^p = \hbar^{p-1}\widetilde{z^{[p]}} \equiv 0 \mod \hbar\,\mathbf{u}_\hbar(\mathfrak{g})'$; therefore, from $\mathbf{u}_\hbar(\mathfrak{g}) = T_R(\mathfrak{g}) \Big/ \big(\big\{\, x\,y - y\,x - [x, y]\,,\, z^p - z^{[p]} \,\big|\, x, y, z \in \mathfrak{g} \,\big\}\big)$ we get

$$\mathbf{u}_\hbar(\mathfrak{g})' = \langle \tilde{\mathfrak{g}} \rangle \xrightarrow{\hbar \to 0} \widetilde{\mathbf{u}(\mathfrak{g})} = T_{\Bbbk}(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{ \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x}\,,\ \tilde{z}^p \ \Big|\ \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{\mathfrak{g}} \Big\} \Big) =$$

$$= T_{\Bbbk}(\mathfrak{g}) \Big/ \big(\{ x\,y - y\,x\,,\ z^p \mid x, y, z \in \mathfrak{g} \} \big) = S_{\Bbbk}(\mathfrak{g}) \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) = F[\mathfrak{g}^\star] \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big)$$

that is $\widetilde{\mathbf{u}(\mathfrak{g})} := G_{\underline{D}}\big(\mathbf{u}(\mathfrak{g})\big) = \mathbf{u}_\hbar(\mathfrak{g})' \big/ \hbar\, \mathbf{u}_\hbar(\mathfrak{g})' \cong F[\mathfrak{g}^\star] \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big)$ *as Poisson Hopf algebras.* In particular, *this means that* $\widetilde{\mathbf{u}(\mathfrak{g})}$ *is the function algebra of, and* $\mathbf{u}_\hbar(\mathfrak{g})'$ *is a QFA (at* $\hbar$*) for, a non-reduced algebraic Poisson group of dimension 0 and height 1, whose cotangent Lie bialgebra is* $\mathfrak{g}$*, hence which is dual to* $\mathfrak{g}$; thus, in a sense, part *(c)* of Theorem 2.2 is still valid in this case too.

Remark: Note that this last result reminds the classical formulation of the analogue of Lie's Third Theorem in the context of group-schemes: *Given a restricted Lie algebra* $\mathfrak{g}$*, there exists a group-scheme* G *of dimension 0 and height 1 whose tangent Lie algebra is* $\mathfrak{g}$ (see e.g. [DG]). Here we have just given sort of a "dual Poisson-theoretic version" of this fact, in that our result sounds as follows: *Given a restricted Lie algebra* $\mathfrak{g}$*, there exists a Poisson group-scheme* G *of dimension 0 and height 1 whose cotangent Lie algebra is* $\mathfrak{g}$.

As a byproduct, since $U_\hbar(\mathfrak{g}) = \mathbf{u}_\hbar\big(\mathfrak{g}^{[p]^\infty}\big)$ we have also $U_\hbar(\mathfrak{g})' = \mathbf{u}_\hbar\big(\mathfrak{g}^{[p]^\infty}\big)'$, whence

$$U_\hbar(\mathfrak{g})' = \mathbf{u}_\hbar\big(\mathfrak{g}^{[p]^\infty}\big)' \xrightarrow{\hbar \to 0} S_{\Bbbk}\Big(\mathfrak{g}^{[p]^\infty}\Big) \Big/ \Big(\{ z^p \}_{z \in \mathfrak{g}^{[p]^\infty}} \Big) = F\Big[\big(\mathfrak{g}^{[p]^\infty}\big)^\star \Big] \Big/ \Big(\{ z^p \}_{z \in \mathfrak{g}^{[p]^\infty}} \Big) .$$

Furthermore, $\mathbf{u}_\hbar(\mathfrak{g})' = \langle \tilde{\mathfrak{g}} \rangle$ implies that $I_{\mathbf{u}_\hbar(\mathfrak{g})'}$ is generated (as an ideal) by $\hbar\, R \cdot 1_{\mathbf{u}_\hbar(\mathfrak{g})} + R\, \tilde{\mathfrak{g}}$, hence $\hbar^{-1} I_{\mathbf{u}_\hbar(\mathfrak{g})'}$ is generated by $R \cdot 1 + R\,\mathfrak{g}$, therefore

$$\big(\mathbf{u}_\hbar(\mathfrak{g})'\big)^\vee := \bigcup_{n \geq 0} \big(\hbar^{-1} I_{\mathbf{u}_\hbar(\mathfrak{g})'}\big)^n = \bigcup_{n \geq 0} \big(R \cdot 1 + R\,\mathfrak{g}\big)^n = \mathbf{u}_\hbar(\mathfrak{g}) .$$

This means that also part *(b)* of Theorem 2.2 is still valid, though now *Char* $(\Bbbk) > 0$.

When *Char* $(\Bbbk) = 0$ and we look at $U(\mathfrak{g})$, the like argument applies: $\underline{D}$ coincides with the standard filtration of $U(\mathfrak{g})$ provided by the total degree, via the PBW theorem. This and (3.2) imply $U(\mathfrak{g})' = \langle \tilde{\mathfrak{g}} \rangle = \langle \hbar\,\mathfrak{g} \rangle$, so that from the presentation $U_\hbar(\mathfrak{g}) = T_R(\mathfrak{g}) \Big/ \big(\{ x\,y - y\,x - [x,y] \}_{x,y,z \in \mathfrak{g}} \big)$ we get $U_\hbar(\mathfrak{g})' = T_R(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{ \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} - \hbar \cdot \widetilde{[x,y]} \Big\}_{\tilde{x},\tilde{y} \in \tilde{\mathfrak{g}}} \Big)$, whence we get at once

$$U_\hbar(\mathfrak{g})' \xrightarrow{\hbar \to 0} \widetilde{U(\mathfrak{g})} \cong T_{\Bbbk}(\tilde{\mathfrak{g}}) \Big/ \big(\{ \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} \mid \tilde{x}, \tilde{y} \in \tilde{\mathfrak{g}} \} \big) \cong S_{\Bbbk}(\mathfrak{g}) = F[\mathfrak{g}^\star]$$

i.e. $\widetilde{U(\mathfrak{g})} := G_{\underline{D}}\big(U(\mathfrak{g})\big) = U_\hbar(\mathfrak{g})' \big/ \hbar\, U_\hbar(\mathfrak{g})' \cong F[\mathfrak{g}^\star]$ as *Poisson Hopf* algebras, as predicted by Theorem 2.2*(c)*. Moreover, $U_\hbar(\mathfrak{g})' = \langle \tilde{\mathfrak{g}} \rangle = T(\tilde{\mathfrak{g}}) \Big/ \big(\{ \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} = \hbar \cdot \widetilde{[x,y]} \mid \tilde{x}, \tilde{y} \in \tilde{\mathfrak{g}} \} \big)$ implies that $I_{U_\hbar(\mathfrak{g})'}$ is generated by $\hbar\, R \cdot 1_{U_\hbar(\mathfrak{g})} + R\, \tilde{\mathfrak{g}}$: thus $\hbar^{-1} I_{U_\hbar(\mathfrak{g})'}$ is generated by $R \cdot 1_{U_\hbar(\mathfrak{g})} + R\,\mathfrak{g}$, so $\big(U_\hbar(\mathfrak{g})'\big)^\vee := \bigcup\limits_{n \geq 0} \big(\hbar^{-1} I_{U_\hbar(\mathfrak{g})'}\big)^n = \bigcup\limits_{n \geq 0} \big(R \cdot 1_{U_\hbar(\mathfrak{g})} + R\,\mathfrak{g}\big)^n = U_\hbar(\mathfrak{g})$, agreeing with Theorem 2.2*(b)*.

What for the functor $(\ )^\vee$? This heavily depends on the $\mathfrak{g}$ we start from!

First assume *Char* $(\Bbbk) = 0$. Let $\mathfrak{g}_{(1)} := \mathfrak{g}$, $\mathfrak{g}_{(k)} := \big[\mathfrak{g}, \mathfrak{g}_{(k-1)}\big]$ ($k \in \mathbb{N}_+$), be the *lower central series* of $\mathfrak{g}$. Pick subsets $B_1, B_2, \dots, B_k, \dots$ ($\subseteq \mathfrak{g}$) such that $B_k \mod \mathfrak{g}_{(k+1)}$ be a $\Bbbk$–basis of

$\mathfrak{g}_{(k)} \big/ \mathfrak{g}_{(k+1)}$ (for all $k \in \mathbb{N}_+$), pick also a $\Bbbk$–basis B_∞ of $\mathfrak{g}_{(\infty)} := \bigcap_{k\in\mathbb{N}_+}$, and set $\partial(b) := k$ for any $b \in B_k$ and each $k \in \mathbb{N}_+ \cup \{\infty\}$. Then $B := \left(\bigcup_{k\in\mathbb{N}_+} B_k\right) \cup B_\infty$ is a $\Bbbk$–basis of $\mathfrak{g}$; we fix a total order on it. Applying the PBW theorem to this ordered basis of $\mathfrak{g}$ we get that J^n has basis the set of ordered monomials $\left\{ b_1^{e_1} b_2^{e_2} \cdots b_s^{e_s} \,\middle|\, s \in \mathbb{N}_+ , b_r \in B \, , \sum_{r=1}^s b_r \, \partial(b_r) \geq n \right\}$. Then one finds that $U_\hbar(\mathfrak{g})^\vee$ is generated by $\left\{ \hbar^{-1} b \,\middle|\, b \in B_1 \setminus B_2 \right\}$ (as a unital R–algebra) and it is the direct sum

$$U_\hbar(\mathfrak{g})^\vee = \left(\oplus_{\substack{s\in\mathbb{N}_+ \\ b_r\in B\setminus B_\infty}} R \left(\hbar^{-\partial(b_1)} b_1\right)^{e_1} \cdots \left(\hbar^{-\partial(b_s)} b_s\right)^{e_s} \right) \bigoplus \left(\oplus_{\substack{s\in\mathbb{N}_+ ,\, b_r\in B \\ \exists\, \bar{r} \,:\, b_{\bar{r}}\in B_\infty}} R\big[\hbar^{-1}\big] \, b_1^{e_1} \cdots b_s^{e_s} \right)$$

From this it follows at once that $U_\hbar(\mathfrak{g})^\vee \Big/ \hbar \, U_\hbar(\mathfrak{g})^\vee \cong U\left(\mathfrak{g} \big/ \mathfrak{g}_{(\infty)}\right)$ via an isomorphism which maps $\hbar^{-\partial(b)} b \mod \hbar \, U_\hbar(\mathfrak{g})^\vee$ to $b \mod \mathfrak{g}_{(\infty)} \in \mathfrak{g}\big/\mathfrak{g}_{(\infty)} \subset U\left(\mathfrak{g}\big/\mathfrak{g}_{(\infty)}\right)$ for all $b \in B \setminus B_\infty$ and maps $\hbar^{-n} b \mod \hbar \, U_\hbar(\mathfrak{g})^\vee$ to 0 for all $b \in B \setminus B_\infty$ and all $n \in \mathbb{N}$.

Now assume *Char* $(\Bbbk) = p > 0$. Then in addition to the previous considerations one has to take into account the filtration of $\mathbf{u}(\mathfrak{g})$ induced by both the lower central series of $\mathfrak{g}$ *and* the p–filtration of $\mathfrak{g}$, that is $\mathfrak{g} \supseteq \mathfrak{g}^{[p]} \supseteq \mathfrak{g}^{[p]^2} \supseteq \cdots \supseteq \mathfrak{g}^{[p]^n} \supseteq \cdots$, where $\mathfrak{g}^{[p]^n}$ is the restricted Lie subalgebra generated by $\left\{ x^{[p]^n} \,\middle|\, x \in \mathfrak{g} \right\}$ and $x \mapsto x^{[p]}$ is the p–operation in $\mathfrak{g}$: these encode the J–filtration of $\mathbf{u}(\mathfrak{g})$, hence of $\mathbf{u}_\hbar(\mathfrak{g})$, so permit to describe $\mathbf{u}_\hbar(\mathfrak{g})^\vee$.

In detail, for any restricted Lie algebra $\mathfrak{h}$, let $\mathfrak{h}_n := \left\langle \bigcup_{(m\, p^k \geq n} \left(\mathfrak{h}_{(m)}\right)^{[p^k]} \right\rangle$ for all $n \in \mathbb{N}_+$ (where $\langle X \rangle$ denotes the Lie subalgebra of $\mathfrak{h}$ generated by X) and $\mathfrak{h}_\infty := \bigcap_{n\in\mathbb{N}_+} \mathfrak{h}_n$: we call $\left\{\mathfrak{h}_n\right\}_{n\in\mathbb{N}_+}$ *the p–lower central series of* $\mathfrak{h}$. It is a *strongly central series* of $\mathfrak{h}$, i.e. a central series of $\mathfrak{h}$ such that $[\mathfrak{h}_m, \mathfrak{h}_n] \leq \mathfrak{h}_{m+n}$ for all m, n, and $\mathfrak{h}_n^{[p]} \leq \mathfrak{h}_{n+1}$ for all n.

Applying these tools to $\mathfrak{g} \subseteq \mathbf{u}(\mathfrak{g})$ the very definitions give $\mathfrak{g}_n \subseteq J^n$ (for all $n \in \mathbb{N}$) where $J := J_{\mathbf{u}(\mathfrak{g})}$: more precisely, if B is an ordered basis of $\mathfrak{g}$ then the (restricted) PBW theorem for $\mathbf{u}(\mathfrak{g})$ implies that $J^n \big/ J^{n+1}$ admits as $\Bbbk$–basis the set of ordered monomials of the form $x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_s}^{e_s}$ such that $\sum_{r=1}^s e_r \partial(x_{i_r}) = n$ where $\partial(x_{i_r}) \in \mathbb{N}$ is uniquely determined by the condition $x_{i_r} \in \mathfrak{g}_{\partial(x_{i_r})} \setminus \mathfrak{g}_{\partial(x_{i_r})+1}$ and each x_{i_k} is a fixed lift in $\mathfrak{g}$ of an element of a fixed ordered basis of $\mathfrak{g}_{\partial(x_{i_k})} \Big/ \mathfrak{g}_{\partial(x_{i_k})+1}$. This yields an explicit description of $\underline{J}$, hence of $\mathbf{u}(\mathfrak{g})^\vee$ and $\mathbf{u}_\hbar(\mathfrak{g})^\vee$, like before: in particular

$$\widehat{\mathbf{u}_\hbar(\mathfrak{g})} \; := \; \mathbf{u}_\hbar(\mathfrak{g})^\vee \Big/ \hbar \, \mathbf{u}_\hbar(\mathfrak{g})^\vee \; \cong \; \mathbf{u}\left(\mathfrak{g}\big/\mathfrak{g}_\infty\right) \; .$$

Definition 3.8. *We call pre-restricted universal enveloping algebra (=PrUEA) any $H \in \mathcal{HA}_\Bbbk$ which is down-filtered by $\underline{J}$ (i.e., $\bigcap_{n\in\mathbb{N}} J^n = \{0\}$), and $\mathcal{P}r\mathcal{UEA}$ the full subcategory of $\mathcal{HA}_\Bbbk$ of all PrUEAs. We call pre-function algebra (=PFA) any $H \in \mathcal{HA}_\Bbbk$ which is up-filtered by $\underline{D}$ (i.e., $\bigcup_{n\in\mathbb{N}} D_n = H$), and $\mathcal{PFA}$ the full subcategory of $\mathcal{HA}_\Bbbk$ of all PFAs.*

Theorem 3.9 *("The Crystal Duality Principle")*

(a) $H \mapsto H^\vee := H \big/ J_H^\infty$ and $H \mapsto H' := \bigcup_{n\in\mathbb{N}} D_n$ define functors $(\ )^\vee \colon \mathcal{HA}_\Bbbk \longrightarrow \mathcal{HA}_\Bbbk$ and $(\ )' \colon \mathcal{HA}_\Bbbk \longrightarrow \mathcal{HA}_\Bbbk$ respectively whose image are $\mathcal{P}r\mathcal{UEA}$ and $\mathcal{PFA}$ respectively.

(b) Let $H \in \mathcal{HA}_\Bbbk$. Then $\widehat{H} := G_{\underline{J}}(H) \cong \mathcal{U}(\mathfrak{g})$ as graded co-Poisson Hopf algebras, for some restricted Lie bialgebra $\mathfrak{g}$ which is graded as a Lie algebra. In particular, if Char $(\Bbbk) = 0$ and $\dim(H) \in \mathbb{N}$ then $\widehat{H} = \Bbbk \cdot 1$ and $\mathfrak{g} = \{0\}$.

More in general, the same holds if $H = B$ *is a* $\Bbbk$*–bialgebra.*

(c) Let $H \in \mathcal{HA}_{\Bbbk}$. *Then* $\widetilde{H} := G_{\underline{D}}(H) \cong F[G]$, *as graded Poisson Hopf algebras, for some connected algebraic Poisson group* G *whose variety of closed points form a (pro)affine space. If Char* $(\Bbbk) = 0$ *then* $F[G] = \widetilde{H}$ *is a polynomial algebra, i.e.* $F[G] = \Bbbk\big[\{x_i\}_{i\in\mathcal{I}}\big]$ *(for some set* $\mathcal{I}$*); in particular, if* $\dim(H) \in \mathbb{N}$ *then* $\widetilde{H} = \Bbbk \cdot 1$ *and* $G = \{1\}$. *If* $p :=$ *Char* $(\Bbbk) > 0$ *then* G *has dimension 0 and height 1, and if* $\Bbbk$ *is perfect then* $F[G] = \widetilde{H}$ *is a truncated polynomial algebra, i.e.* $F[G] = \Bbbk\big[\{x_i\}_{i\in\mathcal{I}}\big] \Big/ \big(\{x_i^p\}_{i\in\mathcal{I}}\big)$ *(for some set* $\mathcal{I}$*).*

More in general, the same holds if $H = B$ *is a* $\Bbbk$*–bialgebra.*

(d) For every $H \in \mathcal{HA}_{\Bbbk}$, *there exist two 1-parameter families* $(H^\vee)_\hbar^\vee = \mathcal{R}_{\underline{J}}^\hbar(H^\vee)$ *and* $\big((H^\vee)_\hbar^\vee\big)'$ *in* $\mathcal{HA}_{\Bbbk}$ *giving deformations of* $H^\vee$ *with regular fibers*

$$\left.\begin{array}{ll}\text{if Char}\,(\Bbbk)=0\,, & U(\mathfrak{g}_-)\\ \text{if Char}\,(\Bbbk)>0\,, & \mathbf{u}(\mathfrak{g}_-)\end{array}\right\} = \widehat{H} \xleftarrow[(H^\vee)_\hbar^\vee]{0\leftarrow\hbar\rightarrow 1} H^\vee \xleftarrow[((H^\vee)_\hbar^\vee)']{1\leftarrow\hbar\rightarrow 0} \begin{cases} F[K_-] = F[G_-^\star] \\ F[K_-] \end{cases}$$

and two 1-parameter families $H_\hbar' = \mathcal{R}_{\underline{D}}^\hbar(H')$ *and* $(H_\hbar')^\vee$ *in* $\mathcal{HA}_{\Bbbk}$ *giving deformations*

$$F[G_+] = \widetilde{H} \xleftarrow[H_\hbar']{0\leftarrow\hbar\rightarrow 1} H' \xleftarrow[(H_\hbar')^\vee]{1\leftarrow\hbar\rightarrow 0} \begin{cases} U(\mathfrak{k}_+) = U(\mathfrak{g}_+^\times) & \text{if Char}\,(\Bbbk)=0 \\ \mathbf{u}(\mathfrak{k}_+) & \text{if Char}\,(\Bbbk)>0 \end{cases}$$

of H' *with regular fibers, where* G_+ *is like* G *in (c),* K_- *is a connected algebraic Poisson group,* $\mathfrak{g}_-$ *is like* $\mathfrak{g}$ *in (b),* $\mathfrak{k}_+$ *is a (restricted, if Char* $(\Bbbk) > 0$*) Lie bialgebra,* $\mathfrak{g}_+^\times$ *is the cotangent Lie bialgebra to* G_+ *and* $G_-^\star$ *is a connected algebraic Poisson group with cotangent Lie bialgebra* $\mathfrak{g}_-$.

(e) If $H = F[G]$ *is the function algebra of an algebraic Poisson group* G*, then* $\widehat{F[G]}$ *is a bi-Poisson Hopf algebra (cf. [KT], §1), namely*

$$\widehat{F[G]} \;\cong\; S(\mathfrak{g}^\times)\Big/\Big(\Big\{\overline{x}^{\,p^{n(x)}}\Big\}_{x\in\mathcal{N}_{F[G]}}\Big) \;\cong\; U(\mathfrak{g}^\times)\Big/\Big(\Big\{\overline{x}^{\,p^{n(x)}}\Big\}_{x\in\mathcal{N}_{F[G]}}\Big)$$

where $\mathcal{N}_{F[G]}$ *is the nilradical of* $F[G]$, $p^{n(x)}$ *is the order of nilpotency of* $x \in \mathcal{N}_{F[G]}$ *and the bi-Poisson Hopf structure of* $S(\mathfrak{g}^\times)\Big/\Big(\Big\{\overline{x}^{\,p^{n(x)}}\Big\}_{x\in\mathcal{N}_{F[G]}}\Big)$ *is the quotient one from* $S(\mathfrak{g}^\times)$*; in particular if the group* G *is reduced then* $\widehat{F[G]} \cong S(\mathfrak{g}^\times) \cong U(\mathfrak{g}^\times)$.

(f) If Char $(\Bbbk) = 0$ *and* $H = U(\mathfrak{g})$ *is the universal enveloping algebra of some Lie bialgebra* $\mathfrak{g}$, *then* $\widetilde{U(\mathfrak{g})}$ *is a bi-Poisson Hopf algebra, namely* $\widetilde{U(\mathfrak{g})} \cong S(\mathfrak{g}) = F[\mathfrak{g}^\star]$ *where the bi-Poisson Hopf structure on* $S(\mathfrak{g})$ *is the canonical one.*

If Char $(\Bbbk) = p > 0$ *and* $H = \mathbf{u}(\mathfrak{g})$ *is the restricted universal enveloping algebra of some restricted Lie bialgebra* $\mathfrak{g}$, *then* $\widetilde{\mathbf{u}(\mathfrak{g})}$ *is a bi-Poisson Hopf algebra, namely we have* $\widetilde{\mathbf{u}(\mathfrak{g})} \cong S(\mathfrak{g})\Big/\big(\{x^p \,|\, x \in \mathfrak{g}\}\big) = F[G^\star]$ *where the bi-Poisson Hopf structure on* $S(\mathfrak{g})\Big/\big(\{x^p \,|\, x \in \mathfrak{g}\}\big)$ *is induced by the canonical one on* $S(\mathfrak{g})$*, and* $G^\star$ *is a connected algebraic Poisson group of dimension 0 and height 1 whose cotangent Lie bialgebra is* $\mathfrak{g}$.

(g) Let $H, K \in \mathcal{HA}_{\Bbbk}$ *and let* $\pi\colon\, H \times K \longrightarrow \Bbbk$ *be a Hopf pairing. Then* π *induce a filtered Hopf pairing* $\pi_f :\, H^\vee \times K' \longrightarrow \Bbbk$, *a graded Hopf pairing* $\pi_{_G} :\, \widehat{H} \times \widetilde{K} \longrightarrow \Bbbk$, *both perfect on the*

right, and Hopf pairings over $\Bbbk[\hbar]$ *(notation of §3.1)* $H_\hbar \times K_\hbar \longrightarrow \Bbbk[\hbar]$ *and* $H_\hbar^\vee \times K_\hbar' \longrightarrow \Bbbk[\hbar]$, *the latter being perfect on the right. If in addition the pairing* $\pi_f : H^\vee \times K' \longrightarrow \Bbbk$ *is perfect, then all other induced pairings are perfect as well, and* $H_\hbar^\vee$ *and* $K_\hbar'$ *are dual to each other.*

The left-right symmetrical results hold too.

Proof. Everything follows from the previous analysis, but for *(g)*, to be found in [Ga5] or [Ga6]. $\square$

Remarks 3.10. *(a)* Though usually introduced in a different way, H' is an object pretty familiar to Hopf algebraists: it is the *connected component* of H (see [Ga6] for a proof); in particular, H is a PFA iff it is connected. Nevertheless, the remarkable properties of $\widetilde{H} = G_{\underline{D}}(H)$ in Theorem 3.9*(c)* seems to have been unknown so far. Similarly, the "dual" construction of $H^\vee$ and the important properties of $\widehat{H} = G_{\underline{J}}(H)$ in Theorem 3.9*(b)* seem to be new.

(b) Theorem 3.9*(f)* reminds the classical formulation of the analogue of Lie's Third Theorem for group-schemes, i.e.: *Given a restricted Lie algebra* $\mathfrak{g}$, *there exists a group-scheme* G *of dimension 0 and height 1 whose tangent Lie algebra is* $\mathfrak{g}$ (see e.g. [DG]). Our result gives just sort of a "dual Poisson-theoretic version" of this fact, in that it sounds as follows: *Given a restricted Lie algebra* $\mathfrak{g}$, *there exists a Poisson group-scheme* G *of dimension 0 and height 1 whose cotangent Lie algebra is* $\mathfrak{g}$.

(c) Part *(d)* of Theorem 3.9 is quite interesting for applications in physics. In fact, let H be a Hopf algebra which describes the symmetries of some physically meaningful system, but has no geometrical meaning, and assume also $H' = H = H^\vee$. Then Theorem 3.9*(d)* yields a recipe to deform H to four Hopf algebras with geometrical content, which means having two Poisson groups and two Lie bialgebras attached to H, hence a rich "Poisson geometrical symmetry" underlying the physical system. As $\mathbb{R}$ (the typical ground field) has zero characteristic, we have in fact two pairs of mutually dual Poisson groups along with their tangent Lie bialgebras. A nice application is in [Ga7].

3.11. The hyperalgebra case. Let G be an algebraic group, which for simplicity we assume to be finite-dimensional. Let *Hyp* (G) be the hyperalgebra of G (cf. §1.1), which is connected cocommutative. Recall also the Hopf algebra morphism $\Phi : U(\mathfrak{g}) \longrightarrow \mathit{Hyp}(G)$; if $\mathit{Char}(\Bbbk) = 0$ then Φ is an isomorphism, so *Hyp* (G) identifies to $U(\mathfrak{g})$; if $\mathit{Char}(\Bbbk) > 0$ then Φ factors through $\mathbf{u}(\mathfrak{g})$ and the induced morphism $\overline{\Phi} : \mathbf{u}(\mathfrak{g}) \longrightarrow \mathit{Hyp}(G)$ is injective, so that $\mathbf{u}(\mathfrak{g})$ identifies with a Hopf subalgebra of *Hyp* (G). Now we study $\mathit{Hyp}(G)'$, $\mathit{Hyp}(G)^\vee$, $\widetilde{\mathit{Hyp}(G)}$, $\widehat{\mathit{Hyp}(G)}$, the key tool being the existence of a perfect (= non-degenerate) Hopf pairing between $F[G]$ and *Hyp* (G).

One can prove (see [Ga6]) that a Hopf $\Bbbk$–algebra H is connected iff $H = H'$. As *Hyp* (G) is connected, we have $\mathit{Hyp}(G) = \mathit{Hyp}(G)'$. Now, Theorem 3.9*(c)* gives $\widetilde{\mathit{Hyp}(G)} := G_{\underline{D}}\big(\mathit{Hyp}(G)\big) = F[\varGamma\,]$ for some connected algebraic Poisson group $\varGamma$; Theorem 3.9*(e)* yields

$$\widehat{F[G]} \cong S(\mathfrak{g}^*) \Big/ \Big(\Big\{ \overline{x}^{\,p^{n(x)}} \Big\}_{x \in \mathcal{N}_{F[G]}} \Big) = \mathbf{u}\bigg(P\bigg(S(\mathfrak{g}^*) \Big/ \Big(\Big\{ \overline{x}^{\,p^{n(x)}} \Big\}_{x \in \mathcal{N}_{F[G]}} \Big) \bigg) \bigg) = \mathbf{u}\Big(\big(\mathfrak{g}^*\big)^{p^\infty} \Big)$$

with $\big(\mathfrak{g}^*\big)^{p^\infty} := \mathit{Span}\Big(\Big\{ x^{p^n} \,\Big|\, x \in \mathfrak{g}^*, n \in \mathbb{N} \Big\} \Big) \subseteq \widehat{F[G]}$, and noting that $\mathfrak{g}^\times = \mathfrak{g}^*$. On the other hand, exactly like for $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$ respectively in case $\mathit{Char}(\Bbbk) = 0$ and $\mathit{Char}(\Bbbk) > 0$, the filtration $\underline{D}$ of *Hyp* (G) is nothing but the natural filtration given by the order of differential operators:

this implies immediately $\mathit{Hyp}(G)_\hbar{}' := \big(\Bbbk[\hbar] \otimes_\Bbbk \mathit{Hyp}(G)\big)' = \big\langle \big\{ \hbar^n x^{(n)} \,\big|\, x \in \mathfrak{g} \,, n \in \mathbb{N} \big\} \big\rangle$, where $x^{(n)}$ denotes the n–th divided power of $x \in \mathfrak{g}$ (recall that $\mathit{Hyp}(G)$ is generated as an algebra by all the $x^{(n)}$'s, some of which might be zero). It is then immediate to check that the graded Hopf pairing between $\mathit{Hyp}(G)_\hbar{}' \Big/ \hbar\, \mathit{Hyp}(G)_\hbar{}' = \widetilde{\mathit{Hyp}(G)} = F[\varGamma]$ and $\widehat{F[G]}$ from Theorem 3.9*(f)* is perfect. From this one argues that the cotangent Lie bialgebra of $\varGamma$ is isomorphic to $\Big(\big(\mathfrak{g}^*\big)^{p^\infty}\Big)^*$.

As for $\mathit{Hyp}(G)^\vee$ and $\widehat{\mathit{Hyp}(G)}$, the situation is much like for $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$, in that it strongly depends on the algebraic nature of G (cf. §3.7).

3.12 The CDP on group algebras and their duals. In this section, G is any abstract group. We divide the subsequent material in several subsections.

Group-related algebras. For any commutative unital ring $\mathbb{A}$, by $\mathbb{A}[G]$ we mean the group algebra of G over $\mathbb{A}$; when G is *finite*, we denote by $A_\mathbb{A}(G) := \mathbb{A}[G]^*$ (the linear dual of $\mathbb{A}[G]$) the function algebra of G over $\mathbb{A}$. Our aim is to apply the Crystal Duality Principle to $\Bbbk[G]$ and $A_\Bbbk(G)$ with their standard Hopf algebra structure: hereafter $\Bbbk$ is a field and $R := \Bbbk[\hbar]$ as in §5.1, with $p := \mathit{Char}(\Bbbk)$.

Recall that $H := \mathbb{A}[G]$ admits G itself as a distinguished basis, with Hopf algebra structure given by $g \cdot_{{}_H} \gamma := g \cdot_{{}_G} \gamma$, $1_{{}_H} := 1_{{}_G}$, $\Delta(g) := g \otimes g$, $\epsilon(g) := 1$, $S(g) := g^{-1}$, for all $g, \gamma \in G$. Dually, $H := A_\mathbb{A}(G)$ has basis $\big\{ \varphi_g \,\big|\, g \in G \big\}$ dual to the basis G of $\mathbb{A}[G]$, with $\varphi_g(\gamma) := \delta_{g,\gamma}$ for all $g, \gamma \in G$; its Hopf algebra structure is given by $\varphi_g \cdot \varphi_\gamma := \delta_{g,\gamma} \varphi_g$, $1_{{}_H} := \sum_{g \in G} \varphi_g$, $\Delta(\varphi_g) := \sum_{\gamma \cdot \ell = g} \varphi_\gamma \otimes \varphi_\ell$, $\epsilon(\varphi_g) := \delta_{g,1_G}$, $S(\varphi_g) := \varphi_{g^{-1}}$, for all $g, \gamma \in G$. In particular, $R[G] = R \otimes_\Bbbk \Bbbk[G]$ and $A_R[G] = R \otimes_\Bbbk A_\Bbbk[G]$. Our first result is

Theorem A: $\big(\Bbbk[G]\big)_\hbar{}' = R \cdot 1$, $\Bbbk[G]' = \Bbbk \cdot 1$ *and* $\widetilde{\Bbbk[G]} = \Bbbk \cdot 1 = F\big[\{*\}\big]$.

Proof. The claim follows easily from the formula $\delta_n(g) = (g-1)^{\otimes n}$, for $g \in G$, $n \in \mathbb{N}$. $\square$

$R[G]^\vee$, $\Bbbk[G]^\vee$, $\widehat{\Bbbk[G]}$ and the dimension subgroup problem. In contrast with the triviality result in Theorem A above, things are more interesting for $R[G]^\vee = \big(\Bbbk[G]\big)_\hbar^\vee$, $\Bbbk[G]^\vee$ and $\widehat{\Bbbk[G]}$. Note however that since $\Bbbk[G]$ is cocommutative the induced Poisson cobracket on $\widehat{\Bbbk[G]}$ is trivial, hence the Lie cobracket of $\mathfrak{k}_G := P\Big(\widehat{\Bbbk[G]}\Big)$ is trivial as well.

Studying $\Bbbk[G]^\vee$ and $\widehat{\Bbbk[G]}$ amounts to study the filtration $\big\{ J^n \big\}_{n \in \mathbb{N}}$, with $J := \mathit{Ker}(\epsilon_{{}_{\Bbbk[G]}})$, which is a classical topic. Indeed, for $n \in \mathbb{N}$ let $D_n(G) := \big\{ g \in G \,\big|\, (g-1) \in J^n \big\}$: this is a characteristic subgroup of G, called *the n^{th} dimension subgroup of G*. All these form a filtration inside G: characterizing it in terms of G is the *dimension subgroup problem*, which (for group algebras over fields) is completely solved (see [Pa], Ch. 11, §1, and [HB], and references therein); this also gives a description of $\big\{ J^n \big\}_{n \in \mathbb{N}_+}$. Thus we find ourselves within the domain of classical group theory: now we use the results which solve the dimension subgroup problem to argue a description of $\Bbbk[G]^\vee$, $\widehat{\Bbbk[G]}$ and $R[G]^\vee$, and later on we'll get from this a description of $\big(R[G]^\vee\big)'$ and its semiclassical limit too.

By construction, J has $\Bbbk$–basis $\big\{ \eta_g \,\big|\, g \in G \setminus \{1_{{}_G}\} \big\}$, where $\eta_g := (g-1)$. Then $\Bbbk[G]^\vee$ is generated by $\big\{ \eta_g \mod J^\infty \,\big|\, g \in G \setminus \{1_{{}_G}\} \big\}$, and $\widehat{\Bbbk[G]}$ by $\big\{ \overline{\eta_g} \,\big|\, g \in G \setminus \{1_{{}_G}\} \big\}$: hereafter

$\overline{x} := x \mod J^{n+1}$ for all $x \in J^n$, that is $\overline{x}$ is the element in $\widehat{\Bbbk[G]}$ which corresponds to $x \in \Bbbk[G]$. Moreover, $\overline{g} = \overline{1+\eta_g} = \overline{1}$ for all $g \in G$; also, $\Delta\big(\overline{\eta_g}\big) = \overline{\eta_g} \otimes \overline{g} + 1 \otimes \overline{\eta_g} = \overline{\eta_g} \otimes 1 + 1 \otimes \overline{\eta_g}$: thus $\overline{\eta_g}$ is primitive, so $\big\{ \overline{\eta_g} \,\big|\, g \in G \setminus \{1_G\} \big\}$ generates $\mathfrak{k}_G := P\Big(\widehat{\Bbbk[G]}\Big)$.

The Jennings-Hall theorem. The description of $D_n(G)$ is given by the Jennings-Hall theorem, which we now recall. The construction involved strongly depends on whether $p := \mathit{Char}\,(\Bbbk)$ is zero or not, so we shall distinguish these two cases.

First assume $p = 0$. Let $G_{(1)} := G$, $G_{(k)} := \big(G, G_{(k-1)}\big)$ ($k \in \mathbb{N}_+$), form the *lower central series* of G; hereafter (X,Y) is the commutator subgroup of G generated by the set of commutators $\big\{ (x,y) := x\,y\,x^{-1}y^{-1} \,\big|\, x \in X, y \in Y \big\}$: this is a *strongly central series* in G, which means a central series $\big\{G_k\big\}_{k\in\mathbb{N}_+}$ (= decreasing filtration of normal subgroups, each one centralizing the previous one) of G such that $(G_m, G_n) \leq G_{m+n}$ for all m, n. Then let $\sqrt{G_{(n)}} := \big\{ x \in G \,\big|\, \exists\, s \in \mathbb{N}_+ : x^s \in G_{(n)} \big\}$ for all $n \in \mathbb{N}_+$: these form a descending series of characteristic subgroups in G, such that each composition factor $A_{(n)}^G := \sqrt{G_{(n)}} \Big/ \sqrt{G_{(n+1)}}$ is torsion-free Abelian. Therefore $\mathcal{L}_0(G) := \bigoplus_{n\in\mathbb{N}_+} A_{(n)}^G$ is a graded Lie ring, with Lie bracket $\big[\overline{g}, \overline{\ell}\,\big] := \overline{(g,\ell\,)}$ for all *homogeneous* $\overline{g}, \overline{\ell} \in \mathcal{L}_0(G)$, with obvious notation. It is easy to see that the map $\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_0(G) \longrightarrow \mathfrak{k}_G$, $\overline{g} \mapsto \overline{\eta_g}$, is an epimorphism *of graded Lie rings*: thus *the Lie algebra* $\mathfrak{k}_G$ *is a quotient of* $\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_0(G)$; in fact, the above is an isomorphism, see below. We use notation $\partial(g) := n$ for all $g \in \sqrt{G_{(n)}} \setminus \sqrt{G_{(n+1)}}$.

For each $k \in \mathbb{N}_+$ pick in $A_{(k)}^G$ a subset $\overline{B}_k$ which is a $\mathbb{Q}$–basis of $\mathbb{Q} \otimes_{\mathbb{Z}} A_{(k)}^G$; for each $\overline{b} \in \overline{B}_k$, choose a fixed $b \in \sqrt{G_{(k)}}$ such that its coset in $A_{(k)}^G$ be $\overline{b}$, and denote by B_k the set of all such elements b. Let $B := \bigcup_{k\in\mathbb{N}_+} B_k$: we call such a set *t.f.l.c.s.-net* (= "torsion-free-lower-central-series-net") on G. Clearly $B_k = \Big(B \cap \sqrt{G_{(k)}} \Big) \setminus \Big(B \cap \sqrt{G_{(k+1)}} \Big)$ for all k. By an *ordered t.f.l.c.s.-net* is meant a t.f.l.c.s.-net B which is totally ordered in such a way that: *(i)* if $a \in B_m$, $b \in B_n$, $m < n$, then $a \preceq b$; *(ii)* for each k, every non-empty subset of B_k has a greatest element. As a matter of fact, an ordered t.f.l.c.s.-net always exists.

Now assume instead $p > 0$. The situation is similar, but we must also consider the p–power operation in the group G and in the restricted Lie algebra $\mathfrak{k}_G$. Starting from the lower central series $\big\{G_{(k)}\big\}_{k\in\mathbb{N}_+}$, define $G_{[n]} := \prod_{kp^\ell \geq n} \big(G_{(k)}\big)^{p^\ell}$ for all $n \in \mathbb{N}_+$ (hereafter, for any group $\varGamma$ we denote $\varGamma^{p^e}$ the subgroup generated by $\big\{ \gamma^{p^e} \,\big|\, \gamma \in \varGamma \big\}$): this gives another strongly central series $\big\{G_{[n]}\big\}_{n\in\mathbb{N}_+}$ in G, with the additional property that $\big(G_{[n]}\big)^p \leq G_{[n+1]}$ for all n, called *the p–lower central series of* G. Then $\mathcal{L}_p(G) := \oplus_{n\in\mathbb{N}_+} G_{[n]} \big/ G_{[n+1]}$ is a graded restricted Lie algebra over $\mathbb{Z}_p := \mathbb{Z} \big/ p\,\mathbb{Z}$, with operations $\overline{g} + \overline{\ell} := \overline{g \cdot \ell}$, $\big[\overline{g}, \overline{\ell}\,\big] := \overline{(g,\ell\,)}$, $\overline{g}^{\,[p]} := \overline{g^p}$, for all $g, \ell \in G$ (cf. [HB], Ch. VIII, §9). Like before, we consider the map $\Bbbk \otimes_{\mathbb{Z}_p} \mathcal{L}_p(G) \longrightarrow \mathfrak{k}_G$, $\overline{g} \mapsto \overline{\eta_g}$, which now is an epimorphism *of graded restricted Lie $\mathbb{Z}_p$–algebras*, whose image spans $\mathfrak{k}_G$ over $\Bbbk$: therefore $\mathfrak{k}_G$ *is a quotient of* $\Bbbk \otimes_{\mathbb{Z}_p} \mathcal{L}_p(G)$; in fact, the above is an isomorphism, see below. Finally, we introduce also the notation $d(g) := n$ for all $g \in G_{[n]} \setminus G_{[n+1]}$.

For each $k \in \mathbb{N}_+$ choose a $\mathbb{Z}_p$–basis $\overline{B}_k$ of the $\mathbb{Z}_p$–vector space $G_{[k]} \big/ G_{[k+1]}$; for each $\overline{b} \in \overline{B}_k$, fix $b \in G_{[k]}$ such that $\overline{b} = b\,G_{[k+1]}$, and let B_k be the set of all such elements b. Let $B :=$

$\bigcup_{k \in \mathbb{N}_+} B_k$: such a set will be called a *p-l.c.s.-net* (= "p-lower-central-series-net"; the terminology in [HB] is "κ-net") on G. Of course $B_k = \big(B \cap G_{[k]}\big) \setminus \big(B \cap G_{[k+1]}\big)$ for all k. By an *ordered p-l.c.s.-net* we mean a p-l.c.s.-net B which is totally ordered in such a way that: *(i)* if $a \in B_m$, $b \in B_n$, $m < n$, then $a \preceq b$; *(ii)* for each k, every non-empty subset of B_k has a greatest element (like for $p = 0$). Again, it is known that p-l.c.s.-nets always do exist.

We can now describe each $D_n(G)$, hence also each graded summand $J^n \big/ J^{n+1}$ of $\widehat{\Bbbk[G]}$, in terms of the lower central series or the p–lower central series of G, more precisely in terms of a fixed ordered t.f.l.c.s.-net or p-l.c.s.-net. To unify notations, set $G_n := G_{(n)}$, $\theta(g) := \partial(g)$ if $p=0$, and $G_n := G_{[n]}$, $\theta(g) := d(g)$ if $p>0$, set $G_\infty := \bigcap_{n \in N_+} G_n$, let $B := \bigcup_{k \in \mathbb{N}_+} B_k$ be an ordered t.f.l.c.s.-net or p-l.c.s.-net according to whether $p=0$ or $p>0$, and set $\ell(0) := +\infty$ and $\ell(p) := p$ for $p > 0$. The key result we need is

<u>Jennings-Hall theorem</u> *(cf. [HB], [Pa] and references therein). Let $p :=$ Char $(\Bbbk)$.*

(a) For all $g \in G$, $\eta_g \in J^n \Longleftrightarrow g \in G_n$. Therefore $D_n(G) = G_n$ for all $n \in \mathbb{N}_+$.

(b) For any $n \in \mathbb{N}_+$, the set of ordered monomials

$$\mathbb{B}_n := \Big\{ \overline{\eta_{b_1}}^{\,e_1} \cdots \overline{\eta_{b_r}}^{\,e_r} \,\Big|\, b_i \in B_{d_i},\ e_i \in \mathbb{N}_+,\ e_i < \ell(p),\ b_1 \precneqq \cdots \precneqq b_r,\ \textstyle\sum_{i=1}^r e_i\, d_i = n \Big\}$$

is a $\Bbbk$–basis of $J^n \big/ J^{n+1}$, and $\mathbb{B} := \{1\} \cup \bigcup_{n \in \mathbb{N}} \mathbb{B}_n$ is a $\Bbbk$–basis of $\widehat{\Bbbk[G]}$.

(c) $\big\{ \overline{\eta_b} \,\big|\, b \in B_n \big\}$ is a $\Bbbk$–basis of the n–th graded summand $\mathfrak{k}_G \cap \big(J^n \big/ J^{n+1}\big)$ of the graded restricted Lie algebra $\mathfrak{k}_G$, and $\big\{ \overline{\eta_b} \,\big|\, b \in B \big\}$ is a $\Bbbk$–basis of $\mathfrak{k}_G$.

(d) $\big\{ \overline{\eta_b} \,\big|\, b \in B_1 \big\}$ is a minimal set of generators of the (restricted) Lie algebra $\mathfrak{k}_G$.

(e) The map $\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_p(G) \longrightarrow \mathfrak{k}_G$, $\overline{g} \mapsto \overline{\eta_g}$, is an isomorphism of graded restricted Lie algebras. Therefore $\widehat{\Bbbk[G]} \cong \mathcal{U}\big(\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_p(G)\big)$ as Hopf algebras.

(f) $J^\infty =$ Span$\big(\{ \eta_g \,|\, g \in G_\infty \}\big)$, whence $\Bbbk[G]^\vee \cong \bigoplus_{\overline{g} \in G/G_\infty} \Bbbk \cdot \overline{g} \cong \Bbbk\big[G/G_\infty\big]$. □

Recall that $A\big[x, x^{-1}\big]$ (for any A) has A–basis $\big\{(x-1)^n x^{-[n/2]} \,\big|\, n \in \mathbb{N}\big\}$, where $[q]$ is the integer part of $q \in \mathbb{Q}$. Then from Jennings-Hall theorem and (5.2) we argue

<u>Proposition B.</u> *Let $\chi_g := \hbar^{-\theta(g)} \eta_g$, for all $g \in \{G\} \setminus \{1\}$. Then*

$$R[G]^\vee = \Big(\bigoplus_{\substack{b_i \in B,\, 0 < e_i < \ell(p) \\ r \in \mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot \chi_{b_1}^{e_1}\, b_1^{-[e_1/2]} \cdots \chi_{b_r}^{e_r}\, b_r^{-[e_r/2]} \Big) \bigoplus R\big[\hbar^{-1}\big] \cdot J^\infty =$$

$$= \Big(\bigoplus_{\substack{b_i \in B,\, 0 < e_i < \ell(p) \\ r \in \mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot \chi_{b_1}^{e_1}\, b_1^{-[e_1/2]} \cdots \chi_{b_r}^{e_r}\, b_r^{-[e_r/2]} \Big) \bigoplus \Big(\textstyle\sum_{\gamma \in G_\infty} R\big[\hbar^{-1}\big] \cdot \eta_\gamma \Big) ;$$

If $J^\infty = J^n$ for some $n \in \mathbb{N}$ (iff $G_\infty = G_n$) we can drop the factors $b_1^{-[e_1/2]}, \dots, b_r^{-[e_r/2]}$. □

<u>Poisson groups from $\Bbbk[G]$.</u> The previous discussion attached to the abstract group G the (maybe restricted) Lie algebra $\mathfrak{k}_G$ which, by Jennings-Hall theorem, is just the scalar extension of the Lie ring $\mathcal{L}_{Char(\Bbbk)}$ associated to G via the central series of the G_n's; in particular the functor $G \mapsto \mathfrak{k}_G$ is one considered since long in group theory. Now, by Theorem 5.8*(d)* we know that $\big(R[G]^\vee\big)'$ is a QFA, with $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[\varGamma_G\big]$ for some connected Poisson group $\varGamma_G$. This defines a functor

$G \mapsto \varGamma_G$ from abstract groups to connected Poisson groups, of dimension zero and height 1 if $p > 0$; in particular, this $\varGamma_G$ *is a new invariant for abstract groups.*

The description of $R[G]^\vee$ in Proposition B above leads us to an explicit description of $\big(R[G]^\vee\big)'$, hence of $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[\varGamma_G\big]$ and of $\varGamma_G$. Indeed direct inspection gives $\delta_n\big(\chi_g\big) = \hbar^{(n-1)\theta(g)} \chi_g^{\otimes n}$, so $\psi_g := \hbar\, \chi_g = \hbar^{1-\theta(g)} \eta_g \in \big(R[G]^\vee\big)' \setminus \hbar\,\big(R[G]^\vee\big)'$ for each $g \in G \setminus G_\infty$, whilst for $\gamma \in G_\infty$ we have $\eta_\gamma \in J^\infty$ which implies $\eta_\gamma \in \big(R[G]^\vee\big)'$ and even $\eta_\gamma \in \bigcap_{n\in\mathbb{N}} \hbar^n \big(R[G]^\vee\big)'$. Thus $\big(R[G]^\vee\big)'$ is generated by $\big\{\, \psi_g \,\big|\, g \in G \setminus \{1\} \big\} \cup \big\{\, \eta_\gamma \,\big|\, \gamma \in G_\infty \big\}$. Moreover, $g = 1 + \hbar^{\theta(g)-1} \psi_g \in \big(R[G]^\vee\big)'$ for every $g \in G \setminus G_\infty$, and $\gamma = 1 + (\gamma - 1) \in 1 + J^\infty \subseteq \big(R[G]^\vee\big)'$ for $\gamma \in G_\infty$. This and the previous analysis along with Proposition B prove next result, which in turn is the basis for Theorem D below.

Proposition C.

$$\big(R[G]^\vee\big)' \;=\; \Big(\bigoplus\nolimits_{\substack{b_i \in B,\; 0<e_i<\ell(p) \\ r\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_r}} R\cdot \psi_{b_1}^{e_1}\, b_1^{-[e_1/2]} \cdots \psi_{b_r}^{e_r}\, b_r^{-[e_r/2]} \Big) \bigoplus R\big[\hbar^{-1}\big] \cdot J^\infty \;=$$

$$=\; \Big(\bigoplus\nolimits_{\substack{b_i \in B,\; 0<e_i<\ell(p) \\ r\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_r}} R\cdot \psi_{b_1}^{e_1}\, b_1^{-[e_1/2]} \cdots \psi_{b_r}^{e_r}\, b_r^{-[e_r/2]} \Big) \bigoplus \Big(\sum\nolimits_{\gamma\in G_\infty} R\big[\hbar^{-1}\big] \cdot \eta_\gamma \Big) \,.$$

In particular, $\big(R[G]^\vee\big)' = R[G]$ *if and only if* $G_2 = \{1\} = G_\infty$ *. If in addition* $J^\infty = J^n$ *for some* $n \in \mathbb{N}$ *(iff* $G_\infty = G_n$*) then we can drop the factors* $b_1^{-[e_1/2]}, \dots, b_r^{-[e_r/2]}$. □

Theorem D. *Let* $x_g := \psi_g \mod \hbar\, \big(R[G]^\vee\big)'$, $z_g := g \mod \hbar\, \big(R[G]^\vee\big)'$ *for all* $g \neq 1$, *and* $B_1 := \big\{\, b \in B \,\big|\, \theta(b) = 1 \big\}$, $B_> := \big\{\, b \in B \,\big|\, \theta(b) > 1 \big\}$.

(a) If $p = 0$, *then* $F\big[\varGamma_G\big] = \big(R[G]^\vee\big)'\Big|_{\hbar=0}$ *is a polynomial/Laurent polynomial algebra, namely* $F\big[\varGamma_G\big] = \Bbbk\big[\{x_b\}_{b\in B_>} \cup \big\{z_b^{\pm 1}\big\}_{b\in B_1} \big]$, *the* x_b*'s being primitive and the* z_b*'s being group-like. In particular* $\varGamma_G \cong \big(\mathbb{G}_a^{\times B_>}\big) \times \big(\mathbb{G}_m^{\times B_1}\big)$ *as algebraic groups, i.e.* $\varGamma_G$ *is a (pro)affine space times a torus.*

(b) If $p > 0$, *then* $F\big[\varGamma_G\big] = \big(R[G]^\vee\big)'\Big|_{\hbar=0}$ *is a truncated polynomial/Laurent polynomial algebra, namely* $F\big[\varGamma_G\big] = \Bbbk\big[\{x_b\}_{b\in B_>} \cup \big\{z_b^{\pm 1}\big\}_{b\in B_1} \big] \Big/ \big(\{x_b^p\}_{b\in B_>} \cup \{z_b^p - 1\} \big)$, *the* x_b*'s being primitive and the* z_b*'s being group-like. In particular* $\varGamma_G \cong \big(\boldsymbol{\alpha}_p^{\times B_>}\big) \times \big(\boldsymbol{\mu}_p^{\times B_1}\big)$ *as algebraic groups of dimension zero and height 1.*

(c) The Poisson group $\varGamma_G$ *has cotangent Lie bialgebra* $\mathfrak{k}_G$ *, that is* $\mathit{coLie}\,(\varGamma_G) = \mathfrak{k}_G$ *.*

Proof. (a) The very definitions give $\partial(g\,\ell\,) \geq \partial(g) + \partial(\ell\,)$ for all $g, \ell \in G$, so that $[\psi_g, \psi_\ell] = \hbar^{1-\partial(g)-\partial(\ell)+\partial((g,\ell))}\, \psi_{(g,\ell)}\, g\,\ell \in \hbar \cdot \big(R[G]^\vee\big)'$, which proves (directly) that $\big(R[G]^\vee\big)'\Big|_{\hbar=0}$ is commutative! Moreover, the relation $1 = g^{-1} g = g^{-1}\big(1 + \hbar^{\partial(g)-1} \psi_g\big)$ (for any $g \in G$) yields $z_{g^{-1}} = z_g^{-1}$ iff $\partial(g) = 1$ and $z_{g^{-1}} = 1$ iff $\partial(g) > 1$. Noting also that $J^\infty \equiv 0 \mod \hbar\,\big(R[G]^\vee\big)'$ and $g = 1 + \hbar^{\partial(g)-1} \psi_g \equiv 1 \mod \hbar\,\big(R[G]^\vee\big)'$ for $g \in G \setminus G_\infty$, and also $\gamma = 1 + (\gamma - 1) \in 1 + J^\infty \equiv 1 \mod \hbar\,\big(R[G]^\vee\big)'$ for $\gamma \in G_\infty$, Proposition C gives

$$F\big[\varGamma_G\big] \;=\; \big(R[G]^\vee\big)'\Big|_{\hbar=0} \;=\; \Big(\bigoplus\nolimits_{\substack{b_i\in B_>,\; e_i\in\mathbb{N}_+ \\ r\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_r}} \Bbbk \cdot x_{b_1}^{e_1} \cdots x_{b_r}^{e_r} \Big) \bigoplus \Big(\bigoplus\nolimits_{\substack{b_i\in B_1,\; a_i\in\mathbb{Z} \\ s\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_s}} \Bbbk \cdot z_{b_1}^{a_1} \cdots z_{b_s}^{a_s} \Big)$$

so $F\big[\varGamma_G\big]$ is a polynomial-Laurent polynomial algebra as claimed. Similarly $\Delta(z_g) = z_g \otimes z_g$ for all $g \in G$ and $\Delta(x_g) = x_g \otimes 1 + 1 \otimes x_g$ iff $\partial(g) > 1$; so the z_b's are group-like and the x_b's primitive.

(b) The definition of d implies $d(g\,\ell\,) \geq d(g) + d(\ell\,)$ $(g, \ell \in G)$, whence we get $[\psi_g, \psi_\ell] = \hbar^{1-d(g)-d(\ell)+d((g,\ell))}\, \psi_{(g,\ell)}\, g\,\ell \,\in\, \hbar \cdot \big(R[G]^\vee\big)'$, proving that $\big(R[G]^\vee\big)'\Big|_{\hbar=0}$ is commutative. In addition $d(g^p) \geq p\; d(g)$, so $\psi_g^{\,p} = \hbar^{p\,(1-d(g))}\, \eta_g^{\,p} = \hbar^{p-1+d(g^p)-p\,d(g)}\, \psi_{g^p} \in \hbar \cdot \big(R[G]^\vee\big)'$, whence $\big(\psi_g^{\,p}\big|_{\hbar=0}\big)^p = 0$ inside $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[\varGamma_G\big]$, which proves that $\varGamma_G$ has dimension 0 and height 1. Finally $b^p = (1+\psi_b)^p = 1 + \psi_b^p \equiv 1 \mod \hbar\,\big(R[G]^\vee\big)'$ for all $b \in B_1$, so $b^{-1} \equiv b^{p-1} \mod \hbar\,\big(R[G]^\vee\big)'$. Thus letting $x_g := \psi_g \mod \hbar\,\big(R[G]^\vee\big)'$ (for $g \neq 1$) we get

$$F\big[\varGamma_G\big] \;=\; \big(R[G]^\vee\big)'\Big|_{\hbar=0} \;=\; \Big(\bigoplus\nolimits_{\substack{b_i \in B_>,\; 0<e_i<p \\ r\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_r}} \Bbbk \cdot x_{b_1}^{e_1} \cdots x_{b_r}^{e_r}\Big) \bigoplus \Big(\bigoplus\nolimits_{\substack{b_i \in B_1,\; 0<e_i<p \\ s\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_s}} \Bbbk \cdot z_{b_1}^{e_1} \cdots z_{b_s}^{e_s}\Big)$$

just like for *(a)* and also taking care that $z_b = x_b + 1$ and $z_b^{\,p} = 1$ for $b \in B_1$. Therefore $\big(R[G]^\vee\big)'\Big|_{\hbar=0}$ is a truncated polynomial/Laurent polynomial algebra as claimed. The properties of the x_b's and the z_b's w.r.t. the Hopf structure are then proved like for *(a)* again.

(c) The augmentation ideal $\mathfrak{m}_e$ of $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[\varGamma_G\big]$ is generated by $\{x_b\}_{b\in B}$; then $\hbar^{-1}\,[\psi_g, \psi_\ell] \;=\; \hbar^{\theta((g,\ell))-\theta(g)-\theta(\ell)}\, \psi_{(g,\ell)}\, \big(1 + \hbar^{\theta(g)-1} \psi_g\big)\, \big(1 + \hbar^{\theta(\ell)-1} \psi_\ell\,\big)$ by the previous computation, whence at $\hbar = 0$ one has $\big\{x_g\,, x_\ell\big\} \equiv x_{(g,\ell)} \mod \mathfrak{m}_e^2$ if $\theta\big((g,\ell\,)\big) = \theta(g) + \theta(\ell\,)$, and $\big\{x_g\,, x_\ell\big\} \equiv 0 \mod \mathfrak{m}_e^2$ if $\theta\big((g,\ell\,)\big) > \theta(g) + \theta(\ell\,)$. This means that the cotangent Lie bialgebra $\mathfrak{m}_e \Big/ \mathfrak{m}_e^2$ of $\varGamma_G$ is isomorphic to $\mathfrak{k}_G$, as claimed. $\square$

<u>Remarks</u>: *(a)* Theorem D claims that the connected Poisson group $K_G^\star := \varGamma_G$ is *dual* to $\mathfrak{k}_G$ in the sense of §1.1. Since $R[G]^\vee\Big|_{\hbar=0} = \mathcal{U}(\mathfrak{k}_G)$ and $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[K_G^\star\big]$, *this gives a close analogue, in positive characteristic, of the second half of Theorem 2.2(c).*

(b) Theorem D provides functorial recipes to attach to each abstract group G and each field $\Bbbk$ a connected Abelian algebraic Poisson group over $\Bbbk$, namely $G \mapsto \varGamma_G$, explicitly described as algebraic group and such that *coLie* $(K_G^\star) = \mathfrak{k}_G$. *Every such $\varGamma_G$ (for given $\Bbbk$) is then an invariant of G, a new one to the author's knowledge. Indeed, it is perfectly equivalent to the well-known invariant* $\mathfrak{k}_G$ (over the same $\Bbbk$), because clearly $G_1 \cong G_2$ implies $\mathfrak{k}_{G_1} \cong \mathfrak{k}_{G_2}$, whereas $\mathfrak{k}_{G_1} \cong \mathfrak{k}_{G_2}$ implies that G_1 and G_2 are isomorphic as algebraic groups — by Theorem D*(a–b)* — and bear isomorphic Poisson structures — by part *(c)* of Theorem D — whence $G_1 \cong G_2$ as algebraic Poisson groups.

<u>*The case of $A_\Bbbk(G)$*</u>. Let's now dwell upon $H := A_\Bbbk(G)$, for a *finite* group G.

Let $\mathbb{A}$ be a commutative unital ring, and $\Bbbk$, $R := \Bbbk[\hbar]$ be as before. Since $A_\mathbb{A}(G) := \mathbb{A}[G]^*$, we have $\mathbb{A}[G] = A_\mathbb{A}(G)^*$, so there is a perfect Hopf pairing $A_\mathbb{A}(G) \times \mathbb{A}[G] \longrightarrow \mathbb{A}$. Our first result is

<u>*Theorem E*</u>. $A_R(G)^\vee = R \cdot 1 \oplus R\big[\hbar^{-1}\big]\, J = \big(A_R(G)^\vee\big)'$, $A_\Bbbk(G)^\vee = \Bbbk \cdot 1$, $\widehat{A_\Bbbk(G)} = A_R(G)^\vee\Big|_{\hbar=0} = \Bbbk \cdot 1 = \mathcal{U}(\mathbf{0})$ *and* $\big(A_R(G)^\vee\big)'\Big|_{\hbar=0} = \Bbbk \cdot 1 = F\big[\{*\}\big]$.

Proof. By construction $J := \mathit{Ker}\,(\epsilon_{A_\Bbbk(G)})$ has $\Bbbk$–basis $\big\{\varphi_g\big\}_{g\in G\setminus\{1_G\}} \cup \big\{\varphi_{1_G} - 1_{A_\Bbbk(G)}\big\}$, and since $\varphi_g = \varphi_g^{\,2}$ for all g and $(\varphi_{1_G} - 1)^2 = -(\varphi_{1_G} - 1)$ we have $J = J^\infty$, so $A_\Bbbk(G)^\vee = \Bbbk \cdot 1$ and $\widehat{A_\Bbbk(G)} = \Bbbk \cdot 1$. Similarly, $A_R(G)^\vee$ is generated by $\big\{\hbar^{-1}\varphi_g\big\}_{g\in G\setminus\{1_G\}} \cup \big\{\hbar^{-1}(\varphi_{1_G} - 1_{A_R(G)})\big\}$; moreover, $J = J^\infty$ implies $\hbar^n J \subseteq A_R(G)^\vee$ for all n, whence $A_R(G)^\vee = R\,1 \oplus R[\hbar^{-1}]J$.

Then $J_{A_R(G)^\vee} = R\big[\hbar^{-1}\big] J \subseteq \hbar\, A_R(G)$, which implies $\big(A_R(G)^\vee\big)' = A_R(G)^\vee$: in particular, $\big(A_R(G)^\vee\big)'\Big|_{\hbar=0} = A_R(G)^\vee\Big|_{\hbar=0} = \Bbbk \cdot 1$, as claimed. $\square$

Poisson groups from $A_\Bbbk(G)$. Now we look at $A_R(G)'$, $A_\Bbbk(G)'$ and $\widetilde{A_\Bbbk(G)}$. By construction $A_R(G)$ and $R[G]$ are in perfect Hopf pairing, and are free R–modules of *finite rank*. In this case, using a general result about the relation between Drinfeld's functors and Hopf pairings (namely, Proposition 4.4 in [Ga5]) one finds $A_R(G)' = \big(R[G]^\vee\big)^\bullet = \big(R[G]^\vee\big)^*$: thus $A_R(G)'$ is the *dual* Hopf algebra to $R[G]^\vee$. Then from Proposition B we can argue an explicit description of $A_R(G)'$, whence also of $\big(A_R(G)'\big)^\vee$. Now, in proving Theorem 3.9*(g)* one also shows that $A_\Bbbk(G)' = \big(J_{\Bbbk[G]}^\infty\big)^\perp$; therefore there is a perfect filtered Hopf pairing $\Bbbk[G]^\vee \times A_\Bbbk(G)' \longrightarrow \Bbbk$ and a perfect graded Hopf pairing $\widetilde{A_\Bbbk(G)} \times \widehat{\Bbbk[G]} \longrightarrow \Bbbk$. Thus $A_\Bbbk(G)' \cong \big(\Bbbk[G]^\vee\big)^*$ as filtered Hopf algebras and $\widetilde{A_\Bbbk(G)} \cong \big(\widehat{\Bbbk[G]}\big)^*$ as graded Hopf algebras. If $p=0$ then $J = J^\infty$, as each $g \in G$ has finite order and $g^n = 1$ implies $g \in G_\infty$: then $\Bbbk[G]^\vee = \Bbbk \cdot 1 = \widehat{\Bbbk[G]}$, so $A_\Bbbk(G)' = \Bbbk \cdot 1 = \widetilde{A_\Bbbk(G)}$. If $p > 0$ instead, this analysis gives $\widetilde{A_\Bbbk(G)} = \big(\widehat{\Bbbk[G]}\big)^* = \big(\mathbf{u}(\mathfrak{k}_G)\big)^* = F[K_G]$, where K_G is a connected Poisson group of dimension 0, height 1 and tangent Lie bialgebra $\mathfrak{k}_G$. Thus

Theorem F.

(a) There is a second functorial recipe to attach to each finite abstract group a connected algebraic Poisson group of dimension zero and height 1 over any field $\Bbbk$ with $Char(\Bbbk) > 0$, namely $G \mapsto K_G := Spec\Big(\widetilde{A_\Bbbk(G)}\Big)$. This K_G is Poisson dual to $\varGamma_G$ of Theorem D in the sense of §1.1, in that $Lie(K_G) = \mathfrak{k}_G = coLie(\varGamma_G)$.

(b) If $p := Char(\Bbbk) > 0$, then $\big(A_R(G)'\big)^\vee\Big|_{\hbar=0} = \mathbf{u}\big(\mathfrak{k}_G^\times\big) = S\big(\mathfrak{k}_G^\times\big)\Big/\big(\{x^p \,|\, x \in \mathfrak{k}_G^\times\}\big)$.

Proof. Claim *(a)* is the outcome of the discussion above. Part *(b)* instead requires an explicit description of $\big(A_R(G)'\big)^\vee$. Since $A_R(G)' \cong \big(R[G]^\vee\big)^*$, from Proposition B we get $A_R(G)' = \Big(\bigoplus_{\substack{b_i \in B,\, 0<e_i<p \\ r\in\mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot \rho_{b_1,\dots,b_r}^{e_1,\dots,e_r}\Big)$ where each $\rho_{b_1,\dots,b_r}^{e_1,\dots,e_r}$ is defined by

$$\Big\langle \rho_{b_1,\dots,b_r}^{e_1,\dots,e_r} \,,\, \chi_{\beta_1}^{\varepsilon_1}\, \beta_1^{-[\varepsilon_1/2]} \cdots \chi_{\beta_s}^{\varepsilon_s}\, \beta_s^{-[\varepsilon_s/2]} \Big\rangle \;=\; \delta_{r,s} \prod\nolimits_{i=1}^r \delta_{b_i,\beta_i} \delta_{e_i,\varepsilon_i}$$

(for all $b_i, \beta_j \in B$ and $0 < e_i, \varepsilon_j < p$). Now, using notation of §1.3, $K_\infty \subseteq K'$ for any $K \in \mathcal{HA}$, whence $K' = \pi^{-1}\big(\overline{K}\,'\big)$ where $\pi : K \twoheadrightarrow K\big/K_\infty =: \overline{K}$ is the canonical projection. So let $K := R[G]^\vee$, $H := A_R(G)'$; Proposition B gives $K_\infty = R\big[\hbar^{-1}\big] \cdot J^\infty$ and provides at once a description of $\overline{K}$; from this and the previous description of H one sees also that in the present case K_∞ is exactly the right kernel of the natural pairing $H \times K \longrightarrow R$, which is perfect on the left, so that the induced pairing $H \times \overline{K} \longrightarrow R$ is perfect. By construction its specialization at $\hbar = 0$ is the natural pairing $F[K_G] \times \mathbf{u}(\mathfrak{k}_G) \longrightarrow \Bbbk$, which is perfect too. Then one applies Proposition 4.4*(c)* of [Ga5] (with $\overline{K}$ playing the rôle of K therein), which yields $\overline{K}\,' = \big(H^\vee\big)^\bullet = \Big(\big(A_R(G)'\big)^\vee\Big)^\bullet$. By construction $\overline{K}\,' = \big(R[G]^\vee\big)'\Big/\big(R\big[\hbar^{-1}\big] \cdot J^\infty\big)$, and Proposition C describes the latter as $\overline{K}\,' = \Big(\bigoplus_{\substack{b_i \in B,\, 0<e_i<p \\ r\in\mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot \overline{\psi}_{b_1}^{\,e_1} \cdots \overline{\psi}_{b_r}^{\,e_r}\Big)$, where $\overline{\psi}_{b_i} := \psi_{b_i} \mod R\big[\hbar^{-1}\big] \cdot J^\infty$ for all i; since $\overline{K}\,' = \Big(\big(A_R(G)'\big)^\vee\Big)^\bullet$ and

$\psi_g = \hbar^{+1} \chi_g$, this yields $\big(A_R(G)'\big)^\vee = \Big(\bigoplus_{\substack{b_i \in B,\, 0 < e_i < p \\ r \in \mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot \hbar^{-\sum_i e_i d(b_i)} \rho_{b_1, \dots, b_r}^{e_1, \dots, e_r} \Big) \cong \big(\overline{K}\,'\big)^*$, whence we get $\big(A_R(G)'\big)^\vee \Big|_{\hbar=0} \cong \big(\overline{K}\,'\big)^* \Big|_{\hbar=0} = \big(K'\big|_{\hbar=0}\big)^* = \Big(\big(R[G]^\vee\big)' \big|_{\hbar=0} \Big)^* \cong F\big[\varGamma_G\big]^* = \mathbf{u}\big(\mathfrak{k}_G^\times\big) = S\big(\mathfrak{k}_G^\times\big) \Big/ \big(\big\{ x^p \,\big|\, x \in \mathfrak{k}_G^\times \big\} \big)$ as claimed, the latter identity being trivial (for $\mathfrak{k}_G^\times$ is Abelian). □

Remarks: (a) This K_G is another invariant for G, but again equivalent to $\mathfrak{k}_G$.

(b) Theorem F(b) is a positive characteristic analogue for $F_\hbar[G] = A_R(G)'$ of the first half of Theorem 2.2(c).

Examples:

(1) Finite Abelian p–groups. Let p be a prime number and $G := \mathbb{Z}_{p^{e_1}} \times \mathbb{Z}_{p^{e_2}} \times \cdots \times \mathbb{Z}_{p^{e_k}}$ ($k, e_1, \dots, e_k \in \mathbb{N}$), with $e_1 \geq e_2 \geq \cdots \geq e_k$. Let $\Bbbk$ be a field with *Char* $(\Bbbk) = p > 0$, and $R := \Bbbk[\hbar]$ as above, so that $\Bbbk[G]_\hbar = R[G]$.

First, $\mathfrak{k}_G$ is Abelian, because G is. Let g_i be a generator of $\mathbb{Z}_{p^{e_i}}$ (for all i), identified with its image in G. Since G is Abelian we have $G_{[n]} = G^{p^n}$ (for all n), and an ordered p-l.c.s.-net is $B := \bigcup_{r \in \mathbb{N}_+} B_r$ with $B_r := \Big\{ g_1^{p^r}, g_2^{p^r}, \dots, g_{j_r}^{p^r} \Big\}$ where j_r is uniquely defined by $e_{j_r} > r$, $e_{j_r+1} \leq r$. Then $\mathfrak{k}_G$ has $\Bbbk$–basis $\big\{ \overline{\eta_{g_i^{p^{s_i}}}} \big\}_{1 \leq i \leq k;\, 0 \leq s_i < e_i}$, and minimal set of generators (as a restricted Lie algebra) $\big\{ \overline{\eta_{g_1}}, \overline{\eta_{g_2}}, \dots, \overline{\eta_{g_k}} \big\}$, for the p–operation of $\mathfrak{k}_G$ is $\big(\overline{\eta_{g_i^{p^s}}}\big)^{[p]} = \overline{\eta_{g_i^{p^{s+1}}}}$, and the order of nilpotency of each $\overline{\eta_{g_i}}$ is exactly p^{e_i}, i.e. the order of g_i. In addition $J^\infty = \{0\}$ so $\Bbbk[G]^\vee = \Bbbk[G]$. The outcome is $\Bbbk[G]^\vee = \Bbbk[G]$ and

$$\widehat{\Bbbk[G]} \;=\; \mathbf{u}(\mathfrak{k}_G) \;=\; U(\mathfrak{k}_G) \Big/ \bigg(\Big\{ \big(\overline{\eta_{g_i^{p^s}}}\big)^p - \overline{\eta_{g_i^{p^{s+1}}}} \Big\}_{1 \leq i \leq k}^{0 \leq s < e_i} \bigcup \Big\{ \big(\overline{\eta_{g_i^{p^{e_i-1}}}}\big)^p \Big\}_{1 \leq i \leq k} \bigg)$$

whence $\widehat{\Bbbk[G]} \cong \Bbbk[x_1, \dots, x_k] \Big/ \Big(\Big\{ x_i^{p^{e_i}} \;\Big|\; 1 \leq i \leq k \Big\} \Big)$, via $\overline{\eta_{g_i^{p^s}}} \mapsto x_i^{p^s}$ (for all i, s).

As for $\Bbbk[G]_\hbar^\vee$, for all $r < e_i$ we have $d\big(g_i^{p^r}\big) = p^r$ and so $\chi_{g_i^{p^r}} = \hbar^{-p^r}\big(g_i^{p^r} - 1\big)$ and $\psi_{g_i^{p^r}} = \hbar^{1-p^r}\big(g_i^{p^r} - 1\big)$; since $G_{[\infty]} = \{1\}$ (or, equivalently, $J^\infty = \{0\}$) and everything is Abelian, from the general theory we conlude that both $\Bbbk[G]_\hbar^\vee$ and $\Big(\Bbbk[G]_\hbar^\vee\Big)'$ are truncated-polynomial algebras, in the $\chi_{g_i^{p^r}}$'s and in the $\psi_{g_i^{p^r}}$'s respectively, namely

$$\Bbbk[G]_\hbar{}^\vee \;=\; \Bbbk[\hbar]\Big[\big\{ \chi_{g_i^{p^s}} \big\}_{1 \leq i \leq k\,;\, 0 \leq s < e_i}\Big] \;\cong\; \Bbbk[\hbar]\big[y_1, \dots, y_k\big] \Big/ \Big(\Big\{ y_i^{p^{e_i}} \;\Big|\; 1 \leq i \leq k \Big\} \Big)$$

$$\Big(\Bbbk[G]_\hbar^\vee\Big)' \;=\; \Bbbk[\hbar]\Big[\big\{ \psi_{g_i^{p^s}} \big\}_{1 \leq i \leq k\,;\, 0 \leq s < e_i}\Big] \;\cong\; \Bbbk[\hbar]\Big[\big\{ z_{i,s} \big\}_{1 \leq i \leq k\,;\, 0 \leq s < e_i}\Big] \Big/ \Big(\Big\{ z_{i,s}{}^p \;\Big|\; 1 \leq i \leq k \Big\} \Big)$$

via the isomorphisms given by $\overline{\chi_{g_i^{p^s}}} \mapsto y_i^{p^s}$ and $\overline{\psi_{g_i^{p^s}}} \mapsto z_{i,s}$ (for all i, s). When $e_1 > 1$ this implies $\big(\Bbbk[G]_\hbar^\vee\big)' \supsetneqq \Bbbk[G]_\hbar$, that is a *counterexample* to Theorem 2.2*(b)*. Setting $\overline{\psi_{g_i^{p^s}}} := \psi_{g_i^{p^s}}$ mod $\hbar\, \big(\Bbbk[G]_\hbar^\vee\big)'$ (for all $1 \leq i \leq k$, $0 \leq s < e_i$) we have

$$F\big[K_G^\star\big] \;=\; \big(\Bbbk[G]_\hbar^\vee\big)'\Big|_{\hbar=0} = \Bbbk\Big[\big\{\overline{\psi_{g_i^{p^s}}}\big\}_{1 \leq i \leq k}^{0 \leq s < e_i}\Big] \;\cong\; \Bbbk\,\Big[\big\{ w_{i,s} \big\}_{1 \leq i \leq k}^{0 \leq s < e_i}\Big] \Big/ \Big(\Big\{ w_{i,s}^p \Big|\, 1 \leq i \leq k \Big\} \Big)$$

(via $\overline{\psi_{g_i^{p^s}}} \mapsto w_{i,s}$) as a $\Bbbk$–algebra. The Poisson bracket trivial, and the $w_{i,s}$'s are primitive for $s > 1$ and $\Delta(w_{i,1}) = w_{i,1} \otimes 1 + 1 \otimes w_{i,1} + w_{i,1} \otimes w_{i,1}$ for all $1 \leq i \leq k$. If instead $e_1 = \cdots = e_k = 1$, then $\big(\Bbbk[G]_\hbar^\vee\big)' = \Bbbk[G]_\hbar$; this is an analogue of Theorem 2.2*(b)*, though now *Char* $(\Bbbk) > 0$, in that $\Bbbk[G]_\hbar$ is a QFA, with $\Bbbk[G]_\hbar\Big|_{\hbar=0} = \Bbbk[G] = F\big[\,\widehat{G}\,\big]$ where $\widehat{G}$ is the *group of characters* of G. But then $F\big[\,\widehat{G}\,\big] = \Bbbk[G] = \Bbbk[G]_\hbar\Big|_{\hbar=0} = \big(\Bbbk[G]_\hbar^\vee\big)'\Big|_{\hbar=0} = F\big[K_G^\star\big]$ (by our general analysis) so $\widehat{G}$ can be realized as a finite, connected, Poisson group-scheme of dimension 0 and height 1 dual to $\mathfrak{k}_G$, namely $K_G^\star$.

Finally, a direct easy calculation shows that — letting $\chi_g^* := \hbar^{\,d(g)}\,(\varphi_g - \varphi_1) \in A_\Bbbk(G)_\hbar'$ and $\psi_g^* := \hbar^{\,d(g)-1}\,(\varphi_g - \varphi_1) \in \big(A_\Bbbk(G)'\big)_\hbar^\vee$ (for all $g \in G \setminus \{1\}$) — we have also

$$A_\Bbbk(G)_\hbar' = \Bbbk[\hbar]\Big[\big\{\chi^*_{g_i^{p^s}}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big] \cong \Bbbk[\hbar]\Big[\big\{Y_{i,j}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big]\Big/\Big(\big\{Y_{i,j}^{\,p}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big)$$

$$\big(A_\Bbbk(G)_\hbar'\big)^\vee = \Bbbk[\hbar]\Big[\big\{\psi^*_{g_i^{p^s}}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big] \cong \Bbbk[\hbar]\Big[\big\{Z_{i,s}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big]\Big/\Big(\big\{Z_{i,s}^{\,p} - Z_{i,s}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big)$$

via the isomorphisms given by $\chi^*_{g_i^{p^s}} \mapsto Y_{i,s}$ and $\psi^*_{g_i^{p^s}} \mapsto Z_{i,s}$, from which one also gets the analogous descriptions of $A_\Bbbk(G)_\hbar'\Big|_{\hbar=0} = \widetilde{A_\Bbbk(G)} = F[K_G]$ and of $\big(A_\Bbbk(G)_\hbar'\big)^\vee\Big|_{\hbar=0} = \mathbf{u}(\mathfrak{k}_G^\times)$.

(2) A non-Abelian p–group. Let p be a prime number, $\Bbbk$ be a field with *Char* $(\Bbbk) = p > 0$, and $R := \Bbbk[\hbar]$ as above, so that $\Bbbk[G]_\hbar = R[G]$.

Let $G := \mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$, that is the group with generators ν, τ and relations $\nu^p = 1$, $\tau^{p^2} = 1$, $\nu\,\tau\,\nu^{-1} = \tau^{1+p}$. In this case, $G_{[2]} = \cdots = G_{[p]} = \big\{1, \tau^p\big\}$, $G_{[p+1]} = \{1\}$, so we can take $B_1 = \{\nu\,,\tau\}$ and $B_p = \big\{\tau^p\big\}$ to form an ordered p-l.c.s.-net $B := B_1 \cup B_p$ w.r.t. the ordering $\nu \preceq \tau \preceq \tau^p$. Noting also that $J^\infty = \{0\}$ (for $G_{[\infty]} = \{1\}$), we have

$$\Bbbk[G]_\hbar^\vee = \bigoplus\nolimits_{a,b,c=0}^{p-1} \Bbbk[\hbar]\cdot \chi_\nu^a\,\chi_\tau^b\,\chi_{\tau^p}^c = \bigoplus\nolimits_{a,b,c=0}^{p-1} \Bbbk[\hbar]\,\hbar^{-a-b-c\,p}\cdot(\nu-1)^a\,(\tau-1)^b\,\big(\tau^p-1\big)^c$$

as $\Bbbk[\hbar]$–modules, since $d(\nu) = 1 = d(\tau)$ and $d\big(\tau^p\big)) = p$, with $\Delta(\chi_g) = \chi_g \otimes 1 + 1 \otimes \chi_g + \hbar^{d(g)}\,\chi_g \otimes \chi_g$ for all $g \in B$. As a direct consequence we have also

$$\bigoplus\nolimits_{a,b,c=0}^{p-1} \Bbbk\cdot\overline{\chi_\nu}^{\,a}\,\overline{\chi_\tau}^{\,b}\,\overline{\chi_{\tau^p}}^{\,c} = \Bbbk[G]_\hbar^\vee\Big|_{\hbar=0} \cong \widehat{\Bbbk[G]} = \bigoplus\nolimits_{a,b,c=0}^{p-1} \Bbbk\cdot\overline{\eta_\nu}^{\,a}\,\overline{\eta_\tau}^{\,b}\,\overline{\eta_{\tau^p}}^{\,c}\,.$$

The two relations $\nu^p = 1$ and $\tau^{p^2} = 1$ within G yield trivial relations inside $\Bbbk[G]$ and $\Bbbk[G]_\hbar$; instead, the relation $\nu\,\tau\,\nu^{-1} = \tau^{1+p}$ turns into $[\eta_\nu, \eta_\tau] = \eta_{\tau^p}\cdot\tau\,\nu$, which gives $[\chi_\nu, \chi_\tau] = \hbar^{p-2}\,\chi_{\tau^p}\cdot\tau\,\nu$ in $\Bbbk[G]_\hbar^\vee$. Therefore $\big[\,\overline{\chi_\nu}\,,\overline{\chi_\tau}\,\big] = \delta_{p,2}\,\overline{\chi_{\tau^p}}$. Since $\big[\,\overline{\chi_\tau}\,,\overline{\chi_{\tau^p}}\,\big] = 0 = \big[\,\overline{\chi_\nu}\,,\overline{\chi_{\tau^p}}\,\big]$ (because $\nu\,\tau^p\,\nu^{-1} = \big(\tau^{1+p}\big)^p = \tau^{p+p^2} = \tau^p$) and $\big\{\overline{\chi_\nu}\,,\overline{\chi_\tau}\,,\overline{\chi_{\tau^p}}\big\}$ is a $\Bbbk$–basis of $\mathfrak{k}_G = \mathcal{L}_p(G)$, we conclude that the latter has trivial or non-trivial Lie bracket according to whether $p \neq 2$ or $p = 2$. In addition, we have the relations $\chi_\nu^{\,p} = 0$, $\chi_{\tau^p}^{\,p} = 0$ and $\chi_\tau^{\,p} = \chi_{\tau^p}$: these give analogous relations in $\Bbbk[G]_\hbar^\vee\Big|_{\hbar=0}$, which define the p–operation of $\mathfrak{k}_G$, namely $\overline{\chi_\nu}^{\,[p]} = 0$, $\overline{\chi_{\tau^p}}^{\,[p]} = 0$, $\overline{\chi_\tau}^{\,[p]} = \chi_{\tau^p}$.

To sum up, we have a complete presentation for $R[G]^\vee$ by generators and relations, that is

$$\Bbbk[G]_\hbar^\vee \cong \Bbbk[\hbar]\big\langle v_1, v_2, v_3\big\rangle\Big/\left(\begin{matrix} v_1\,v_2 - v_2\,v_1 - \hbar^{p-2}\,v_3\,(1+\hbar\,v_2)\,(1+\hbar\,v_1) \\ v_1\,v_3 - v_3\,v_1\,, \quad v_1^{\,p}\,, \quad v_2^{\,p} - v_3\,, \quad v_3^{\,p}\,, \quad v_2\,v_3 - v_3\,v_2 \end{matrix}\right)$$

via $\chi_\nu \mapsto v_1$, $\chi_\tau \mapsto v_2$, $\chi_{\tau^p} \mapsto v_3$. Similarly (as a consequence) we have the presentation

$$\widehat{\Bbbk[G]} = \Bbbk[G]_\hbar^\vee\Big|_{\hbar=0} \cong \Bbbk\,\big\langle y_1, y_2, y_3\big\rangle\Big/\left(\begin{matrix} y_1\,y_2 - y_2\,y_1 - \delta_{p,2}\,y_3\,, \qquad y_2^{\,p} - y_3 \\ y_1\,y_3 - y_3\,y_1\,, \quad y_1^{\,p}\,, \quad y_3^{\,p}\,, \quad y_2\,y_3 - y_3\,y_2 \end{matrix}\right)$$

via $\overline{\chi_\nu} \mapsto y_1$, $\overline{\chi_\tau} \mapsto y_2$, $\overline{\chi_{\tau^p}} \mapsto y_3$, with p–operation as above and the y_i's being primitive.

Remark: if $p \neq 2$ exactly the same result holds for $G = \mathbb{Z}_p \times \mathbb{Z}_{p^2}$, i.e. $\mathfrak{k}_{\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}} = \mathfrak{k}_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$: this shows that the restricted Lie bialgebra $\mathfrak{k}_G$ may be not enough to recover the group G.

As for $\big(\Bbbk[G]_\hbar{}^\vee\big)'$, it is generated by $\psi_\nu = \nu - 1$, $\psi_\tau = \tau - 1$, $\psi_{\tau^p} = \hbar^{1-p}\big(\tau^p - 1\big)$, with relations $\psi_\nu^{\,p} = 0$, $\psi_\tau^{\,p} = \hbar^{p-1}\psi_{\tau^p}$, $\psi_{\tau^p}^{\,p} = 0$, $\psi_\nu\,\psi_\tau - \psi_\tau\,\psi_\nu = \hbar^{p-1}\psi_{\tau^p}\,(1+\psi_\tau)\,(1+\psi_\nu)$, $\psi_\tau\,\psi_{\tau^p} - \psi_{\tau^p}\,\psi_\tau = 0$, and $\psi_\nu\,\psi_{\tau^p} - \psi_{\tau^p}\,\psi_\nu = 0$. In particular $\big(\Bbbk[G]_\hbar{}^\vee\big)' \supsetneqq \Bbbk[G]_\hbar$, and

$$\big(\Bbbk[G]_\hbar{}^\vee\big)' \;\cong\; \Bbbk[\hbar]\,\big\langle u_1, u_2, u_3 \big\rangle \bigg/ \left(\begin{matrix} u_1\,u_3 - u_3\,u_1\,, \quad u_2^{\,p} - \hbar^{1-p}\,u_3\,, \quad u_2\,u_3 - u_3\,u_2 \\ u_1^{\,p}\,, \quad u_1\,u_2 - u_2\,u_1 - \hbar^{p-1}\,u_3\,(1+u_2)\,(1+u_1)\,, \quad u_3^{\,p} \end{matrix} \right)$$

via $\psi_\nu \mapsto u_1$, $\psi_\tau \mapsto u_2$, $\psi_{\tau^p} \mapsto u_3$. Letting $z_1 := \psi_\nu\big|_{\hbar=0}+1$, $z_2 := \psi_\tau\big|_{\hbar=0}+1$ and $x_3 := \psi_{\tau^p}\big|_{\hbar=0}$ this gives $\big(\Bbbk[G]_\hbar{}^\vee\big)'\Big|_{\hbar=0} = \Bbbk\big[z_1, z_2, x_3\big] \Big/ \big(z_1^{\,p}-1, z_2^{\,p}-1, x_3^{\,p}\big)$ as a $\Bbbk$–algebra, with the z_i's group-like, x_3 primitive (cf. Theorem D *(b)*), and Poisson bracket given by $\big\{z_1, z_2\big\} = \delta_{p,2}\,z_1\,z_2\,x_3$, $\big\{z_2, x_3\big\} = 0$ and $\big\{z_1, x_3\big\} = 0$. Thus $\big(\Bbbk[G]_\hbar{}^\vee\big)'\Big|_{\hbar=0} = F[\varGamma_G]$ with $\varGamma_G \cong \boldsymbol{\mu}_p \times \boldsymbol{\mu}_p \times \boldsymbol{\alpha}_p$ as algebraic groups, with Poisson structure such that *coLie* $(\varGamma_G) \cong \mathfrak{k}_G$.

Since $G_\infty = \{1\}$ the general theory ensures that $A_\Bbbk(G)' = A_\Bbbk(G)$. We leave to the interested reader the task of computing the filtration $\underline{D}$ of $A_\Bbbk(G)$, and consequently describe $A_R(G)'$, $\big(A_R(G)'\big)^\vee$, $\widetilde{A_\Bbbk(G)}$ and the connected Poisson group $K_G :=$ *Spec* $\big(\widetilde{A_\Bbbk(G)}\big)$.

(3) An Abelian infinite group. Let $G = \mathbb{Z}^n$ (written multiplicatively with generators $e_1, \dots, e_n$), then $\Bbbk[G] = \Bbbk[\mathbb{Z}^n] = \Bbbk\big[e_1^{\pm 1}, \dots, e_n^{\pm 1}\big]$ (the ring of Laurent polynomials). This is the function algebra of the algebraic group $\mathbb{G}_m^{\;n}$ — the n–dimensional torus on $\Bbbk$ — which is exactly the character group of $\mathbb{Z}^n$, thus we get back to the function algebra case.

§ 4 First example: the Kostant-Kirillov structure

4.1 Classical and quantum setting. Let $\mathfrak{g}$ and $\mathfrak{g}^\star$ be as in §3.7, consider $\mathfrak{g}$ as a Lie bialgebra with trivial Lie cobracket and look at $\mathfrak{g}^\star$ as its dual Poisson group, whose Poisson structure then is exactly the Kostant-Kirillov one. Take as ground ring $R := \Bbbk[\nu]$ (a PID, hence a 1dD): we shall consider the primes $\hbar = \nu$ and $\hbar = \nu - 1$, and we'll find quantum groups at either of them for both $\mathfrak{g}$ and $\mathfrak{g}^\star$.

To begin with, we assume *Char* $(\Bbbk) = 0$, and postpone to §4.4 the case *Char* $(\Bbbk) > 0$.

Let $\mathfrak{g}_\nu := \mathfrak{g}[\nu] = \Bbbk[\nu] \otimes_\Bbbk \mathfrak{g}$, endow it with the unique $\Bbbk[\nu]$–linear Lie bracket $[\ ,\ ]_\nu$ given by $[x,y]_\nu := \nu\,[x,y]$ for all $x, y \in \mathfrak{g}$, and define

$$H := U_{\Bbbk[\nu]}(\mathfrak{g}_\nu) = T_{\Bbbk[\nu]}(\mathfrak{g}_\nu) \Big/ \Big(\big\{\, x \cdot y - y \cdot x - \nu\,[x,y] \;\big|\; x, y \in \mathfrak{g} \,\big\} \Big)$$

the universal enveloping algebra of the Lie $\Bbbk[\nu]$–algebra $\mathfrak{g}_\nu$, endowed with its natural structure of Hopf algebra. Then H is a free $\Bbbk[\nu]$–algebra, so that $H \in \mathcal{HA}$ and $H_F := \Bbbk(\nu) \otimes_{\Bbbk[\nu]} H \in \mathcal{HA}_F$ (see §1.3); its specializations at $\nu = 1$ and at $\nu = 0$ are $H \Big/ (\nu - 1)\,H = U(\mathfrak{g})$, as a *co-Poisson* Hopf algebra, and $H \Big/ \nu\,H = S(\mathfrak{g}) = F[\mathfrak{g}^\star]$, as a *Poisson* Hopf algebra. In a more suggesting way, we can also express this with notation like $H \xrightarrow{\nu\to 1} U(\mathfrak{g})$, $H \xrightarrow{\nu\to 0} F[\mathfrak{g}^\star]$. So H *is a QrUEA at* $\hbar := (\nu - 1)$ *and a QFA at* $\hbar := \nu$; so we'll consider Drinfeld's functors for H at $(\nu - 1)$ and at (ν).

4.2 Drinfeld's functors at (ν)**.** Let $(\ )^{\vee(\nu)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ and $(\ )'^{(\nu)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ be the Drinfeld's functors at $(\nu)\ \big(\in \mathit{Spec}\big(\Bbbk[\nu]\big)\big)$. By definitions $J := \mathit{Ker}\big(\epsilon : H \longrightarrow \Bbbk[\nu]\big)$ is nothing but the 2-sided ideal of $H := U(\mathfrak{g}_\nu)$ generated by $\mathfrak{g}_\nu$ itself; thus $H^{\vee(\nu)}$, which by definition is the unital $\Bbbk[\nu]$–subalgebra of H_F generated by $J^{\vee(\nu)} := \nu^{-1} J$, is just the unital $\Bbbk[\nu]$–subalgebra of H_F generated by ${\mathfrak{g}_\nu}^{\vee(\nu)} := \nu^{-1}\mathfrak{g}_\nu$. Now consider the $\Bbbk[\nu]$–module isomorphism $(\ )^{\vee(\nu)} : \mathfrak{g}_\nu \xrightarrow{\cong} {\mathfrak{g}_\nu}^{\vee(\nu)} := \nu^{-1}\mathfrak{g}_\nu$ given by $z \mapsto z^\vee := \nu^{-1} z \in {\mathfrak{g}_\nu}^{\vee(\nu)}$ for all $z \in \mathfrak{g}_\nu$; consider on $\mathfrak{g}_\nu := \Bbbk[\nu] \otimes_\Bbbk \mathfrak{g}$ the natural Lie algebra structure (with trivial Lie cobracket), given by scalar extension from $\mathfrak{g}$, and push it over ${\mathfrak{g}_\nu}^{\vee(\nu)}$ via $(\ )^{\vee(\nu)}$, so that ${\mathfrak{g}_\nu}^{\vee(\nu)}$ is isomorphic to $\mathfrak{g}_\nu^{\mathit{nat}}$ (i.e. $\mathfrak{g}_\nu$ carrying the natural Lie bialgebra structure) as a Lie bialgebra. Consider $x^\vee, y^\vee \in {\mathfrak{g}_\nu}^{\vee(\nu)}$ (with $x, y \in \mathfrak{g}_\nu$): then $H^{\vee(\nu)} \ni \big(x^\vee y^\vee - y^\vee x^\vee\big) = \nu^{-2}\big(x\,y - y\,x\big) = \nu^{-2}\,[x,y]_\nu = \nu^{-2}\,\nu\,[x,y] = \nu^{-1}\,[x,y] = [x,y]^\vee =: \big[x^\vee, y^\vee\big] \in {\mathfrak{g}_\nu}^{\vee(\nu)}$. Therefore we can conclude at once that $H^{\vee(\nu)} = U\big({\mathfrak{g}_\nu}^{\vee(\nu)}\big) \cong U\big(\mathfrak{g}_\nu^{\mathit{nat}}\big)$.

As a first consequence, $\big(H^{\vee(\nu)}\big)\Big|_{\nu=0} \cong U\big(\mathfrak{g}_\nu^{\mathit{nat}}\big)\Big/\nu\, U\big(\mathfrak{g}_\nu^{\mathit{nat}}\big) = U\Big(\mathfrak{g}_\nu^{\mathit{nat}}\big/\nu\,\mathfrak{g}_\nu^{\mathit{nat}}\Big) = U(\mathfrak{g})$, that is $H^{\vee(\nu)} \xrightarrow{\nu\to 0} U(\mathfrak{g})$, thus agreeing with the second half of Theorem 2.2*(c)*.

Second, look at $\big(H^{\vee(\nu)}\big)'^{(\nu)}$. Since $H^{\vee(\nu)} = U\big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)$, and $\delta_n(\eta) = 0$ for all $\eta \in U\big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)$ such that $\partial(\eta) < n$ (cf. Lemma 4.2*(d)* in [Ga5]), it is easy to see that

$$\big(H^{\vee(\nu)}\big)'^{(\nu)} = \big\langle \nu\, {\mathfrak{g}_\nu}^{\vee(\nu)} \big\rangle = \big\langle \nu\,\nu^{-1}\mathfrak{g}_\nu \big\rangle = U(\mathfrak{g}_\nu) = H$$

(hereafter $\langle S \rangle$ is the subalgebra generated by S), so $\big(H^{\vee(\nu)}\big)'^{(\nu)} = H$, which agrees with Theorem 2.2*(b)*. Finally, proceeding as in §3.7 we see that $H'^{(\nu)} = U(\nu\,\mathfrak{g}_\nu)$, whence $\big(H'^{(\nu)}\big)\Big|_{\nu=0} = \big(U(\nu\,\mathfrak{g}_\nu)\big)\Big|_{\nu=0} \cong S(\mathfrak{g}_{ab}) = F\big[\mathfrak{g}^\star_{\delta-ab}\big]$ where $\mathfrak{g}_{ab}$, resp. $\mathfrak{g}^\star_{\delta-ab}$, is simply $\mathfrak{g}$, resp. $\mathfrak{g}^\star$, endowed with the trivial Lie bracket, resp. cobracket, so that $\big(H'^{(\nu)}\big)\Big|_{\nu=0} \cong S(\mathfrak{g}_{ab}) = F\big[\mathfrak{g}^\star_{\delta-ab}\big]$ has trivial Poisson bracket. Iterating this procedure one finds that all further images $\Big(\cdots\big((H)'^{(\nu)}\big)'^{(\nu)}\cdots\Big)'^{(\nu)}$ of the functor $(\ )'^{(\nu)}$ applied many times to H are pairwise isomorphic; thus in particular they all have the same specialization at (ν), namely $\left(\Big(\cdots\big((H)'^{(\nu)}\big)'^{(\nu)}\cdots\Big)'^{(\nu)}\right)\Bigg|_{\nu=0} \cong S(\mathfrak{g}_{ab}) = F\big[\mathfrak{g}^\star_{\delta-ab}\big]$.

4.3 Drinfeld's functors at $(\nu - 1)$**.** Now we consider the non-zero prime $(\nu - 1)\ \big(\in \mathit{Spec}\big(\Bbbk[\nu]\big)\big)$; let $(\ )^{\vee(\nu-1)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ and $(\ )'^{(\nu-1)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ be the corresponding Drinfeld's functors. Set ${\mathfrak{g}_\nu}'^{(\nu-1)} := (\nu - 1)\,\mathfrak{g}_\nu$, let $: \mathfrak{g}_\nu \xrightarrow{\cong} {\mathfrak{g}_\nu}'^{(\nu-1)} := (\nu - 1)\,\mathfrak{g}_\nu$ be the $\Bbbk[\nu]$–module isomorphism given by $z \mapsto z' := (\nu - 1)\, z \in {\mathfrak{g}_\nu}'^{(\nu-1)}$ for all $z \in \mathfrak{g}_\nu$, and push over via it the Lie bialgebra structure of $\mathfrak{g}_\nu$ to an isomorphic Lie bialgebra structure on ${\mathfrak{g}_\nu}'^{(\nu-1)}$, whose Lie bracket will be denoted by $[\ ,\ ]_*$. Notice then that we have Lie bialgebra isomorphisms $\mathfrak{g} \cong \mathfrak{g}_\nu\big/(\nu - 1)\,\mathfrak{g}_\nu \cong {\mathfrak{g}_\nu}'^{(\nu-1)}\big/(\nu - 1)\,{\mathfrak{g}_\nu}'^{(\nu-1)}$.

Since $H := U(\mathfrak{g}_\nu)$ it is easy to see by direct computation that

$$H'^{(\nu-1)} = \big\langle (\nu - 1)\,\mathfrak{g}_\nu \big\rangle = U\big({\mathfrak{g}_\nu}'^{(\nu-1)}\big) \tag{4.1}$$

where ${\mathfrak{g}_\nu}'^{(\nu-1)}$ is seen as a Lie $\Bbbk[\nu]$–subalgebra of $\mathfrak{g}_\nu$. Now, if $x', y' \in {\mathfrak{g}_\nu}'^{(\nu-1)}$ (with $x, y \in \mathfrak{g}_\nu$), then

$$x' \, y' - y' \, x' = (\nu-1)^2 \big(x \, y - y \, x \big) = (\nu-1)^2 \, [x,y]_\nu = (\nu-1) \, {[x,y]_\nu}' = (\nu-1) \, \big[x',y'\big]_* \, . \qquad (4.2)$$

This and (4.1) show at once that $\left(H'^{(\nu-1)}\right)\Big|_{(\nu-1)=0} = S\Big({\mathfrak{g}_\nu}'^{(\nu-1)} \Big/ (\nu-1) \, {\mathfrak{g}_\nu}'^{(\nu-1)}\Big)$ as Hopf algebras, and also as *Poisson* algebras: indeed, the latter holds because the Poisson bracket $\{\ ,\ \}$ of $S\Big({\mathfrak{g}_\nu}'^{(\nu-1)} \Big/ (\nu-1) \, {\mathfrak{g}_\nu}'^{(\nu-1)}\Big)$ inherited from $H'^{(\nu-1)}$ (by specialization) is uniquely determined by its restriction to ${\mathfrak{g}_\nu}'^{(\nu-1)} \Big/ (\nu-1) \, {\mathfrak{g}_\nu}'^{(\nu-1)}$, and on the latter space we have $\{\ ,\ \} = [\ ,\ ]_*$ $\mod (\nu-1) \, {\mathfrak{g}_\nu}'^{(\nu-1)}$ (by (4.2)). Finally, since ${\mathfrak{g}_\nu}'^{(\nu-1)} \Big/ (\nu-1) \, {\mathfrak{g}_\nu}'^{(\nu-1)} \cong \mathfrak{g}$ as Lie algebras we have $\left(H'^{(\nu-1)}\right)\Big|_{(\nu-1)=0} = S(\mathfrak{g}) = F[\mathfrak{g}^\star]$ as Poisson Hopf algebras, or, in short, $H'^{(\nu-1)} \xrightarrow{\ \nu \to 1\ } F[\mathfrak{g}^\star]$, as prescribed by the "first half" of Theorem 2.2*(c)*.

Second, look at $\left(H'^{(\nu-1)}\right)^{\vee(\nu-1)}$. Since $H'^{(\nu-1)} = U\big({\mathfrak{g}_\nu}'^{(\nu-1)}\big)$, the kernel $\mathit{Ker}\big(\epsilon : H'^{(\nu-1)} \longrightarrow \Bbbk[\nu]\big) =: J'^{(\nu-1)}$ is just the 2-sided ideal of $H'^{(\nu-1)} = U\big({\mathfrak{g}_\nu}'^{(\nu-1)}\big)$ generated by ${\mathfrak{g}_\nu}'^{(\nu-1)}$. Therefore $\left(H'^{(\nu-1)}\right)^{\vee(\nu-1)}$, generated by $\left(J'^{(\nu-1)}\right)^{\vee(\nu-1)} := (\nu-1)^{-1} J'^{(\nu-1)}$ as a unital $\Bbbk[\nu]$–subalgebra of $\left(H'^{(\nu-1)}\right)_F = H_F$, is just the unital $\Bbbk[\nu]$–subalgebra of H_F generated by $(\nu-1)^{-1} {\mathfrak{g}_\nu}'^{(\nu-1)} = (\nu-1)^{-1} (\nu-1) \, \mathfrak{g}_\nu = \mathfrak{g}_\nu$, that is $\left(H'^{(\nu-1)}\right)^{\vee(\nu-1)} = U(\mathfrak{g}_\nu) = H$, confirming Theorem 2.2*(b)*.

Finally, for $H^{\vee(\nu-1)}$ one has essentially the same feature as in §3.7, and the analysis therein can be repeated; the final result then will depend on the nature of $\mathfrak{g}$, in particular on its lower central series.

4.4 The case of positive characteristic. Let us consider now a field $\Bbbk$ such that *Char* $(\Bbbk) = p > 0$. Starting from $\mathfrak{g}$ and $R := \Bbbk[\nu]$ as in §4.1, define $\mathfrak{g}_\nu$ like therein, and consider $H := U_{\Bbbk[\nu]}(\mathfrak{g}_\nu) = U_R(\mathfrak{g}_\nu)$. Then we have $H \Big/ (\nu-1) \, H = U(\mathfrak{g}) = \mathbf{u}\Big(\mathfrak{g}^{[p]^\infty}\Big)$ as a *co-Poisson* Hopf algebra and $H \Big/ \nu \, H = S(\mathfrak{g}) = F[\mathfrak{g}^\star]$ as a *Poisson* Hopf algebra; therefore H is a QrUEA at $\hbar := (\nu-1)$ (for $\mathbf{u}\Big(\mathfrak{g}^{[p]^\infty}\Big)$) and is a QFA at $\hbar := \nu$ (for $F[\mathfrak{g}^\star]$). Now we go and study Drinfeld's functors for H at $(\nu-1)$ and at (ν).

Exactly the same procedure as before shows again that $H^{\vee(\nu)} = U\big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)$, from which it follows that $\left(H^{\vee(\nu)}\right)\Big|_{\nu=0} \cong U(\mathfrak{g})$, i.e. in short $H^{\vee(\nu)} \xrightarrow{\ \nu \to 0\ } U(\mathfrak{g})$, which is a result quite "parallel" to the second half of Theorem 2.2*(c)*. Changes occur when looking at $\left(H^{\vee(\nu)}\right)'^{(\nu)}$: since $H^{\vee(\nu)} = U\big({\mathfrak{g}_\nu}^{\vee(\nu)}\big) = \mathbf{u}\Big(\big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)^{[p]^\infty}\Big)$ we have $\delta_n(\eta) = 0$ for all $\eta \in \mathbf{u}\Big(\big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)^{[p]^\infty}\Big)$ such that $\partial(\eta) < n$ w.r.t. the standard filtration of $\mathbf{u}\Big(\big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)^{[p]^\infty}\Big)$ (cf. the proof of Lemma 4.2*(d)* in [Ga5], which clearly adapts to the present situation): this implies

$$\left(H^{\vee(\nu)}\right)'^{(\nu)} = \Big\langle \nu \cdot \big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)^{[p]^\infty} \Big\rangle \qquad \Big(\subset \mathbf{u}\Big(\nu \cdot \big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)^{[p]^\infty} \Big) \Big)$$

which is strictly *bigger* than H, because we have $\Big\langle \nu \cdot \big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)^{[p]^\infty} \Big\rangle = \Big\langle \sum_{n \geq 0} \nu \cdot \big({\mathfrak{g}_\nu}^{\vee(\nu)}\big)^{[p]^n} \Big\rangle =$
$= \Big\langle \mathfrak{g}_\nu + \nu^{1-p} \left\{ x^p \,\middle|\, x \in \mathfrak{g}_\nu \right\} + \nu^{1-p^2} \Big\{ x^{p^2} \,\Big|\, x \in \mathfrak{g}_\nu \Big\} + \cdots \Big\rangle \supsetneqq U(\mathfrak{g}_\nu) = H$.

Finally, proceeding as above it is easy to see that $H'^{(\nu)} = \Big\langle \nu \, P\big(U(\mathfrak{g}_\nu)\big) \Big\rangle = \Big\langle \nu \, \mathfrak{g}^{[p]^\infty} \Big\rangle$ whence, letting $\tilde{\mathfrak{g}} := \nu \, \mathfrak{g}$ and $\tilde{x} := \nu \, x$ for all $x \in \mathfrak{g}$, we have

$$H'^{(\nu)} \; = \; T_R(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{ \, \tilde{x} \, \tilde{y} - \tilde{y} \, \tilde{x} - \nu^2 \, \widetilde{[x,y]} \, , \, \tilde{z}^p - \nu^{p-1} \widetilde{z^{[p]}} \; \Big| \; x, y, z \in \mathfrak{g} \, \Big\} \Big)$$

so that $H'^{(\nu)} \xrightarrow{\nu \to 0} T_{\Bbbk}(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{ \, \tilde{x} \, \tilde{y} - \tilde{y} \, \tilde{x} \, , \, \tilde{z}^p \; \Big| \; \tilde{x}, \tilde{y}, \tilde{z} \, \in \, \tilde{\mathfrak{g}} \, \Big\} \Big) \; = \; S_{\Bbbk}(\mathfrak{g}_{ab}) \big/ \big(\big\{ \, z^p \; \big| \; z \, \in \, \mathfrak{g} \, \big\} \big) \; = \; = \; F[\mathfrak{g}^\star_{\delta-ab}] \big/ \big(\big\{ \, z^p \; \big| \; z \, \in \, \mathfrak{g} \, \big\} \big)$, that is $H'^{(\nu)}\Big|_{\nu=0} \cong F[\mathfrak{g}^\star_{\delta-\mathrm{ab}}] \big/ \big(\big\{ \, z^p \; \big| \; z \, \in \, \mathfrak{g} \, \big\} \big)$ *as Poisson Hopf algebras*, where $\mathfrak{g}_{ab}$ and $\mathfrak{g}^\star_{\delta-ab}$ are as above. Therefore $H'^{(\nu)}$ is a QFA (at $\hbar = \nu$) for a non-reduced, zero-dimensional algebraic Poisson group of height 1, whose cotangent Lie bialgebra is the vector space $\mathfrak{g}$ with trivial Lie bialgebra structure: this again yields somehow an analogue of part *(c)* of Theorem 2.2 for the present case. If we iterate, we find that all further images $\Big(\cdots \big((H)'^{(\nu)} \big)'^{(\nu)} \cdots \Big)'^{(\nu)}$ of the functor $(\ )'^{(\nu)}$ applied to H are pairwise isomorphic, so that

$$\Big(\cdots \big((H)'^{(\nu)} \big)'^{(\nu)} \cdots \Big)'^{(\nu)} \bigg|_{\nu=0} \; \cong \; S(\mathfrak{g}_{ab}) \big/ \big(\big\{ \, z^p \; \big| \; z \in \mathfrak{g} \, \big\} \big) = F\big[\mathfrak{g}^\star_{\delta-ab}\big] \big/ \big(\big\{ \, z^p \; \big| \; z \in \mathfrak{g} \, \big\} \big) \; .$$

Now for Drinfeld's functors at $(\nu - 1)$. Up to minor changes, with the same procedure and notations as in §4.3 we get analogous results. First of all, a result analogous to (4.1) holds, namely $H'^{(\nu-1)} = \Big\langle (\nu - 1) \cdot P\big(U(\mathfrak{g}_\nu)\big) \Big\rangle = \Big\langle (\nu - 1) \, (\mathfrak{g}_\nu)^{[p]^\infty} \Big\rangle = \Big\langle \Big((\mathfrak{g}_\nu)^{[p]^\infty} \Big)'^{(\nu-1)} \Big\rangle$, which yields

$$H'^{(\nu-1)} \; = \; T_R \bigg(\Big((\mathfrak{g}_\nu)^{[p]^\infty} \Big)'^{(\nu-1)} \bigg) \bigg/ \bigg(\Big\{ \, x' \, y' - y' \, x' - (\nu - 1) \, \big[x', y'\big]_* \, , \, (x')^p - (\nu - 1)^{p-1} \big(x^{[p]}\big)' \; \Big| \; x, y \in (\mathfrak{g}_\nu)^{[p]^\infty} \, \Big\} \bigg)$$

and consequently $H'^{(\nu-1)}\Big|_{(\nu-1)=0} \cong S_{\Bbbk}(\mathfrak{g}) \big/ \big(\big\{ \, x^p \; \big| \; x \in \mathfrak{g} \, \big\} \big) \; = \; F[\mathfrak{g}^\star] \big/ \big(\big\{ \, x^p \; \big| \; x \in \mathfrak{g} \, \big\} \big)$ as Poisson Hopf algebras: in a nutshell, $H'^{(\nu-1)} \xrightarrow{\nu \to 1} F[\mathfrak{g}^\star] \big/ \big(\big\{ \, x^p \; \big| \; x \in \mathfrak{g} \, \big\} \big)$.

Iterating, one finds again that all $\Big(\cdots \big((H)'^{(\nu)} \big)'^{(\nu-1)} \cdots \Big)'^{(\nu)}$ are pairwise isomorphic, so

$$\Big(\cdots \big((H)'^{(\nu-1)} \big)'^{((\nu-1)} \cdots \Big)'^{(\nu-1)} \bigg|_{(\nu-1)=0} \; \cong \; S(\mathfrak{g}_{ab}) \big/ \big(\big\{ z^p \; \big| \; z \in \mathfrak{g} \big\} \big) = F\big[\mathfrak{g}^\star_{\delta-ab}\big] \big/ \big(\big\{ z^p \; \big| \; z \in \mathfrak{g} \, \big\} \big) \; .$$

Further on, one has $\big(H'^{(\nu-1)}\big)^{\vee(\nu-1)} \; = \; \Big\langle (\nu - 1) \, (\mathfrak{g}_\nu)^{[p]^\infty} \Big\rangle^{\vee(\nu-1)} \; = \; \big\langle (\nu - 1)^{-1} \cdot (\nu - 1) \, \mathfrak{g}_\nu \big\rangle = = \big\langle \mathfrak{g}_\nu \big\rangle = U_R(\mathfrak{g}_\nu) =: H$, which perfectly agrees with Theorem 2.2*(b)*. Finally, for $H^{\vee(\nu-1)}$ one has again the same feature as in §3.7: one has to apply the analysis therein, however the p–filtration in this case is "harmless", since it is "encoded" in the standard filtration of $U(\mathfrak{g})$. In any case the final result will depend on the lower central series of $\mathfrak{g}$.

Second, we assume in addition that $\mathfrak{g}$ be a *restricted* Lie algebra and consider $H := \mathbf{u}_{\Bbbk[\nu]}(\mathfrak{g}_\nu) = \mathbf{u}_R(\mathfrak{g}_\nu)$. In this case we have $H \Big/ (\nu - 1) \, H \; = \; \mathbf{u}(\mathfrak{g})$ as a *co-Poisson* Hopf algebra, and $H \Big/ \nu \, H \; = \; S(\mathfrak{g}) \big/ \big(\big\{ \, z^p \; \big| \; z \in \mathfrak{g} \, \big\} \big) \; = \; F[\mathfrak{g}^\star] \big/ \big(\big\{ \, z^p \; \big| \; z \in \mathfrak{g} \, \big\} \big)$ as a *Poisson* Hopf algebra, which means that H

is a QrUEA at $\hbar := (\nu - 1)$ (for $\mathbf{u}(\mathfrak{g})$) and is a QFA at $\hbar := \nu$ (for $F[\mathfrak{g}^\star]\Big/\big(\{\, z^p \mid z \in \mathfrak{g} \,\}\big)$). Then we go and study Drinfeld's functors for H at $(\nu - 1)$ and at (ν) .

As for $H^{\vee(\nu)}$, it depends again on the p–operation of $\mathfrak{g}$, in short because the I–filtration of $\mathbf{u}_\nu(\mathfrak{g})$ depends on the p–filtration of $\mathfrak{g}$. In the previous case — i.e. when $\mathfrak{g} = \mathfrak{h}^{[p]^\infty}$ for some Lie algebra $\mathfrak{h}$ — the solution was plain, because the p–filtration of $\mathfrak{g}$ is "encoded" in the standard filtration of $U(\mathfrak{h})$; but the general case will be more complicated, and in consequence also the situation for $\big(H^{\vee(\nu)}\big)'^{(\nu)}$, since $H^{\vee(\nu)}$ will be different according to the nature of $\mathfrak{g}$. Instead, proceeding exactly like before one finds $H'^{(\nu)} = \Big\langle \nu\, P\big(u(\mathfrak{g}_\nu)\big) \Big\rangle = \langle \nu\, \mathfrak{g} \rangle$, whence, letting $\tilde{\mathfrak{g}} := \nu\, \mathfrak{g}$ and $\tilde{x} := \nu\, x$ for all $x \in \mathfrak{g}$, we have

$$H'^{(\nu)} \;=\; T_{\Bbbk[\nu]}(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{\, \tilde{x}\, \tilde{y} - \tilde{y}\, \tilde{x} - \nu^2\, \widetilde{[x,y]} \,,\; \tilde{z}^p - \nu^{p-1} \widetilde{z^{[p]}} \;\Big|\; x,y,z \in \mathfrak{g} \,\Big\} \Big)$$

so that $H'^{(\nu)} \xrightarrow{\;\nu \to 0\;} T_\Bbbk(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{\, \tilde{x}\, \tilde{y} - \tilde{y}\, \tilde{x} \,,\; \tilde{z}^p \;\Big|\; \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{\mathfrak{g}} \,\Big\} \Big) = S_\Bbbk(\mathfrak{g}_{ab}) \Big/ \big(\{\, z^p \mid z \in \mathfrak{g} \,\}\big) = = F[\mathfrak{g}^\star_{\delta - ab}] \Big/ \big(\{\, z^p \mid z \in \mathfrak{g} \,\}\big)$, that is $H'^{(\nu)}\Big|_{\nu=0} \cong F[\mathfrak{g}^\star_{\delta-\mathrm{ab}}] \Big/ \big(\{\, z^p \mid z \in \mathfrak{g} \,\}\big)$ *as Poisson Hopf algebras* (using notation as before). Thus $H'^{(\nu)}$ is a QFA (at $\hbar = \nu$) for a non-reduced, zero-dimensional algebraic Poisson group of height 1, whose cotangent Lie bialgebra is $\mathfrak{g}$ with the trivial Lie bialgebra structure: so again we get an analogue of part of Theorem 2.2*(c)*. Moreover, iterating again one finds that all $\Big(\cdots \big((H)'^{(\nu)}\big)'^{(\nu-1)} \cdots \Big)'^{(\nu-1)}$ are pairwise isomorphic, so

$$\Big(\cdots \big((H)'^{(\nu-1)}\big)'^{((\nu-1)} \cdots \Big)'^{(\nu-1)} \bigg|_{(\nu-1)=0} \cong\; S(\mathfrak{g}_{ab}) \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \}\big) \;=\; F\big[\mathfrak{g}^\star_{\delta-ab}\big] \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \,\}\big) \;.$$

As for Drinfeld's functors at $(\nu - 1)$, the situation is more similar to the previous case of $H = U_R(\mathfrak{g}_\nu)$. First $H'^{(\nu-1)} = \Big\langle (\nu - 1) \cdot P\big(\mathbf{u}(\mathfrak{g}_\nu)\big) \Big\rangle = \big\langle (\nu - 1)\, \mathfrak{g}_\nu \big\rangle =: \big\langle {\mathfrak{g}_\nu}'^{(\nu-1)} \big\rangle$, hence

$$H'^{(\nu-1)} = T_R\Big({\mathfrak{g}_\nu}'^{(\nu-1)}\Big) \Big/ \Big(\Big\{\, x'\, y' - y'\, x' - (\nu - 1)\, \big[x', y'\big]_* \,,\; (x')^p - (\nu - 1)^{p-1} \big(x^{[p]}\big)' \Big\}_{x,y \in \mathfrak{g}_\nu} \Big)$$

thus again $H'^{(\nu-1)}\Big|_{(\nu-1)=0} \cong S_\Bbbk(\mathfrak{g}) \Big/ \big(\{\, x^p \mid x \in \mathfrak{g} \,\}\big) = F[\mathfrak{g}^\star] \Big/ \big(\{\, x^p \mid x \in \mathfrak{g} \,\}\big)$ as Poisson Hopf algebras, that is $H'^{(\nu-1)} \xrightarrow{\;\nu \to 1\;} F[\mathfrak{g}^\star] \Big/ \big(\{\, x^p \mid x \in \mathfrak{g} \,\}\big)$. Iteration then shows that all $\Big(\cdots \big((H)'^{(\nu)}\big)'^{(\nu-1)} \cdots \Big)'^{(\nu)}$ are pairwise isomorphic, so that again

$$\Big(\cdots \big((H)'^{(\nu-1)}\big)'^{((\nu-1)} \cdots \Big)'^{(\nu-1)} \bigg|_{(\nu-1)=0} \cong\; S(\mathfrak{g}_{ab}) \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \}\big) \;=\; F\big[\mathfrak{g}^\star_{\delta-ab}\big] \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \,\}\big) \;.$$

Further, we have $\big(H'^{(\nu-1)}\big)^{\vee(\nu-1)} = \big\langle (\nu - 1)\, \mathfrak{g}_\nu \big\rangle^{\vee(\nu-1)} = \big\langle \mathfrak{g}_\nu \big\rangle = \mathbf{u}_R(\mathfrak{g}_\nu) =: H$, which agrees at all with Theorem 2.2*(b)*. Finally, $H^{\vee(\nu-1)}$ again has the same feature as in §3.7: in particular, the outcome strongly depends on the properties of *both* the lower central series *and* of the p–filtration of $\mathfrak{g}$.

4.5 The hyperalgebra case. Let $\Bbbk$ be again a field with *Char* $(\Bbbk) = p > 0$. Like in §3.11, let G be an algebraic group (finite-dimensional, for simplicity), and let *Hyp* $(G) := \big(F[G]^\circ\big)_\epsilon = \big\{\, \phi \in F[G]^\circ \,\big|\, \phi(\mathfrak{m}_e{}^n) = 0 \,,\, \forall\, n \gg 0 \,\big\}$ be the hyperalgebra associated to G (see §1.1).

For each $\nu \in \Bbbk$, let $\mathfrak{g}_\nu := \big(\mathfrak{g}\,,[\;,\;]_\nu\big)$ be the Lie algebra given by $\mathfrak{g}$ endowed with the rescaled Lie bracket $[\;,\;]_\nu := \nu\,[\;,\;]_{\mathfrak{g}}$. By general theory, the algebraic group G is uniquely determined by a neighborhood of the identity together with the formal group law uniquely determined by $[\;,\;]_{\mathfrak{g}}$. Similarly, a neighborhood of the identity of G together with $[\;,\;]_\nu$ uniquely determines a new connected algebraic group G_ν, whose hyperalgebra $\mathit{Hyp}\,(G_\nu)$ is an algebraic deformation of $\mathit{Hyp}\,(G)$; moreover, G_ν is birationally equivalent to G, and for $\nu \neq 0$ they are also isomorphic as algebraic groups, via an isomorphism induced by $\mathfrak{g} \cong \mathfrak{g}_\nu$, $x \mapsto \nu^{-1}x$ (however, this may not be the case when $\nu = 0$). Note that $\mathit{Hyp}\,(G_0)$ is clearly commutative, because G_0 is Abelian: indeed, we have

$$\mathit{Hyp}\,(G_0) \;=\; S_\Bbbk\Big(\mathfrak{g}^{(p)^\infty}\Big)\Big/\Big(\big\{\,x^p\,\big\}_{x\in\mathfrak{g}^{(p)^\infty}}\Big) \;=\; F\Big[\Big(\mathfrak{g}^{(p)^\infty}\Big)^{\star}\Big]\Big/\Big(\big\{\,y^p\,\big\}_{y\in\mathfrak{g}^{(p)^\infty}}\Big)$$

where $\mathfrak{g}^{(p)^\infty} := \mathit{Span}\Big(\Big\{\,x^{(p^n)}\,\Big|\,x\in\mathfrak{g}\,,n\in\mathbb{N}\Big\}\Big)$; here as usual $x^{(n)}$ denotes the n–th divided power of $x \in \mathfrak{g}$ (recall that $\mathit{Hyp}\,(G)$, hence also $\mathit{Hyp}\,(G_\nu)$, is generated as an algebra by all the $x^{(n)}$'s, some of which might be zero). So $\mathit{Hyp}\,(G_0) = F[\varGamma]$ where $\varGamma$ is a connected algebraic group of dimension zero and height 1: moreover, $\varGamma$ is a Poisson group, with cotangent Lie bialgebra $\mathfrak{g}^{(p)^\infty}$ and Poisson bracket induced by the Lie bracket of $\mathfrak{g}$.

Now think at ν as a parameter in $R := \Bbbk[\nu]$ (as in §4.1), and set $H := \Bbbk[\nu] \otimes_\Bbbk \mathit{Hyp}\,(G_\nu)$. Then we find a situation much similar to that of §4.1, which we shall shortly describe.

Namely, H is a free $\Bbbk[\nu]$–algebra, thus $H \in \mathcal{HA}$ and $H_F := \Bbbk(\nu) \otimes_{\Bbbk[\nu]} H \in \mathcal{HA}_F$ (see §1.3); its specialization at $\nu = 1$ is $H\big/(\nu-1)\,H \;=\; \mathit{Hyp}\,(G_1) \;=\; \mathit{Hyp}\,(G)$, and at $\nu = 0$ is $H\big/\nu\,H \;=\; \mathit{Hyp}\,(G_0) \;=\; F[\varGamma]$ (as a *Poisson* Hopf algebra), or $H \xrightarrow{\;\nu\to1\;} \mathit{Hyp}\,(G)$ and $H \xrightarrow{\;\nu\to0\;} F[\varGamma]$, i.e. H is a "quantum hyperalgebra" at $\hbar := (\nu-1)$ and a QFA at $\hbar := \nu$. Now we study Drinfeld's functors for H at $\hbar = (\nu-1)$ and at $\hbar = \nu$.

First, a straightforward analysis like in §4.2 yields $H^{\vee_{(\nu)}} \cong \Bbbk[\nu] \otimes_\Bbbk \mathit{Hyp}\,(G)$ (induced by $\mathfrak{g} \cong \mathfrak{g}_\nu$, $x \mapsto \nu^{-1}x$) whence in particular $\big(H^{\vee_{(\nu)}}\big)\Big|_{\nu=0} \cong \mathit{Hyp}\,(G)$, that is $H^{\vee_{(\nu)}} \xrightarrow{\;\nu\to0\;} \mathit{Hyp}\,(G)$. Second, one can also see (essentially, *mutatis mutandis*, like in §4.2) that $\big(H^{\vee_{(\nu)}}\big)'_{(\nu)} = H$, whence $\big(H^{\vee_{(\nu)}}\big)'_{(\nu)}\Big|_{\nu=0} = H\Big|_{\nu=0} = \mathit{Hyp}\,(G_0) = F[\varGamma]$ follows.

At $\hbar = (\nu-1)$, we can see by direct computation that $H'_{(\nu-1)} = \Big\langle\Big(\mathfrak{g}^{(p)^\infty}\Big)'_{(\nu-1)}\Big\rangle$ where $\Big(\mathfrak{g}^{(p)^\infty}\Big)'_{(\nu-1)} := \mathit{Span}\Big(\Big\{\,(\nu-1)^{p^n}x^{(p^n)}\,\Big|\,x\in\mathfrak{g}\,,n\in\mathbb{N}\Big\}\Big)$. Indeed the structure of $H'_{(\nu-1)}$ depends only on the coproduct of H, in which ν plays no role; therefore we can do the same analysis as in the trivial deformation case (see §3.11): the filtration $\underline{D}$ of $\mathit{Hyp}\,(G_\nu)$ is just the natural filtration given by the order (of divided powers), and this yields the previous description of $H'_{(\nu-1)}$. At $\nu = 1$ we find

$$H'_{(\nu-1)}\Big/(\nu-1)\,H'_{(\nu-1)} \;\cong\; S_\Bbbk\Big(\mathfrak{g}^{(p)^\infty}\Big)\Big/\Big(\big\{\,x^p\,\big\}_{x\in\mathfrak{g}^{(p)^\infty}}\Big) \;=\; \mathit{Hyp}\,(G_0) \;=\; F[\varGamma]$$

as Poisson Hopf algebras: in short, $H'^{(\nu-1)}$ is a QFA, at $\hbar = \nu - 1$, for the Poisson group Γ. Similarly $H'^{(\nu)} = \left\langle \left(\mathfrak{g}^{(p)^\infty}\right)'^{(\nu)} \right\rangle$ with $\left(\mathfrak{g}^{(p)^\infty}\right)'^{(\nu)} := \mathit{Span}\left(\left\{ \nu^{p^n} x^{(p^n)} \,\middle|\, x \in \mathfrak{g}\,,\, n \in \mathbb{N} \right\}\right)$; therefore

$$H'^{(\nu)} \Big/ \nu\, H'^{(\nu)} \;\cong\; S_{\Bbbk}\Big(\mathfrak{g}_{ab}^{(p)^\infty}\Big) \Big/ \Big(\{ x^p \}_{x \in \mathfrak{g}^{(p)^\infty}} \Big) \;=\; F\big[\Gamma_{ab}\big]$$

where $\mathfrak{g}_{ab}$ is simply $\mathfrak{g}$ with trivialized Lie bracket and Γ_{ab} is the *same algebraic group* as Γ but with *trivial* Poisson bracket: this comes essentially like in §4.2, roughly because $\left\{ \overline{\nu\, x}\,, \overline{\nu\, y} \right\} := \left(\nu^{-1}[\nu\, x\,, \nu\, y\,]\right)\Big|_{\nu=0} = \left(\nu^{-1} \cdot \nu^3 [x,y]_{\mathfrak{g}}\right)\Big|_{\nu=0} = \left(\nu \cdot \nu\,[x,y]_{\mathfrak{g}}\right)\Big|_{\nu=0} = 0$ (for all $x, y \in \mathfrak{g}$).

Finally, we have $\left(H'^{(\nu-1)}\right)^{\vee(\nu-1)} = \left\langle \left\{ (\nu-1)^{p^n-1} x^{(p^n)} \,\middle|\, x \in \mathfrak{g}\,,\, n \in \mathbb{N} \right\} \right\rangle \subsetneqq H$ and $\left(H'^{(\nu)}\right)^{\vee(\nu)} = \left\langle \left\{ \nu^{p^n-1} x^{(p^n)} \,\middle|\, x \in \mathfrak{g}\,,\, n \in \mathbb{N} \right\} \right\rangle \subsetneqq H$, by direct computation. For $H^{\vee(\nu-1)}$ we have the same features as in §3.7: the analysis therein can be repeated, with the final upshot depending on the nature of G (or of $\mathfrak{g}$, essentially, in particular on its p–lower central series).

§ 5 Second example: SL_2, SL_n and the semisimple case

5.1 The classical setting. Let $\Bbbk$ be any field of characteristic $p \geq 0$. Let $G := SL_2(\Bbbk) \equiv SL_2$; its tangent Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ is generated by f, h, e (the *Chevalley generators*) with relations $[h,e] = 2\,e$, $[h,f] = -2f$, $[e,f] = h$. The formulas $\delta(f) = h \otimes f - f \otimes h$, $\delta(h) = 0$, $\delta(e) = h \otimes e - e \otimes h$, define a Lie cobracket on $\mathfrak{g}$ which makes it into a Lie bialgebra, corresponding to a structure of Poisson group on G. These formulas give also a presentation of the co-Poisson Hopf algebra $U(\mathfrak{g})$ (with the standard Hopf structure). If $p > 0$, the p–operation in $\mathfrak{sl}_2$ is given by $e^{[p]} = 0$, $f^{[p]} = 0$, $h^{[p]} = h$.

On the other hand, $F[SL_2]$ is the unital associative commutative $\Bbbk$–algebra with generators a, b, c, d and the relation $ad - bc = 1$, and Poisson Hopf structure given by

$$\Delta(a) = a \otimes a + b \otimes c\,,\ \ \Delta(b) = a \otimes b + b \otimes d\,,\ \ \Delta(c) = c \otimes a + d \otimes c\,,\ \ \Delta(d) = c \otimes b + d \otimes d$$

$$\epsilon(a) = 1\,,\ \ \epsilon(b) = 0\,,\ \ \epsilon(c) = 0\,,\ \ \epsilon(d) = 1\,,\quad S(a) = d\,,\ \ S(b) = -b\,,\ \ S(c) = -c\,,\ \ S(d) = a$$

$$\{a,b\} = b\,a\,,\quad \{a,c\} = c\,a\,,\quad \{b,c\} = 0\,,\quad \{d,b\} = -b\,d\,,\quad \{d,c\} = -c\,d\,,\quad \{a,d\} = 2\,b\,c\,.$$

The dual Lie bialgebra $\mathfrak{g}^* = \mathfrak{sl}_2^{\,*}$ is the Lie algebra with generators $\mathrm{f}, \mathrm{h}, \mathrm{e}$, and relations $[\mathrm{h},\mathrm{e}] = \mathrm{e}$, $[\mathrm{h},\mathrm{f}\,] = \mathrm{f}$, $[\mathrm{e},\mathrm{f}\,] = 0$, with Lie cobracket given by $\delta(\mathrm{f}\,) = 2(\mathrm{f} \otimes \mathrm{h} - \mathrm{h} \otimes \mathrm{f}\,)$, $\delta(\mathrm{h}) = \mathrm{e} \otimes \mathrm{f} - \mathrm{f} \otimes \mathrm{e}$, $\delta(\mathrm{e}) = 2(\mathrm{h} \otimes \mathrm{e} - \mathrm{e} \otimes \mathrm{h})$ (we choose as generators $\mathrm{f} := f^*$, $\mathrm{h} := h^*$, $\mathrm{e} := e^*$, where $\left\{f^*, h^*, e^*\right\}$ is the basis of $\mathfrak{sl}_2^{\,*}$ which is the dual of the basis $\{f, h, e\}$ of $\mathfrak{sl}_2$). This again yields also a presentation of $U\left(\mathfrak{sl}_2^{\,*}\right)$. If $p > 0$, the p–operation in $\mathfrak{sl}_2^{\,*}$ is given by $\mathrm{e}^{[p]} = 0$, $\mathrm{f}^{\,[p]} = 0$, $\mathrm{h}^{[p]} = \mathrm{h}$. The simply connected algebraic Poisson group whose tangent Lie bialgebra is $\mathfrak{sl}_2^{\,*}$ can be realized as the group of pairs of matrices (the left subscript s meaning "simply connected")

$${}_sSL_2^{\,*} = \left\{ \left(\begin{pmatrix} z^{-1} & 0 \\ y & z \end{pmatrix}, \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \right) \,\middle|\, x, y \in k, z \in \Bbbk \setminus \{0\} \right\} \;\leq\; SL_2 \times SL_2\;.$$

This group has centre $Z := \big\{ (I,I), (-I,-I) \big\}$, so there is only one other (Poisson) group sharing the same Lie (bi)algebra, namely the quotient ${}_aSL_2^{\,*} := {}_sSL_2^{\,*} \Big/ Z$ (the adjoint of ${}_sSL_2^{\,*}$, as the left subscript a means). Therefore $F\big[{}_sSL_2^{\,*}\big]$ is the unital associative commutative $\Bbbk$–algebra with generators x, $z^{\pm 1}$, y, with Poisson Hopf structure given by

$$\Delta(x) = x \otimes z^{-1} + z \otimes x \, , \qquad \Delta\big(z^{\pm 1}\big) = z^{\pm 1} \otimes z^{\pm 1} \, , \qquad \Delta(y) = y \otimes z^{-1} + z \otimes y$$
$$\epsilon(x) = 0 \, , \quad \epsilon\big(z^{\pm 1}\big) = 1 \, , \quad \epsilon(y) = 0 \, , \qquad S(x) = -x \, , \quad S\big(z^{\pm 1}\big) = z^{\mp 1} \, , \quad S(y) = -y$$
$$\{x, y\} = \big(z^2 - z^{-2}\big)\big/2 \, , \qquad \big\{z^{\pm 1}, x\big\} = \pm\, x\, z^{\pm 1} \, , \qquad \big\{z^{\pm 1}, y\big\} = \mp\, z^{\pm 1} y$$

(Remark: with respect to this presentation, we have $\mathrm{f} = \partial_y\big|_e$, $\mathrm{h} = z\,\partial_z\big|_e$, $\mathrm{e} = \partial_x\big|_e$, where e is the identity element of ${}_sSL_2^{\,*}$). Moreover, $F\big[{}_aSL_2^{\,*}\big]$ can be identified with the Poisson Hopf subalgebra of $F\big[{}_sSL_2^{\,*}\big]$ spanned by products of an even number of generators — i.e. monomials of even degree: this is generated, as a unital subalgebra, by xz, $z^{\pm 2}$, and $z^{-1}y$.

In general, we shall consider $\mathfrak{g} = \mathfrak{g}^\tau$ a semisimple Lie algebra, endowed with the Lie cobracket — depending on the parameter τ — given in [Ga1], §1.3; in the following we shall also retain from [*loc. cit.*] all the notation we need: in particular, we denote by Q, resp. P, the root lattice, resp. the weight lattice, of $\mathfrak{g}$, and by r the rank of $\mathfrak{g}$.

5.2 The[3] QrUEAs $U_q(\mathfrak{g})$. We turn now to quantum groups, starting with the $\mathfrak{sl}_2$ case. Let R be any 1dD, $\hbar \in R \setminus \{0\}$ a prime such that $R \big/ \hbar\, R = \Bbbk$; moreover, letting $q := \hbar + 1$ we assume that q be invertible in R, i.e. there exists $q^{-1} = (\hbar + 1)^{-1} \in R$. For instance, one can pick $R := \Bbbk[q, q^{-1}]$ for an indeterminate q and $\hbar := q - 1$, then $F(R) = \Bbbk(q)$.

Let $\mathbb{U}_q(\mathfrak{g}) = \mathbb{U}_q(\mathfrak{sl}_2)$ be the associative unital $F(R)$–algebra with (Chevalley-like) generators F, $K^{\pm 1}$, E, and relations

$$KK^{-1} = 1 = K^{-1}K \ , \ \ K^{\pm 1}F = q^{\mp 2}FK^{\pm 1} \ , \ \ K^{\pm 1}E = q^{\pm 2}EK^{\pm 1} \ , \ \ EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \ .$$

This is a Hopf algebra, with Hopf structure given by

$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F \, , \qquad \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1} \, , \qquad \Delta(E) = E \otimes 1 + K \otimes E$$
$$\epsilon(F) = 0 \, , \ \epsilon\big(K^{\pm 1}\big) = 1 \, , \ \epsilon(E) = 0 \, , \qquad S(F) = -FK, \ S\big(K^{\pm 1}\big) = K^{\mp 1}, \ S(E) = -K^{-1}E \, .$$

Then let $U_q(\mathfrak{g})$ be the R–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ generated by F, $H := \dfrac{K-1}{q-1}$, $\varGamma := \dfrac{K - K^{-1}}{q - q^{-1}}$, $K^{\pm 1}$, E. From the definition of $\mathbb{U}_q(\mathfrak{g})$ one gets a presentation of $U_q(\mathfrak{g})$ as the associative unital algebra with generators F, H, $\varGamma$, $K^{\pm 1}$, E and relations

$$KK^{-1} = 1 = K^{-1}K \, , \qquad K^{\pm 1}H = HK^{\pm 1} \, , \qquad K^{\pm 1}\varGamma = \varGamma K^{\pm 1} \, , \qquad H\varGamma = \varGamma H$$
$$(q-1)H = K - 1 \, , \quad \big(q - q^{-1}\big)\varGamma = K - K^{-1} \, , \quad H\big(1 + K^{-1}\big) = \big(1 + q^{-1}\big)\varGamma \, , \quad EF - FE = \varGamma$$

[3] In §§5–7 we should use notation $U_{q-1}(\mathfrak{g})$ and $F_{q-1}[G]$, after Remark 1.5 (for $\hbar = q - 1$); instead, we write $U_q(\mathfrak{g})$ and $F_q[G]$ to be consistent with the standard notation in use for these quantum algebras.

$$K^{\pm 1} F = q^{\mp 2} F K^{\pm 1}\,, \quad H F = q^{-2} F H - (q+1) F\,, \quad \varGamma F = q^{-2} F \varGamma - \big(q + q^{-1}\big) F$$
$$K^{\pm 1} E = q^{\pm 2} E K^{\pm 1}\,, \quad H E = q^{+2} E H + (q+1) E\,, \quad \varGamma E = q^{+2} E \varGamma + \big(q + q^{-1}\big) E$$

and with a Hopf structure given by the same formulas as above for F, $K^{\pm 1}$, and E plus

$$\Delta(\varGamma) = \varGamma \otimes K + K^{-1} \otimes \varGamma\,, \qquad \epsilon(\varGamma) = 0\,, \qquad S(\varGamma) = -\varGamma$$
$$\Delta(H) = H \otimes 1 + K \otimes H\,, \qquad \epsilon(H) = 0\,, \qquad S(H) = -K^{-1} H\,.$$

Note also that $K = 1 + (q-1)H$ and $K^{-1} = K - \big(q - q^{-1}\big)\varGamma = 1 + (q-1)H - \big(q - q^{-1}\big)\varGamma$, hence $U_q(\mathfrak{g})$ is generated even by F, H, $\varGamma$ and E alone. Further, notice also that

$$\mathbb{U}_q(\mathfrak{g}) \;=\; \text{free } F(R)\text{–module over } \Big\{ F^a K^z E^d \,\Big|\, a, d \in \mathbb{N}, z \in \mathbb{Z} \Big\} \tag{5.1}$$
$$U_q(\mathfrak{g}) \;=\; R\text{–span of } \Big\{ F^a H^b \varGamma^c E^d \,\Big|\, a, b, c, d \in \mathbb{N} \Big\} \text{ inside } \mathbb{U}_q(\mathfrak{g}) \tag{5.2}$$

which implies that $F(R) \otimes_R U_q(\mathfrak{g}) = \mathbb{U}_q(\mathfrak{g})$. Moreover, definitions imply at once that $U_q(\mathfrak{g})$ is torsion-free, and also that it is a Hopf R–subalgebra of $\mathbb{U}_q(\mathfrak{g})$. Therefore $U_q(\mathfrak{g}) \in \mathcal{HA}$, and in fact $U_q(\mathfrak{g})$ is even a QrUEA, whose semiclassical limit is $U(\mathfrak{g}) = U(\mathfrak{sl}_2)$, with the generators F, $K^{\pm 1}$, H, $\varGamma$, E respectively mapping to f, 1, h, h, $e \in U(\mathfrak{sl}_2)$.

It is also possible to define a “simply connected” version of $\mathbb{U}_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$, obtained from the previous ones — called “adjoint” — as follows. For $\mathbb{U}_q(\mathfrak{g})$, one adds a square root of $K^{\pm 1}$, call it $L^{\pm 1}$, as new generator; for $U_q(\mathfrak{g})$ one adds the new generators $L^{\pm 1}$ and also $D := \dfrac{L-1}{q-1}$. Then the same analysis as before shows that $U_q(\mathfrak{g})$ is another quantization (containing the “adjoint” one) of $U(\mathfrak{g})$.

In the general case of semisimple $\mathfrak{g}$, let $\mathbb{U}_q(\mathfrak{g})$ be the Lusztig-like quantum group — over R — associated to $\mathfrak{g} = \mathfrak{g}^\tau$ as in [Ga1], namely $\mathbb{U}_q(\mathfrak{g}) := U_{q,\varphi}^M(\mathfrak{g})$ with respect to the notation in [*loc. cit.*], where M is any intermediate lattice such that $Q \leq M \leq P$ (this is just a matter of choice, of the type mentioned in the statement of Theorem 2.2*(c)*): this is a Hopf algebra over $F(R)$, generated by elements F_i, M_i, E_i for $i = 1, \dots, r =: \textit{rank}\,(\mathfrak{g})$. Then let $U_q(\mathfrak{g})$ be the unital R–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ generated by the elements F_i, $H_i := \dfrac{M_i - 1}{q-1}$, $\varGamma_i := \dfrac{K_i - K_i^{-1}}{q - q^{-1}}$, $M_i^{\pm 1}$, E_i, where the $K_i = M_{\alpha_i}$ are suitable product of M_j's, defined as in [Ga1], §2.2 (whence K_i, $K_i^{-1} \in U_q(\mathfrak{g})$). From [Ga1], §§2.5, 3.3, we have that $\mathbb{U}_q(\mathfrak{g})$ is the free $F(R)$–module with basis the set of monomials

$$\bigg\{ \prod_{\alpha \in \Phi^+} F_\alpha^{f_\alpha} \cdot \prod_{i=1}^n K_i^{z_i} \cdot \prod_{\alpha \in \Phi^+} E_\alpha^{e_\alpha} \,\bigg|\, f_\alpha, e_\alpha \in \mathbb{N}\,,\ z_i \in \mathbb{Z}\,,\ \ \forall\, \alpha \in \Phi^+,\ i = 1, \dots, n \bigg\}$$

while $U_q(\mathfrak{g})$ is the R–span inside $\mathbb{U}_q(\mathfrak{g})$ of the set of monomials

$$\bigg\{ \prod_{\alpha \in \Phi^+} F_\alpha^{f_\alpha} \cdot \prod_{i=1}^n H_i^{t_i} \cdot \prod_{j=1}^n \varGamma_j^{c_j} \cdot \prod_{\alpha \in \Phi^+} E_\alpha^{e_\alpha} \,\bigg|\, f_\alpha, t_i, c_j, e_\alpha \in \mathbb{N} \ \ \forall\, \alpha \in \Phi^+,\ i, j = 1, \dots, n \bigg\}$$

(hereafter, Φ^+ is the set of positive roots of $\mathfrak{g}$, each E_α, resp. F_α, is a root vector attached to $\alpha \in \Phi^+$, resp. to $-\alpha \in (-\Phi^+)$, and the products of factors indexed by Φ^+ are ordered with respect to a fixed convex order of Φ^+, see [Ga1]), whence (as for $n = 2$) $U_q(\mathfrak{g})$ is a free R–module. In this case again $U_q(\mathfrak{g})$ is a QrUEA, with semiclassical limit $U(\mathfrak{g})$.

5.3 Computation of $U_q(\mathfrak{g})'$ and specialization $U_q(\mathfrak{g})' \xrightarrow{q \to 1} F[G^\star]$. We begin with the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. From the definition of $U_q(\mathfrak{g}) = U_q(\mathfrak{sl}_2)$ we have $\delta_n(E) = (\mathrm{id} - \epsilon)^{\otimes n}\big(\Delta^n(E)\big) = (\mathrm{id} - \epsilon)^{\otimes n}\left(\sum_{s=1}^{n} K^{\otimes (s-1)} \otimes E \otimes 1^{\otimes (n-s)}\right) = (q-1)^{n-1} H^{\otimes (n-1)} \otimes E$ from which $\delta_n\big((q-1)E\big) \in (q-1)^n U_q(\mathfrak{g}) \setminus (q-1)^{n+1} U_q(\mathfrak{g})$ (for all $n \in \mathbb{N}$), whence $(q-1)E \in U_q(\mathfrak{g})'$, whereas $E \notin U_q(\mathfrak{g})'$. Similarly, $(q-1)F \in U_q(\mathfrak{g})'$, whilst $F \notin U_q(\mathfrak{g})'$. As for generators H, $\varGamma$, $K^{\pm 1}$, we have $\Delta^n(H) = \sum_{s=1}^n K^{\otimes (s-1)} \otimes H \otimes 1^{\otimes (n-s)}$, $\Delta^n\big(K^{\pm 1}\big) = \big(K^{\pm 1}\big)^{\otimes n}$, $\Delta^n(\varGamma) = \sum_{s=1}^n K^{\otimes (s-1)} \otimes \varGamma \otimes \big(K^{-1}\big)^{\otimes (n-s)}$, hence for $\delta_n = (\mathrm{id} - \epsilon)^{\otimes n} \circ \Delta^n$ we have $\delta_n(H) = (q-1)^{n-1} \cdot H^{\otimes n}$, $\delta^n\big(K^{-1}\big) = (q-1)^n \cdot \big(-K^{-1}H\big)^{\otimes n}$, $\delta^n(K) = (q-1)^n \cdot H^{\otimes n}$, $\delta^n(\varGamma) = (q-1)^{n-1} \cdot \sum_{s=1}^{n} (-1)^{n-s} H^{\otimes (s-1)} \otimes \varGamma \otimes \big(HK^{-1}\big)^{\otimes (n-s)}$ for all $n \in \mathbb{N}$, so that $(q-1)H$, $(q-1)\varGamma$, $K^{\pm 1} \in U_q(\mathfrak{g})' \setminus (q-1) U_q(\mathfrak{g})'$. Therefore $U_q(\mathfrak{g})'$ contains the subalgebra U' generated by $(q-1)F$, K, K^{-1}, $(q-1)H$, $(q-1)\varGamma$, $(q-1)E$. On the other hand, using (5.2) a thorough — but straightforward — computation along the same lines as above shows that any element in $U_q(\mathfrak{g})'$ does necessarily lie in U' (details are left to the reader: everything follows from definitions and the formulas above for Δ^n). Thus $U_q(\mathfrak{g})'$ is nothing but the subalgebra of $U_q(\mathfrak{g})$ generated by $\dot{F} := (q-1)F$, K, K^{-1}, $\dot{H} := (q-1)H$, $\dot{\varGamma} := (q-1)\varGamma$, $\dot{E} := (q-1)E$; notice also that the generator $\dot{H}$ is unnecessary, for $\dot{H} = K - 1$. Then $U_q(\mathfrak{g})'$ can be presented as the unital associative R–algebra with generators $\dot{F}$, $\dot{\varGamma}$, $K^{\pm 1}$, $\dot{E}$ and relations

$$KK^{-1} = 1 = K^{-1}K, \;\; K^{\pm 1}\dot{\varGamma} = \dot{\varGamma} K^{\pm 1}, \;\; \big(1 + q^{-1}\big)\dot{\varGamma} = K - K^{-1}, \;\; \dot{E}\dot{F} - \dot{F}\dot{E} = (q-1)\dot{\varGamma}$$
$$K - K^{-1} = \big(1 + q^{-1}\big)\dot{\varGamma}, \quad K^{\pm 1}\dot{F} = q^{\mp 2}\dot{F}K^{\pm 1}, \quad K^{\pm 1}\dot{E} = q^{\pm 2}\dot{E}K^{\pm 1}$$
$$\dot{\varGamma}\dot{F} = q^{-2}\dot{F}\dot{\varGamma} - (q-1)\big(q + q^{-1}\big)\dot{F}, \qquad \dot{\varGamma}\dot{E} = q^{+2}\dot{E}\dot{\varGamma} + (q-1)\big(q + q^{-1}\big)\dot{E}$$

with Hopf structure given by

$$\begin{array}{lll} \Delta\big(\dot{F}\big) = \dot{F} \otimes K^{-1} + 1 \otimes \dot{F}, & \epsilon\big(\dot{F}\big) = 0, & S\big(\dot{F}\big) = -\dot{F}K \\ \Delta\big(\dot{\varGamma}\big) = \dot{\varGamma} \otimes K + K^{-1} \otimes \dot{\varGamma}, & \epsilon\big(\dot{\varGamma}\big) = 0, & S\big(\dot{\varGamma}\big) = -\dot{\varGamma} \\ \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}, & \epsilon\big(K^{\pm 1}\big) = 1, & S\big(K^{\pm 1}\big) = K^{\mp 1} \\ \Delta\big(\dot{E}\big) = \dot{E} \otimes 1 + K \otimes \dot{E}, & \epsilon\big(\dot{E}\big) = 0, & S\big(\dot{E}\big) = -K^{-1}\dot{E}. \end{array}$$

When $q \to 1$, a direct computation shows that this gives a presentation of $F\big[{}_aSL_2^{\,*}\big]$, and the Poisson structure that $F\big[{}_aSL_2^{\,*}\big]$ inherits from this quantization process is exactly the one coming from the Poisson structure on ${}_aSL_2^{\,*}$: in fact, there is a Poisson Hopf algebra isomorphism

$$U_q(\mathfrak{g})' \Big/ (q-1)\, U_q(\mathfrak{g})' \xrightarrow{\;\cong\;} F\big[{}_aSL_2^{\,*}\big] \quad \Big(\subseteq F\big[{}_sSL_2^{\,*}\big] \Big)$$

given by: $\dot{E} \mod (q-1) \mapsto x\,z$, $K^{\pm 1} \mod (q-1) \mapsto z^{\pm 2}$, $\dot{H} \mod (q-1) \mapsto z^2 - 1$, $\dot{\varGamma} \mod (q-1) \mapsto \big(z^2 - z^{-2}\big)\big/2$, $\dot{F} \mod (q-1) \mapsto z^{-1} y$. In other words, $U_q(\mathfrak{g})'$ specializes to $F\big[{}_aSL_2^{\,*}\big]$ as a Poisson Hopf algebra. *Note that this was predicted by Theorem 2.2(c) when* $Char\,(\Bbbk) = 0$, *but our analysis now proves it also for* $Char\,(\Bbbk) > 0$.

Note that we got the *adjoint* Poisson group dual of $G = SL_2$, that is ${}_aSL_2^{\,*}$; a different choice of the initial QrUEA leads us to the *simply connected* one, i.e. ${}_sSL_2^{\,*}$. Indeed, if we start from the "simply connected" version of $U_q(\mathfrak{g})$ (see §5.2) the same analysis shows that $U_q(\mathfrak{g})'$ is like above but for containing also the new generators $L^{\pm 1}$, and similarly when specializing q at 1: thus we get the function algebra of a Poisson group which is a double covering of ${}_aSL_2{}^*$, namely ${}_sSL_2{}^*$. So changing the QrUEA quantizing $\mathfrak{g}$ we get two different QFAs, one for each of the two connected Poisson algebraic groups dual of SL_2, i.e. with tangent Lie bialgebra $\mathfrak{sl}_2{}^*$; this shows the dependence of $G^\star$ (here denoted G^* since $\mathfrak{g}^\times = \mathfrak{g}^*$) in Theorem 2.2*(c)* on the choice of the QrUEA $U_q(\mathfrak{g})$, for fixed $\mathfrak{g}$.

With a bit more careful study, exploiting the analysis in [Ga1], one can treat the general case too: we sketch briefly our arguments — restricting to the simply laced case, to simplify the exposition — leaving to the reader the straightforward task of filling in details.

So now let $\mathfrak{g} = \mathfrak{g}^\tau$ be a semisimple Lie algebra, as in §5.1, and let $U_q(\mathfrak{g})$ be the QrUEA introduced in §5.2: our aim again is to compute the QFA $U_q(\mathfrak{g})'$.

The same computations as for $\mathfrak{g} = \mathfrak{sl}(2)$ show that $\delta_n(H_i) = (q-1)^{n-1} \cdot H_i^{\otimes n}$ and $\delta^n(\varGamma_i) = (q-1)^{n-1} \cdot \sum_{s=1}^n (-1)^{n-s} H_i^{\otimes(s-1)} \otimes \varGamma_i \otimes \big(H_i K_i^{-1}\big)^{\otimes(n-s)}$, which gives

$$\dot{H}_i := (q-1)H_i \in U_q(\mathfrak{g})' \setminus (q-1)\, U_q(\mathfrak{g})' \quad \text{and} \quad \dot{\varGamma}_i := (q-1)\,\varGamma_i \in U_q(\mathfrak{g})' \setminus (q-1)\, U_q(\mathfrak{g})' .$$

As for root vectors, let $\dot{E}_\gamma := (q-1)E_\gamma$ and $\dot{F}_\gamma := (q-1)F_\gamma$ for all $\gamma \in \Phi^+$: using the same type of arguments as in [Ga1][4] §5.16, we can prove that $E_\alpha \notin U_q(\mathfrak{g})'$ but $\dot{E}_\alpha \in U_q(\mathfrak{g})' \setminus (q-1)\, U_q(\mathfrak{g})'$. In fact, let $\mathbb{U}_q(\mathfrak{b}_+)$ and $\mathbb{U}_q(\mathfrak{b}_-)$ be quantum Borel subalgebras, and $\mathfrak{U}_{\varphi,\geq}^M$, $\mathcal{U}_{\varphi,\geq}^M$, $\mathfrak{U}_{\varphi,\leq}^M$, $\mathcal{U}_{\varphi,\leq}^M$ their R–subalgebras defined in [Ga1], §2: then both $\mathbb{U}_q(\mathfrak{b}_+)$ and $\mathbb{U}_q(\mathfrak{b}_-)$ are Hopf subalgebras of $\mathbb{U}_q(\mathfrak{g})$. In addition, letting M' be the lattice between Q and P dual of M (in the sense of [Ga1], §1.1, there exists an $F(R)$–valued perfect Hopf pairing between $\mathbb{U}_q(\mathfrak{b}_\pm)$ and $\mathbb{U}_q(\mathfrak{b}_\mp)$ — one built up on M and the other on M' — such that $\mathfrak{U}_{\varphi,\geq}^M = \Big(\mathcal{U}_{\varphi,\leq}^{M'}\Big)^\bullet$, $\mathfrak{U}_{\varphi,\leq}^M = \Big(\mathcal{U}_{\varphi,\geq}^{M'}\Big)^\bullet$, $\mathcal{U}_{\varphi,\geq}^M = \Big(\mathfrak{U}_{\varphi,\leq}^{M'}\Big)^\bullet$, and $\mathcal{U}_{\varphi,\leq}^M = \Big(\mathfrak{U}_{\varphi,\geq}^{M'}\Big)^\bullet$. Now, $\big(q-q^{-1}\big)E_\alpha \in \mathcal{U}_{\varphi,\geq}^M = \Big(\mathfrak{U}_{\varphi,\leq}^{M'}\Big)^\bullet$, hence — since $\mathfrak{U}_{\varphi,\leq}^{M'}$ is an algebra — we have $\Delta\Big(\big(q-q^{-1}\big)E_\alpha\Big) \in \Big(\mathfrak{U}_{\varphi,\leq}^{M'} \otimes \mathfrak{U}_{\varphi,\leq}^{M'}\Big)^\bullet = \Big(\mathfrak{U}_{\varphi,\leq}^{M'}\Big)^\bullet \otimes \Big(\mathfrak{U}_{\varphi,\leq}^{M'}\Big)^\bullet = \mathcal{U}_{\varphi,\geq}^M \otimes \mathcal{U}_{\varphi,\geq}^M$. Therefore, by definition of $\mathcal{U}_{\varphi,\geq}^M$ and by the PBW theorem for it and for $\mathfrak{U}_{\varphi,\leq}^{M'}$ (cf. [Ga1], §2.5) we have that $\Delta\Big(\big(q-q^{-1}\big)E_\alpha\Big)$ is an R–linear combination like $\Delta\Big(\big(q-q^{-1}\big)E_\alpha\Big) = \sum_r A_r^{(1)} \otimes A_r^{(2)}$ in which the $A_r^{(j)}$'s are monomials in the M_j's and in the $\overline{E}_\gamma$'s, where $\overline{E}_\gamma := \big(q-q^{-1}\big)E_\gamma$ for all $\gamma \in \Phi^+$: iterating, we find that $\Delta^\ell\Big(\big(q-q^{-1}\big)E_\alpha\Big)$ is an R–linear combination

$$\Delta^\ell\Big(\big(q-q^{-1}\big)E_\alpha\Big) \;=\; \textstyle\sum_r A_r^{(1)} \otimes A_r^{(2)} \otimes \cdots \otimes A_r^{(\ell)} \tag{5.3}$$

in which the $A_r^{(j)}$'s are again monomials in the M_j's and in the $\overline{E}_\gamma$'s. Now, we distinguish two cases: either $A_r^{(j)}$ does contain some $\overline{E}_\gamma$ $\big(\in \big(q-q^{-1}\big)\, U_q(\mathfrak{g})\big)$, thus $\epsilon\Big(A_r^{(j)}\Big) = A_r^{(j)} \in (q-1)\, U_q(\mathfrak{g})$ whence $(\mathrm{id}-\epsilon)\Big(A_r^{(j)}\Big) = 0$; or $A_r^{(j)}$ does not contain any $\overline{E}_\gamma$ and is only a monomial in the M_t's, say $A_r^{(j)} = \prod_{t=1}^n M_t^{m_t}$: then $(\mathrm{id}-\epsilon)\Big(A_r^{(j)}\Big) = \prod_{t=1}^n M_t^{m_t} - 1 = \prod_{t=1}^n \big((q-1)\, H_t + 1\big)^{m_t} - 1 \in (q-1)\, U_q(\mathfrak{g})$. In addition, for some "Q–grading reasons" (as in [Ga1], §3.16), in each one of the summands in (5.3)

[4] In [Ga1] one assumes *Char* $(\Bbbk) = 0$: however, *this is not necesary* for the present analysis.

the sum of all the γ's such that the (rescaled) root vectors $\overline{E}_\gamma$ occur in any of the factors $A_r^{(1)}$, $A_r^{(2)}$, ..., $A_r^{(n)}$ must be equal to α: therefore, in each of these summands at least one factor $\overline{E}_\gamma$ does occur. The conclusion is that $\delta_\ell\big(\overline{E}_\alpha\big) \in \big(1+q^{-1}\big)(q-1)^\ell \, U_q(\mathfrak{g})^{\otimes \ell}$ (the factor $\big(1+q^{-1}\big)$ being there because at least one rescaled root vector $\overline{E}_\gamma$ occurs in each summand of $\delta_\ell\big(\overline{E}_\alpha\big)$, thus providing a coefficient $\big(q-q^{-1}\big)$ the term $\big(1+q^{-1}\big)$ is factored out of), whence $\delta_\ell\big(\dot{E}_\alpha\big) \in (q-1)^\ell \, U_q(\mathfrak{g})^{\otimes \ell}$. More precisely, we have also $\delta_\ell\big(\dot{E}_\alpha\big) \notin (q-1)^{\ell+1} \, U_q(\mathfrak{g})^{\otimes \ell}$, for we can easily check that $\Delta^\ell\big(\dot{E}_\alpha\big)$ is the sum of $M_\alpha \otimes M_\alpha \otimes \cdots \otimes M_\alpha \otimes \dot{E}_\alpha$ plus other summands which are R–linearly independent of this first term: but then $\delta_\ell\big(\dot{E}_\alpha\big)$ is the sum of $(q-1)^{\ell-1} H_\alpha \otimes H_\alpha \otimes \cdots \otimes H_\alpha \otimes \dot{E}_\alpha$ (where $H_\alpha := \frac{M_\alpha - 1}{q-1}$ is equal to an R–linear combination of products of M_j's and H_t's) plus other summands which are R–linearly independent of the first one, and since $H_\alpha \otimes H_\alpha \otimes \cdots \otimes H_\alpha \otimes \dot{E}_\alpha \notin (q-1)^2 \, U_q(\mathfrak{g})^{\otimes \ell}$ we can conclude as claimed. Therefore $\delta_\ell\big(\dot{E}_\alpha\big) \in (q-1)^\ell \, U_q(\mathfrak{g})^{\otimes \ell} \setminus (q-1)^{\ell+1} \, U_q(\mathfrak{g})^{\otimes \ell}$, whence we get $\dot{E}_\alpha := (q-1) E_\alpha \in U_q(\mathfrak{g})' \setminus (q-1) \, U_q(\mathfrak{g})' \;\; \forall \, \alpha \in \Phi^+$. An entirely similar analysis yields also $\dot{F}_\alpha := (q-1) F_\alpha \in U_q(\mathfrak{g})' \setminus (q-1) \, U_q(\mathfrak{g})' \;\; \forall \, \alpha \in \Phi^+$.

Summing up, we have found that $U_q(\mathfrak{g})'$ contains the subalgebra U' generated by $\dot{F}_\alpha$, $\dot{H}_i$, $\dot{\Gamma}_i$, $\dot{E}_\alpha$ for all $\alpha \in \Phi^+$ and all $i = 1, \dots, n$. On the other hand, using (5.2) a thorough — but straightforward — computation along the same lines as above shows that any element in $U_q(\mathfrak{g})'$ must lie in U' (details are left to the reader). Thus finally $U_q(\mathfrak{g})' = U'$, so we have a concrete description of $U_q(\mathfrak{g})'$.

Now compare $U' = U_q(\mathfrak{g})'$ with the algebra $\mathcal{U}_\varphi^M(\mathfrak{g})$ in [Ga1], §3.4 (for $\varphi = 0$), the latter being just the R–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ generated by the set $\big\{ \overline{F}_\alpha, M_i, \overline{E}_\alpha \,\big|\, \alpha \in \Phi^+, i = 1, \dots, n \big\}$. First of all, by definition, we have $\mathcal{U}_\varphi^M(\mathfrak{g}) \subseteq U' = U_q(\mathfrak{g})'$; moreover, $\dot{F}_\alpha \equiv \frac{1}{2} \overline{F}_\alpha$, $\dot{E}_\alpha \equiv \frac{1}{2} \overline{E}_\alpha$, $\dot{\Gamma}_i \equiv \frac{1}{2}\big(K_i - K_i^{-1}\big) \mod (q-1) \, \mathcal{U}_\varphi^M(\mathfrak{g})$ for all α, i. Then

$$\big(U_q(\mathfrak{g})'\big)_1 \; := \; U_q(\mathfrak{g})' \Big/ (q-1) \, U_q(\mathfrak{g})' \; = \; \mathcal{U}_\varphi^M(\mathfrak{g}) \Big/ (q-1) \, \mathcal{U}_\varphi^M(\mathfrak{g}) \; \cong \; F\big[G_M^*\big]$$

where G_M^* is the Poisson group dual of $G = G^\tau$ with centre $Z(G_M^*) \cong M/Q$ and fundamental group $\pi_1(G_M^*) \cong P/M$, and the isomorphism (of Poisson Hopf algebras) on the right is given by [Ga1], Theorem 7.4 (see also references therein for the original statement and proof). In other words, $U_q(\mathfrak{g})'$ specializes to $F\big[G_M^*\big]$ *as a Poisson Hopf algebra*, as prescribed by Theorem 2.2. By the way, notice that in the present case the dependence of the dual group $G^\star = G_M^*$ on the choice of the initial QrUEA (for fixed $\mathfrak{g}$) — mentioned in the last part of the statement of Theorem 2.2*(c)* — is evident.

By the way, the previous discussion applies as well to the case of $\mathfrak{g}$ *an untwisted affine Kac-Moody algebra*, just replacing quotations from [Ga1] — referring to results about *finite* Kac-Moody algebras — with similar quotations from [Ga3] — referring to untwisted *affine* Kac-Moody algebras.

5.4 The identity $\big(U_q(\mathfrak{g})'\big)^\vee = U_q(\mathfrak{g})$. In this section we check the part of Theorem 2.2*(b)* claiming that, when $p = 0$, one has $H \in \mathcal{Q}r\mathcal{UEA} \Longrightarrow \big(H'\big)^\vee = H$ for $H = U_q(\mathfrak{g})$ as above. In addition, our proof now will work for $p > 0$ as well. Of course, we start once again from $\mathfrak{g} = \mathfrak{sl}_2$.

Since $\epsilon\big(\dot{F}\big) = \epsilon\big(\dot{H}\big) = \epsilon\big(\dot{\Gamma}\big) = \epsilon\big(\dot{E}\big) = 0$, the ideal $J := Ker\big(\epsilon : \, U_q(\mathfrak{g})' \longrightarrow R\big)$ is generated by $\dot{F}$, $\dot{H}$, $\dot{\Gamma}$, and $\dot{E}$. This implies that J is the R–span of $\Big\{ \dot{F}^\varphi \dot{H}^\kappa \dot{\Gamma}^\gamma \dot{E}^\eta \,\Big|\, (\varphi, \kappa, \gamma, \eta) \in \mathbb{N}^4 \setminus$

$\{(0,0,0,0)\}\big\}$. Therefore $\big(U_q(\mathfrak{g})'\big)^\vee := \sum_{n\geq 0} \Big((q-1)^{-1}J\Big)^n$ is generated, as a unital R-subalgebra of $\mathbb{U}_q(\mathfrak{g})$, by the elements $(q-1)^{-1}\dot{F} = F$, $(q-1)^{-1}\dot{H} = H$, $(q-1)^{-1}\dot{\varGamma} = \varGamma$, $(q-1)^{-1}\dot{E} = E$, hence it coincides with $U_q(\mathfrak{g})$, q.e.d. A similar analysis works in the "adjoint" case as well, and also for the general semisimple or affine Kac-Moody case.

5.5 The quantum hyperalgebra $\mathit{Hyp}_q(\mathfrak{g})$. Let G be a semisimple (affine) algebraic group, with Lie algebra $\mathfrak{g}$, and let $\mathbb{U}_q(\mathfrak{g})$ be the quantum group considered in the previous sections. Lusztig introduced (cf. [Lu1-2]) a "quantum hyperalgebra", i.e. a Hopf subalgebra of $\mathbb{U}_q(\mathfrak{g})$ over $\mathbb{Z}\big[q,q^{-1}\big]$ whose specialization at $q=1$ is exactly the Kostant's $\mathbb{Z}$–integer form $U_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$ from which one gets the hyperalgebra $\mathit{Hyp}(\mathfrak{g})$ over any field $\Bbbk$ of characteristic $p>0$ by scalar extension, namely $\mathit{Hyp}(\mathfrak{g}) = \Bbbk \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g})$. In fact, to be precise one needs a suitable enlargement of the algebra given by Lusztig, which is provided in [DL], §3.4, and denoted by $\varGamma(\mathfrak{g})$. Now we study Drinfeld's functors (at $\hbar = q-1$) on $\mathit{Hyp}_q(\mathfrak{g}) := R \otimes_{\mathbb{Z}[q,q^{-1}]} \varGamma(\mathfrak{g})$ (with R like in §5.2), taking as sample the case $\mathfrak{g} = \mathfrak{sl}_2$.

Let $\mathfrak{g} = \mathfrak{sl}_2$. Let $\mathit{Hyp}_q^{\mathbb{Z}}(\mathfrak{g})$ be the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ (say the one of "adjoint type" defined like above *but over* $\mathbb{Z}\big[q,q^{-1}\big]$) generated the "quantum divided powers"

$$F^{(n)} := F^n \Big/ [n]_q! \quad , \qquad \binom{K\,;\,c}{n} := \prod_{s=1}^{n} \frac{q^{c+1-s}K-1}{q^s-1} \quad , \qquad E^{(n)} := E^n \Big/ [n]_q!$$

(for all $n \in \mathbb{N}$, $c \in \mathbb{Z}$) and by K^{-1}, where $[n]_q! := \prod_{s=1}^n [s]_q$ and $[s]_q = \big(q^s - q^{-s}\big)\Big/\big(q-q^{-1}\big)$ for all $n, s \in \mathbb{N}$. Then (cf. [DL]) this is a Hopf subalgebra of $\mathbb{U}_q(\mathfrak{g})$, and $\mathit{Hyp}_q^{\mathbb{Z}}(\mathfrak{g})\Big|_{q=1} \cong U_{\mathbb{Z}}(\mathfrak{g})$; therefore $\mathit{Hyp}_q(\mathfrak{g}) := R \otimes_{\mathbb{Z}[q,q^{-1}]} \mathit{Hyp}_q^{\mathbb{Z}}(\mathfrak{g})$ (for any R like in §5.2, with $\Bbbk := R\big/\hbar R$ and $p := \mathit{Char}(\Bbbk)$) specializes at $q=1$ to the $\Bbbk$–hyperalgebra $\mathit{Hyp}(\mathfrak{g})$. Moreover, among all the $\binom{K;c}{n}$'s it is enough to take only those with $c=0$. *From now on we assume* $p>0$.

Using formulas for the iterated coproduct in [DL], Corollary 3.3 (which uses the opposite coproduct than ours, but this doesn't matter), and exploiting the PBW-like theorem for $\mathit{Hyp}_q(\mathfrak{g})$ (see [DL] again) we see by direct inspection that $\mathit{Hyp}_q(\mathfrak{g})'$ is the unital R–subalgebra of $\mathit{Hyp}_q(\mathfrak{g})$ generated by K^{-1} and the "rescaled quantum divided powers" $(q-1)^n F^{(n)}$, $(q-1)^n \binom{K;0}{n}$ and $(q-1)^n E^{(n)}$ for all $n \in \mathbb{N}$. Since $[n]_q!\Big|_{q=1} = n! = 0$ iff $p \,\Big|\, n$, we argue that $\mathit{Hyp}_q(\mathfrak{g})'\Big|_{q=1}$ is generated by the corresponding specializations of $(q-1)^{p^s} F^{(p^s)}$, $(q-1)^{p^s} \binom{K;0}{p^s}$ and $(q-1)^{p^s} E^{(p^s)}$ for all $s \in \mathbb{N}$: in particular this shows that the spectrum of $\mathit{Hyp}_q(\mathfrak{g})'\Big|_{q=1}$ has dimension 0 and height 1, and its cotangent Lie algebra $J\Big/J^2$ — where J is the augmentation ideal of $\mathit{Hyp}_q(\mathfrak{g})'\Big|_{q=1}$ — has basis $\Big\{(q-1)^{p^s}F^{(p^s)},\, (q-1)^{p^s}\binom{K;0}{p^s},\, (q-1)^{p^s}E^{(p^s)} \mod (q-1)\,\mathit{Hyp}_q(\mathfrak{g})', \mod J^2 \,\Big|\, s\in\mathbb{N}\Big\}$. Furthermore, $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee$ is generated by the elements $(q-1)^{p^s-1}F^{(p^s)}$, $(q-1)^{p^s-1}\binom{K;0}{p^s}$, K^{-1} and $(q-1)^{p^s-1}E^{(p^s)}$ for all $s \in \mathbb{N}$: in particular we have that $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee \subsetneqq \mathit{Hyp}_q(\mathfrak{g})$, and $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee\Big|_{q=1}$ is generated by the cosets modulo $(q-1)$ of the previous elements, which do form a basis of the restricted Lie bialgebra $\mathfrak{k}$ such that $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee\Big|_{q=1} = \mathbf{u}(\mathfrak{k})$.

We performed the previous study using the "adjoint" version of $U_q(\mathfrak{g})$ as starting point: instead, we can use as well its "simply connected" version, thus obtaining a "simply connected version of $\mathit{Hyp}_q(\mathfrak{g})$" which is defined like before but for using $L^{\pm 1}$ instead of $K^{\pm 1}$; up to these changes, the analysis and its outcome will be exactly the same. Note that all quantum objects involved — namely, $\mathit{Hyp}_q(\mathfrak{g})$, $\mathit{Hyp}_q(\mathfrak{g})'$ and $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee$ — will strictly contain the corresponding "adjoint" quantum objects; on the other hand, the semiclassical limit is the same in the case of $\mathit{Hyp}_q(\mathfrak{g})$ (giving $\mathit{Hyp}(\mathfrak{g})$, in both cases) and in the case of $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee$ (giving $\mathbf{u}(\mathfrak{k})$, in both cases), whereas the semiclassical limit of $\mathit{Hyp}_q(\mathfrak{g})'$ in the "simply connected" case is a (countable) covering of the "adjoint" one.

The general case of semisimple or affine Kac-Moody $\mathfrak{g}$ can be dealt with similarly, with analogous outcome. Indeed, $\mathit{Hyp}_q^{\mathbb{Z}}(\mathfrak{g})$ is defined as the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ (defined like before *but over* $\mathbb{Z}\big[q,q^{-1}\big]$) generated by K_i^{-1} and the "quantum divided powers" (in the above sense) $F_i^{(n)}$, $\binom{K_i\,;\,c}{n}$, $E_i^{(n)}$ for all $n \in \mathbb{N}$, $c \in \mathbb{Z}$ and $i = 1, \dots, \mathit{rank}(\mathfrak{g})$ (notation of §5.2, but now each divided power relative to i is built upon q_i, see [Ga1]). Then (cf. [DL]) this is a Hopf subalgebra of $\mathbb{U}_q(\mathfrak{g})$ with $\mathit{Hyp}_q^{\mathbb{Z}}(\mathfrak{g})\Big|_{q=1} \cong U_{\mathbb{Z}}(\mathfrak{g})$, so $\mathit{Hyp}_q(\mathfrak{g}) := R \otimes_{\mathbb{Z}[q,q^{-1}]} \mathit{Hyp}_q^{\mathbb{Z}}(\mathfrak{g})$ (for any R like before) specializes at $q=1$ to the $\Bbbk$–hyperalgebra $\mathit{Hyp}(\mathfrak{g})$; and among the $\binom{K_i\,;\,c}{n}$'s it is enough to take those with $c=0$.

Again a PBW-like theorem holds for $\mathit{Hyp}_q(\mathfrak{g})$ (see [DL]), where powers of root vectors are replaced by quantum divided powers like $F_\alpha^{(n)}$, $\binom{K_i\,;\,c}{n} \cdot K_i^{-\mathit{Ent}(n/2)}$ and $E_\alpha^{(n)}$, for all positive roots α of $\mathfrak{g}$ (each divided power being relative to q_α, see [Ga1]) both in the finite and in the affine case. Using this and the same type of arguments as in §5.3 — i.e. the perfect graded Hopf pairing between quantum Borel subalgebras — we see by direct inspection that $\mathit{Hyp}_q(\mathfrak{g})'$ is the unital R–subalgebra of $\mathit{Hyp}_q(\mathfrak{g})$ generated by the K_i^{-1}'s and the "rescaled quantum divided powers" $(q_\alpha - 1)^n F_\alpha^{(n)}$, $(q_i-1)^n \binom{K_i\,;\,0}{n}$ and $(q_\alpha - 1)^n E_\alpha^{(n)}$ for all $n \in \mathbb{N}$. Since $[n]_{q_\alpha}!\Big|_{q=1} = n! = 0$ iff $p \Big| n$, one argues like before that $\mathit{Hyp}_q(\mathfrak{g})'\Big|_{q=1}$ is generated by the corresponding specializations of $(q_\alpha - 1)^{p^s} F_\alpha^{(p^s)}$, $(q_i - 1)^{p^s} \binom{K_i\,;\,0}{p^s}$ and $(q_\alpha - 1)^{p^s} E_\alpha^{(p^s)}$ for all $s \in \mathbb{N}$ and all positive roots α. Again, this shows that the spectrum of $\mathit{Hyp}_q(\mathfrak{g})'\Big|_{q=1}$ has (dimension 0 and) height 1, and its cotangent Lie algebra $J \big/ J^2$ (where J is the augmentation ideal of $\mathit{Hyp}_q(\mathfrak{g})'\Big|_{q=1}$) has basis $\Big\{ (q_\alpha - 1)^{p^s} F_\alpha^{(p^s)},\, (q_i - 1)^{p^s} \binom{K_i\,;\,0}{p^s},\, (q_\alpha - 1)^{p^s} E^{(p^s)} \mod (q-1)\mathit{Hyp}_q(\mathfrak{g})' \mod J^2 \,\Big|\, s \in \mathbb{N} \Big\}$. Moreover, $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee$ is generated by $(q_\alpha - 1)^{p^s - 1} F_\alpha^{(p^s)}$, $(q_i - 1)^{p^s - 1} \binom{K_i\,;\,0}{p^s}$, K_i^{-1} and $(q_\alpha - 1)^{p^s - 1} E_\alpha^{(p^s)}$ for all s, i and α: in particular $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee \subsetneqq \mathit{Hyp}_q(\mathfrak{g})$, and $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee\Big|_{q=1}$ is generated by the cosets modulo $(q-1)$ of the previous elements, which in fact do form a basis of the restricted Lie bialgebra $\mathfrak{k}$ such that $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee\Big|_{q=1} = \mathbf{u}(\mathfrak{k})$.

5.6 The QFA $F_q[G]$. In this and the following sections we pass to look at Theorem 2.2 the other way round: namely, we start from QFAs and produce QrUEAs.

We begin with $G = SL_n$, with the standard Poisson structure, for which an especially explicit description of the QFA is available. Namely, let $F_q[SL_n]$ be the unital associative R–algebra generated by $\{\rho_{ij} \mid i,j=1,\dots,n\}$ with relations

$$\begin{array}{lll} \rho_{ij}\rho_{ik} = q\,\rho_{ik}\rho_{ij}\ , & \rho_{ik}\rho_{hk} = q\,\rho_{hk}\rho_{ik} & \forall\, j<k,\ i<h \\ \rho_{il}\rho_{jk} = \rho_{jk}\rho_{il}\ , & \rho_{ik}\rho_{jl} - \rho_{jl}\rho_{ik} = \left(q-q^{-1}\right)\,\rho_{il}\rho_{jk} & \forall\, i<j,\ k<l \\ det_q(\rho_{ij}) := \sum_{\sigma\in S_n} (-q)^{l(\sigma)} \rho_{1,\sigma(1)}\rho_{2,\sigma(2)}\cdots\rho_{n,\sigma(n)} = 1\ . & & \end{array}$$

This is a Hopf algebra, with comultiplication, counit and antipode given by

$$\Delta(\rho_{ij}) = \sum_{k=1}^{n} \rho_{ik}\otimes\rho_{kj}\ ,\quad \epsilon(\rho_{ij}) = \delta_{ij}\ ,\quad S(\rho_{ij}) = (-q)^{i-j}\, det_q\Big((\rho_{hk})_{h\neq j}^{k\neq i} \Big)$$

for all $i,j=1,\dots,n$. Let $\mathbb{F}_q[SL_n] := F(R)\otimes_R F_q[SL_n]$. The set of ordered monomials

$$M \ := \ \left\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} \rho_{hk}^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \ \middle|\ N_{st}\in\mathbb{N}\ \ \forall\, s,t\ ;\ \min\big\{N_{1,1},\dots,N_{n,n}\big\} = 0 \right\} \tag{5.4}$$

is an R–basis of $F_q[SL_n]$ and an $F(R)$–basis of $\mathbb{F}_q[SL_n]$ (cf. [Ga2], Theorem 7.4, suitably adapted to $F_q[SL_n]$). Moreover, $F_q[SL_n]$ is a QFA (at $\hbar = q-1$), with $F_q[SL_n] \xrightarrow{\ q\to 1\ } F[SL_n]$.

5.7 Computation of $F_q[G]^\vee$ and specialization $F_q[G]^\vee \xrightarrow{\ q\to 1\ } U(\mathfrak{g}^\times)$. In this section we compute $F_q[G]^\vee$ and its semiclassical limit (= specialization at $q=1$). Note that

$$M' \ := \ \left\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} (\rho_{hk}-1)^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \ \middle|\ N_{st}\in\mathbb{N}\ \ \forall\, s,t\ ;\ \min\big\{N_{1,1},\dots,N_{n,n}\big\} = 0 \right\}$$

is an R–basis of $F_q[SL_n]$ and an $F(R)$–basis of $\mathbb{F}_q[SL_n]$; then, from the definition of the counit, it follows that $M'\setminus\{1\}$ is an R–basis of *Ker*$\big(\epsilon : F_q[SL_n] \longrightarrow R\big)$. Now, by definition $I := Ker\Big(F_q[SL_n] \xrightarrow{\ \epsilon\ }\!\!\!\!\twoheadrightarrow R \xrightarrow{\ q\mapsto 1\ }\!\!\!\!\twoheadrightarrow \Bbbk \Big)$, whence $I = Ker(\epsilon) + (q-1)\cdot F_q[SL_n]$; therefore $\big(M'\setminus\{1\}\big)\cup\big\{(q-1)\cdot 1\big\}$ is an R–basis of I, hence $(q-1)^{-1} I$ has R–basis $(q-1)^{-1}\cdot\big(M'\setminus\{1\}\big)\cup\{1\}$. The outcome is that $F_q[SL_n]^\vee := \sum_{n\geq 0} \Big((q-1)^{-1} I\Big)^n$ is just the unital R–subalgebra of $\mathbb{F}_q[SL_n]$ generated by $\left\{ r_{ij} := \dfrac{\rho_{ij}-\delta_{ij}}{q-1} \ \middle|\ i,j=1,\dots,n \right\}$. Then one can directly show that this is a Hopf algebra, and that $F_q[SL_n]^\vee \xrightarrow{\ q\to 1\ } U(\mathfrak{sl}_n{}^*)$ as predicted by Theorem 2.2. Details can be found in [Ga2], §§ 2, 4, looking at the algebra $\widetilde{F}_q[SL_n]$ considered therein, up to the following changes. The algebra which is considered in [*loc. cit.*] has generators $\left(1+q^{-1}\right)^{\delta_{ij}} \dfrac{\rho_{ij}-\delta_{ij}}{q-q^{-1}}$ $(i,j=1,\dots,n)$ instead of our r_{ij}'s (they coincide iff $i=j$) and also generators $\rho_{ii} = 1+(q-1)\,r_{ii}$ $(i=1,\dots,n)$; then the presentation in §2.8 of [*loc. cit.*] must be changed accordingly; computing the specialization then goes exactly the same, and gives the same result — specialized generators are rescaled, though, compared with the standard ones given in [*loc. cit.*], §1.

We sketch the case of $n=2$ (see also [FG]).

Using notation $\mathrm{a} := \rho_{1,1}$, $\mathrm{b} := \rho_{1,2}$, $\mathrm{c} := \rho_{2,1}$, $\mathrm{d} := \rho_{2,2}$, we have the relations

$$\begin{array}{llll} \mathrm{a\,b} = q\,\mathrm{b\,a}\,, & \mathrm{a\,c} = q\,\mathrm{c\,a}\,, & \mathrm{b\,d} = q\,\mathrm{d\,b}\,, & \mathrm{c\,d} = q\,\mathrm{d\,c}\,, \\ \mathrm{b\,c} = \mathrm{c\,b}\,, & \mathrm{a\,d} - \mathrm{d\,a} = \left(q-q^{-1}\right)\mathrm{b\,c}\,, & \mathrm{a\,d} - q\,\mathrm{b\,c} = 1 & \end{array}$$

holding in $F_q[SL_2]$ and in $\mathbb{F}_q[SL_2]$, with

$$\Delta(\mathrm{a}) = \mathrm{a} \otimes \mathrm{a} + \mathrm{b} \otimes \mathrm{c}\,,\ \Delta(\mathrm{b}) = \mathrm{a} \otimes \mathrm{b} + \mathrm{b} \otimes \mathrm{d}\,,\ \Delta(\mathrm{c}) = \mathrm{c} \otimes \mathrm{a} + \mathrm{d} \otimes \mathrm{c}\,,\ \Delta(\mathrm{d}) = \mathrm{c} \otimes \mathrm{b} + \mathrm{d} \otimes \mathrm{d}$$
$$\epsilon(\mathrm{a}) = 1\,,\ \epsilon(\mathrm{b}) = 0\,,\ \epsilon(\mathrm{c}) = 0\,,\ \epsilon(\mathrm{d}) = 1\,,\quad S(\mathrm{a}) = \mathrm{d}\,,\ S(\mathrm{b}) = -q^{-1}\mathrm{b}\,,\ S(\mathrm{c}) = -q^{+1}\mathrm{c}\,, S(\mathrm{d}) = \mathrm{a}\,.$$

Then the elements $H_+ := r_{1,1} = \dfrac{\mathrm{a}-1}{q-1}$, $E := r_{1,2} = \dfrac{\mathrm{b}}{q-1}$, $F := r_{2,1} = \dfrac{\mathrm{c}}{q-1}$ and $H_- := r_{2,2} = \dfrac{\mathrm{d}-1}{q-1}$ generate $F_q[SL_2]^\vee$. Moreover, these generators have relations

$$H_+E = q\,EH_+ + E\,,\ \ H_+F = q\,FH_+ + F\,,\ \ EH_- = q\,H_-E + E\,,\ \ FH_- = q\,H_-F + F\,,$$
$$EF = FE\,,\quad H_+H_- - H_-H_+ = \big(q-q^{-1}\big)EF\,,\quad H_- + H_+ = (q-1)\big(q\,EF - H_+H_-\big)$$

and Hopf operations given by

$$\Delta(H_+) = H_+ \otimes 1 + 1 \otimes H_+ + (q-1)\big(H_+ \otimes H_+ + E \otimes F\big)\,,\quad \epsilon(H_+) = 0\,,\quad S(H_+) = H_-$$
$$\Delta(E) = E \otimes 1 + 1 \otimes E + (q-1)\big(H_+ \otimes E + E \otimes H_-\big)\,,\qquad \epsilon(E) = 0\,,\qquad S(E) = -q^{-1}E$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F + (q-1)\big(F \otimes H_+ + H_- \otimes F\big)\,,\qquad \epsilon(F) = 0\,,\qquad S(F) = -q^{+1}F$$
$$\Delta(H_-) = H_- \otimes 1 + 1 \otimes H_- + (q-1)\big(H_- \otimes H_- + F \otimes E\big)\,,\quad \epsilon(H_-) = 0\,,\quad S(H_-) = H_+$$

from which one easily checks that $F_q[SL_2]^\vee \xrightarrow{\ q\to 1\ } U(\mathfrak{sl}_2^{\,*})$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$F_q[SL_2]^\vee \Big/ (q-1)\,F_q[SL_2]^\vee \xrightarrow{\ \cong\ } U(\mathfrak{sl}_2^{\,*})$$

exists, given by: $H_\pm \mod (q-1) \mapsto \pm\mathrm{h}$, $E \mod (q-1) \mapsto \mathrm{e}$, $F \mod (q-1) \mapsto \mathrm{f}$; that is, $F_q[SL_2]^\vee$ specializes to $U(\mathfrak{sl}_2^{\,*})$ *as a co-Poisson Hopf algebra*, q.e.d.

Finally, the general case of any semisimple group $G = G^\tau$, with the Poisson structure induced from the Lie bialgebra structure of $\mathfrak{g} = \mathfrak{g}^\tau$, can be treated in a different way. Following [Ga1], §§5–6, $\mathbb{F}_q[G]$ can be embedded into a (topological) Hopf algebra $\mathbb{U}_q(\mathfrak{g}^*) = \mathbb{U}_{q,\varphi}^M(\mathfrak{g}^*)$, so that the image of the integer form $F_q[G]$ lies into a suitable (topological) integer form $\mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)$ of $\mathbb{U}_q(\mathfrak{g}^*)$. Now, the analysis given in [*loc. cit.*], when carefully read, shows that $F_q[G]^\vee = \mathbb{F}_q[G] \cap \mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)^\vee$; moreover, the latter (intersection) algebra "almost" coincides — it is its closure in a suitable topology — with the integer form $\mathcal{F}_q[G]$ considered in [*loc. cit.*]: in particular, they have the same specialization at $q = 1$. Since in addition $\mathcal{F}_q[G]$ does specialize to $U(\mathfrak{g}^*)$, the same is true for $F_q[G]^\vee$, q.e.d.

The last point to stress is that, once more, the whole analysis above is valid for $p := \mathit{Char}\,(\Bbbk) \geq 0$, i.e. also for $p > 0$, which was not granted by Theorem 2.2.

5.8 The identity $\big(F_q[G]^\vee\big)' = F_q[G]$. In this section we verify the validity of that part of Theorem 2.2*(b)* claiming that $H \in \mathcal{QFA} \Longrightarrow \big(H^\vee\big)' = H$ for $H = F_q[G]$ as above; moreover we show that this holds for $p > 0$ too. We begin with $G = SL_n$.

From $\Delta(\rho_{ij}) = \sum_{k=1}^{n} \rho_{i,k} \otimes \rho_{k,j}$, we get $\Delta^N(\rho_{ij}) = \sum_{k_1,\dots,k_{N-1}=1}^{n} \rho_{i,k_1} \otimes \rho_{k_1,k_2} \otimes \cdots \otimes \rho_{k_{N-1},j}$, by repeated iteration, whence a simple computation yields

$$\delta_N(r_{ij}) = \sum_{k_1,\dots,k_{N-1}=1}^{n} (q-1)^{-1} \cdot \big((q-1)\, r_{i,k_1} \otimes (q-1)\, r_{k_1,k_2} \otimes \cdots \otimes (q-1)\, r_{k_{N-1},j}\big) \qquad \forall\ i,j$$

so that

$$\delta_N\big((q-1)r_{ij}\big) \in (q-1)^N F_q[SL_n]^\vee \setminus (q-1)^{N+1} F_q[SL_n]^\vee \qquad \forall\ i,j\,. \tag{5.5}$$

Now consider $M' := \Big\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} (\rho_{hk}-1)^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \,\Big|\, N_{st} \in \mathbb{N}\ \forall\, s,t\,;\ \min_i \big\{N_{i,i}\big\} = 0 \Big\}$: since this is an R–basis of $F_q[SL_n]$, we have also that

$$M'' := \Big\{ \prod_{i>j} r_{ij}^{N_{ij}} \prod_{h=k} r_{hk}^{N_{hk}} \prod_{l<m} r_{lm}^{N_{lm}} \,\Big|\, N_{st} \in \mathbb{N}\ \ \forall\, s,t\,;\ \min\big\{ N_{1,1}, \dots, N_{n,n} \big\} = 0 \Big\}$$

is an R–basis of $F_q[SL_n]^\vee$. This and (5.5) above imply that $\big(F_q[SL_n]^\vee\big)'$ is the unital R–subalgebra of $\mathbb{F}_q[SL_n]$ generated by the set $\big\{ (q-1)r_{ij} \,\big|\, i,j = 1,\dots,n \big\}$; since $(q-1)\, r_{ij} = \rho_{ij} - \delta_{ij}$, the latter algebra does coincide with $F_q[SL_n]$, as expected.

For the general case of any semisimple group $G = G^\tau$, the result can be obtained again by looking at the immersions $\mathbb{F}_q[G] \subseteq \mathbb{U}_q(\mathfrak{g}^*)$ and $F_q[G] \subseteq \mathcal{U}_{q,\varphi}^{M}(\mathfrak{g}^*)$, and at the identity $F_q[G]^\vee = \mathbb{F}_q[G] \cap \mathcal{U}_{q,\varphi}^{M}(\mathfrak{g}^*)^\vee$ (cf. §5.6). If we go and compute $\Big(\mathcal{U}_{q,\varphi}^{M}(\mathfrak{g}^*)^\vee\Big)'$ (noting that $\big(\mathcal{U}_{q,\varphi}^{M}(\mathfrak{g}^*)\big)^\vee$ is a QrUEA), we have to apply the like methods as for $U_q(\mathfrak{g})'$, thus finding a similar result; this and the identity $F_q[G]^\vee = \mathbb{F}_q[G] \cap \mathcal{U}_{q,\varphi}^{M}(\mathfrak{g}^*)^\vee$ eventually yield $\Big(F_q[G]^\vee\Big)' = F_q[G]$.

Is is worth pointing out once more that the previous analysis is valid for $p := \mathit{Char}(\Bbbk) \geq 0$, i.e. also for $p > 0$, so the outcome is stronger than what ensured by Theorem 2.2.

Remark: Formula (5.4) gives an explicit R–basis M of $F_q[SL_2]$. By direct computation one sees that $\delta_n(\mu) \in F_q[SL_2]^{\otimes n} \setminus (q-1)\, F_q[SL_2]^{\otimes n}$ for all $\mu \in M \setminus \{1\}$ and $n \in \mathbb{N}$, whence $F_q[SL_2]' = R \cdot 1$, which implies $\big(F_q[SL_2]'\big)_F = F(R) \cdot 1 \subsetneqq \mathbb{F}_q[SL_2]$ and also $\big(F_q[SL_2]'\big)^\vee = R \cdot 1 \subsetneqq F_q[SL_2]$. This yields a counterexample to part of Theorem 2.2*(b)*.

5.9 Drinfeld's functors and L–operators in $U_q(\mathfrak{g})$ for classical $\mathfrak{g}$. Let now $\Bbbk$ have zero characteristic, and let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\Bbbk$ whose simple Lie subalgebra are all of classical type. It is known from [FRT2] that in this case $\mathbb{U}_q^P(\mathfrak{g})$ (where the subscript P means that we are taking a "simply-connected" quantum group) admits an alternative presentation, in which the generators are the so-called L–operators, denoted $l_{i,j}^{(\varepsilon)}$ with $\varepsilon = \pm 1$ and i, j ranging in a suitable set of indices (see [FRT2], §2). Now, if we consider instead the subalgebra of $\mathbb{U}_q^P(\mathfrak{g})$, call it H, generated by the L–operators *over* R, we get at once from the very description of the relations between the $l_{i,j}^{(\varepsilon)}$'s given in [FRT2] that H is a *Hopf* R–subalgebra of $\mathbb{U}_q^P(\mathfrak{g})$, and more precisely it is a QFA for the connected simply-connected dual Poisson group $G^\star$.

When computing $H^\vee$, it is generated by the elements $(q-1)^{-1} l_{i,j}^{(\varepsilon)}$; even more, the elements $(q-1)^{-1} l_{i,i+1}^{(+)}$ and $(q-1)^{-1} l_{i+1,i}^{(-)}$ are enough to generate. Now, Theorem 12 in [FRT2] shows that these latter generators are simply multiples of the Chevalley generators of $U_q^P(\mathfrak{g})$ (in the sense of Jimbo, Drinfeld, etc.), by a coefficient of type $\pm q^s \big(1+q^{-1}\big)$ for some $s \in \mathbb{Z}$: this proves directly that $H^\vee$ is a QrUEA associated to $\mathfrak{g}$, that is the dual Lie bialgebra of $G^\star$, as prescribed by Theorem 2.2. Conversely, if we start from $U_q^P(\mathfrak{g})$, again Theorem 12 of [FRT2] shows that the $\big(q-q^{-1}\big)^{-1} l_{i,j}^{(\varepsilon)}$'s are quantum root vectors in $U_q^P(\mathfrak{g})$. Then when computing $U_q^P(\mathfrak{g})'$ we can shorten a lot the analysis in §5.3, because the explicit expression of the coproduct on the L–operators given in [FRT2] — roughly, Δ is given on them by a standard "matrix coproduct" — tells us directly that all the $\big(1+q^{-1}\big)^{-1} l_{i,j}^{(\varepsilon)}$'s do belong to $U_q^P(\mathfrak{g})'$, and again by a PBW argument we conclude that $U_q^P(\mathfrak{g})'$ is generated by these rescaled L–operators, i.e. the $\big(1+q^{-1}\big)^{-1} l_{i,j}^{(\varepsilon)}$.

Therefore, we can say in short that shifting from H to $H^\vee$ or from $U_q^P(\mathfrak{g})$ to $U_q^P(\mathfrak{g})'$ essentially amounts — up to rescaling by irrelevant factors (in that they do not vanish at $q=1$) — to switching from the presentation of $\mathbb{U}_q^P(\mathfrak{g})$ via L–operators (after [FRT2]) to the presentation of Serre-Chevalley type (after Drinfeld and Jimbo), and conversely. See also [Ga8] for the cases $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathfrak{g} = \mathfrak{sl}_n$.

5.10 The cases $U_q(\mathfrak{gl}_n)$, $F_q[GL_n]$ and $F_q[M_n]$. In [Ga2], §3.2, a certain algebra $U_q(\mathfrak{gl}_n)$ is considered as a quantization of $\mathfrak{gl}_n$; due to their strict relationship, from the analysis for $\mathfrak{sl}_n$ one argues a description of $U_q(\mathfrak{gl}_n)'$ and its specialization at $q=1$, and also verifies that $\big(U_q(\mathfrak{gl}_n)'\big)^\vee = U_q(\mathfrak{gl}_n)$.

Similarly, we can consider the unital associative R–algebra $F_q[M_n]$ with generators ρ_{ij} ($i, j = 1, \dots, n$) and relations $\rho_{ij}\rho_{ik} = q\,\rho_{ik}\rho_{ij}$, $\rho_{ik}\rho_{hk} = q\,\rho_{hk}\rho_{ik}$ (for all $j<k$, $i<h$), $\rho_{il}\rho_{jk} = \rho_{jk}\rho_{il}$, $\rho_{ik}\rho_{jl} - \rho_{jl}\rho_{ik} = \big(q-q^{-1}\big)\,\rho_{il}\rho_{jk}$ (for all $i<j$, $k<l$) — i.e. like for SL_n, but for skipping the last relation. This is the celebrated standard quantization of $F[M_n]$, the function algebra of the variety M_n of $(n \times n)$–matrices over $\Bbbk$: it is a $\Bbbk$–bialgebra, whose structure is given by formulas $\Delta(\rho_{ij}) = \sum_{k=1}^n \rho_{ik} \otimes \rho_{kj}$, $\epsilon(\rho_{ij}) = \delta_{ij}$ (for all $i, j = 1, \dots, n$) again, but it is *not a Hopf algebra*. The quantum determinant $det_q(\rho_{ij}) := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} \rho_{1,\sigma(1)}\, \rho_{2,\sigma(2)} \cdots \rho_{n,\sigma(n)}$ is central in $F_q[M_n]$, so by standard theory we can extend $F_q[M_n]$ by adding a formal inverse to $det_q(\rho_{ij})$, thus getting a larger algebra $F_q[GL_n] := F_q[M_n]\big[det_q(\rho_{ij})^{-1}\big]$: this is now a Hopf algebra, with antipode $S(\rho_{ij}) = (-q)^{i-j}\, det_q\Big(\big(\rho_{hk}\big)_{h\neq j}^{k\neq i}\Big)$ (for all $i, j = 1, \dots, n$), the well-known standard quantization of $F[GL_n]$, due to Manin (see [Ma]).

Applying Drinfeld's functor $(\ )^\vee$ w.r.t. $\hbar := (q-1)$ at $F_q[GL_n]$ we can repeat step by step the analysis made for $F_q[SL_n]$: then $F_q[GL_n]^\vee$ is generated by the r_{ij}'s and $(q-1)^{-1}\big(det_q(\rho_{ij})-1\big)$, the sole real difference being the lack of the relation $det_q(\rho_{ij}) = 1$, which implies one relation less among the r_{ij}'s inside $F_q[GL_n]^\vee$, hence also one relation less among their cosets modulo $(q-1)$. The outcome is pretty similar, in particular $F_q[GL_n]^\vee\Big|_{q=1} = U(\mathfrak{gl}_n^*)$ (cf. [Ga2], §6.2). Even more, we can do the same with $F_q[M_n]$: things are even easier, because we have only the r_{ij}'s alone which generate $F_q[M_n]^\vee$, with no relation coming from the relation $det_q(\rho_{ij}) = 1$; nevertheless at $q=1$ the relations among the cosets of the r_{ij}'s are exactly the same as in the case of $F_q[GL_n]^\vee\Big|_{q=1}$, whence we get $F_q[M_n]^\vee\Big|_{q=1} = U(\mathfrak{gl}_n^{\,*})$. *In particular,* $F_q[M_n]^\vee\Big|_{q=1}$ *is a Hopf algebra, although both* $F_q[M_n]$ *and* $F_q[M_n]^\vee$ *are only bialgebras, not Hopf algebras: this gives a non-trivial explicit example of how Theorem 2.2 may be improved.* The general result in this sense is Theorem 4.9 in [Ga5].

Finally, an analysis of the relationship between Drinfeld functors and L–operators about $\mathbb{U}_q^P(\mathfrak{gl}_n)$ can be done again, exactly like in §7.9, leading to entirely similar results.

§ 6 Third example: the three-dimensional Euclidean group E_2

6.1 The classical setting. Let $\Bbbk$ be any field of characteristic $p \geq 0$. Let $G := E_2(\Bbbk) \equiv E_2$, the three-dimensional Euclidean group; its tangent Lie algebra $\mathfrak{g} = \mathfrak{e}_2$ is generated by f , h , e with relations $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = 0$. The formulas $\delta(f) = h \otimes f - f \otimes h$, $\delta(h) = 0$, $\delta(e) = h \otimes e - e \otimes h$, make $\mathfrak{e}_2$ into a Lie bialgebra, hence E_2 into a Poisson group. These also give a presentation of the co-Poisson Hopf algebra $U(\mathfrak{e}_2)$ (with standard Hopf structure). If $p > 0$, we consider on $\mathfrak{e}_2$ the p–operation given by $e^{[p]} = 0$, $f^{[p]} = 0$, $h^{[p]} = h$.

On the other hand, the function algebra $F[E_2]$ is the unital associative commutative $\Bbbk$–algebra with generators b, $a^{\pm 1}$, c, with Poisson Hopf algebra structure given by

$$\Delta(b) = b \otimes a^{-1} + a \otimes b \,, \qquad \Delta\big(a^{\pm 1}\big) = a^{\pm 1} \otimes a^{\pm 1} \,, \qquad \Delta(c) = c \otimes a + a^{-1} \otimes c$$

$$\epsilon(b) = 0 \,, \;\; \epsilon\big(a^{\pm 1}\big) = 1 \,, \;\; \epsilon(c) = 0 \,, \qquad S(b) = -b \,, \;\; S\big(a^{\pm 1}\big) = a^{\mp 1} \,, \;\; S(c) = -c$$

$$\big\{a^{\pm 1}, b\big\} \, = \, \pm\, a^{\pm 1} b \,, \qquad \big\{a^{\pm 1}, c\big\} \, = \, \pm\, a^{\pm 1} c \,, \qquad \{b, c\} \, = \, 0$$

We can realize E_2 as $E_2 = \big\{ (b,a,c) \,\big|\, b\,,c \in k\,, a \in \Bbbk \setminus \{0\} \big\}$, with group operation

$$(b_1, a_1, c_1) \cdot (b_2, a_2, c_2) \; = \; \big(b_1 a_2^{-1} + a_1 b_2 \,,\; a_1 a_2 \,,\; c_1 a_2 + a_1^{-1} c_2 \big) \;;$$

in particular the centre of E_2 is simply $Z := \big\{ (0,1,0), (0,-1,0) \big\}$, so there is only one other connected Poisson group having $\mathfrak{e}_2$ as Lie bialgebra, namely the adjoint group ${}_aE_2 := E_2 \Big/ Z$ (the left subscript a stands for “adjoint”). Then $F[{}_aE_2]$ coincides with the Poisson Hopf subalgebra of $F[{}_aE_2]$ spanned by products of an even number of generators, i.e. monomials of even degree: as a unital subalgebra, this is generated by $b\,a$, $a^{\pm 2}$, and $a^{-1}c$.

The dual Lie bialgebra $\mathfrak{g}^* = \mathfrak{e}_2{}^*$ is the Lie algebra with generators $\mathrm{f}, \mathrm{h}, \mathrm{e}$, and relations $[\mathrm{h},\mathrm{e}] = 2\mathrm{e}$, $[\mathrm{h},\mathrm{f}\,] = 2\mathrm{f}$, $[\mathrm{e},\mathrm{f}\,] = 0$, with Lie cobracket given by $\delta(\mathrm{f}\,) = \mathrm{f} \otimes \mathrm{h} - \mathrm{h} \otimes \mathrm{f}$, $\delta(\mathrm{h}) = 0$, $\delta(\mathrm{e}) = \mathrm{h} \otimes \mathrm{e} - \mathrm{e} \otimes \mathrm{h}$ (we choose as generators $\mathrm{f} := f^*$, $\mathrm{h} := 2h^*$, $\mathrm{e} := e^*$, where $\big\{ f^*, h^*, e^* \big\}$ is the basis of $\mathfrak{e}_2{}^*$ which is dual to the basis $\{f, h, e\}$ of $\mathfrak{e}_2$). If $p > 0$, the p–operation of $\mathfrak{e}_2{}^*$ reads $\mathrm{e}^{[p]} = 0$, $\mathrm{f}^{[p]} = 0$, $\mathrm{h}^{[p]} = \mathrm{h}$. This again gives a presentation of $U\left(\mathfrak{e}_2{}^*\right)$ too. The simply connected algebraic Poisson group with tangent Lie bialgebra $\mathfrak{e}_2{}^*$ can be realized as the group of pairs of matrices

$${}_sE_2{}^\star \;\equiv\; {}_sE_2{}^* := \left\{ \left(\begin{pmatrix} z^{-1} & 0 \\ y & z \end{pmatrix}, \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \right) \,\middle|\, x, y \in k, z \in \Bbbk \setminus \{0\} \right\} ;$$

this group has centre $Z := \big\{(I,I), (-I,-I)\big\}$, so there is only one other (Poisson) group with Lie (bi)algebra ${\mathfrak{e}_2}^*$, namely the adjoint group ${}_a{E_2}^* := {}_s{E_2}^* \Big/ Z$.

Therefore $F\big[{}_s{E_2}^*\big]$ is the unital associative commutative $\Bbbk$–algebra with generators x, $z^{\pm 1}$, y, with Poisson Hopf structure given by

$$\Delta(x) = x \otimes z^{-1} + z \otimes x \; , \qquad \Delta\big(z^{\pm 1}\big) = z^{\pm 1} \otimes z^{\pm 1} \; , \qquad \Delta(y) = y \otimes z^{-1} + z \otimes y$$
$$\epsilon(x) = 0 \; , \;\; \epsilon\big(z^{\pm 1}\big) = 1 \; , \;\; \epsilon(y) = 0 \; , \qquad S(x) = -x \; , \;\; S\big(z^{\pm 1}\big) = z^{\mp 1} \; , \;\; S(y) = -y$$
$$\{x,y\} \, = \, 0 \; , \qquad \big\{z^{\pm 1}, x\big\} \, = \, \pm\, z^{\pm 1} x \; , \qquad \big\{z^{\pm 1}, y\big\} \, = \, \mp\, z^{\pm 1} y$$

(Remark: with respect to this presentation, we have $\mathrm{f} = \partial_y\big|_e$, $\mathrm{h} = z\,\partial_z\big|_e$, $\mathrm{e} = \partial_x\big|_e$, where e is the identity element of ${}_s{E_2}^*$). Moreover, $F\big[{}_a{E_2}^*\big]$ can be identified with the Poisson Hopf subalgebra of $F\big[{}_s{E_2}^*\big]$ spanned by products of an even number of generators, i.e. monomials of even degree: this is generated, as a unital subalgebra, by $x\,z$, $z^{\pm 2}$ and $z^{-1}y$.

6.2 The QrUEAs $U_q^s(\mathfrak{e}_2)$ and $U_q^a(\mathfrak{e}_2)$. We turn now to quantizations: the situation is much similar to the case of $\mathfrak{sl}_2$, so we follow the same pattern; nevertheless, now we stress a bit more the occurrence of different groups sharing the same tangent Lie bialgebra.

Let R be a 1dD, and let $\hbar \in R \setminus \{0\}$ and $q := \hbar + 1 \in R$ be like in §5.2.

Let $\mathbb{U}_q(\mathfrak{g}) = \mathbb{U}_q^s(\mathfrak{e}_2)$ (where the superscript s stands for "simply connected") be the associative unital $F(R)$–algebra with generators F, $L^{\pm 1}$, E, and relations

$$LL^{-1} = 1 = L^{-1}L \; , \quad L^{\pm 1}F = q^{\mp 1} F L^{\pm 1} \; , \quad L^{\pm 1}E = q^{\pm 1} E L^{\pm 1} \; , \quad EF = FE \; .$$

This is a Hopf algebra, with Hopf structure given by

$$\Delta(F) = F \otimes L^{-2} + 1 \otimes F \; , \quad \Delta\big(L^{\pm 1}\big) = L^{\pm 1} \otimes L^{\pm 1} \; , \quad \Delta(E) = E \otimes 1 + L^2 \otimes E$$
$$\epsilon(F) = 0 \; , \; \epsilon\big(L^{\pm 1}\big) = 1 \; , \; \epsilon(E) = 0 \; , \quad S(F) = -FL^2 \; , \; S\big(L^{\pm 1}\big) = L^{\mp 1} \; , \; S(E) = -L^{-2}E \; .$$

Then let $U_q^s(\mathfrak{e}_2)$ be the R–subalgebra of $\mathbb{U}_q^s(\mathfrak{e}_2)$ generated by F, $D_\pm := \dfrac{L^{\pm 1} - 1}{q - 1}$, E. From the definition of $\mathbb{U}_q^s(\mathfrak{e}_2)$ one gets a presentation of $U_q^s(\mathfrak{e}_2)$ as the associative unital algebra with generators F, $D_\pm$, E and relations

$$D_+E = qED_+ + E \; , \quad FD_+ = qD_+F + F \; , \quad ED_- = qD_-E + E \; , \quad D_-F = qFD_- + F$$
$$EF = FE \; , \qquad D_+D_- = D_-D_+ \; , \qquad D_+ + D_- + (q-1)D_+D_- = 0$$

with a Hopf structure given by

$$\Delta(E) = E \otimes 1 + 1 \otimes E + 2(q-1)D_+ \otimes E + (q-1)^2 \cdot D_+^2 \otimes E$$
$$\Delta(D_\pm) = D_\pm \otimes 1 + 1 \otimes D_\pm + (q-1) \cdot D_\pm \otimes D_\pm$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F + 2(q-1) F \otimes D_- + (q-1)^2 \cdot F \otimes D_-^2$$

$$\epsilon(E) = 0 \,, \qquad S(E) = -E - 2(q-1)D_-E - (q-1)^2 D_-^2 E$$
$$\epsilon(D_\pm) = 0 \,, \qquad S(D_\pm) = D_\mp$$
$$\epsilon(F) = 0 \,, \qquad S(F) = -F - 2(q-1)FD_+ - (q-1)^2 FD_+^2 \,.$$

The "adjoint version" of $\mathbb{U}_q^s(\mathfrak{e}_2)$ is the unital subalgebra $\mathbb{U}_q^a(\mathfrak{e}_2)$ generated by F, $K^{\pm 1} := L^{\pm 2}$, E, which is clearly a Hopf subalgebra. It also has an R–integer form $U_q^a(\mathfrak{e}_2)$, the unital R–subalgebra generated by F, $H_\pm := \dfrac{K^{\pm 1} - 1}{q-1}$, E: this has relations

$$EF = FE \,,\;\; H_+E = q^2 EH_+ + (q+1)E \,,\;\; FH_+ = q^2 H_+F + (q+1)F \,,\;\; H_+H_- = H_-H_+$$
$$EH_- = q^2 H_-E + (q+1)E \,,\;\; H_-F = q^2 FH_- + (q+1)F \,,\;\; H_+ + H_- + (q-1)H_+H_- = 0$$

and it is a Hopf subalgebra, with Hopf operations given by

$$\Delta(E) = E \otimes 1 + 1 \otimes E + (q-1) \cdot H_+ \otimes E \,, \qquad \epsilon(E) = 0 \,, \qquad S(E) = -E - (q-1)H_-E$$
$$\Delta(H_\pm) = H_\pm \otimes 1 + 1 \otimes H_\pm + (q-1) \cdot H_\pm \otimes H_\pm \,, \qquad \epsilon(H_\pm) = 0 \,, \qquad S(H_\pm) = H_\mp$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F + (q-1) \cdot F \otimes H_- \,, \qquad \epsilon(F) = 0 \,, \qquad S(F) = -F - (q-1)FH_+ \,.$$

It is easy to check that $U_q^s(\mathfrak{e}_2)$ is a QrUEA, whose semiclassical limit is $U(\mathfrak{e}_2)$: in fact, mapping the generators $F \mod (q-1)$, $D_\pm \mod (q-1)$, $E \mod (q-1)$ respectively to f, $\pm h\big/2$, $e \in U(\mathfrak{e}_2)$ gives an isomorphism $U_q^s(\mathfrak{e}_2)\Big/(q-1)\,U_q^s(\mathfrak{e}_2) \xrightarrow{\;\cong\;} U(\mathfrak{e}_2)$ of co-Poisson Hopf algebras. Similarly, $U_q^a(\mathfrak{e}_2)$ is a QrUEA too, with semiclassical limit $U(\mathfrak{e}_2)$ again: here a co-Poisson Hopf algebra isomorphism $U_q^a(\mathfrak{e}_2)\Big/(q-1)\,U_q^a(\mathfrak{e}_2) \cong U(\mathfrak{e}_2)$ is given mapping $F \mod (q-1)$, $H_\pm \mod (q-1)$, $E \mod (q-1)$ respectively to f, $\pm h$, $e \in U(\mathfrak{e}_2)$.

6.3 Computation of $U_q(\mathfrak{e}_2)'$ and specialization $U_q(\mathfrak{e}_2)' \xrightarrow{\;q\to 1\;} F\big[E_2^\star\big]$. This section is devoted to compute $U_q^s(\mathfrak{e}_2)'$ and $U_q^a(\mathfrak{e}_2)'$, and their specialization at $q=1$: everything goes on as in §5.3, so we can be more sketchy. From definitions we have, for any $n \in \mathbb{N}$, $\Delta^n(E) = \sum_{s=1}^n K^{\otimes (s-1)} \otimes E \otimes 1^{\otimes (n-s)}$, so $\delta_n(E) = (K-1)^{\otimes (n-1)} \otimes E = (q-1)^{n-1} \cdot H_+^{\otimes (n-1)} \otimes E$, whence $\delta_n\big((q-1)E\big) \in (q-1)^n\, U_q^a(\mathfrak{e}_2) \setminus (q-1)^{n+1}\, U_q^a(\mathfrak{e}_2)$ thus $(q-1)E \in U_q^a(\mathfrak{e}_2)'$, whereas $E \notin U_q^a(\mathfrak{e}_2)'$. Similarly, we have $(q-1)F$, $(q-1)H_\pm \in U_q^a(\mathfrak{e}_2)' \setminus (q-1)\,U_q^a(\mathfrak{e}_2)'$. Therefore $U_q^a(\mathfrak{e}_2)$ contains the subalgebra U' generated by $\dot{F} := (q-1)F$, $\dot{H}_\pm := (q-1)H_\pm$, $\dot{E} := (q-1)E$. On the other hand, $U_q^a(\mathfrak{e}_2)'$ is clearly the R–span of the set $\Big\{\, F^a H_+^b H_-^c E^d \,\Big|\, a,b,c,d \in \mathbb{N} \,\Big\}$: to be precise, the set

$$\Big\{\, F^a H_+^b K^{-[b/2]} E^d \,\Big|\, a,b,d \in \mathbb{N} \,\Big\} \;=\; \Big\{\, F^a H_+^b \big(1 + (q-1)H_-\big)^{[b/2]} E^d \,\Big|\, a,b,d \in \mathbb{N} \,\Big\}$$

is an R–basis of $U_q^a(\mathfrak{e}_2)'$; therefore, a straightforward computation shows that any element in $U_q^a(\mathfrak{e}_2)'$ does necessarily lie in U', thus $U_q^a(\mathfrak{e}_2)'$ coincides with U'. Moreover, since $\dot{H}_\pm = K^{\pm 1} - 1$, the unital algebra $U_q^a(\mathfrak{e}_2)'$ is generated by $\dot{F}$, $K^{\pm 1}$ and $\dot{E}$ as well.

The previous analysis — *mutatis mutandis* — ensures also that $U_q^s(\mathfrak{e}_2)'$ coincides with the unital R–subalgebra U'' of $\mathbb{U}_q^s(\mathfrak{e}_2)$ generated by $\dot{F} := (q-1)F$, $\dot{D}_\pm := (q-1)D_\pm$, $\dot{E} := (q-1)E$; in particular, $U_q^s(\mathfrak{e}_2)' \supset U_q^a(\mathfrak{e}_2)'$. Moreover, as $\dot{D}_\pm = L^{\pm 1} - 1$, the unital algebra $U_q^s(\mathfrak{e}_2)'$ is generated by $\dot{F}$, $L^{\pm 1}$ and $\dot{E}$ as well. Thus $U_q^s(\mathfrak{e}_2)'$ is the unital associative R–algebra with generators $\mathcal{F} := L\dot{F}$, $\mathcal{L}^{\pm 1} := L^{\pm 1}$, $\mathcal{E} := \dot{E}L^{-1}$ and relations

$$\mathcal{L}\mathcal{L}^{-1} = 1 = \mathcal{L}^{-1}\mathcal{L}\ , \qquad \mathcal{E}\,\mathcal{F} = \mathcal{F}\,\mathcal{E}\ , \qquad \mathcal{L}^{\pm 1}\mathcal{F} = q^{\mp 1}\mathcal{F}\mathcal{L}^{\pm 1}\ , \qquad \mathcal{L}^{\pm 1}\mathcal{E} = q^{\pm 1}\mathcal{E}\mathcal{L}^{\pm 1}$$

with Hopf structure given by

$$\Delta(\mathcal{F}) = \mathcal{F} \otimes \mathcal{L}^{-1} + \mathcal{L} \otimes \mathcal{F}\ , \qquad \Delta\big(\mathcal{L}^{\pm 1}\big) = \mathcal{L}^{\pm 1} \otimes \mathcal{L}^{\pm 1}\ , \qquad \Delta(\mathcal{E}) = \mathcal{E} \otimes \mathcal{L}^{-1} + \mathcal{L} \otimes \mathcal{E}$$
$$\epsilon(\mathcal{F}) = 0\ ,\ \ \epsilon\big(\mathcal{L}^{\pm 1}\big) = 1\ ,\ \ \epsilon(\mathcal{E}) = 0\ , \qquad S(\mathcal{F}) = -\mathcal{F}\ ,\ \ S\big(\mathcal{L}^{\pm 1}\big) = \mathcal{L}^{\mp 1}\ ,\ \ S(\mathcal{E}) = -\mathcal{E}\ .$$

As $q \to 1$, this yields a presentation of the function algebra $F\big[{}_sE_2^{\,*}\big]$, and the Poisson bracket that $F\big[{}_sE_2^{\,*}\big]$ earns from this quantization process coincides with the one coming from the Poisson structure on ${}_sE_2^{\,*}$: namely, there is a Poisson Hopf algebra isomorphism

$$U_q^s(\mathfrak{e}_2)' \Big/ (q-1)\, U_q^s(\mathfrak{e}_2)' \xrightarrow{\ \cong\ } F\big[{}_sE_2^{\,*}\big]$$

given by $\mathcal{E} \mod (q-1) \mapsto x$, $\mathcal{L}^{\pm 1} \mod (q-1) \mapsto z^{\pm 1}$, $\mathcal{F} \mod (q-1) \mapsto y$. That is, $U_q^s(\mathfrak{e}_2)'$ specializes to $F\big[{}_sE_2^{\,*}\big]$ *as a Poisson Hopf algebra*, as predicted by Theorem 2.2.

In the "adjoint case", from the definition of U' and from $U_q^a(\mathfrak{e}_2)' = U'$ we find that $U_q^a(\mathfrak{e}_2)'$ is the unital associative R–algebra with generators $\dot{F}$, $K^{\pm 1}$, $\dot{E}$ and relations

$$KK^{-1} = 1 = K^{-1}K\ , \quad \dot{E}\dot{F} = \dot{F}\dot{E}\ , \quad K^{\pm 1}\dot{F} = q^{\mp 2}\dot{F}K^{\pm 1}\ , \quad K^{\pm 1}\dot{E} = q^{\pm 2}\dot{E}K^{\pm 1}$$

with Hopf structure given by

$$\Delta\big(\dot{F}\big) = \dot{F} \otimes K^{-1} + 1 \otimes \dot{F}\ , \qquad \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\ , \qquad \Delta\big(\dot{E}\big) = \dot{E} \otimes 1 + K \otimes \dot{E}$$
$$\epsilon\big(\dot{F}\big) = 0\ ,\ \epsilon\big(K^{\pm 1}\big) = 1\ ,\ \epsilon\big(\dot{E}\big) = 0\ , \quad S\big(\dot{F}\big) = -\dot{F}K\ ,\ S\big(K^{\pm 1}\big) = K^{\mp 1}\ ,\ S\big(\dot{E}\big) = -K^{-1}\dot{E}\ .$$

The outcome is that there is a Poisson Hopf algebra isomorphism

$$U_q^a(\mathfrak{e}_2)' \Big/ (q-1)\, U_q^a(\mathfrak{e}_2)' \xrightarrow{\ \cong\ } F\big[{}_aE_2^{\,*}\big] \quad \Big(\subset F\big[{}_sE_2^{\,*}\big]\Big)$$

given by $\dot{E} \mod (q-1) \mapsto x\,z$, $K^{\pm 1} \mod (q-1) \mapsto z^{\pm 2}$, $\dot{F} \mod (q-1) \mapsto z^{-1}y$, which means $U_q^a(\mathfrak{e}_2)'$ specializes to $F\big[{}_aE_2^{\,*}\big]$ *as a Poisson Hopf algebra*, according to Theorem 2.2.

To finish with, note that *all this analysis (and its outcome) is entirely characteristic-free.*

6.4 The identity $\big(U_q(\mathfrak{e}_2)'\big)^\vee = U_q(\mathfrak{e}_2)$. This section goal is to check the part of Theorem 2.2*(b)* claiming that $H \in \mathcal{QrUEA} \implies \big(H'\big)^\vee = H$ both for $H = U_q^s(\mathfrak{e}_2)$ and $H = U_q^a(\mathfrak{e}_2)$. *In addition, our analysis work for all $p :=$ Char $(\Bbbk)$, thus giving a stronger result than Theorem 2.2(b).*

First, $U_q^s(\mathfrak{e}_2)'$ is clearly a free R–module, with basis $\Big\{ \mathcal{F}^a\mathcal{L}^d\mathcal{E}^c \,\Big|\, a, c \in \mathbb{N}, d \in \mathbb{Z} \Big\}$, hence $\mathbb{B} := \Big\{ \mathcal{F}^a\big(\mathcal{L}^{\pm 1} - 1\big)^b\mathcal{E}^c \,\Big|\, a, b, c \in \mathbb{N} \Big\}$, is an R–basis as well. Second, since $\epsilon(\mathcal{F}) = \epsilon\big(\mathcal{L}^{\pm 1} - 1\big) = \epsilon(\mathcal{E}) = 0$, the ideal $J :=$ *Ker*$\Big(\epsilon : U_q^s(\mathfrak{e}_2)' \longrightarrow R\Big)$ is the span of $\mathbb{B} \setminus \{1\}$. Therefore

$\big(U_q^s(\mathfrak{e}_2)'\big)^\vee = \sum_{n\geq 0}\Big((q-1)^{-1}J\Big)^n$ is generated by $(q-1)^{-1}\mathcal{F} = LF$, $(q-1)^{-1}(\mathcal{L}-1) = D_+$, $(q-1)^{-1}\big(\mathcal{L}^{-1}-1\big) = D_-$, $(q-1)^{-1}\mathcal{E} = EL^{-1}$, hence by F, $D_\pm$, E, so it coincides with $U_q^s(\mathfrak{e}_2)$.

The situation is entirely similar for the adjoint case: one simply has to change $\mathcal{F}$, $\mathcal{L}^{\pm 1}$, $\mathcal{E}$ respectively with $\dot{F}$, $K^{\pm 1}$, $\dot{E}$, and $D_\pm$ with $H_\pm$, then everything goes through as above.

6.5 The quantum hyperalgebra $\mathit{Hyp}_q(\mathfrak{e}_2)$. Like for semisimple groups, we can define "quantum hyperalgebras" for $\mathfrak{e}_2$ mimicking what done in §5.5. Namely, we can first define a Hopf $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbb{U}_q^s(\mathfrak{e}_2)$ whose specialization at $q=1$ is the Kostant-like $\mathbb{Z}$–integer form $U_\mathbb{Z}(\mathfrak{e}_2)$ of $U(\mathfrak{e}_2)$ (generated by divided powers, and giving the hyperalgebra *Hyp* $(\mathfrak{e}_2)$ over any field $\Bbbk$ by scalar extension, namely *Hyp* $(\mathfrak{e}_2) = \Bbbk \otimes_\mathbb{Z} U_\mathbb{Z}(\mathfrak{e}_2)$), and then take its scalar extension over R.

To be precise, let $\mathit{Hyp}_q^{s,\mathbb{Z}}(\mathfrak{e}_2)$ be the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbb{U}_q^s(\mathfrak{e}_2)$ (defined like above *but over* $\mathbb{Z}\big[q,q^{-1}\big]$) generated by the "quantum divided powers"

$$F^{(n)} := F^n\Big/[n]_q! \; , \qquad \binom{L\,;\,c}{n} := \prod_{r=1}^n \frac{q^{c+1-r}L-1}{q^r-1} \; , \qquad E^{(n)} := E^n\Big/[n]_q!$$

(for all $n\in\mathbb{N}$ and $c\in\mathbb{Z}$, with notation of §5.5) and by L^{-1}. Comparing with the case of $\mathfrak{sl}_2$ one easily sees that this is a Hopf subalgebra of $\mathbb{U}_q^s(\mathfrak{e}_2)$, and $\mathit{Hyp}_q^{s,\mathbb{Z}}(\mathfrak{e}_2)\Big|_{q=1} \cong U_\mathbb{Z}(\mathfrak{e}_2)$; thus $\mathit{Hyp}_q^s(\mathfrak{e}_2) := R\otimes_{\mathbb{Z}[q,q^{-1}]}\mathit{Hyp}_q^{s,\mathbb{Z}}(\mathfrak{e}_2)$ (for any R like in §6.2, with $\Bbbk := R\big/\hbar R$ and $p := \mathit{Char}(\Bbbk)$) specializes at $q=1$ to the $\Bbbk$–hyperalgebra *Hyp* $(\mathfrak{e}_2)$. In addition, among all the $\binom{L\,;\,c}{n}$'s it is enough to take only those with $c=0$. *From now on we assume* $p>0$.

Again a strict comparison with the $\mathfrak{sl}_2$ case shows us that $\mathit{Hyp}_q^s(\mathfrak{e}_2)'$ is the unital R–subalgebra of $\mathit{Hyp}_q^s(\mathfrak{e}_2)$ generated by L^{-1} and the "rescaled quantum divided powers" $(q-1)^nF^{(n)}$, $(q-1)^n\binom{L\,;\,0}{n}$ and $(q-1)^nE^{(n)}$ for all $n\in\mathbb{N}$. It follows that $\mathit{Hyp}_q^s(\mathfrak{e}_2)'\Big|_{q=1}$ is generated by the corresponding specializations of $(q-1)^{p^r}F^{(p^r)}$, $(q-1)^{p^r}\binom{L\,;\,0}{p^r}$ and $(q-1)^{p^r}E^{(p^r)}$ for all $r\in\mathbb{N}$: this proves that the spectrum of $\mathit{Hyp}_q^s(\mathfrak{e}_2)'\Big|_{q=1}$ has dimension 0 and height 1, and its cotangent Lie algebra has basis $\Big\{\, (q-1)^{p^r}F^{(p^r)},\, (q-1)^{p^r}\binom{L\,;\,0}{p^r},\, (q-1)^{p^r}E^{(p^r)} \mod (q-1)\,\mathit{Hyp}_q^s(\mathfrak{g})' \mod J^2 \;\Big|\; r\in\mathbb{N}\Big\}$ (where J is the augmentation ideal of $\mathit{Hyp}_q^s(\mathfrak{e}_2)'\Big|_{q=1}$, so that $J\big/J^2$ is the aforementioned cotangent Lie bialgebra). Moreover, $\big(\mathit{Hyp}_q^s(\mathfrak{e}_2)'\big)^\vee$ is generated by $(q-1)^{p^r-1}F^{(p^r)}$, $(q-1)^{p^r-1}\binom{L\,;\,0}{p^r}$, L^{-1} and $(q-1)^{p^r-1}E^{(p^r)}$ (for all $r\in\mathbb{N}$): in particular $\big(\mathit{Hyp}_q^s(\mathfrak{e}_2)'\big)^\vee \subsetneqq \mathit{Hyp}_q^s(\mathfrak{e}_2)$, and finally $\big(\mathit{Hyp}_q^s(\mathfrak{e}_2)'\big)^\vee\Big|_{q=1}$ is generated by the cosets modulo $(q-1)$ of the elements above, which in fact form a basis of the restricted Lie bialgebra $\mathfrak{k}$ such that $\big(\mathit{Hyp}_q^s(\mathfrak{e}_2)'\big)^\vee\Big|_{q=1} = \mathbf{u}(\mathfrak{k})$.

All this analysis was made starting from $\mathbb{U}_q^s(\mathfrak{e}_2)$, which gave "simply connected quantum objects". If we start instead from $\mathbb{U}_q^a(\mathfrak{e}_2)$, we get "adjoint quantum objects" following the same pattern but for replacing everywhere $L^{\pm 1}$ by $K^{\pm 1}$: apart from these changes, the analysis and its outcome will be exactly the same. Like for $\mathfrak{sl}_2$ (cf. §5.5), all the adjoint quantum objects — i.e. $\mathit{Hyp}_q^a(\mathfrak{e}_2)$, $\mathit{Hyp}_q^a(\mathfrak{e}_2)'$

and $\big(Hyp_q^a(\mathfrak{e}_2)'\big)^\vee$ — will be strictly contained in the corresponding simply connected quantum objects; nevertheless, the semiclassical limits will be the same in the case of $Hyp_q(\mathfrak{e}_2)$ (always yielding $Hyp(\mathfrak{e}_2)$) and in the case of $\big(Hyp_q(\mathfrak{e}_2)'\big)^\vee$ (giving $\mathbf{u}(\mathfrak{k})$, in both cases), while the semiclassical limit of $Hyp_q(\mathfrak{e}_2)'$ in the simply connected case will be a (countable) covering of that in the adjoint case.

6.6 The QFAs $F_q[E_2]$ and $F_q[{}_aE_2]$. In this and the following sections we look at Theorem 2.2 starting from QFAs, to get QrUEAs out of them.

We begin by introducing a QFA for the Euclidean groups E_2 and ${}_aE_2$. Let $F_q[E_2]$ be the unital associative R–algebra with generators $\mathrm{a}^{\pm 1}, \mathrm{b}, \mathrm{c}$ and relations

$$\mathrm{a\,b} = q\,\mathrm{b\,a}\,, \qquad \mathrm{a\,c} = q\,\mathrm{c\,a}\,, \qquad \mathrm{b\,c} = \mathrm{c\,b}$$

endowed with the Hopf algebra structure given by

$$\Delta\big(\mathrm{a}^{\pm 1}\big) = \mathrm{a}^{\pm 1} \otimes \mathrm{a}^{\pm 1}\,, \quad \Delta(\mathrm{b}) = \mathrm{b} \otimes \mathrm{a}^{-1} + \mathrm{a} \otimes \mathrm{b}\,, \quad \Delta(\mathrm{c}) = \mathrm{c} \otimes \mathrm{a} + \mathrm{a}^{-1} \otimes \mathrm{c}$$

$$\epsilon\big(\mathrm{a}^{\pm 1}\big) = 1\,,\ \epsilon(\mathrm{b}) = 0\,,\ \epsilon(\mathrm{c}) = 0\,, \quad S\big(\mathrm{a}^{\pm 1}\big) = \mathrm{a}^{\mp 1}\,,\ S(\mathrm{b}) = -q^{-1}\,\mathrm{b}\,,\ S(\mathrm{c}) = -q^{+1}\,\mathrm{c}\,.$$

Define $F_q[{}_aE_2]$ as the R–submodule of $F_q[E_2]$ spanned by the products of an even number of generators, i.e. monomials of even degree in $\mathrm{a}^{\pm 1}$, b, c: this is a unital subalgebra of $F_q[E_2]$, generated by $\beta := \mathrm{b\,a}$, $\alpha^{\pm 1} := \mathrm{a}^{\pm 2}$, and $\gamma := \mathrm{a}^{-1}\mathrm{c}$. Let also $\mathbb{F}_q[E_2] := \big(F_q[E_2]\big)_F$ and $\mathbb{F}_q[{}_aE_2] := \big(F_q[{}_aE_2]\big)_F$, having the same presentation than $F_q[E_2]$ and $F_q[{}_aE_2]$ but over $F(R)$. By construction $F_q[E_2]$ and $F_q[{}_aE_2]$ are QFAs (at $\hbar = q-1$), with semiclassical limit $F[E_2]$ and $F[{}_aE_2]$ respectively.

6.7 Computation of $F_q[E_2]^\vee$ and $F_q[{}_aE_2]^\vee$ and specializations $F_q[E_2]^\vee \xrightarrow{q\to 1} U(\mathfrak{g}^\times)$ and $F_q[{}_aE_2]^\vee \xrightarrow{q\to 1} U(\mathfrak{g}^\times)$. In this section we go and compute $F_q[G]^\vee$ and its semiclassical limit (i.e. its specialization at $q=1$), both for $G = E_2$ and $G = {}_aE_2$.

First, $F_q[E_2]$ is free over R, with basis $\Big\{ \mathrm{b}^b \mathrm{a}^a \mathrm{c}^c \,\Big|\, a \in \mathbb{Z}, b, c \in \mathbb{N} \Big\}$, and so also the set $\mathbb{B}_s := \Big\{ \mathrm{b}^b (\mathrm{a}^{\pm 1} - 1)^a \mathrm{c}^c \,\Big|\, a, b, c \in \mathbb{N} \Big\}$ is an R–basis. Second, as $\epsilon(\mathrm{b}) = \epsilon\big(\mathrm{a}^{\pm 1} - 1\big) = \epsilon(\mathrm{c}) = 0$, the ideal $J := Ker\Big(\epsilon :\ F_q[E_2] \longrightarrow R\Big)$ is the span of $\mathbb{B}_s \setminus \{1\}$. Then $F_q[E_2]^\vee = \sum_{n \geq 0} \Big((q-1)^{-1} J\Big)^n$ is the unital R–algebra with generators $D_\pm := \dfrac{\mathrm{a}^{\pm 1} - 1}{q-1}$, $E := \dfrac{\mathrm{b}}{q-1}$, and $F := \dfrac{\mathrm{c}}{q-1}$ and relations

$$D_+E = qED_+ + E\,, \quad D_+F = qFD_+ + F\,, \quad ED_- = qD_-E + E\,, \quad FD_- = qD_-F + F$$

$$EF = FE\,, \qquad D_+D_- = D_-D_+\,, \qquad D_+ + D_- + (q-1)D_+D_- = 0$$

with a Hopf structure given by

$$\Delta(E) = E \otimes 1 + 1 \otimes E + (q-1)\big(E \otimes D_- + D_+ \otimes E\big)\,, \quad \epsilon(E) = 0\,, \quad S(E) = -q^{-1}E$$

$$\Delta(D_\pm) = D_\pm \otimes 1 + 1 \otimes D_\pm + (q-1)\cdot D_\pm \otimes D_\pm\,, \qquad \epsilon(D_\pm) = 0\,, \quad S(D_\pm) = D_\mp$$

$$\Delta(F) = F \otimes 1 + 1 \otimes F + (q-1)\big(F \otimes D_+ + D_- \otimes F\big)\,, \quad \epsilon(F) = 0\,, \quad S(F) = -q^{+1}F\,.$$

This implies that $F_q[E_2]^\vee \xrightarrow{q\to 1} U(\mathfrak{e}_2{}^*)$ as co-Poisson Hopf algebras, for an isomorphism

$$F_q[E_2]^\vee \Big/ (q-1)\, F_q[E_2]^\vee \xrightarrow{\ \cong\ } U(\mathfrak{e}_2{}^*)$$

of co-Poisson Hopf algebra exists, given by $D_\pm \mod (q-1) \mapsto \pm\,\mathsf{h}\big/2$, $E \mod (q-1) \mapsto \mathsf{e}$, and $F \mod (q-1) \mapsto \mathsf{f}$; so $F_q[E_2]^\vee$ does specialize to $U(\mathfrak{e}_2{}^*)$ *as a co-Poisson Hopf algebra*, q.e.d.

Similarly, if we consider $F_q[{}_aE_2]$ the same analysis works again. In fact, $F_q[{}_aE_2]$ is free over R, with basis $\mathbb{B}_a := \Big\{ \beta^b(\alpha^{\pm 1}-1)^a\gamma^c \,\Big|\, a,b,c \in \mathbb{N} \Big\}$; then as above $J := \mathit{Ker}\Big(\epsilon :\, F_q[{}_aE_2] \to R\Big)$ is the span of $\mathbb{B}_a \setminus \{1\}$. $F_q[{}_aE_2]^\vee = \sum_{n\geq 0}\Big((q-1)^{-1}J\Big)^n$ is nothing but the unital R–algebra (inside $\mathbb{F}_q[{}_aE_2]$) with generators $H_\pm := \dfrac{\alpha^{\pm 1}-1}{q-1}$, $E' := \dfrac{\beta}{q-1}$, and $F' := \dfrac{\gamma}{q-1}$ and relations

$$E'F' = q^{-2}F'E',\; H_+E' = q^2E'H_+ + (q+1)E',\; H_+F' = q^2F'H_+ + (q+1)F',\; H_+H_- = H_-H_+$$
$$E'H_- = q^2H_-E' + (q+1)E',\; F'H_- = q^2H_-F' + (q+1)F',\; H_+ + H_- + (q-1)H_+H_- = 0$$

with a Hopf structure given by

$$\Delta(E') = E'\otimes 1 + 1\otimes E' + (q-1)\cdot H_+\otimes E', \qquad \epsilon(E') = 0, \qquad S(E') = -E' - (q-1)H_-E'$$
$$\Delta(H_\pm) = H_\pm\otimes 1 + 1\otimes H_\pm + (q-1)\cdot H_\pm\otimes H_\pm, \qquad \epsilon(H_\pm) = 0, \qquad S(H_\pm) = H_\mp$$
$$\Delta(F') = F'\otimes 1 + 1\otimes F' + (q-1)\cdot H_-\otimes F', \qquad \epsilon(F') = 0, \qquad S(F') = -F' - (q-1)H_+F'.$$

This implies that $F_q[{}_aE_2]^\vee \xrightarrow{\,q\to 1\,} U(\mathfrak{e}_2{}^*)$ as co-Poisson Hopf algebras, for an isomorphism

$$F_q[{}_aE_2]^\vee\Big/(q-1)\,F_q[{}_aE_2]^\vee \xrightarrow{\;\cong\;} U(\mathfrak{e}_2{}^*)$$

of co-Poisson Hopf algebras is given by $H_\pm \mod (q-1) \mapsto \pm\mathsf{h}$, $E' \mod (q-1) \mapsto \mathsf{e}$, and $F' \mod (q-1) \mapsto \mathsf{f}$; so $F_q[{}_aE_2]^\vee$ too specializes to $U(\mathfrak{e}_2{}^*)$ *as a co-Poisson Hopf algebra*, as expected.

We finish noting that, once more, *this analysis (and its outcome) is characteristic-free.*

6.8 The identities $\big(F_q[E_2]^\vee\big)' = F_q[E_2]$ and $\big(F_q[{}_aE_2]^\vee\big)' = F_q[{}_aE_2]$. In this section we verify for the QFAs $H = F_q[E_2]$ and $H = F_q[{}_aE_2]$ the validity of the part of Theorem 2.2*(b)* claiming that $H \in \mathcal{QFA} \implies \big(H^\vee\big)' = H$. Once more, our arguments will prove this result for *Char* $(\Bbbk) \geq 0$, thus going beyond what forecasted by Theorem 2.2.

Formulas $\Delta^n(E) = \sum_{r+s+1=n} \mathsf{a}^{\otimes r}\otimes E\otimes \big(\mathsf{a}^{-1}\big)^{\otimes s}$, $\Delta^n(D_\pm) = \sum_{r+s+1=n} \big(\mathsf{a}^{\pm 1}\big)^{\otimes r}\otimes D_\pm\otimes 1^{\otimes s}$ and $\Delta^n(F) = \sum_{r+s+1=n} \big(\mathsf{a}^{-1}\big)^{\otimes r}\otimes E\otimes \mathsf{a}^{\otimes s}$ are found by induction. These identities imply the following

$$\begin{aligned}
\delta_n(E) &= \sum_{r+s+1=n} (\mathsf{a}-1)^{\otimes r}\otimes E\otimes \big(\mathsf{a}^{-1}-1\big)^{\otimes s} = (q-1)^{n-1}\sum_{r+s+1=n} D_+{}^{\otimes r}\otimes E\otimes D_-{}^{\otimes s}\\
\delta_n(D_\pm) &= \big(\mathsf{a}^{\pm 1}-1\big)^{\otimes(n-1)}\otimes D_\pm = (q-1)^{n-1}D_\pm{}^{\otimes n}\\
\delta_n(F) &= \sum_{r+s+1=n} \big(\mathsf{a}^{-1}-1\big)^{\otimes r}\otimes E\otimes (\mathsf{a}-1)^{\otimes s} = (q-1)^{n-1}\sum_{r+s+1=n} D_-{}^{\otimes r}\otimes E\otimes D_+{}^{\otimes s}
\end{aligned}$$

which give $\dot{E} := (q-1)E$, $\dot{D}_\pm := (q-1)D_\pm$, $\dot{F} := (q-1)F \in \big(F_q[E_2]^\vee\big)' \setminus (q-1)\cdot\big(F_q[E_2]^\vee\big)'$. So $\big(F_q[E_2]^\vee\big)'$ contains the unital R–subalgebra A' generated (inside $\mathbb{F}_q[E_2]$) by $\dot{E}$, $\dot{D}_\pm$ and $\dot{F}$; but $\dot{E} = \mathsf{b}$, $\dot{D}_\pm = \mathsf{a}^{\pm 1}-1$, and $\dot{F} = \mathsf{c}$, thus A' is just $F_q[E_2]$. Since $F_q[E_2]^\vee$ is the R–span of $\Big\{ E^eD_+^{d_+}D_-^{d_-}F^f \,\Big|\, e,d_+,d_-,f \in \mathbb{N} \Big\}$, one easily sees — using the previous formulas for Δ^n — that in fact $\big(F_q[E_2]^\vee\big)' = A' = F_q[E_2]$, q.e.d.

When dealing with the adjoint case, the previous arguments go through again: in fact, $\big(F_q[{}_aE_2]^\vee\big)'$ turns out to coincide with the unital R–subalgebra A'' generated (inside $\mathbb{F}_q[{}_aE_2]$) by $\dot{E}' := (q-1)E' = \beta$, $\dot{H}_\pm := (q-1)H_\pm = \alpha^{\pm 1} - 1$, and $\dot{F}' := (q-1)F' = \gamma$; but this is also generated by β , $\alpha^{\pm 1}$ and γ , thus it coincides with $F_q[{}_aE_2]$, q.e.d.

§ 7 Fourth example: the Heisenberg group H_n

7.1 The classical setting. Let $\Bbbk$ be any field of characteristic $p \geq 0$. Let $G := H_n(\Bbbk) = H_n$, the $(2\,n+1)$–dimensional Heisenberg group; its tangent Lie algebra $\mathfrak{g} = \mathfrak{h}_n$ is generated by $\{\, f_i, h, e_i \,|\, i = 1, \dots, n \,\}$ with relations $[e_i, f_j] = \delta_{ij} h$, $[e_i, e_j] = [f_i, f_j] = [h, e_i] = [h, f_j] = 0$ ($\forall\, i, j = 1, \dots n$). The formulas $\delta(f_i) = h \otimes f_i - f_i \otimes h$, $\delta(h) = 0$, $\delta(e_i) = h \otimes e_i - e_i \otimes h$ ($\forall\, i = 1, \dots n$) make $\mathfrak{h}_n$ into a Lie bialgebra, which provides H_n with a structure of Poisson group; these same formulas give also a presentation of the co-Poisson Hopf algebra $U(\mathfrak{h}_n)$ (with the standard Hopf structure). When $p > 0$ we consider on $\mathfrak{h}_n$ the p–operation uniquely defined by $e_i^{[p]} = 0$, $f_i^{[p]} = 0$, $h^{[p]} = h$ (for all $i = 1, \dots, n$), which makes it into a restricted Lie bialgebra. The group H_n is usually realized as the group of all square matrices $\big(a_{ij}\big)_{i,j=1,\dots,n+2;}$ such that $a_{ii} = 1 \;\forall\, i$ and $a_{ij} = 0 \;\forall\, i, j$ such that either $i > j$ or $1 \neq i < j$ or $i < j \neq n+2$; it can also be realized as $H_n = \Bbbk^n \times \Bbbk \times \Bbbk^n$ with group operation given by $\big(\underline{a}', c', \underline{b}'\big) \cdot \big(\underline{a}'', c'', \underline{b}''\big) = \big(\underline{a}' + \underline{a}'', c' + c'' + \underline{a}' * \underline{b}'', \underline{b}' + \underline{b}''\big)$, where we use vector notation $\underline{v} = (v_1, \dots, v_n) \in k^n$ and $\underline{a}' * \underline{b}'' := \sum_{i=1}^n a'_i b''_i$ is the standard scalar product in k^n ; in particular the identity of H_n is $e = (\underline{0}, 0, \underline{0})$ and the inverse of a generic element is given by $\big(\underline{a}, c, \underline{b}\big)^{-1} = \big(-\underline{a}\,, -c + \underline{a} * \underline{b}\,, -\underline{b}\big)$. Therefore $F[H_n]$ is the unital associative commutative $\Bbbk$–algebra with generators $a_1, \dots, a_n$, c , $b_1, \dots, b_n$, and with Poisson Hopf structure given by

$$\Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i \;, \quad \Delta(c) = c \otimes 1 + 1 \otimes c + \textstyle\sum_{\ell=1}^n a_\ell \otimes b_\ell \;, \quad \Delta(b_i) = b_i \otimes 1 + 1 \otimes b_i$$

$$\epsilon(a_i) = 0 \;, \;\; \epsilon(c) = 0 \;, \;\; \epsilon(b_i) = 0 \;, \qquad S(a_i) = -a_i \;, \;\; S(c) = -c + \textstyle\sum_{\ell=1}^n a_\ell b_\ell \;, \;\; S(b_i) = -b_i$$

$$\{a_i, a_j\} = 0 \;, \qquad \{a_i, b_j\} = 0 \;, \qquad \{b_i, b_j\} = 0 \;, \qquad \{c\,, a_i\} = a_i \;, \qquad \{c\,, b_i\} = b_i$$

for all $i, j = 1, \dots, n$. (Remark: with respect to this presentation, we have $f_i = \partial_{b_i}\big|_e$, $h = \partial_c\big|_e$, $e_i = \partial_{a_i}\big|_e$, where e is the identity element of H_n). The dual Lie bialgebra $\mathfrak{g}^* = {\mathfrak{h}_n}^*$ is the Lie algebra with generators f_i , h , e_i , and relations $[\mathrm{h}, \mathrm{e}_i] = \mathrm{e}_i$, $[\mathrm{h}, \mathrm{f}_i] = \mathrm{f}_i$, $[\mathrm{e}_i, \mathrm{e}_j] = [\mathrm{e}_i, \mathrm{f}_j] = [\mathrm{f}_i, \mathrm{f}_j] = 0$, with Lie cobracket given by $\delta(\mathrm{f}_i) = 0$, $\delta(\mathrm{h}) = \sum_{j=1}^n (\mathrm{e}_j \otimes \mathrm{f}_j - \mathrm{f}_j \otimes \mathrm{e}_j)$, $\delta(\mathrm{e}_i) = 0$ for all $i = 1, \dots, n$ (we take $\mathrm{f}_i := f_i^*$, $\mathrm{h} := h^*$, $\mathrm{e}_i := e_i^*$, where $\big\{\, f_i^*, h^*, e_i^* \,|\, i = 1, \dots, n \,\big\}$ is the basis of ${\mathfrak{h}_n}^*$ which is the dual of the basis $\{\, f_i, h, e_i \,|\, i = 1, \dots, n \,\}$ of $\mathfrak{h}_n$). This again gives a presentation of $U({\mathfrak{h}_n}^*)$ too. If $p > 0$ then ${\mathfrak{h}_n}^*$ is a restricted Lie bialgebra with respect to the p–operation given by $\mathrm{e}_i^{[p]} = 0$, $\mathrm{f}_i^{[p]} = 0$, $\mathrm{h}^{[p]} = \mathrm{h}$ (for all $i = 1, \dots, n$). The simply connected algebraic Poisson group with tangent Lie bialgebra ${\mathfrak{h}_n}^*$ can be realized (with $\Bbbk^\star := \Bbbk \setminus \{0\}$) as ${}_sH_n^{\,*} = \Bbbk^n \times \Bbbk^\star \times \Bbbk^n$, with group operation $\big(\underline{\dot{\alpha}}, \dot{\gamma}, \underline{\dot{\beta}}\big) \cdot \big(\underline{\check{\alpha}}, \check{\gamma}, \underline{\check{\beta}}\big) = \big(\check{\gamma}\underline{\dot{\alpha}} + \dot{\gamma}^{-1}\underline{\check{\alpha}}, \dot{\gamma}\check{\gamma}, \check{\gamma}\underline{\dot{\beta}} + \dot{\gamma}^{-1}\underline{\check{\beta}}\big)$; so the identity of ${}_sH_n^{\,*}$ is $e = (\underline{0}, 1, \underline{0})$ and the inverse is given by $\big(\underline{\alpha}, \gamma, \underline{\beta}\big)^{-1} = \big(-\underline{\alpha}, \gamma^{-1}, -\underline{\beta}\big)$. Its centre is

$Z\big({}_sH_n{}^*\big) = \big\{(\underline{0},1,\underline{0}),(\underline{0},-1,\underline{0})\big\} =: Z$, so there is only one other (Poisson) group with tangent Lie bialgebra $\mathfrak{h}_n{}^*$, that is the adjoint group ${}_aH_n{}^* := {}_sH_n{}^* \Big/ Z$.

It is clear that $F\big[{}_sH_n{}^*\big]$ is the unital associative commutative $\Bbbk$–algebra with generators $\alpha_1, \dots, \alpha_n, \gamma^{\pm 1}, \beta_1, \dots, \beta_n$, and with Poisson Hopf algebra structure given by

$$\Delta(\alpha_i) = \alpha_i \otimes \gamma + \gamma^{-1} \otimes \alpha_i \, , \qquad \Delta\big(\gamma^{\pm 1}\big) = \gamma^{\pm 1} \otimes \gamma^{\pm 1} \, , \qquad \Delta(\beta_i) = \beta_i \otimes \gamma + \gamma^{-1} \otimes \beta_i$$
$$\epsilon(\alpha_i) = 0 \, , \ \epsilon\big(\gamma^{\pm 1}\big) = 1 \, , \ \epsilon(\beta_i) = 0 \, , \qquad S(\alpha_i) = -\alpha_i \, , \ S\big(\gamma^{\pm 1}\big) = \gamma^{\mp 1} \, , \ S(\beta_i) = -\beta_i$$
$$\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta_j\} = \{\beta_i, \beta_j\} = \{\alpha_i, \gamma\} = \{\beta_i, \gamma\} = 0 \, , \qquad \{\alpha_i, \beta_j\} = \delta_{ij}\big(\gamma^2 - \gamma^{-2}\big)\big/2$$

for all $i, j = 1, \dots, n$ (Remark: with respect to this presentation, we have $\mathrm{f}_i = \partial_{\beta_i}\big|_e$, $\mathrm{h} = \frac{1}{2}\,\gamma\,\partial_\gamma\big|_e$, $\mathrm{e}_i = \partial_{\alpha_i}\big|_e$, where e is the identity element of ${}_sH_n{}^*$), and $F\big[{}_aH_n{}^*\big]$ can be identified — as in the case of the Euclidean group — with the Poisson Hopf subalgebra of $F\big[{}_aH_n{}^*\big]$ which is spanned by products of an even number of generators: this is generated by $\alpha_i\,\gamma$, $\gamma^{\pm 2}$, and $\gamma^{-1}\,\beta_i$ $(i = 1, \dots, n)$.

7.2 The QrUEAs $U_q^s(\mathfrak{h}_n)$ and $U_q^a(\mathfrak{h}_n)$. We switch now to quantizations. Once again, let R be a 1dD and let $\hbar \in R \setminus \{0\}$ and assume $q := 1 + \hbar \in R$ be invertible, like in §5.2.

Let $\mathbb{U}_q(\mathfrak{g}) = \mathbb{U}_q^s(\mathfrak{h}_n)$ be the unital associative $F(R)$–algebra with generators F_i , $L^{\pm 1}$, E_i (for $i = 1, \dots, n$) and relations

$$LL^{-1} = 1 = L^{-1}L \, , \quad L^{\pm 1}F = FL^{\pm 1} \, , \quad L^{\pm 1}E = EL^{\pm 1} \, , \quad E_iF_j - F_jE_i = \delta_{ij}\frac{L^2 - L^{-2}}{q - q^{-1}}$$

for all $i, j = 1, \dots, n$; we give it a structure of Hopf algebra, by setting ($\forall\, i, j = 1, \dots, n$)

$$\Delta(E_i) = E_i \otimes 1 + L^2 \otimes E_i \, , \quad \Delta\big(L^{\pm 1}\big) = L^{\pm 1} \otimes L^{\pm 1} \, , \quad \Delta(F_i) = F_i \otimes L^{-2} + 1 \otimes F_i$$
$$\epsilon(E_i) = 0 \, , \quad \epsilon\big(L^{\pm 1}\big) = 1 \, , \quad \epsilon(F_i) = 0 \, , \quad S(E_i) = -L^{-2}E_i \, , \quad S\big(L^{\pm 1}\big) = L^{\mp 1} \, , \quad S(F_i) = -F_iL^2$$

Note then that $\Big\{ \prod_{i=1}^n F_i^{a_i} \cdot L^z \cdot \prod_{i=1}^n E_i^{d_i} \,\Big|\, z \in \mathbb{Z}, \, a_i, d_i \in \mathbb{N}, \, \forall\, i \Big\}$ is an $F(R)$–basis of $\mathbb{U}_q^s(\mathfrak{h}_n)$.

Now, let $U_q^s(\mathfrak{h}_n)$ be the unital R–subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$ generated by the elements $F_1, \dots, F_n$, $D := \dfrac{L-1}{q-1}$, $\varGamma := \dfrac{L - L^{-2}}{q - q^{-1}}$, $E_1, \dots, E_n$. Then $U_q^s(\mathfrak{h}_n)$ can be presented as the associative unital algebra with generators $F_1, \dots, F_n, L^{\pm 1}, D, \varGamma, E_1, \dots, E_n$ and relations

$$DX = XD \, , \qquad L^{\pm 1}X = XL^{\pm 1} \, , \qquad \varGamma X = X\varGamma \, , \qquad E_iF_j - F_jE_i = \delta_{ij}\varGamma$$
$$L = 1 + (q-1)D \, , \quad L^2 - L^{-2} = \big(q - q^{-1}\big)\varGamma \, , \quad D(L+1)\big(1 + L^{-2}\big) = \big(1 + q^{-1}\big)\varGamma$$

for all $X \in \big\{F_i, L^{\pm 1}, D, \varGamma, E_i\big\}_{i=1,\dots,n}$ and $i, j = 1, \dots, n$; furthermore, $U_q^s(\mathfrak{h}_n)$ is a Hopf subalgebra (over R), with

$$\Delta(\varGamma) = \varGamma \otimes L^2 + L^{-2} \otimes \varGamma \, , \qquad \epsilon(\varGamma) = 0 \, , \qquad S(\varGamma) = -\varGamma$$
$$\Delta(D) = D \otimes 1 + L \otimes D \, , \qquad \epsilon(D) = 0 \, , \qquad S(D) = -L^{-1}D \, .$$

Moreover, from relations $L = 1 + (q-1)D$ and $L^{-1} = L^3 - \big(q - q^{-1}\big)L\varGamma$ it follows that

$$U_q^s(\mathfrak{h}_n) \ = \ R\text{–span of } \left\{ \prod_{i=1}^n F_i^{a_i} \cdot D^b \varGamma^c \cdot \prod_{i=1}^n E_i^{d_i} \,\middle|\, a_i, b, c, d_i \in \mathbb{N}, \, \forall\, i = 1, \dots, n \right\} \tag{7.1}$$

The “adjoint version” of $\mathbb{U}_q^s(\mathfrak{h}_n)$ is the subalgebra $\mathbb{U}_q^a(\mathfrak{h}_n)$ generated by F_i, $K^{\pm 1} := L^{\pm 2}$, E_i $(i = 1, \dots, n)$, which is a *Hopf* subalgebra too. It also has an R–integer form $U_q^a(\mathfrak{h}_n)$, the R–subalgebra generated by $F_1, \dots, F_n, K^{\pm 1}, H := \dfrac{K-1}{q-1}, \varGamma := \dfrac{K-K^{-1}}{q-q^{-1}}, E_1, \dots, E_n$: this has relations

$$HX = XH\,, \qquad K^{\pm 1}X = XK^{\pm 1}\,, \qquad \varGamma X = X\varGamma\,, \qquad E_iF_j - F_jE_i = \delta_{ij}\varGamma$$
$$K = 1 + (q-1)H\,, \quad K - K^{-1} = \big(q - q^{-1}\big)\varGamma\,, \quad H\big(1 + K^{-1}\big) = \big(1 + q^{-1}\big)\varGamma$$

for all $X \in \big\{F_i, K^{\pm 1}, H, \varGamma, E_i\big\}_{i=1,\dots,n}$ and $i,j = 1, \dots, n$, and Hopf operations given by

$$\begin{array}{ccc}
\Delta(E_i) = E_i \otimes 1 + K \otimes E_i\,, & \epsilon(E_i) = 0\,, & S(E_i) = -K^{-1}E_i \\
\Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, & \epsilon\big(K^{\pm 1}\big) = 1\,, & S\big(K^{\pm 1}\big) = K^{\mp 1} \\
\Delta(H) = H \otimes 1 + K \otimes H\,, & \epsilon(H) = 0\,, & S(H) = -K^{-1}H \\
\Delta(\varGamma) = \varGamma \otimes K^{-1} + K \otimes \varGamma\,, & \epsilon(\varGamma) = 0\,, & S(\varGamma) = -\varGamma \\
\Delta(F_i) = F_i \otimes K^{-1} + 1 \otimes F_i\,, & \epsilon(F_i) = 0\,, & S(F_i) = -F_iK^{+1}
\end{array}$$

for all $i = 1, \dots, n$. One can easily check that $U_q^s(\mathfrak{h}_n)$ is a QrUEA, with $U(\mathfrak{h}_n)$ as semiclassical limit: in fact, mapping the generators $F_i \mod (q-1)$, $L^{\pm 1} \mod (q-1)$, $D \mod (q-1)$, $\varGamma \mod (q-1)$, $E_i \mod (q-1)$ respectively to $f_i, 1, h/2, h, e_i \in U(\mathfrak{h}_n)$ yields a co-Poisson Hopf algebra isomorphism between $U_q^s(\mathfrak{h}_n)\Big/(q-1)\,U_q^s(\mathfrak{h}_n)$ and $U(\mathfrak{h}_n)$. Similarly, $U_q^a(\mathfrak{h}_n)$ is a QrUEA too, again with limit $U(\mathfrak{h}_n)$, for a co-Poisson Hopf algebra isomorphism between $U_q^a(\mathfrak{h}_n)\Big/(q-1)\,U_q^a(\mathfrak{h}_n)$ and $U(\mathfrak{h}_n)$ is given by mapping the generators $F_i \mod (q-1)$, $K^{\pm 1} \mod (q-1)$, $H \mod (q-1)$, $\varGamma \mod (q-1)$, $E_i \mod (q-1)$ respectively to $f_i, 1, h, h, e_i \in U(\mathfrak{h}_n)$.

7.3 Computation of $U_q(\mathfrak{h}_n)'$ and specialization $U_q(\mathfrak{h}_n)' \xrightarrow{\,q\to 1\,} F\big[H_n^\star\big]$. Here we compute $U_q^s(\mathfrak{h}_n)'$ and $U_q^a(\mathfrak{h}_n)'$, and their semiclassical limits, along the pattern of §5.3.

Definitions give, for any $n \in \mathbb{N}$, $\Delta^n(E_i) = \sum_{s=1}^n \big(L^2\big)^{\otimes(s-1)} \otimes E_i \otimes 1^{\otimes(n-s)}$, hence $\delta_n(E_i) = (q-1)^{n-1} \cdot D^{\otimes(n-1)} \otimes E_i$ so $\delta_n\big((q-1)E\big) \in (q-1)^n\,U_q^s(\mathfrak{h}_n) \setminus (q-1)^{n+1}\,U_q^s(\mathfrak{h}_n)$ whence $\dot{E}_i := (q-1)\,E_i \in U_q^s(\mathfrak{h}_n)'$, whereas $E_i \notin U_q^s(\mathfrak{h}_n)'$; similarly, we have $\dot{F}_i := (q-1)\,F_i$, $L^{\pm 1}$, $\dot{D} := (q-1)\,D = L-1$, $\dot{\varGamma} := (q-1)\,\varGamma \in U_q^s(\mathfrak{h}_n)' \setminus (q-1)\,U_q^s(\mathfrak{h}_n)'$, for all $i = 1, \dots, n$. Thus $U_q^s(\mathfrak{h}_n)'$ contains the subalgebra U' generated by $\dot{F}_i, L^{\pm 1}, \dot{D}, \dot{\varGamma}, \dot{E}_i$; we argue that $U_q^s(\mathfrak{h}_n)' = U'$: this is easily seen — like for SL_2 and for E_2 — using the formulas above along with (7.1). Therefore $U_q^s(\mathfrak{h}_n)'$ is the unital R–algebra with generators $\dot{F}_1, \dots, \dot{F}_n, L^{\pm 1}, \dot{D}, \dot{\varGamma}, \dot{E}_1, \dots, \dot{E}_n$ and relations

$$\dot{D}\dot{X} = \dot{X}\dot{D}\,, \qquad L^{\pm 1}\dot{X} = \dot{X}L^{\pm 1}\,, \qquad \dot{\varGamma}\dot{X} = \dot{X}\dot{\varGamma}\,, \qquad \dot{E}_i\dot{F}_j - \dot{F}_j\dot{E}_i = \delta_{ij}(q-1)\dot{\varGamma}$$
$$L = 1 + \dot{D}\,, \quad L^2 - L^{-2} = \big(1 + q^{-1}\big)\dot{\varGamma}\,, \quad \dot{D}(L+1)\big(1 + L^{-2}\big) = \big(1 + q^{-1}\big)\dot{\varGamma}$$

for all $\dot{X} \in \big\{\dot{F}_i, L^{\pm 1}, \dot{D}, \dot{\varGamma}, \dot{E}_i\big\}_{i=1,\dots,n}$ and $i,j = 1, \dots, n$, with Hopf structure given by

$$\begin{array}{cccc}
\Delta\big(\dot{E}_i\big) = \dot{E}_i \otimes 1 + L^2 \otimes \dot{E}_i\,, & \epsilon\big(\dot{E}_i\big) = 0\,, & S\big(\dot{E}_i\big) = -L^{-2}\dot{E}_i & \forall\, i = 1, \dots, n \\
\Delta\big(L^{\pm 1}\big) = L^{\pm 1} \otimes L^{\pm 1}\,, & \epsilon\big(L^{\pm 1}\big) = 1\,, & S\big(L^{\pm 1}\big) = L^{\mp 1} & \\
\Delta\big(\dot{\varGamma}\big) = \dot{\varGamma} \otimes L^2 + L^{-2} \otimes \dot{\varGamma}\,, & \epsilon\big(\dot{\varGamma}\big) = 0\,, & S\big(\dot{\varGamma}\big) = -\varGamma &
\end{array}$$

$$\Delta\big(\dot{D}\big) = \dot{D} \otimes 1 + L \otimes \dot{D} \; , \qquad \epsilon\big(\dot{D}\big) = 0 \; , \qquad S\big(\dot{D}\big) = -L^{-1} \dot{D}$$
$$\Delta\big(\dot{F}_i\big) = \dot{F}_i \otimes L^{-2} + 1 \otimes \dot{F}_i \; , \qquad \epsilon\big(F_i\big) = 0 \; , \qquad S\big(\dot{F}_i\big) = -\dot{F}_i L^2 \qquad \forall \; i = 1, \dots, n \; .$$

A similar analysis shows that $U_q^a(\mathfrak{h}_n)'$ is the unital R–subalgebra U'' of $U_q^a(\mathfrak{h}_n)$ generated by $\dot{F}_i$, $K^{\pm 1}$, $\dot{H} := (q-1)\, H$, $\dot{\varGamma}$, $\dot{E}_i$ $(i = 1, \dots, n)$; in particular, $U_q^a(\mathfrak{h}_n)' \subset U_q^s(\mathfrak{h}_n)'$. Thus $U_q^a(\mathfrak{h}_n)'$ is the unital associative R–algebra with generators $\dot{F}_1, \dots, \dot{F}_n, \dot{H}, K^{\pm 1}, \dot{\varGamma}, \dot{E}_1, \dots, \dot{E}_n$ and relations

$$\dot{H}\dot{X} = \dot{X}\dot{H} \; , \qquad K^{\pm 1}\dot{X} = \dot{X}K^{\pm 1} \; , \qquad \dot{\varGamma}\dot{X} = \dot{X}\dot{\varGamma} \; , \qquad \dot{E}_i\dot{F}_j - \dot{F}_j\dot{E}_i = \delta_{ij}(q-1)\dot{\varGamma}$$
$$K = 1 + \dot{H} \; , \quad K - K^{-1} = \big(1 + q^{-1}\big)\dot{\varGamma} \; , \quad \dot{H}\big(1 + K^{-1}\big) = \big(1 + q^{-1}\big)\dot{\varGamma}$$

for all $\dot{X} \in \big\{\dot{F}_i, K^{\pm 1}, \dot{K}, \dot{\varGamma}, \dot{E}_i\big\}_{i=1,\dots,n}$ and $i, j = 1, \dots, n$, with Hopf structure given by

$$\Delta\big(\dot{E}_i\big) = \dot{E}_i \otimes 1 + K \otimes \dot{E}_i \; , \qquad \epsilon\big(\dot{E}_i\big) = 0 \; , \qquad S\big(\dot{E}_i\big) = -K^{-1}\dot{E}_i \qquad \forall \; i = 1, \dots, n$$
$$\Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1} \; , \qquad \epsilon\big(K^{\pm 1}\big) = 1 \; , \qquad S\big(K^{\pm 1}\big) = K^{\mp 1}$$
$$\Delta\big(\dot{\varGamma}\big) = \dot{\varGamma} \otimes K + K^{-1} \otimes \dot{\varGamma} \; , \qquad \epsilon\big(\dot{\varGamma}\big) = 0 \; , \qquad S\big(\dot{\varGamma}\big) = -\varGamma$$
$$\Delta\big(\dot{H}\big) = \dot{H} \otimes 1 + K \otimes \dot{H} \; , \qquad \epsilon\big(\dot{H}\big) = 0 \; , \qquad S\big(\dot{H}\big) = -K^{-1}\dot{H}$$
$$\Delta\big(\dot{F}_i\big) = \dot{F}_i \otimes K^{-1} + 1 \otimes \dot{F}_i \; , \qquad \epsilon\big(F_i\big) = 0 \; , \qquad S\big(\dot{F}_i\big) = -\dot{F}_i K \qquad \forall \; i = 1, \dots, n \; .$$

As $q \to 1$, the presentation above provides an isomorphism of Poisson Hopf algebras

$$U_q^s(\mathfrak{h}_n)' \Big/ (q-1)\, U_q^s(\mathfrak{h}_n)' \xrightarrow{\;\cong\;} F\big[{}_sH_n{}^*\big]$$

given by $\dot{E}_i \mod (q-1) \mapsto \alpha_i\, \gamma^{+1}$, $L^{\pm 1} \mod (q-1) \mapsto \gamma^{\pm 1}$, $\dot{D} \mod (q-1) \mapsto \gamma - 1$, $\dot{\varGamma} \mod (q-1) \mapsto \big(\gamma^2 - \gamma^{-2}\big)\Big/2$, $\dot{F}_i \mod (q-1) \mapsto \gamma^{-1}\beta_i$. In other words, the semiclassical limit of $U_q^s(\mathfrak{h}_n)'$ is $F\big[{}_sH_n{}^*\big]$, as predicted by Theorem 2.2*(c)* for $p = 0$. Similarly, when considering the "adjoint case", we find a Poisson Hopf algebra isomorphism

$$U_q^a(\mathfrak{h}_n)' \Big/ (q-1)\, U_q^a(\mathfrak{h}_n)' \xrightarrow{\;\cong\;} F\big[{}_aH_n{}^*\big] \quad \Big(\subset F\big[{}_sH_n{}^*\big]\Big)$$

given by $\dot{E}_i \mod (q-1) \mapsto \alpha_i\, \gamma^{+1}$, $K^{\pm 1} \mod (q-1) \mapsto \gamma^{\pm 2}$, $\dot{H} \mod (q-1) \mapsto \gamma^2 - 1$, $\dot{\varGamma} \mod (q-1) \mapsto \big(\gamma^2 - \gamma^{-2}\big)\Big/2$, $\dot{F}_i \mod (q-1) \mapsto \gamma^{-1}\beta_i$. That is to say, $U_q^a(\mathfrak{h}_n)'$ has semiclassical limit $F\big[{}_aH_n{}^*\big]$, as predicted by Theorem 2.2*(c)* for $p = 0$.

We stress the fact that *this analysis is characteristic-free*, so we get in fact that its outcome does hold for $p > 0$ as well, thus "improving" Theorem 2.2*(c)* (like in §§5–6).

7.4 The identity $\big(U_q(\mathfrak{h}_n)'\big)^\vee = U_q(\mathfrak{h}_n)$. In this section we verify the part of Theorem 2.2*(b)* claiming, for $p = 0$, that $H \in \mathcal{Q}r\mathcal{UEA} \implies \big(H'\big)^\vee = H$, both for $H = U_q^s(\mathfrak{h}_n)$ and for $H = U_q^a(\mathfrak{h}_n)$. *In addition, the same arguments will prove such a result for $p > 0$ too.*

To begin with, using (7.1) and the fact that $\dot{F}_i$, $\dot{D}$, $\dot{\varGamma}$, $\dot{E}_i \in Ker\Big(\epsilon : U_q^s(\mathfrak{h}_n)' \twoheadrightarrow R\Big)$ we get that $J := Ker(\epsilon)$ is the R–span of $\mathbb{M} \setminus \{1\}$, where $\mathbb{M}$ is the set in the right-hand-side of (7.1). Since $\big(U_q^s(\mathfrak{h}_n)'\big)^\vee := \sum_{n \geq 0} \Big((q-1)^{-1} J\Big)^n$, we have that $\big(U_q^s(\mathfrak{h}_n)'\big)^\vee$ is generated — as a unital R–subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$ — by $(q-1)^{-1}\dot{F}_i = F_i$, $(q-1)^{-1}\dot{D} = D$, $(q-1)^{-1}\dot{\varGamma} = \varGamma$, $(q-1)^{-1}\dot{E}_i =$

E_i $(i=1,\dots,n)$, so it coincides with $U_q^s(\mathfrak{h}_n)$, q.e.d. In the adjoint case the procedure is similar: one changes $L^{\pm 1}$, resp. $\dot{D}$, with $K^{\pm 1}$, resp. $\dot{H}$, and everything works as before.

7.5 The quantum hyperalgebra $Hyp_q(\mathfrak{h}_n)$. Like in §§5.5 and 6.5, we can define "quantum hyperalgebras" associated to $\mathfrak{h}_n$. Namely, first we define a Hopf subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$ over $\mathbb{Z}\big[q,q^{-1}\big]$ whose specialization at $q=1$ is the natural Kostant-like $\mathbb{Z}$–integer form $U_{\mathbb{Z}}(\mathfrak{h}_n)$ of $U(\mathfrak{h}_n)$ (generated by divided powers, and giving the hyperalgebra $Hyp(\mathfrak{h}_n)$ over any field $\Bbbk$ by scalar extension), and then take its scalar extension over R.

To be precise, let $Hyp_q^{s,\mathbb{Z}}(\mathfrak{h}_n)$ be the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$ (defined like above *but over* $\mathbb{Z}\big[q,q^{-1}\big]$) generated by the "quantum divided powers"

$$F_i^{(m)} := F_i^m \Big/ [m]_q! \ , \qquad \binom{L\,;\,c}{m} := \prod_{r=1}^{n} \frac{q^{c+1-r}L-1}{q^r-1} \ , \qquad E_i^{(m)} := E_i^m \Big/ [m]_q!$$

(for all $m\in\mathbb{N}$, $c\in\mathbb{Z}$ and $i=1,\dots,n$, with notation of §5.5) and by L^{-1}. Comparing with the case of $\mathfrak{sl}_2$ — noting that for each i the quadruple (F_i,L,L^{-1},E_i) generates a copy of $\mathbb{U}_q^s(\mathfrak{sl}_2)$ — we see at once that this is a Hopf subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$, and $Hyp_q^{s,\mathbb{Z}}(\mathfrak{h}_n)\Big|_{q=1} \cong U_{\mathbb{Z}}(\mathfrak{h}_n)$; thus $Hyp_q^s(\mathfrak{h}_n) := R\otimes_{\mathbb{Z}[q,q^{-1}]} Hyp_q^{s,\mathbb{Z}}(\mathfrak{h}_n)$ (for any R like in §6.2, with $\Bbbk := R\big/\hbar\,R$ and $p := Char(\Bbbk)$) specializes at $q=1$ to the $\Bbbk$–hyperalgebra $Hyp(\mathfrak{h}_n)$. Moreover, among all the $\binom{L\,;\,c}{n}$'s it is enough to take only those with $c=0$. *From now on we assume* $p>0$.

Pushing forward the close comparison with the case of $\mathfrak{sl}_2$ we also see that $Hyp_q^s(\mathfrak{h}_n)'$ is the unital R–subalgebra of $Hyp_q^s(\mathfrak{h}_n)$ generated by L^{-1} and the "rescaled quantum divided powers" $(q-1)^m F_i^{(m)}$, $(q-1)^m\binom{L\,;\,0}{m}$ and $(q-1)^m E_i^{(m)}$, for all $m\in\mathbb{N}$ and $i=1,\dots,n$. It follows that $Hyp_q^s(\mathfrak{h}_n)'\Big|_{q=1}$ is generated by the specializations at $q=1$ of $(q-1)^{p^r}F_i^{(p^r)}$, $(q-1)^{p^r}\binom{L\,;\,0}{p^r}$ and $(q-1)^{p^r}E_i^{(p^r)}$, for all $r\in\mathbb{N}$, $i=1,\dots,n$: this proves directly that the spectrum of $Hyp_q^s(\mathfrak{h}_n)'\Big|_{q=1}$ has dimension 0 and height 1, and its cotangent Lie algebra $J\Big/J^2$ (where J is the augmentation ideal of $Hyp_q^s(\mathfrak{h}_n)'\Big|_{q=1}$) has basis $\Big\{ (q-1)^{p^r}F_i^{(p^r)},\ (q-1)^{p^r}\binom{L\,;\,0}{p^r},\ (q-1)^{p^r}E_i^{(p^r)} \mod (q-1)\,Hyp_q^s(\mathfrak{g})' \mod J^2 \ \Big|\ r\in\mathbb{N},\ i=1,\dots,n \Big\}$. Finally, $\big(Hyp_q^s(\mathfrak{h}_n)'\big)^\vee$ is generated by $(q-1)^{p^r-1}F_i^{(p^r)}$, $(q-1)^{p^r-1}\binom{L\,;\,0}{p^r}$, L^{-1} and $(q-1)^{p^r-1}E_i^{(p^r)}$ (for all r and i): in particular $\big(Hyp_q^s(\mathfrak{h}_n)'\big)^\vee \subsetneqq Hyp_q^s(\mathfrak{h}_n)$, and $\big(Hyp_q^s(\mathfrak{h}_n)'\big)^\vee\Big|_{q=1}$ is generated by the cosets modulo $(q-1)$ of these elements, which form a basis of the restricted Lie bialgebra $\mathfrak{k}$ such that $\big(Hyp_q^s(\mathfrak{h}_n)'\big)^\vee\Big|_{q=1} = \mathbf{u}(\mathfrak{k})$.

The previous analysis stems from $\mathbb{U}_q^s(\mathfrak{h}_n)$, and so gives "simply connected quantum objects". Instead we can start from $\mathbb{U}_q^a(\mathfrak{h}_n)$, thus getting "adjoint quantum objects", moving along the same pattern but for replacing $L^{\pm 1}$ by $K^{\pm 1}$ throughout: apart from this, the analysis and its outcome are exactly the same. Like for $\mathfrak{sl}_2$ (cf. §5.5), all the adjoint quantum objects — i.e. $Hyp_q^a(\mathfrak{h}_n)$, $Hyp_q^a(\mathfrak{h}_n)'$ and $\big(Hyp_q^a(\mathfrak{h}_n)'\big)^\vee$ — will be strictly contained in the corresponding simply connected quantum objects. However, the semiclassical limits will be the same in the case of $Hyp_q(\mathfrak{g})$ (giving $Hyp(\mathfrak{h}_n)$ in both cases) and in the case of $\big(Hyp_q(\mathfrak{g})'\big)^\vee$ (always yielding $\mathbf{u}(\mathfrak{k})$), whereas the semiclassical limit of $Hyp_q(\mathfrak{g})'$ in the simply connected case will be a (countable) covering of the limit in the adjoint case.

7.6 The QFA $F_q[H_n]$. Now we look at Theorem 2.2 the other way round, i.e. from QFAs to QrUEAs. We begin by introducing a QFA for the Heisenberg group.

Let $F_q[H_n]$ be the unital associative R–algebra with generators $\mathsf{a}_1, \dots, \mathsf{a}_n, \mathsf{c}, \mathsf{b}_1, \dots, \mathsf{b}_n$, and relations (for all $i, j = 1, \dots, n$)

$$\mathsf{a}_i\mathsf{a}_j = \mathsf{a}_j\mathsf{a}_i\,, \quad \mathsf{a}_i\mathsf{b}_j = \mathsf{b}_j\mathsf{a}_i\,, \quad \mathsf{b}_i\mathsf{b}_j = \mathsf{b}_j\mathsf{b}_i\,, \quad \mathsf{c}\,\mathsf{a}_i = \mathsf{a}_i\,\mathsf{c} + (q-1)\,\mathsf{a}_i\,, \quad \mathsf{c}\,\mathsf{b}_j = \mathsf{b}_j\,\mathsf{c} + (q-1)\,\mathsf{b}_j$$

with a Hopf algebra structure given by (for all $i, j = 1, \dots, n$)

$$\Delta(\mathsf{a}_i) = \mathsf{a}_i \otimes 1 + 1 \otimes \mathsf{a}_i\,, \quad \Delta(\mathsf{c}) = \mathsf{c} \otimes 1 + 1 \otimes \mathsf{c} + \sum_{j=1}^n \mathsf{a}_\ell \otimes \mathsf{b}_\ell\,, \quad \Delta(\mathsf{b}_i) = \mathsf{b}_i \otimes 1 + 1 \otimes \mathsf{b}_i$$

$$\epsilon(\mathsf{a}_i) = 0\,, \ \ \epsilon(\mathsf{c}) = 0\,, \ \ \epsilon(\mathsf{b}_i) = 0\,, \qquad S(\mathsf{a}_i) = -\mathsf{a}_i\,, \ \ S(\mathsf{c}) = -\mathsf{c} + \sum_{j=1}^n \mathsf{a}_\ell\,\mathsf{b}_\ell\,, \ \ S(\mathsf{b}_i) = -\mathsf{b}_i$$

and let also $\mathbb{F}_q[H_n]$ be the $F(R)$–algebra obtained from $F_q[H_n]$ by scalar extension. Then $\mathbb{B} := \Big\{ \prod_{i=1}^n \mathsf{a}_i^{a_i} \cdot \mathsf{c}^c \cdot \prod_{j=1}^n \mathsf{b}_j^{b_j} \,\Big|\, a_i, c, b_j \in \mathbb{N} \ \forall\, i, j \Big\}$ is an R–basis of $F_q[H_n]$, hence an $F(R)$–basis of $\mathbb{F}_q[H_n]$. Moreover, $F_q[H_n]$ is a QFA (at $\hbar = q-1$) with semiclassical limit $F[H_n]$.

7.7 Computation of $F_q[H_n]^\vee$ and specialization $F_q[H_n]^\vee \xrightarrow{\,q \to 1\,} U(\mathfrak{h}_n{}^\times)$. This section is devoted to compute $F_q[H_n]^\vee$ and its semiclassical limit (at $q = 1$).

Definitions imply that $\mathbb{B} \setminus \{1\}$ is an R–basis of $J := \mathit{Ker}\Big(\epsilon : \, F_q[H_n] \twoheadrightarrow R\Big)$. Therefore $F_q[H_n]^\vee = \sum_{n \geq 0} \Big((q-1)^{-1} J\Big)^n$ is just the unital R–algebra (subalgebra of $\mathbb{F}_q[H_n]$) with generators $E_i := \dfrac{\mathsf{a}_i}{q-1}$, $H := \dfrac{\mathsf{c}}{q-1}$, and $F_i := \dfrac{\mathsf{b}_i}{q-1}$ ($i = 1, \dots, n$) and relations (for all $i, j = 1, \dots, n$)

$$E_iE_j = E_jE_i\,, \quad E_iF_j = F_jE_i\,, \quad F_iF_j = F_jF_i\,, \quad HE_i = E_iH + E_i\,, \quad HF_j = F_jH + F_j$$

with Hopf algebra structure given by (for all $i, j = 1, \dots, n$)

$$\Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i\,, \ \Delta(H) = H \otimes 1 + 1 \otimes H + (q-1)\sum_{j=1}^n E_j \otimes F_j\,, \ \Delta(F_i) = F_i \otimes 1 + 1 \otimes F_i$$

$$\epsilon(E_i) = \epsilon(H) = \epsilon(F_i) = 0\,, \quad S(E_i) = -E_i\,, \quad S(H) = -H + (q-1)\sum_{j=1}^n E_jF_j\,, \quad S(F_i) = -F_i\,.$$

At $q = 1$ this implies that $F_q[H_n]^\vee \xrightarrow{\,q \to 1\,} U(\mathfrak{h}_n{}^\star) = U(\mathfrak{h}_n{}^*)$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$F_q[H_n]^\vee \Big/ (q-1)\,F_q[H_n]^\vee \xrightarrow{\;\cong\;} U(\mathfrak{h}_n{}^*)$$

exists, given by $E_i \mod (q-1) \mapsto \mathsf{e}_i$, $H \mod (q-1) \mapsto \mathsf{h}$, $F_i \mod (q-1) \mapsto \mathsf{f}_i$, for all $i, j = 1, \dots, n$. Thus $F_q[H_n]^\vee$ specializes to $U(\mathfrak{h}_n{}^*)$ *as a co-Poisson Hopf algebra*, q.e.d.

7.8 The identity $\big(F_q[H_n]^\vee\big)' = F_q[H_n]$ **.** Finally, we check the validity of the part of Theorem 2.2*(b)* claiming, when $p=0$, that $H \in \mathcal{QFA} \implies \big(H^\vee\big)' = H$ for the QFA $H = F_q[H_n]$. Once more the proof works for all $p \geq 0$, so we do improve Theorem 2.2*(b)*.

First of all, from definitions induction gives, for all $m \in \mathbb{N}$,

$$\Delta^m(E_i) = \sum_{r+s=m-1} 1^{\otimes r} \otimes E_i \otimes 1^{\otimes s}, \quad \Delta^m(F_i) = \sum_{r+s=m-1} 1^{\otimes r} \otimes F_i \otimes 1^{\otimes s} \qquad \forall\, i=1,\dots,n$$

$$\Delta^m(H) = \sum_{r+s=m-1} 1^{\otimes r} \otimes H \otimes 1^{\otimes s} + \sum_{i=1}^{m} \sum_{\substack{j,k=1 \\ j<k}}^{m} 1^{\otimes (j-1)} \otimes E_i \otimes 1^{\otimes (k-j-1)} \otimes F_i \otimes 1^{\otimes (m-k)}$$

so that $\delta_m(E_i) = \delta_\ell(H) = \delta_m(F_i) = 0$ for all $m>1$, $\ell>2$ and $i=1,\dots,n$; moreover, for $\dot{E}_i := (q-1)E_i = \mathrm{a}_i$, $\dot{H} := (q-1)H = \mathrm{c}$, $\dot{F}_i := (q-1)F_i = \mathrm{b}_i$ ($i=1,\dots,n$) one has

$$\delta_1\big(\dot{E}_i\big) = (q-1)E_i\,,\ \delta_1\big(\dot{H}\big) = (q-1)H,\ \delta_1\big(\dot{F}_i\big) = (q-1)F_i \in (q-1)\,F_q[H_n]^\vee \setminus (q-1)^2 F_q[H_n]^\vee$$

$$\delta_2\big(\dot{H}\big) \;=\; (q-1)^2 \textstyle\sum_{i=1}^n E_i \otimes F_i \in (q-1)^2 \big(F_q[H_n]^\vee\big)^{\otimes 2} \setminus (q-1)^3 \big(F_q[H_n]^\vee\big)^{\otimes 2}.$$

The outcome is that $\dot{E}_i = \mathrm{a}_i\,, \dot{H} = \mathrm{c}\,, \dot{F}_i = \mathrm{b}_i \in \big(F_q[H_n]^\vee\big)'$, so the latter algebra contains the one generated by these elements, that is $F_q[H_n]$. Even more, $F_q[H_n]^\vee$ is clearly the R–span of the set $\mathbb{B}^\vee := \Big\{ \prod_{i=1}^n E_i^{a_i} \cdot H^c \cdot \prod_{j=1}^n F_j^{b_j} \,\Big|\, a_i, c, b_j \in \mathbb{N}\ \forall\, i,j \Big\}$, so from this and the previous formulas for Δ^n one gets that $\big(F_q[H_n]^\vee\big)' = F_q[H_n]$, q.e.d.

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