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A SIMPLE PROOF OF THE EXISTENCE OF MODULAR AUTOMORPHISMS IN APPROXIMATELY FINITE DIMENSIONAL VON NEUMANN ALGEBRAS

ROBERTO LONGO

An elementary direct proof of Tomita–Takesaki Theorem for an AFD von Neumann Algebra.

1. Introduction. After that M. Tomita [5] proposed the existence of the modular automorphisms several proofs of Tomita–Takesaki theorem have been given by Takesaki, van Daele, Haagerup (unpublished) and Zsido [4, 6, 7, 8], but none of these is elementary. However a simple proof of the theorem for approximately finite dimensional von Neumann algebras (with a cyclic separating vector) may be extracted by an article of N.M. Hugenholtz and J.D. Wieringa 1], which was published very soon after the appearance of Tomita's original preprint. Motivated by the great interest that approximately finite dimensional von Neumann algebras have in Mathematics and in Physics, we present a simplified shorter version of the proof of Hugenholtz and Wieringa.

2. Statement and Proof. Let $\mathcal{R}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ a cyclic separating vector for $\mathcal{R}$ and then also for its commutant $\mathcal{R}'$. As usual we introduce the antilinear operators

$$S_0\colon A\xi, A \in \mathcal{R}, \to A^*\xi, \ \mathcal{D}(S_0) = \mathcal{R}\xi,$$
$$F_0\colon B\xi, B \in \mathcal{R}', \to B^*\xi, \quad \mathcal{D}(F_0) = \mathcal{R}'\xi;$$

S_0 (and F_0) is a closable operator: in fact if $A \in \mathcal{R}$ and $B \in \mathcal{R}'$

$$(S_0A\xi, B\xi) = (A^*\xi, B\xi) = (\xi, AB\xi) = (\xi, BA\xi)$$
$$= (B^*\xi, A\xi) = (F_0B\xi, A\xi)$$

so that $S_0^* \supset F_0$ and $\mathcal{D}(S_0^*)$ is dense.

In what follows we call $F = S_0^*$ the adjoint of S_0, $S = F^*$ the closure of S_0 and $\Delta = FS$ the modular operator which is non singular and positive. For the moment we suppose $\mathcal{R}$ finite dimensional; then there exists a faithful tracial state τ and for each state ω of $\mathcal{R}$ there exists a positive operator $H \in \mathcal{R}$ *s.t.*

$$\omega(A) = \tau(AH) = \tau(H^{1/2}AH^{1/2}), \qquad A \in \mathcal{R},$$

moreover H is invertible iff ω is faithful. Let $\pi: \mathcal{R} \to \mathcal{B}(\mathcal{H}_\tau)$ be the *GNS* representation given by τ and ω a faithful state of $\mathcal{R}$: we have $\mathcal{H}_\tau = \mathcal{R}$, $\pi(A)B = AB$ if $A, B \in \mathcal{R}$ and

$$\omega(A) = (\pi(A)H^{1/2}, H^{1/2}), \qquad A \in \mathcal{R},$$

where $H^{1/2} \in \mathcal{R}$ is a cyclic separating vector for $\pi(\mathcal{R})$. It is easily seen the operator Δ of $\pi(\mathcal{R})$ relative to the vector $H^{1/2}$ is given by

$$\Delta: A \in \mathcal{H}_\tau \to HAH^{-1} \in \mathcal{H}_\tau$$

from which it follows

$$\Delta^{-it}\pi(A)\Delta^{it} = \pi(H^{-it}AH^{it}), \qquad A \in \acute{\mathcal{R}}, t \in \mathbf{R},$$

and then

$$\Delta^{-it}\pi(\mathcal{R})\Delta^{it} = \pi(\mathcal{R}), \qquad t \in \mathbf{R}.$$

By the uniqueness of the *GNS* representation we then see that for each finite dimensional von Neumann algebra $\mathcal{R}^{\cdot}$ the modular operator Δ relative to a cyclic separating vector is such that

$$\Delta^{-it}\mathcal{R}\Delta^{it} = \mathcal{R}, \qquad t \in \mathbf{R},$$

which is a particular case of Tomita–Takesaki theory.

Next step is proving the theorem when $\mathcal{R}$ is approximately finite dimensional, in the sense that there exists an increasing sequence $\mathcal{M}_n \subset \mathcal{R}$ of finite dimensional von Newmann algebras s.t.

$$\mathcal{R} = \left(\bigcup_{n=1}^{\infty} \mathcal{M}_n\right)''.$$

Then we have to prove:

THEOREM 1. *Let $\mathcal{R}$ be an approximately finite dimensional von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ a cyclic separating vector for $\mathcal{R}$. The modular operator Δ relative to ξ is such that*

$$\Delta^{-it}\mathcal{R}\Delta^{it} = \mathcal{R}, \qquad t \in \mathbf{R}.$$

Our proof requires some lemmas. Let $\mathcal{M}_n$ be an increasing sequence of finite dimensional von Neumann algebras generating $\mathcal{R}$ and put $\mathfrak{A} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$ so that $\mathfrak{A}$ is a weakly dense $*$ subalgebra of $\mathcal{R}$.

LEMMA 1. *The linear subspace of* $\mathcal{H}$ $\mathfrak{A}\xi = \{A\xi \mid A \in \mathfrak{A}\}$ *is a core for* S.

Proof. It is enough to show that for each $A \in \mathcal{R}$ there exists a sequence $A_n \in \mathfrak{A}$ s.t.

$$A_n\xi \to A\xi \quad \text{and} \quad A_n^*\xi \to A^*\xi$$

and this follows because the selfadjoint elements of $\mathfrak{A}$ are dense in the selfadjoint elements of $\mathcal{R}$ in the strong topology.

Now we call $\mathcal{H}'$ the domain of S with scalar product

$$(x, y)' = (x, y) + (Sy, Sx), \qquad x, y \in \mathcal{D}(S).$$

As the topology of $\mathcal{H}'$ is that of the graph of S, we see that $\mathcal{H}'$ is a Hilbert space and by lemma 1 $\mathfrak{A}\xi$ is a dense linear subspace of $\mathcal{H}'$.
Now the sesquilinear form (x, y), $x, y \in \mathcal{H}'$, is bounded in $\mathcal{H}'$

$$|(x, y)| \leqq \|x\| \; \|y\| \leqq \|x\|' \; \|y\|', \qquad x, y \in \mathcal{H}'$$

($\|x\|' = (x, x)'^{1/2}$) and therefore there exists a linear operator $T \in \mathcal{B}(\mathcal{H}')$ of norm less than 1 s.t.

$$(1) \qquad (Tx, y)' = (x, y), \qquad x, y \in \mathcal{H}'$$

Let $E_n \in \mathcal{M}_n'$ be the selfadjoint projection of $\mathcal{H}$ onto $\mathcal{M}_n\xi$ and $\mathcal{M}_{nE_n} = \{A\,|_{E_n(\mathcal{H})} \mid A \in \mathcal{M}_n\}$ the von Neumann algebra $\mathcal{M}_n$ cut down by E_n. The application

$$(2) \qquad \pi_n : A \in \mathcal{M}_n \to A\,|_{\mathcal{M}_n\xi} \in \mathcal{M}_{nE_n}$$

is *a* $*$ isomorphism between $\mathcal{M}_n$ and $\mathcal{M}_{nE_n}$ because ξ is a separating vector for $\mathcal{M}_n$; moreover ξ is a cyclic separating vector for $\mathcal{M}_{nE_n}$ and therefore if S_n is the antilinear operator

$$S_n : A\xi = \pi_n(A)\xi \to A^*\xi = \pi_n(A)^*\xi, \qquad A \in \mathcal{M}_n$$

then, by what we know, the modular operator $\Delta_n = S_n^* S_n$ is s.t.

$$\Delta_n^{-it}\mathcal{M}_{nE_n}\Delta_n^{it} = \mathcal{M}_{nE_n}, \qquad t \in \mathbf{R}.$$

We see also that $S_n = S|_{\mathcal{M}_{n\xi}}$ and if $\mathcal{H}'_n$ is the linear space $\mathcal{D}(S_n)$ with scalar product

$$(x, y)' = (x, y) + (S_n y, S_n x), \qquad x, y \in \mathcal{D}(S_n)$$

then $\mathcal{H}'_n$ is a Hilbert subspace of $\mathcal{H}'$; as in (1) there exists $T_n \in \mathcal{B}(\mathcal{H}'_n)$ of norm less that 1 s.t.

$$(3) \qquad (T_n x, y)' = (x, y) \qquad x, y \in \mathcal{H}'_n.$$

LEMMA 2. *Let $P_n \in \mathcal{B}(\mathcal{H}')$ be the selfadjoint projection of $\mathcal{H}'$ onto $\mathcal{H}'_n$. The operators $\tilde{T}_n = T_n P_n + (I - P_n) \in \mathcal{B}(\mathcal{H}')$ are* s.t.

$$\| Tx - \tilde{T}_n x \|' \to 0, \qquad \forall x \in \mathcal{H}'.$$

Proof. As $\bigcup_{n=1}^{\infty} \mathcal{H}'_n = \mathfrak{A}\xi$ is dense in $\mathcal{H}'$ by Lemma 1, the orthogonal projections P_n strongly converge to I in $\mathcal{H}'$ (we use the symbol I to indicate both the identity of $\mathcal{H}'$ and the identity of $\mathcal{H}$).

By (1) and (3)

$$(Tx, y) = (T_n x, y) \qquad \text{if} \quad x, y \in \mathcal{H}'_n$$

and therefore

$$\tilde{T}_n = P_n T P_n + (I - P_n);$$

it follows that

$$\| \tilde{T}_n x - Tx \|' \to 0$$

if x belongs to the dense subspace $\bigcup_{n=1}^{\infty} \mathcal{H}'_n$ and then for each $x \in \mathcal{H}'$ because the $\tilde{T}_n$ are equibounded.

We extend the modular operators $\Delta_n = S_n^* S_n$ to the whole space $\mathcal{H}$ by

$$\tilde{\Delta}_n = \Delta_n E_n + I - E_n,$$

then each $\tilde{\Delta}_n$ is a positive invertible operator and we may consider $\tilde{\Delta}_n^{it}$, $t \in \mathbf{R}$.

LEMMA 3. *For each real t, Δ^{it} is the strong limit of $\tilde{\Delta}_n^{it}$ i.e.*

$$\| \tilde{\Delta}_n^{it} x - \Delta^{it} x \| \to 0, \qquad x \in \mathcal{H}.$$

Proof. The lemma is proved if we show that

$$(\tilde{\Delta}_n + I)^{-1} \to (\Delta + I)^{-1} \quad \text{strongly;} \tag{4}$$

in fact by a classical theorem on generalized convergence [**3**, Th. VIII. 20] it follows from (4) that

$$f(\tilde{\Delta}_n) \to f(\Delta) \qquad \text{strongly} \tag{5}$$

for each bounded continuous complex valued function f on the real line; moreover the same argument shows that (5) holds also when f is bounded continuous on an open subset A of the real line of spectral measure 1 for Δ and each $\tilde{\Delta}_n$; in particular for $A = (0, \infty)$ and $f(\lambda) = \lambda^{it}$ the conclusion of the lemma follows from (4).

Note that the range of $(\Delta + I)^{-1}$ is equal to $\mathscr{D}(\Delta) \subset \mathscr{D}(S)$ so that we have by (1), for each $x, y \in \mathscr{D}(S)$,

$$\begin{aligned} ((\Delta + I)^{-1}x, y)' &= ((\Delta + I)^{-1}x, y) + (Sy, S(\Delta + I)^{-1}x) \\ &= ((\Delta + I)^{-1}x, y) + (\Delta(\Delta + I)^{-1}x, y) \\ &= ((\Delta + I)(\Delta + I)^{-1}x, y) \\ &= (x, y) = (Tx, y)' \end{aligned}$$

which implies

$$T = (\Delta + I)^{-1}|_{\mathscr{D}(S)}.$$

By the same argument $T_n = (\Delta_n + I)^{-1}$ and then

$$\tilde{T}_n|_{\mathscr{M}_n\xi} = (\tilde{\Delta}_n + I)^{-1}|_{\mathscr{M}_n\xi}$$

Applying Lemma 2, if $x \in \mathfrak{A}\xi$ we have for large n

$$\begin{aligned} \|(\tilde{\Delta}_n + I)^{-1}x - (\Delta + I)^{-1}x\| &\leqq \|(\tilde{\Delta}_n + I)^{-1}x - (\Delta + I)^{-1}x\|' \\ &= \|\tilde{T}_n x - Tx\|' \to 0, \end{aligned}$$

and as $\|(\tilde{\Delta}_n + I)^{-1}\| \leqq 1$, $n \in \mathbf{N}$, and $\mathfrak{A}\xi$ is dense in $\mathscr{H}$, we obtain the lemma.

By the isomorphism π_n defined in (2) we may define the modular automorphisms σ_t^n, $t \in \mathbf{R}$, of $\mathscr{M}_n$ by

$$\pi_n(\sigma_t^n(A)) = \Delta_n^{-it}\pi_n(A)\Delta_n^{it}, \qquad A \in \mathscr{M}_n, \qquad t \in \mathbf{R}.$$

LEMMA 4. *If $A \in \mathfrak{A}$ then the sequence $\sigma_t^n(A)$, defined above a certain integer, strongly converges to $\Delta^{-it}A\Delta^{it}$, i.e.*

$$\|\sigma_t^n(A)x - \Delta^{-it}A\Delta^{it}x\| \to 0, \qquad x \in \mathcal{H}, t \in \mathbf{R}.$$

Proof. As we suppose $A \in \mathfrak{A}$ there exists $N \in \mathbf{N}$ s.t. $A \in \mathcal{M}_n$, $n \geqq N$. Take $x \in \mathfrak{A}\xi$: there exists $N' \in \mathbf{N}$ s.t. $x \in \mathcal{M}_n\xi$, $n \geqq N'$. Then we have for $n \geqq \max(N, N')$

$$\sigma_t^n(A)x = \pi_n(\sigma_t^n(A))x = \Delta_n^{-it}A\Delta_n^{it}x = \tilde{\Delta}_n^{-it}A\tilde{\Delta}_n^{it}x$$

and Lemma 3 implies

$$\|\sigma_t^n(A)x - \Delta^{-it}A\Delta^{it}x\| \to 0, \ A \in \mathfrak{A}, \ x \in \mathfrak{A}\xi, \ t \in \mathbf{R}.$$

As $\mathfrak{A}\xi$ is dense in $\mathcal{H}$ and $\|\sigma_t^n(A)\| \leqq \|A\|$ is an equibounded sequence the lemma follows.

Proof of Theorem 1. In view of Lemma 4 if $A \in \mathfrak{A}$ then $\Delta^{-it}A\Delta^{it}$, $t \in \mathbf{R}$, belongs to the strong closure of $\mathfrak{A}$ i.e.

$$\Delta^{-it}\mathfrak{A}\Delta^{it} \subset \mathcal{R}, \qquad t \in \mathbf{R};$$

by continuity

$$\Delta^{-it}\mathcal{R}\Delta^{it} \subset \mathcal{R}, \qquad t \in \mathbf{R}$$

and then by symmetry

$$\Delta^{-it}\mathcal{R}\Delta^{it} = \mathcal{R}, \qquad t \in \mathbf{R}.$$

REMARK 1. The essential tool we have used in the proof is the existence of a faithful tracial state on each approximating von Neumann algebra $\mathcal{M}_n$

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ISTITUTO MATEMATICO
G. CASTLNUOVA
UNIVERSITÀ DI ROMA
00100 ROMA