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A BRAUER ALGEBRA THEORETIC PROOF OF LITTLEWOOD'S RESTRICTION RULES

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Abstract. Let U be a complex vector space endowed with an orthogonal or symplectic form, and let G be the subgroup of $GL(U)$ of all the symmetries of this form (resp. $O(U)$ or $Sp(U)$); if M is an irreducible $GL(U)$–module, the Littlewood's restriction rule describes the G–module $M\big|_G^{GL(U)}$. In this paper we give a new representation-theoretic proof of this formula: realizing M in a tensor power $U^{\otimes f}$ and using Schur's duality we reduce to the problem of describing the restriction to an irreducible S_f–module of an irreducible module for the centralizer algebra of the action of G on $U^{\otimes f}$; the latter is a quotient of the Brauer algebra, and we know the kernel of the natural epimorphism, whence we deduce the Littlewood's restriction rule.

"Non potrai dir che quest' è cosa dura:
usando la dualità di Brauer
dimostrazione dar, novella e pura"

N. Barbecue, "Scholia"

Introduction

Let U be a complex vector space, endowed with an orthogonal or symplectic form, and let G be either $O(U)$ or $Sp(U)$ respectively. Consider a simple polynomial $GL(U)$–module V_λ (associated in a standard way to a partition λ), and restrict it to G; if $\lambda_1^t + \lambda_2^t \leq dim(U)$ (in the orthogonal case), λ^t being the dual partition to λ, or $\lambda_1^t \leq dim(U)\big/2$ (in the symplectic case) then its decomposition into simple G–modules is described by the Littlewood's restriction rule (cf. [L]), which gives a formula for the multiplicity in V_λ of each simple G–module. The main aim in this article is to prove this formula.

It is well known (cf. e.g. [W], [H]) that one can realize a copy of V_λ inside the tensor power $U^{\otimes f}$, where f is the sum of parts of λ (i.e. λ is a partition of f); by the general theory of centralizer algebras, a bijection $V_\lambda \longleftrightarrow M_\lambda$ exists between simple $GL(U)$–modules

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and simple modules over $End_{GL(U)}\big(U^{\otimes f}\big)$ (the centralizer algebra of the $GL(U)$–action on $U^{\otimes f}$) occurring in $U^{\otimes f}$, which interchanges dimensions and multiplicities; similarly, a bijection $W_\mu \longleftrightarrow N_\mu$ exists between simple G–modules and simple modules over $End_G\big(U^{\otimes f}\big)$ (the centralizer algebra of the G–action on $U^{\otimes f}$) occurring in $U^{\otimes f}$ (which is now thought of as a G–module), which interchanges dimensions and multiplicities: then we have an identity $\big[V_\lambda : W_\mu\big] = \big[N_\mu : M_\lambda\big]$, thus to get the multiplicity $\big[V_\lambda : W_\mu\big]$ we can compute the above right-hand-side term instead: in other words, instead of studying $V_\lambda\Big|_G^{GL(U)}$ we study $N_\mu\Big|_{End_{GL(U)}(U^{\otimes f})}^{End_G(U^{\otimes f})}$. So if

$$\big[V_\lambda : W_\mu\big] = C_\mu^\lambda \tag{$\star$}$$

is the identity given in Littlewood's restriction formula, our aim is to prove that

$$\big[N_\mu : M_\lambda\big] = C_\mu^\lambda \tag{$\star\star$}$$

Now, one has that $End_{GL(U)}\big(U^{\otimes f}\big) = \mathbb{C}\big[S_f\big]$, with S_f acting on $U^{\otimes f}$ by index permutation; on the other hand, $End_G\big(U^{\otimes f}\big)$ is a quotient of the Brauer algebra $\mathcal{B}_f^{(\epsilon N)}$, where $N = dim_{\mathbb{C}}(U)$ and ϵ is the "sign" of the form on U ("$+$" for orthogonal and "$-$" for symplectic case); the kernel of $\pi_U : \mathcal{B}_f^{(\epsilon N)} \twoheadrightarrow End_G\big(U^{\otimes f}\big)$ is also known, essentially from the Second Fundamental Theorem of Invariant Theory (for the group G). In the stable case (i.e. when $f \leq N/2$ in the symplectic case and $f \leq N$ in the orthogonal case) π_U is an isomorphism, and Littlewood's formula can be proved as a corollary of a suitable description of $V^{\otimes f}$ (cf. [GP]). In the general case a different approach is necessary.

To describe $\mathcal{B}_f^{(x)}$ we can display an explicit basis D_f — whose elements are certain graphs — and assign the multiplication rules for elements in this basis — based on "composition" of graphs. Then from the previously mentioned description of $Ker\big(\pi_U\big)$ we take out an explicit set of linear generators of this kernel.

In addition, the simple G–modules N_μ are quotients of certain $\mathcal{B}_f^{(\varepsilon N)}$–modules N'_μ which have a nice combinatorial description (in terms of graphs related to those of D_f); moreover, we prove that the kernel of the epimorphism $N'_\mu \twoheadrightarrow N_\mu$ is just $Ker\big(\pi_U\big).N'_\mu$. Now, the multiplicity $\big[N'_\mu : M_\lambda\big]$ is exactly equal to the right-hand-side part of ($\star$); then it is enough for us to show that in $Ker\big(\pi_U\big).N'_\mu$, as a $\mathbb{C}\big[S_f\big]$–module, there are no components of type M_λ for λ such that $\lambda_1^t + \lambda_2^t \leq dim(U)$ (in the orthogonal case) or $\lambda_1^t \leq dim(U)/2$ (in the symplectic case): this we deduce from the description of $Ker\big(\pi_U\big)$.

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§1 Reminders of Invariant Theory

1.1 The Fundamental Theorems of Invariant Theory. In this section we recall some well-known facts of Classical Invariant Theory; the general source is [We], nevertheless we shall also mention more specific — and recent — references.

Let $f \in \mathbb{N}_+$ be fixed. Consider $n \in \mathbb{N}$; let V be a $\mathbb{C}$–vector space of dimension n, endowed with a non-degenerate symmetric bilinear form $(\ ,\ )$, and let $O(V)$ be the associated orthogonal group. On the other hand, let W be a $\mathbb{C}$–vector space of dimension $2n$, endowed with a non-degenerate skew-symmetric bilinear form $\langle\ ,\ \rangle$, and let $Sp(W)$ be the associated symplectic group. In this setting, we have canonical isomorphisms $V \xrightarrow{\cong} V^*, v \mapsto (v,\ \cdot\ )$, $W \xrightarrow{\cong} W^*, w \mapsto \langle w,\ \cdot\ \rangle$, which also gives isomorphisms

$$\begin{array}{cc} \Theta_V : V \otimes V \xrightarrow{\qquad\cong\qquad} End(V) & \Theta_W : W \otimes W \xrightarrow{\qquad\cong\qquad} End(W) \\ v_1 \otimes v_2 \mapsto \Theta_V\,(v_1 \otimes v_2)\Big(v \mapsto \big(v_1, v\big) v_2\Big) & w_1 \otimes w_2 \mapsto \Theta_W\,(w_1 \otimes w_2)\Big(w \mapsto \big\langle w_1, w\big\rangle w_2\Big) \end{array}$$

Then $V^{\otimes 2f} \xrightarrow{\cong} \big(V^{\otimes 2f}\big)^*$, $V^{\otimes 2f} = V^{\otimes f} \otimes V^{\otimes f} \xrightarrow{\cong} End\big(V^{\otimes f}\big)$, and $\big(V^{\otimes 2f}\big)^* \xrightarrow{\cong} End\big(V^{\otimes 2f}\big)$, whence also $\Psi_V : \Big(\big(V^{\otimes 2f}\big)^*\Big)^{O(V)} \xrightarrow{\cong} \Big(End\big(V^{\otimes 2f}\big)\Big)^{O(V)} = End_{O(V)}\big(V^{\otimes f}\big)$; and similarly for W, in particular $\Psi_W : \Big(\big(W^{\otimes 2f}\big)^*\Big)^{Sp(W)} \xrightarrow{\cong} \Big(End\big(W^{\otimes 2f}\big)\Big)^{Sp(V)} = End_{Sp(W)}\big(W^{\otimes f}\big)$.

Finally, we define $\psi_V := \Theta_V^{-1}\,(id_V)$, $\psi_W := \Theta_W^{-1}\,(id_W)$.

Definition 1.2. *Fix $f \in \mathbb{N}_+$; for each pair $p, q \in \{1, 2, \dots, f\}$ with $p \neq q$ we define*

(a) a contraction operator $\Phi_{p,q} : V^{\otimes (f+2)} \longrightarrow V^{\otimes f}$ (for $p < q$, say)

$$\Phi_{p,q}\big(v_1 \otimes v_2 \otimes \cdots \otimes v_{f+2}\big) = \big(v_p\,,\, v_q\big) \cdot v_1 \otimes \cdots \widehat{v_p} \otimes \cdots \otimes \widehat{v_q} \otimes \cdots \otimes v_{f+2}\;;$$

(b) an insertion operator $\Psi_{p,q} : V^{\otimes f} \longrightarrow V^{\otimes (f+2)}$, obtained inserting the element ψ_V in the positions p, q;

(c) an operator $\tau_{p,q} : V^{\otimes f} \longrightarrow V^{\otimes f}$ defined by $\tau_{p,q} := \Psi_{p,q} \circ \Phi_{p,q}$.

The same definition with $\langle\ ,\ \rangle$ instead of $(\ ,\ )$ gives operators $\Phi_{p,q} : W^{\otimes (f+2)} \longrightarrow W^{\otimes f}$, $\Psi_{p,q} : W^{\otimes f} \longrightarrow W^{\otimes (f+2)}$, $\tau_{p,q} : W^{\otimes f} \longrightarrow W^{\otimes f}$ in the symplectic case.

In addition, the symmetric group S_f acts on $V^{\otimes f}$ or $W^{\otimes f}$ by

$$\sigma : u_1 \otimes u_2 \otimes \cdots \otimes u_f \mapsto u_{\sigma^{-1}(1)} \otimes u_{\sigma^{-1}(2)} \otimes \cdots \otimes u_{\sigma^{-1}(f)} \qquad \forall\, \sigma \in S_f$$

Theorem 1.3. *(I Fundamental Theorem for $O(V)$ and $Sp(W)$) The operators $\tau_{p,q}$ ($p \neq q$) and σ ($\in S_f$) generate the whole centralizer algebra, $End_{O(V)}\big(V^{\otimes f}\big)$ or $End_{Sp(W)}\big(W^{\otimes f}\big)$.*

Let $\mathcal{P}\left(X^{\oplus f}\right)$ denote the space of polynomial functions on $X^{\oplus f}$, for any vector space X.

Theorem 1.4. *(II Fundamental Theorem for $O(V)$ and $Sp(W)$: cf. [DP], Th. 6.7)*

(a)
$$\Big(\mathcal{P}\big(V^{\oplus f}\big)\Big)^{O(V)} = \mathbb{C}\big[(v_i, v_j)\big]\,.$$

Moreover, the ideal of relations between the generators (v_i, v_j) is generated by the minors of order $(n+1)$ of the $f \times f$ symmetric matrix $\big((v_i, v_j)\big)_{i,j=1,\dots,f}$.

(b)
$$\Big(\mathcal{P}\big(W^{\oplus f}\big)\Big)^{Sp(V)} = \mathbb{C}\big[\langle w_i, w_j\rangle\big]\,.$$

Moreover, the ideal of relations between the generators $\langle v_i, v_j\rangle$ is generated by the Pfaffians of order $2(n+1)$ of the $f \times f$ skew-symmetric matrix $\big(\langle w_i, w_j\rangle\big)_{i,j=1,\dots,f}$.

Now consider the polynomial rings (in the symmetric or antisymmetric variables x_{ij})

$$A^O := \mathbb{C}[x_{ij}]_{i,j=1,i\neq j}^{2f}\Big/ (x_{ij} = x_{ji}) \ , \qquad A^{Sp} := \mathbb{C}[x_{ij}]_{i,j=1,i\neq j}^{2f}\Big/ (x_{ij} = -x_{ji})$$

For $X \in \{O, Sp\}$, define A_f^X (the space of *multilinear elements* in A^X) to be the $\mathbb{C}$–span of all monomials (of degree f) $x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_f j_f}$ such that $(i_1, j_1, i_2, j_2, \dots, i_f, j_f)$ is a permutation of $\{1, 2, 3, 4, \dots, 2f\}$.

Of course A_f^X is an S_{2f}–module, described by the statement below (cf. [LP], Proposition 3.3); hereafter, when dealing with a symmetric group S_h we write $\lambda \vdash h$ to mean that λ is a partition of h ($\in \mathbb{N}$), for given $\lambda \vdash h$ we denote by λ^t the dual partition, and by M_λ the associated irreducible representation of S_h (with the assumption that $M_{(h)}$ is the trivial representation of S_h and $M_{\underbrace{(1,1,\dots,1)}_{h}}$ is the sign (alternating) representation.

Proposition 1.5. *The representation of S_{2f} on A_f^O, resp. A_f^{Sp}, is induced by the trivial, resp. sign, representation of K_f. Moreover, there are isomorphisms of S_{2f}–modules*

$$A_f^O \cong \bigoplus_{\substack{\sigma \vdash 2f \\ \sigma \text{ has even rows}}} M_\sigma \ , \qquad \textit{resp.} \quad A_f^{Sp} \cong \bigoplus_{\substack{\sigma \vdash 2f \\ \sigma \text{ has even columns}}} M_\sigma \ .$$

Now let $\mathbf{i} := (i_1, i_2, \dots, i_f)$, $\mathbf{j} := (j_1, j_2, \dots, j_f)$ be such that $(i_1, j_1, \dots, i_f, j_f)$ is a permutation of $\{1, 2, \dots, 2f-1, 2f\}$. We define $\eta_{\mathbf{i},\,\mathbf{j}} \in \left(V^{\otimes 2f}\right)^*$ and $\eta_{\mathbf{i},\,\mathbf{j}} \in \left(W^{\otimes 2f}\right)^*$, by

$$\eta_{\mathbf{i},\,\mathbf{j}}(v_1 \otimes \cdots \otimes v_{2f}) := \prod_{k=1}^{f} (v_{i_k}, v_{j_k}) \ , \qquad \eta_{\mathbf{i},\,\mathbf{j}}(w_1 \otimes \cdots \otimes w_{2f}) := \prod_{k=1}^{f} \langle w_{i_k}, w_{j_k} \rangle \ ;$$

it is clear that $\eta_{\mathbf{i},\,\mathbf{j}} \in \left(\left(V^{\otimes 2f}\right)^*\right)^{O(V)}$, resp. $\eta_{\mathbf{i},\,\mathbf{j}} \in \left(\left(W^{\otimes 2f}\right)^*\right)^{Sp(W)}$. Remark that both $\left(V^{\otimes 2f}\right)^*$ and $\left(W^{\otimes 2f}\right)^*$ are S_{2f}–modules and, since the action of S_{2f} centralizes that of the form-preserving group, also $\left(\left(V^{\otimes 2f}\right)^*\right)^{O(V)}$ and $\left(\left(W^{\otimes 2f}\right)^*\right)^{Sp(W)}$ are S_{2f}–modules.

Similarly, we shall use the notation $x_{\mathbf{i},\,\mathbf{j}} := x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_f j_f}$.

Proposition 1.6 ([LP], Th. 3.8). *The linear map*

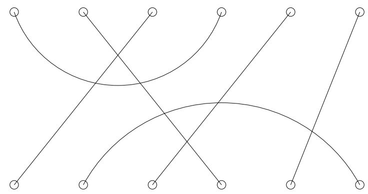
$$\alpha_V : A_f^O \longrightarrow \left(\left(V^{\otimes 2f}\right)^*\right)^{O(V)}, \qquad \textit{resp.} \quad \alpha_W : A_f^{Sp} \longrightarrow \left(\left(W^{\otimes 2f}\right)^*\right)^{Sp(W)}$$

defined by $\alpha_V(x_{\mathbf{i},\,\mathbf{j}}) = \eta_{\mathbf{i},\,\mathbf{j}}$, resp. $\alpha_W(x_{\mathbf{i},\,\mathbf{j}}) = \eta_{\mathbf{i},\,\mathbf{j}}$, is a surjective homomorphism of S_{2f}–modules, whose kernel is the intersection of A_f^O, resp. A_f^{Sp}, with the ideal Min_{n+1}, resp. $Pf_{2(n+1)}$, of A^O, resp. A^{Sp}, generated by the minors of order $n+1$, resp. the Pfaffians of order $2n+2$, of the symmetric, resp. skew-symmetric, matrix $\left(x_{ij}\right)_{i,j=1}^{2f}$, and it corresponds — in the isomorphism of Proposition 1.5 — to the S_{2f}–submodule

$$\bigoplus_{\substack{\sigma \vdash 2f \,,\, l(\sigma) > n \\ \sigma \text{ has even rows}}} M_\sigma \ , \qquad \textit{resp.} \qquad \bigoplus_{\substack{\sigma \vdash 2f \,,\, l(\sigma) > 2n \\ \sigma \text{ has even columns}}} M_\sigma \ .$$

§2 The Brauer algebra

2.1 ***f*–diagrams.** Let $f \in \mathbb{N}_+$ be fixed. Denote by $\mathbb{V}_f$ the datum of $2f$ spots in a plane, arranged in two rows, one upon the other, each of f aligned spots. Then consider the graphs with $\mathbb{V}_f$ as set of vertices and f edges such that each vertex belongs to exactly one edge. The picture below shows an example of such a graph for $f = 6$.



We call such graphs *f–diagrams*, denoting by D_f the set of all of them; in general we shall denote them by bold roman letters, like $\mathbf{d}$. Of course the f–diagrams are as many as the pairings of $2f$ elements, hence $(2f-1)!! := (2f-1)\cdot(2f-3)\cdots 5\cdot 3\cdot 1$ in number.

We shall label the vertices in $\mathbb{V}_f$ in two ways: either we label the spots in the upper row with the numbers 1^+, 2^+, ..., f^+, in their natural order from left to right, and the spots in the lower row with the numbers 1^-, 2^-, ..., f^-, again from left to right, or we label them by setting i for i^+ and $f+j$ for j^- (for all $i,j \in \{1,2,\dots,f\}$). Accordingly, an f–diagram can also be described by simply specifying its set of edges: so for instance the 6–diagram above is given by $\big\{\{1^+,4^+\},\{3^-,5^+\},\{2^+,4^-\},\{5^-,6^+\},\{2^-,6^-\},\{3^+,1^-\}\big\}$. In general, given f–tuples $\mathbf{i} := (i_1,i_2,\dots,i_f)$ and $\mathbf{j} := (j_1,j_2,\dots,j_f)$ such that $\{i_1,\dots,i_f\} \cup \{j_1,\dots,j_f\} = \mathbb{V}_f$, we define $\mathbf{d}_{\mathbf{i},\,\mathbf{j}}$ to be the f–diagram obtained by joining i_k to j_k, for each $k = 1,2,\dots,f$. For instance, the above diagram is $\mathbf{d}_{\mathbf{i},\,\mathbf{j}}$ for $\mathbf{i} = \{1^+,2^+,3^+,5^+,6^+,2^-\}$, $\mathbf{j} = \{4^+,4^-,1^-,3^-,5^-,6^-\}$.

When looking at the edges of an f–diagram, we shall distinguish between those which link two vertices in the same row (upper or lower), which will be called *horizontal edges* or simply *bars*, and those which link two vertices in different rows, to be called *vertical edges*. It is clear that any f–diagram has the same number of bars in the upper row and in the lower one: if this number is k, we shall say that this is a k–bar (f–)diagram. Thus letting $D_{f,k} := \big\{\, \mathbf{d} \in D_f \,\big|\, \mathbf{d} \textit{ is a } k\textit{–bar diagram} \big\}$ we have $D_f = \cup_{k=1}^{[f/2]} D_{f,k}$.

2.2 Bar structure and permutation structure of diagrams. Let $\mathbf{d}$ be an f–diagram. With "bar structure of the upper row", resp. "lower row", of $\mathbf{d}$ we shall mean the datum of the bars in the upper, resp. lower, row of $\mathbf{d}$ (in their positions): to be short we shall also use such terminology as "upper bar structure", resp. "lower bar structure", of $\mathbf{d}$ — to be denoted with $\mathrm{ubs}(\mathbf{d})$, resp. $\mathrm{lbs}(\mathbf{d})$ — and "bar structure of $\mathbf{d}$" — to be denoted with $\mathrm{bs}(\mathbf{d})$ — to mean the datum of both the upper and the lower bar structure of $\mathbf{d}$, i.e. $\mathrm{bs}(\mathbf{d}) := \big(\mathrm{ubs}(\mathbf{d}), \mathrm{lbs}(\mathbf{d})\big)$. Notice that an upper or lower bar structure may be described by a one-row graph of vertices arranged on a horizontal line and some edges (the "bars") joining them pairwise so that every vertex belongs at most to one edge: following Kerov (cf. [Ke]) such a graph will be called a *k–bar f–junction*, or *(f,k)–junction*, where f is its number of vertices and k its number of edges; for instance, here below you find the 1–bar 6–junctions which represent the upper (on the left hand side) and lower (on the right hand side) bar structure of the 6–diagram in §2.1:

We denote the set of (f,k)–junctions by $J_{f,k}$, and by $H_{f,k}$ the $\mathbb{C}$–vector space with basis $J_{f,k}$. It is clear from definitions that $dim(H_{f,k}) = \left|J_{f,k}\right| = \binom{f}{2k}(2k-1)!!$. Finally, for all $\mu \vdash (f-2k)$ $(k \in \{0,1,\dots,[f/2]\})$ we define $H^{\mu}_{f,k} := M_{\mu} \otimes H_{f,k}$.

If $\mathbf{d} \in D_{f,k}$ then it has exactly $f-2k$ vertices in its upper row and $f-2k$ vertices in its lower row which are pairwise joined by its $f-2k$ vertical edges; label with 1, 2, ..., $f-2k$ from left to right the vertices in the upper row, and do the same in the lower row: then we can define a permutation $\sigma = \sigma(\mathbf{d}) \in S_{f-2k}$ — to be called the "permutation structure" (or "symmetric part") of $\mathbf{d}$ — by letting $\sigma(i)$ be the label of the lower row vertex of the vertical edge whose upper row vertex is labelled with i.

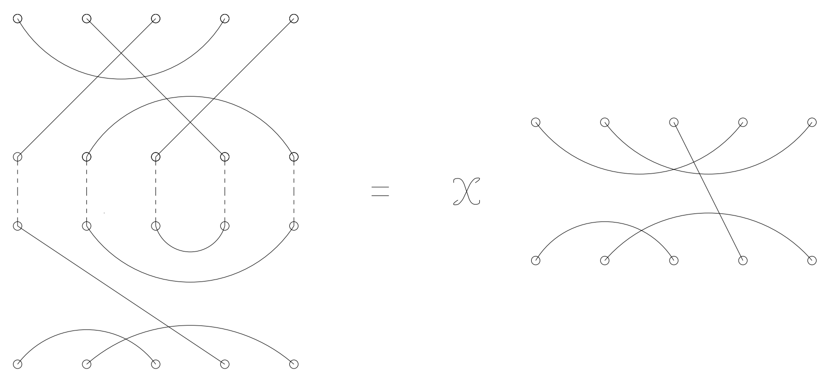
The upshot is that the assignement $\mathbf{d} \mapsto \big(\sigma(\mathbf{d}), \mathrm{bs}(\mathbf{d})\big)$ establishes a bijection

$$D_{f,k} \longrightarrow S_{f-2k} \times \big(J_{f,k} \times J_{f,k}\big) \tag{2.1}$$

and glueing together these maps for all k gives a bijection $D_f \longrightarrow \bigcup_{k=1}^{[f/2]} S_{f-2k} \times \big({J_{f,k}}^{\times 2}\big)$.

2.3 Definition of the Brauer algebra. Fix any field $\mathbb{K}$, and take $x \in \mathbb{K}$. Let $\mathcal{B}_f^{(x)}$ be the $\mathbb{K}$–vector space with basis D_f; we introduce a product in $\mathcal{B}_f^{(x)}$ (which depends on x) by defining the product of f–diagrams and extending by linearity. So for all $\mathbf{a}, \mathbf{b} \in D_f$ define the product $\mathbf{a} \cdot \mathbf{b} = \mathbf{ab}$ as follows: first draw $\mathbf{b}$ below $\mathbf{a}$; second, connect the i–th lower vertex of $\mathbf{a}$ with the i–th upper vertex of $\mathbf{b}$; third, let $C(\mathbf{a},\mathbf{b})$ be the number of cycles in the new graph obtained in (2) and let $\mathbf{c} = \mathbf{a} * \mathbf{b}$ be this graph without the cycles; then $\mathbf{c}$ is an f–diagram, and we set $\mathbf{a}\cdot\mathbf{b} \equiv \mathbf{a}\,\mathbf{b} := x^{C(\mathbf{a},\mathbf{b})}\mathbf{a}*\mathbf{b}$. We denote by $* : D_f \times D_f \to D_f$ the map given by $(\mathbf{a},\mathbf{b}) \mapsto \mathbf{a}*\mathbf{b}$ and $C : D_f \times D_f \to \mathbb{N}$ the map given by $(\mathbf{a},\mathbf{b}) \mapsto C(\mathbf{a},\mathbf{b})$.

The following is a simple example:



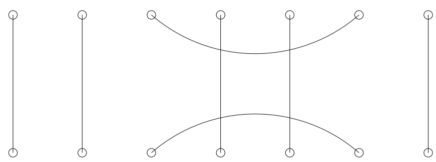
It is well-known that such a definition endows $\mathcal{B}_f^{(x)}$ with a structure of unital associative $\mathbb{K}$–algebra. Notice that, given diagrams $\mathbf{a}$ and $\mathbf{b}$, the upper, resp. lower, bar structure of the diagram $\mathbf{a} * \mathbf{b}$ "contains" that of $\mathbf{a}$, resp. $\mathbf{b}$; in particular if $\mathbf{a} \in D_{f,a}$ and $\mathbf{b} \in D_{f,b}$ this gives $\mathbf{a} * \mathbf{b} \in D_{f,\max(a,b)}$.

One can endow $\mathcal{B}_f^{(x)}$ with several additional structures; in particular, we recall the following ones. The upside down reversing of f–diagrams uniquely defines an antiinvolution $\Omega : \mathcal{B}_f^{(x)} \to \mathcal{B}_f^{(x)}$. The symmetric group S_{2f} acts on $\mathbb{V}_f$, once a numbering of the spots in $\mathbb{V}_f$ is fixed; then it acts also on D_f in the obvious way, and then linear extension gives an action on $\mathcal{B}_f^{(x)}$ too (which does not preserve multiplication, though).

In this paper we consider $\mathbb{K} = \mathbb{C}$ (but the results of this section hold for any $\mathbb{K}$).

2.4 The embedding $S_f \longrightarrow \mathcal{B}_f^{(x)}$. By the very definitions one has that $D_{f,0}$, as a subset of $\mathcal{B}_f^{(x)}$, is closed under the product, i.e. it is a subsemigroup. Now, for any $\sigma \in S_f$ let $\mathbf{d}_\sigma \in D_{f,0}$ be the f–diagram obtained by joining i^+ with $\sigma(i)^-$ (notation of §2.3). Then the map $S_f \to D_{f,0} \subset \mathcal{B}_f^{(x)}$ is a morphism of semigroups, whose image is $D_{f,0}$; thus $\mathcal{B}_f^{(x)}$ contains a copy of S_f (namely $D_{f,0}$) and a copy of the group algebra $\mathbb{C}[S_f]$. Thus restricting the left (right) regular representation of $\mathcal{B}_f^{(x)}$ (on itself) we get a left (right) action of S_f on $\mathcal{B}_f^{(x)}$. Furthermore, the restriction of $\Omega : \mathcal{B}_f^{(x)} \to \mathcal{B}_f^{(x)}$ to $\mathbb{C}[S_f] \, (= \mathbb{C}[D_{f,0}])$ is the antipode, given by $\sigma \mapsto \sigma^{-1}$ for all $\sigma \in S_f$.

2.5 Presentation by generators and relations. Besides the construction above, we can give the Brauer algebra a presentation by generators and relations. From §2.4 we know that $\mathcal{B}_f^{(x)}$ contains a copy of the symmetric group on f elements; moreover, for any pair of distinct indices $i, j \in \{1, 2, \dots, f\}$ we define $\mathbf{h}_{i,j}$ to be the f–diagram with a bar joining i^+ with j^+, a bar joining i^- with j^-, and one vertical edge joining k^+ with k^- for all $k \in \{1, 2, \dots, f\} \setminus \{i, j\}$. By definition, $\mathbf{h}_{i,j} \in D_{f,1}$. For instance, $\mathbf{h}_{3,6} \in D_{7,1}$ is



Theorem 2.6 ([DP], §7). *$\mathcal{B}_f^{(x)}$ is the associative $\mathbb{C}$–algebra with generators $\mathbf{d}_\sigma$, in bijection with elements of S_f, and $\mathbf{h}_{i,j}$, for all $i, j = 1, 2, \dots, f$ and $i \neq j$, and relations (assume all the index sets disjoint)*

$$\mathbf{h}_{i,j} = \mathbf{h}_{j,i} \qquad \mathbf{d}_\sigma \mathbf{h}_{i,j} \mathbf{d}_{\sigma^{-1}} = \mathbf{h}_{\sigma(i),\sigma(j)} \qquad \mathbf{h}_{i,j} \mathbf{h}_{h,k} = \mathbf{h}_{h,k} \mathbf{h}_{i,j}$$
$$\mathbf{h}_{i,j} \mathbf{h}_{j,k} = \mathbf{h}_{i,j} \mathbf{d}_{(i\,k)} \qquad \mathbf{h}_{i,j}^2 = x\, \mathbf{h}_{i,j} \qquad \mathbf{h}_{i,j} = \mathbf{h}_{i,j} \mathbf{d}_{(i\,j)}$$

as well as all relations of the symmetric group S_f among the $\mathbf{d}_\sigma$'s.

2.7 The sign of a diagram. The previous theorem means that $\mathcal{B}_f^{(x)}$ is generated by $D_{f,0}$ and $D_{f,1}$; even more, since $D_{f,1}$ is a single $D_{f,0}$–orbit (i.e. S_f–orbit) it is enough to take only one 1–bar f–diagram, thus $\mathcal{B}_f^{(x)}$ is generated for instance by $D_{f,0} \cup \{\mathbf{h}_{1,2}\}$.

In particular, for any $\mathbf{d} \in D_{f,k}$ there exist unique $\mathbf{d}_\sigma, \mathbf{d}_\rho \in D_{f,0}$ such that $\mathbf{d} = \mathbf{d}_\sigma \, \mathbf{h}_{1,2} \cdots \mathbf{h}_{2k-1,2k} \, \mathbf{d}_\rho$; moreover, we can choose such σ and ρ so that they do not invert any of the pairs $(1,2)$, $(3,4)$, $\dots$, $(2k-1,2k)$. Then given such a factorization of $\mathbf{d}$ we define the *sign of* $\mathbf{d}$ to be $\varepsilon(\mathbf{d}) := sgn(\sigma) \cdot (-1)^k \cdot sgn(\rho)$.

2.8 The standard series. For any $k \in \{1,2,\dots[f/2]\}$, we define $\mathcal{B}_f^{(x)}\langle k \rangle$ to be the vector subspace of $\mathcal{B}_f^{(x)}$ spanned by $D_{f,k}$; then we set $\mathcal{B}_f^{(x)}(k) := \bigoplus_{h \geq k} \mathcal{B}_f^{(x)}\langle h \rangle$. By definition, the $\mathcal{B}_f^{(x)}(k)$'s form a chain of subspaces (the "standard series")

$$\mathcal{B}_f^{(x)} = \mathcal{B}_f^{(x)}(0) \supset \mathcal{B}_f^{(x)}(1) \supset \cdots \supset \mathcal{B}_f^{(x)}(k) \supset \cdots \supset \mathcal{B}_f^{(x)}\big([f/2]\big) \supset 0$$

and each quotient $\mathcal{B}_f^{(x)}[k] := \mathcal{B}_f^{(x)}(k) \Big/ \mathcal{B}_f^{(x)}(k{+}1)$ is well-defined (with $\mathcal{B}_f^{(x)}\big([f/2]{+}1\big) := 0$).

The very definitions imply that each $\mathcal{B}_f^{(x)}(k)$ is a (two-sided) ideal of $\mathcal{B}_f^{(x)}$: therefore every quotient $\mathcal{B}_f^{(x)}[k]$ inherits a structure of associative $\mathbb{C}$–algebra, one of left $\mathcal{B}_f^{(x)}$–module, and one of right $\mathcal{B}_f^{(x)}$–module. Furthermore, since $\mathcal{B}_f^{(x)}(k) = \mathcal{B}_f^{(x)}\langle k \rangle \oplus \mathcal{B}_f^{(x)}(k+1)$, any basis for $\mathcal{B}_f^{(x)}\langle k \rangle$, taken modulo $\mathcal{B}_f^{(x)}(k+1)$, serves as basis for the residue class algebra $\mathcal{B}_f^{(x)}[k]$; in particular we shall use $D_{f,k}$ as a basis of $\mathcal{B}_f^{(x)}[k]$. Note that, since the $\mathcal{B}_f^{(x)}(k)$'s are two sided ideals of $\mathcal{B}_f^{(x)}$, the $\mathcal{B}_f^{(x)}[k]$'s are $\mathcal{B}_f^{(x)}$–bimodules.

2.9 The structure of $\mathcal{B}_f^{(x)}[k]$. Let $k \in \{1,2,\dots[f/2]\}$ be fixed. By inverting (2.1) and extending by linearity two linear isomorphisms

$$\boxtimes : \mathbb{C}\big[S_{f-2k}\big] \otimes \big(H_{f,k} \otimes H_{f,k}\big) \longrightarrow \mathcal{B}_f^{(x)}\langle k \rangle$$
$$\boxtimes : \mathbb{C}\big[S_{f-2k}\big] \otimes \big(H_{f,k} \otimes H_{f,k}\big) \longrightarrow \mathcal{B}_f^{(x)}[k]$$

are defined: more precisely, given any $z \in \mathbb{C}\big[S_{f-2k}\big]$ we can express it as a linear combination of permutations: attaching to all of them the same bar structure we get a linear combination of k–bar f–diagrams, which all share the same bar structure.

From Young's theory, $\mathbb{C}\big[S_{f-2k}\big]$ splits into $\mathbb{C}\big[S_{f-2k}\big] = \bigoplus_{\mu \vdash (f-2k)} I_\mu$, where every I_μ is a two-sided ideal of $\mathbb{C}\big[S_f\big]$ and a simple algebra, namely the algebra of linear endomorphisms of the simple S_{f-2k}–module M_μ, which is a full matrix algebra over $\mathbb{C}$. Then for every $\mu \vdash (f-2k)$ $(k \in \{0,1,\dots,[f/2]\})$ we define $\mathcal{B}_f^{(x)}[k;\mu] := \boxtimes\big(I_\mu \otimes (H_{f,k} \otimes H_{f,k})\big)$.

Theorem 2.10 (cf. [Bw2], §§2.2–3). *Let $\mu \vdash (f-2k)$. Then $\mathcal{B}_f^{(x)}[k;\mu]$ is a two-sided ideal of $\mathcal{B}_f^{(x)}[k]$, and also a $\mathcal{B}_f^{(x)}$–sub-bimodule (of $\mathcal{B}_f^{(x)}[k]$); its semisimple quotient (as an algebra) is simple. Moreover, the various $\mathcal{B}_f^{(x)}[k;\mu]$ (for different μ) are pairwise non-isomorphic, and $\mathcal{B}_f^{(x)}[k]$ splits as a direct sum*

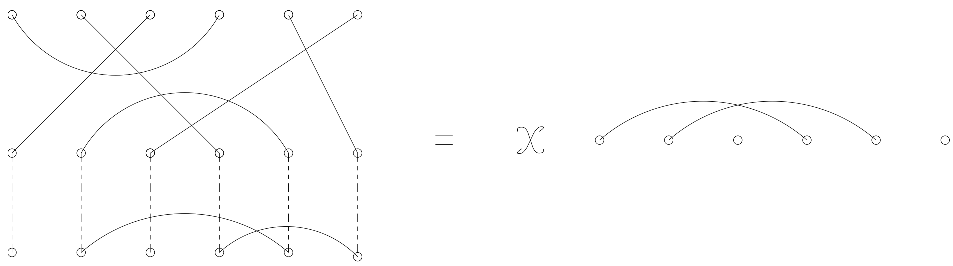
$$\mathcal{B}_f^{(x)}[k] = \bigoplus_{\mu \vdash (f-2k)} \mathcal{B}_f^{(x)}[k;\mu].$$

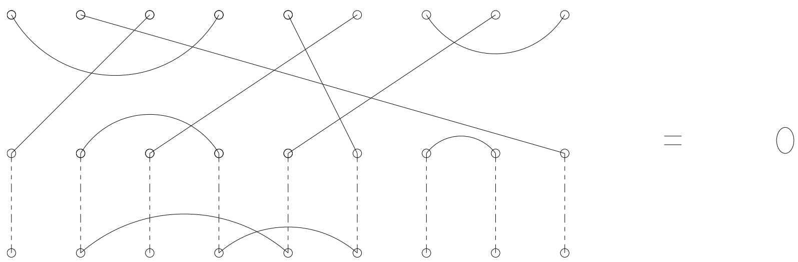
2.11 Representations of $\mathcal{B}_f^{(x)}$. In section 2.2 we defined the vector spaces $H_{f,k}^{\mu}$: now we endow them with a structure of $\mathcal{B}_f^{(x)}$–modules, following Kerov (cf. [Ke], [HW], [GP]).

Let $\mathbf{d}$ be an f–diagram, and let v be an (f,k)–junction; for all $i=1,\dots,f$, connect the i–th lower vertex of $\mathbf{d}$ with the i–th vertex of v: let $C(\mathbf{d},v)$ be the number of loops occurring in the new graph $\Gamma(\mathbf{d},v)$ obtained in this way, and let $a \star v$ be the graph made of the vertices of the upper line of $\mathbf{d}$, connected by an edge iff they are connected (by an edge or a path) in the new graph $\Gamma(\mathbf{d},v)$; then $\mathbf{d}\star v \in J_{f,k'}$, with $k' \geq k$ and $k'=k$ iff each pair of vertices of v which are connected by a path in $\Gamma(\mathbf{d},v)$ are in fact joined by an edge in v: in this case we say that the junction v is *admissible* for the diagram $\mathbf{d}$. We set

$$\mathbf{d}.v := x^{C(\mathbf{d},v)}\, \mathbf{d}\star v \quad \textit{if } v \textit{ is admissible for } \mathbf{d}\,, \qquad \mathbf{d}.v := 0 \quad \textit{otherwise}\,.$$

here are two examples:





To any pair $(\mathbf{d},v)\in D_f\times J_{f,k}$ we can also attach an element $\pi(\mathbf{d},v)\in S_{f-2k}$: this is the permutation which carries — through the graph $\Gamma(\mathbf{d},v)$ — the isolated vertices of v into the isolated vertices of $\mathbf{d}\star v$ (one takes into account only the relative position of the isolated vertices in v, $\mathbf{d}\star v$) in case v is admissible for a, otherwise it is id. In the previous example we have $\pi(\mathbf{d},v)=\left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right)$.

Proposition 2.12 (cf. [Ke], [Bw2]). *Linear extension of the rule* $\mathbf{d}.(u\otimes v) := \pi(\mathbf{d},v).u\otimes \mathbf{d}.v$ *for every* $(\mathbf{d},v)\in D_f\times J_{f,k}$ *endows* $H_{f,k}^{\mu}$ *with a well-defined structure of module over* $\mathcal{B}_f^{(x)}$*; then* $H_{f,k}^{\mu}$ *is also a module over* $\mathcal{B}_f^{(x)}\big/\mathcal{B}_f^{(x)}(k+1)$ *and over* $\mathcal{B}_f^{(x)}[k]$*. The various modules* $H_{f,k}^{\mu}$ *(for different pairs* (k,μ)*) — over any of the previous algebras — are pairwise non-isomorphic. When* $\mathcal{B}_f^{(x)}$ *is semisimple, this module is simple and, conversely, any simple* $\mathcal{B}_f^{(x)}$*–module is isomorphic to one of the* $H_{f,k}^{\mu}$*'s.*

In addition, we prove now something more, namely that the semisimple quotient of $H^{\mu}_{f,k}$ is always simple: indeed, it is the unique simple $\mathcal{B}^{(x)}_f[k;\mu]$–module (by the way, notice that $H^{(f-2k)}_{f,k} = H_{f,k}$). For this we need a closer description of the relationship among $\mathcal{B}^{(x)}_f[k;\mu]$ and $H^{\mu}_{f,k}$. Recall that $H^{\mu}_{f,k} := M_\mu \otimes H_{f,k}$, so $H^{\mu}_{f,k}$ is spanned by tensors $m_\mu \otimes h$ with $m \in M_\mu$ and $h \in H_{f,k}$; moreover, $\mathbb{C}\big[S_{f-2k}\big] \cong \oplus_{\mu \vdash (f-2k)} I_\mu$ and $I_\mu \cong M_\mu \otimes M_\mu$ as S_f–bimodules, hence there exists a monomorphism $\Xi_\mu : M_\mu \otimes M_\mu \hookrightarrow \mathbb{C}\big[S_{f-2k}\big]$. The following statement (whose proof is trivial from definitions) gives the required description.

Lemma 2.13. *Consider on the space $H^{\mu}_{f,k} \otimes H^{\mu}_{f,k}$ the structure of $\mathcal{B}^{(x)}_f$–bimodule given by $(b_1,b_2).(h_1,h_2) := \big(b_1.h_2, \Omega(b_2).h_2\big)$, and on $\mathcal{B}^{(x)}_f[k;\mu]$ the natural structure of $\mathcal{B}^{(x)}_f$–bimodule induced by the left and right regular representations of $\mathcal{B}^{(x)}_f$. Then there exists an isomorphism of $\mathcal{B}^{(x)}_f$–bimodules and of $\mathcal{B}^{(x)}_f[k;\mu]$–bimodules*

$$\Phi_\mu : H^{\mu}_{f,k} \otimes H^{\mu}_{f,k} \xrightarrow{\ \cong\ } \mathcal{B}^{(x)}_f[k;\mu]$$

given by

$$(m_1 \otimes h_1) \otimes (m_2 \otimes h_2) \mapsto \boxtimes\big(\Xi_\mu(m_1 \otimes m_2) \otimes h_1 \otimes h_2\big)$$

Lemma 2.14. *Let A be an algebra, and let M be a left and right A–module such that these two structures are isomorphic, i.e. there exists a linear map $f : M \to M$ such that $f(a.m) = f(m).a$ for all $a \in A$, $m \in M$. Suppose that the semisimple quotient of A is simple, and that $A \cong M \otimes M$ as A–bimodules when A is given the natural A–bimodule structure and $M \otimes M$ is given the bimodule structure given by $(a_1,a_2).(m_1 \otimes m_2) := (a_1.m_1) \otimes (m_2.a_2)$. Then the semisimple quotient of M (both as a left or right A–module) is simple.*

Proof. Let R_A be the radical of A: we know it is the same if we take it to be the radical of A as a left or right A–module. Similarly, since the left and right structures of A–module on M are isomorphic, the left and right radicals of M are equal; then we denote this "common" radical by R_M. Now consider the epimorphism $A \cong M \otimes M \twoheadrightarrow M\big/R_M \otimes M\big/R_M$ defined by $m_1 \otimes m_2 \mapsto (m_1 \mod R_M) \otimes (m_2 \mod R_M)$. Since $M\big/R_M \otimes M\big/R_M$ is semisimple — as an A–bimodule — this epimorphism factors through $A\big/R_A$; by hypothesis the latter is simple, thus the same is true for $M\big/R_M \otimes M\big/R_M$, hence in turn for $M\big/R_M$, too. $\square$

Corollary 2.15. *The semisimple quotient of $H^{\mu}_{f,k}$ is simple.*

Proof. Apply Proposition 2.12, Lemma 2.13, and Lemma 2.14 with $A = \mathcal{B}^{(x)}_f[k;\mu]$ and $M = H^{\mu}_{f,k}$. $\square$

§3 Brauer algebras in Invariant Theory

3.1 Brauer algebras and centralizer algebras. In this section we explain the link between Brauer algebras and the centralizer algebras of §1, and we introduce the basic tools for proving our main result.

Theorem 3.2 (cf. [Br]). *There exist $\mathbb{C}$–algebra epimorphisms uniquely given by*

$$\begin{array}{cc} \pi_V : \mathcal{B}_f^{(n)} \twoheadrightarrow End_{O(V)}\big(V^{\otimes f}\big) & \pi_W : \mathcal{B}_f^{(-2n)} \twoheadrightarrow End_{Sp(W)}\big(W^{\otimes f}\big) \\ \mathbf{d}_\sigma \mapsto \sigma\,,\ \ \mathbf{h}_{p,q} \mapsto \tau_{p,q} & \mathbf{d}_\sigma \mapsto sgn(\sigma)\,\sigma\,,\ \ \mathbf{h}_{p,q} \mapsto -\tau_{p,q} \end{array}$$

When $n \geq f$ these are isomorphisms.

3.3 Diagrammatic minors and diagrammatic Pfaffians. A simple reformulation of Proposition 1.6 will answer the question of what is the kernel of the epimorphisms of Theorem 3.2. To begin with, define vector space isomorphisms

$$\begin{array}{cc} \Phi_V : A_f^O \xrightarrow{\cong} \mathcal{B}_f^{(n)} & \Phi_W : A_f^{Sp} \xrightarrow{\cong} \mathcal{B}_f^{(-2n)} \\ x_{\mathbf{i},\,\mathbf{j}} \mapsto \mathbf{d}_{\mathbf{i},\,\mathbf{j}} & x_{\mathbf{i},\,\mathbf{j}} \mapsto \varepsilon(\mathbf{d}_{\mathbf{i},\,\mathbf{j}}) \cdot \mathbf{d}_{\mathbf{i},\,\mathbf{j}} \end{array}$$

Then, getting through the various maps involved we find that the following diagrams of linear maps are *commutative*

$$\begin{tikzcd} A_f^O \arrow[r, "\Phi_V"] \arrow[d, "\alpha_V"'] & \mathcal{B}_f^{(n)} \arrow[d, "\pi_V"] \\ \Big(\big(V^{\otimes 2f}\big)^*\Big)^{O(V)} \arrow[r, "\Psi_V"'] & End_{O(V)}\big(V^{\otimes f}\big) \end{tikzcd} \qquad \begin{tikzcd} A_f^{Sp} \arrow[r, "\Phi_W"] \arrow[d, "\alpha_W"'] & \mathcal{B}_f^{(-2n)} \arrow[d, "\pi_W"] \\ \Big(\big(W^{\otimes 2f}\big)^*\Big)^{Sp(W)} \arrow[r, "\Psi_W"'] & End_{Sp(W)}\big(W^{\otimes f}\big) \end{tikzcd}$$

Now come back to Proposition 1.6, and look for instance to the orthogonal case. The kernel of α_V is claimed to be the intersection of A_f^O with the ideal Min_{n+1} of A^O generated by the minors of order $n+1$ of the symmetric matrix $\big(x_{ij}\big)_{i,j=1}^{2f}$: more precisely, the last part of the statement ensures that $Ker(\alpha_V)$ is exactly the $\mathbb{C}$–span of the elements of type $\mu_{n+1} x_{i_{n+2}j_{n+2}} x_{i_{n+3}j_{n+3}} \cdots x_{i_f j_f}$, where μ_{n+1} is any minor of $\big(x_{ij}\big)_{i,j=1}^{2f}$ of order $n+1$ such that all rows involved have indices different from those of the columns involved. From the expression of the determinant we get that $Ker(\alpha_V)$ is the $\mathbb{C}$–span of the elements of type

$$\sum_{\sigma \in S_{n+1}} sgn(\sigma) \cdot x_{i_1 j_{\sigma(1)}} x_{i_2 j_{\sigma(2)}} \cdots x_{i_{n+1} j_{\sigma(n+1)}} \cdot x_{i_{n+2} j_{n+2}} x_{i_{n+3} j_{n+3}} \cdots x_{i_{f-1} j_{f-1}} x_{i_f j_f} \tag{3.1}$$

with $\{i_1, \dots, i_{n+1}\} \cup \{j_1, \dots, j_{n+1}\} \cup \{i_{n+2}, \dots, i_f\} \cup \{j_{n+2}, \dots, j_f\} = \{1, 2, 3, \dots, 2f\}$.

Similarly, in the symplectic case Proposition 1.6 tells us that $Ker(\alpha_W)$ is the $\mathbb{C}$–span of the elements of type $\varpi_{n+1} x_{i_{n+2}j_{n+2}} x_{i_{n+3}j_{n+3}} \cdots x_{i_f j_f}$, where ϖ_{n+1} is any Pfaffian of $\big(x_{ij}\big)_{i,j=1}^{2f}$ of order $2n+2$ such that all rows involved have indices different from those of the columns involved. Exploiting the explicit expression of the Pfaffian we get that $Ker(\alpha_W)$ is the $\mathbb{C}$–span of the elements of type

$$\sum_{\substack{h_1<k_1,h_2<k_2,\dots\\ h_1<h_2<h_3<\cdots}} sgn\begin{pmatrix} 1 & 2 & \dots & 2f-1 & 2f \\ h_1 & k_1 & \dots & h_f & k_f \end{pmatrix} \cdot x_{h_1k_1}x_{h_2k_2}\cdots x_{h_{n+1}k_{n+1}} \cdot x_{i_{n+2}j_{n+2}}\cdots x_{i_fj_f} \quad (3.2)$$

with $\{h_1,\dots,h_{n+1}\}\cup\{k_1,\dots,k_{n+1}\}\cup\{i_{n+2},\dots,i_f\}\cup\{j_{n+2},\dots,j_f\}=\{1,2,3,\dots,2f\}$.
This leads us to the following

Definition 3.4 *(a) We call* **(diagrammatic) minor** *of order r ($\in \mathbb{N}_+$) every element of $\mathcal{B}_f^{(x)}$ which is the image through Φ_V of an element of type (3.1) with r instead of $n+1$.*

(b) We call **(diagrammatic) Pfaffian** *of order $2r$ ($\in 2\mathbb{N}_+$) every element of $\mathcal{B}_f^{(x)}$ which is the image through Φ_W of an element of type (3.2) with r instead of $n+1$.*

(c) If X is any given (diagrammatic) minor or Pfaffian, we call **fixed edge** *of X any edge which occurs the same in all the diagrams occurring in the expansion of X; we call* **fixed vertex** *of X any vertex (in $\mathbb{V}_f$) belonging to a fixed edge of X; we call* **fixed part** *of X the datum of all fixed edges and all fixed vertices of X; we call* **moving part** *of X the datum of all vertices (in $\mathbb{V}_f$) which are not fixed in X along with all edges which occur in any diagram in the expansion of X and which are not fixed.*

Remarks 3.5. *(a)* From definitions and Proposition 1.6, it directly follows that a diagrammatic minor is an *alternating sum of f–diagrams*: to be precise, if the minor has order r then it is an S_r–antisymmetric sum of f–diagrams. On the other hand, because of the sign entering in the definition of α_W one has that all diagrams entering in the expansion of a diagrammatic Pfaffian appears there with like sign: that is, up to sign each diagrammatic Pfaffian is just a simple *sum of f–diagrams*.

(b) If δ_r is a minor of order r, the $2r$ vertices in its moving part may be partitioned into two sets I, J (each of r elements) so that, looking at all the diagrams occurring in the expansion of δ_r, no vertex in one of these sets is ever joined to a vertex in the same set, but it is joined to each of the vertices in the other set. Via Φ_V, the sets I and J correspond to the set of rows and the set of columns (or viceversa) in the matrix $\left(x_{ij}\right)_{i,j=1}^{2f}$ on which the minor corresponding to δ_r is computed: therefore, in the sequel we shall use expressions like "v is a row vertex and w is a column vertex" to mean in short that v and w are moving vertices which belong one to I and the other to J, or "v and w are both row vertices" or "column vertices" to mean that they are moving vertices which both belong to I or both belong to J. In fact, the minor δ_r is determined uniquely up to sign by: *(I)* assigning its fixed part; *(II)* assigning the sets I and J, both endowed with a labelling of their vertices by $\{1,2,\dots,r\}$; *(III)* joining every vertex in one set — say I — to a vertex in the other set — say J — according to a permutation $\sigma\in S_r$, so to get an f–diagram $\mathbf{d}(\sigma)$; *(IV)* adding up the diagrams $\mathbf{d}(\sigma)$ with coefficient $sgn(\sigma)$, for all $\sigma\in S_r$: this finally gives $\pm\mathbf{d}_r$ (the sign depends on the choice of the labelling of the vertices in I and in J).

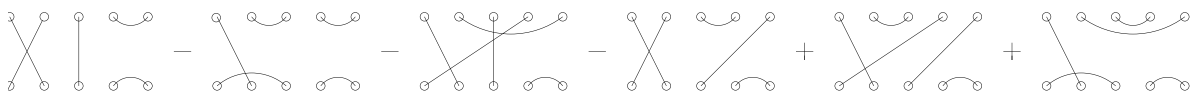
(c) The operation in *(III)* may be better understood as follows: first join every vertex in I with the vertex in J labelled with the same number: this gives the diagram $\mathbf{d}(id)$, which outside the fixed part is given by the r edges $\{i_1,j_1\},\dots,\{i_r,j_r\}$ (with $\{i_1,\dots,i_r\}=I$,

$\{j_1, \dots, j_r\} = J$); second, let S_r act on J, and let $\mathbf{d}[\sigma]$ be the diagram which is equal to $\mathbf{d}(id)$ in the fixed part and outside it is given by the r edges $\big\{i_1, \sigma(j_1)\big\}, \dots, \big\{i_r, \sigma(j_r)\big\}$: then $\mathbf{d}[\sigma] = \mathbf{d}(\sigma)$. Therefore we can also write δ_r as an S_r–antisymmetric sum

$$\delta_r = \sum_{\sigma \in S_r} sgn(\sigma)\, \mathbf{d}(\sigma) = \sum_{\sigma \in S_r} sgn(\sigma)\, \mathbf{d}[\sigma] = \sum_{\sigma \in S_r} sgn(\sigma)\, \sigma.\mathbf{d}[id] \tag{3.3}$$

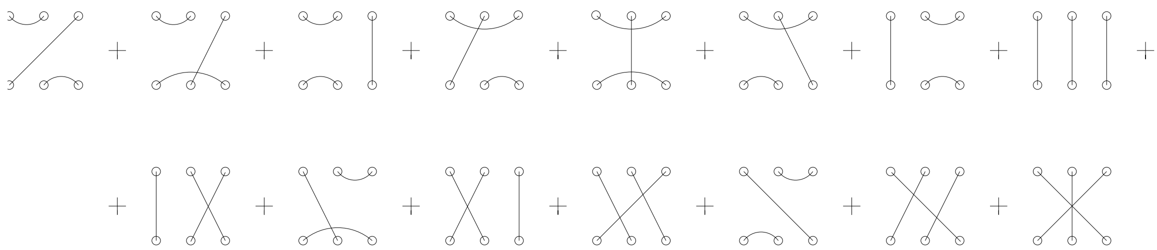
(d) The counterpart for Pfaffians of *(b)* and *(c)* above is that every Pfaffian of order $2r$ is the sum of all diagrams obtained by assigning the fixed part and joining the $2r$ vertices in the moving part with r edges in all possible ways.

Examples 3.6. *(a)* In the picture below we represent the diagrammatic minor ${\Phi_V}^{-1}\big(\mu_3 x_{1^+2^-} x_{4^-5^-}\big)$ $\Big(\in \mathcal{B}_5^{(x)}\Big)$, where μ_3 is the minor (of size 3) of the matrix $\big(x_{ij}\big)_{i,j=1}^{10}$ on the rows 2, 4, 8 and the columns 6, 3, 5, making use (as we shall often do, with f instead of 5) of the identifications $i = i^+$, $j + 5 = j^-$ for all $i, j = 1, \dots, 5$.



The fixed part of this minor is the set of edges $\big\{\{1^+, 2^-\}, \{4^-, 5^-\}\big\}$ and the set of vertices $\{1^+, 2^-, 4^-, 5^-\}$; the moving part is given by the vertices 2^+, 4^+, 3^- — which correspond to rows (or columns) — and 1^-, 3^+, 5^+ — which correspond to columns (or rows).

(b) The next picture represents the (unique, up to sign) Pfaffian of order 6 in $\mathcal{B}_3^{(x)}$; here again we used the identifications $i = i^+$, $j + 3 = j^-$ for all $i, j = 1, \dots, 3$ (note that here there is no fixed part because the order of the Pfaffian equals $2f$).



The importance of diagrammatic minors and Pfaffians lies in the following reformulation of Proposition 1.6 (via §3.3):

Theorem 3.7. *(a) The kernel of* $\pi_V : \mathcal{B}_f^{(n)} \twoheadrightarrow End_{O(V)}\big(V^{\otimes f}\big)$ *is the* $\mathbb{C}$*–span of the set of all diagrammatic minors in* $\mathcal{B}_f^{(n)}$ *of order* $n + 1$*.*

(b) The kernel of $\pi_W : \mathcal{B}_f^{(-2n)} \twoheadrightarrow End_{Sp(W)}\big(W^{\otimes f}\big)$ *is the* $\mathbb{C}$*–span of the set of all diagrammatic Pfaffians in* $\mathcal{B}_f^{(-2n)}$ *of order* $2(n + 1)$*.*

We finish this section by proving some combinatorial results on diagrammatic minors and Pfaffians, to be used in §4.

Lemma 3.8. *(a) Let $\delta_r \, (\in \mathcal{B}_f^{(x)})$ be a diagrammatic minor of order r; let I_ℓ, resp. I_t, be the set of moving row, resp. column, vertices in $\{1^+, 2^+, \dots, f^+\}$ (the upper row of δ_r) and assume $\ell + t \geq r$ (that is, the moving part of δ_r is not larger down than up). Then δ_r may be written as*

$$\delta_r \, = \, Alt_{I_\ell} \, Alt_{I_t} \cdot \sum_{j=0}^{m} \sum_{(p_{j,i}, q_{j,i}) \in V_j} (-1)^j \, \mathbf{h}_{p_{j,1} q_{j,1}} \mathbf{h}_{p_{j,2} q_{j,2}} \cdots \mathbf{h}_{p_{j,j} q_{j,j}} \mathbf{d} \tag{3.4}$$

where m is a suitable nonnegative integer, Alt_{I_ℓ}, resp. Alt_{I_t}, denotes the antisymmetrizer (in $\mathbb{C}[S_f]$) on I_ℓ, resp. on I_t, the V_j's are suitable subsets of $I_\ell \times I_t$, and $\mathbf{d}$ is a suitable f–diagram.

(b) Let $\varpi_r \, (\in \mathcal{B}_f^{(x)})$ be a diagrammatic Pfaffian of order $2r$; let I_t be the subset of moving vertices in $\{1^+, 2^+, \dots, f^+\}$ (the upper row of ϖ_r), and assume $t \geq r$ (that is, the moving part of ϖ_r is not larger down than up). Then ϖ_r may be written as

$$\varpi_r \, = \, Sym_{I_t} \cdot \sum_{j=0}^{m} \sum_{(p_{j,i}, q_{j,i}) \in V_j} \big((h+j)! \, 2^{h+j}\big)^{-1} \, \mathbf{h}_{p_{j,1} q_{j,1}} \mathbf{h}_{p_{j,2} q_{j,2}} \cdots \mathbf{h}_{p_{j,j} q_{j,j}} \mathbf{d} \tag{3.5}$$

where m is a suitable nonnegative integer, Sym_{I_t} denotes the symmetrizer (in $\mathbb{C}[S_f]$) on I_t, the V_j's are suitable subsets of I_t, $\mathbf{d}$ is a suitable f–diagram, and h is the number of bars on vertices of I_t in $\mathbf{d}$.

Proof. (a) As a matter of notation, for all $h \in \{0, 1, \dots, [f/2]\}$ let $\delta_r^{(h)}$ be the part of δ_r which lies in $\mathcal{B}_f^{(x)}(h) \setminus \mathcal{B}_f^{(x)}(h+1)$, i.e. the algebraic sum of those diagrams in the expansion of δ_r (with the signs they have therein) which have exactly h bars in the upper row.

Among the diagrams occurring in the expansion of δ_r, pick one which has the least possible number of bars — to be k, if $\delta_r \in \mathcal{B}_f^{(x)}(k) \setminus \mathcal{B}_f^{(x)}(k+1)$ — and call it $\mathbf{d}$: then we have exactly $\delta_r^{(k)} = (Alt_{I_\ell} \, Alt_{I_t}).\mathbf{d}$.

If $\ell = 0$ or $t = 0$ we have finished, for in this case $\delta_r = \delta_r^{(k)}$. Otherwise, each of the remaining diagrams in δ_r has at least one bar joining a vertex in I_ℓ with a vertex in I_t. Let now $\mathbf{d}'$ be one of the remaining diagrams (if any) having exactly one bar of the previous type; we can choose $\mathbf{d}'$ so that it is equal to $\mathbf{d}$ but on the vertices p^+ and q^+ of this bar and on those vertices u^- and v^- which in (the lower row of) $\mathbf{d}$ are joined to p^+ and q^+: but this simply means that $\mathbf{d}' = \mathbf{h}_{p,q}\mathbf{d}$: then $-(Alt_{I_\ell} \, Alt_{I_t}).\mathbf{d}' = (Alt_{I_\ell} \, Alt_{I_t}).(-\mathbf{h}_{p,q}\mathbf{d})$ is the algebraic sum of those diagrams in the expansion of $\delta_r^{(k+1)}$ which have the bar $u^- \;\; v^-$. Similarly, the other diagrams in $\delta_r^{(k+1)}$ can be obtained by multiplying $\mathbf{d}$ on the left by other suitable $\mathbf{h}_{p',q'}$'s (one each time) for different p' and q'; so finally we find that $\delta_r^{(k+1)} = Alt_{I_\ell} \, Alt_{I_t} \cdot \sum_{(p_{1,1}, q_{1,1}) \in V_1} (-1) \mathbf{h}_{p_{1,1} q_{1,1}} \mathbf{d}$, where V_1 is a suitable subset of $I_\ell \times I_t$. The same procedure applies if we want to describe $\delta_r^{(k+j)}$, for greater j: the only difference is that we have to multiply by exactly j different terms $\mathbf{h}_{p,q}$, choosen in several different ways; thus we find that

$$\delta_r^{(k+j)} = Alt_{I_\ell} \, Alt_{I_t} \cdot \sum_{(p_{j,i}, q_{j,i}) \in V_j} (-1)^j \mathbf{h}_{p_{j,1} q_{j,1}} \mathbf{h}_{p_{j,2} q_{j,2}} \cdots \mathbf{h}_{p_{j,j} q_{j,j}} \mathbf{d} \qquad \forall \, j = 0, 1, \dots, m$$

where V_j is a suitable subset of $I_\ell \times I_t$ and $k+m$ is the maximum number of bars appearing in the upper row of any diagram in the expansion of δ_r. Finally, summing up over j gives us claim *(a)*.

(b) Like in the proof of *(a)*, for all $h \in \{0,1,\dots,[f/2]\}$ we define $\varpi_r^{(h)}$ to be the part of ϖ_r which lies in $\mathcal{B}_f^{(x)}(h) \setminus \mathcal{B}_f^{(x)}(h+1)$, that is the sum of those diagrams in the expansion of ϖ_r which have exactly h bars in the upper row.

Again, choose a diagram $\mathbf{d}$ in the expansion of ϖ_r which has the least possible number of bars, to be k if $\delta_r \in \mathcal{B}_f^{(x)}(k) \setminus \mathcal{B}_f^{(x)}(k+1)$. Then permuting in all possible ways the vertices in I_t we get all the diagrams in the expansion of ϖ_r which have exactly k bars in the upper row; but we get each of them exactly as many times as the cardinality of the stabilizer St of the "bar structure" of I_t; this stabilizer is generated by the stabilizer — a copy of S_2 — of each bar on I_t (in $\mathbf{d}$) and by the whole symmetric group acting on the set of these bars: indeed, we have $St \cong {S_2}^{\times h} \times S_h$ (a hyperoctahedral group) where h is the number of bars on vertices in I_t in the diagram $\mathbf{d}$, so that $\big|St\big| = 2^h \cdot h!$. The upshot is that $\varpi_r^{(k)} = \big(h!\,2^h\big)^{-1} \cdot Sym_{I_t}.\mathbf{d}$. We proceed similarly with the other diagrams in ϖ_r: namely, each of those in $\varpi_r^{(k+j)}$ can be obtained by multiplying $\mathbf{d}$ on the left by j suitable $\mathbf{h}_{p',q'}$'s, the vertices p' and q' being always choosen inside I_t; then using the commutation relations of Theorem 2.10 we can express $\delta_r^{(k+j)}$ as

$$\delta_r^{(k+j)} = \big((h+j)!\,2^{h+j}\big)^{-1} \cdot Sym_{I_t} \cdot \sum_{(p_{j,i},q_{j,i}) \in V_j} \mathbf{h}_{p_{j,1}q_{j,1}} \mathbf{h}_{p_{j,2}q_{j,2}} \cdots \mathbf{h}_{p_{j,j}q_{j,j}} \mathbf{d} \quad \forall\, j = 0,1,\dots,m$$

where V_j is a suitable subset of I_t) and $k+m$ is the maximum number of bars appearing in the upper row of any diagram in the expansion of ϖ_r. Finally summing up over j we get the claim *(b)*. $\square$

<u>*Example*</u>*:* if δ_3 is the minor in Example 3.6*(a)*, then an expression of type (3.4) is for instance $\delta_3 = Alt_{I_\ell} Alt_{I_t} \cdot \big(1 - \mathbf{h}_{2^+3^+}\big)\,\mathbf{d}$ where $I_\ell = \big\{2^+,4^+\big\}$, $I_t = \big\{5^+\big\}$, and $\mathbf{d}$ is the first diagram in the expansion of δ_3 (as it is drawn there); similarly, if ϖ_3 is the Pfaffian in Example 3.6*(b)*, then an expression of type (3.5) is for instance $\varpi_3 = Sym_{I_t} \cdot \big(1 + 2^{-1}(\mathbf{h}_{1^+2^+} + \mathbf{h}_{1^+3^+} + \mathbf{h}_{2^+3^+})\big)\,\mathbf{d}$ where $I_t = \big\{1^+,2^+,3^+\big\}$ and $\mathbf{d}$ is the last diagram in the first row of the expansion of ϖ_3 (as it is drawn there).

Lemma 3.9. *(a) Given $n \in \mathbb{N}_+$, let $\mathbf{d}$ be an f–diagram, and δ_{n+1} $(\in \mathcal{B}_f^{(n)})$ a minor of order $n+1$. Then if $\mathbf{d}$ has a bar r^- — s^-, resp. r^+ — s^+, and r^+ and s^+, resp. r^- and s^-, are moving vertices in δ_{n+1}, then $\mathbf{d} \cdot \delta_{n+1} = 0$, resp. $\delta_{n+1} \cdot \mathbf{d} = 0$. Similarly, if $j \in J_{f,k}$ is an (f,k)–junction (for some k) having a bar r — s and r^- and s^- are moving vertices in δ_{n+1}, then $\delta_{n+1}.j = 0$ in $H^\mu_{f,k}$ for all $\mu \vdash (f-2k)$.*

(b) Given $n \in \mathbb{N}_+$, let $\mathbf{d}$ be an f–diagram, and ϖ_{n+1} $(\in \mathcal{B}_f^{(-2n)})$ a Pfaffian of order $2(n+1)$. Then if $\mathbf{d}$ has a bar r^- — s^-, resp. r^+ — s^+, and r^+ and s^+, resp. r^- and s^-, are moving vertices in ϖ_{n+1}, then $\mathbf{d} \cdot \varpi_{n+1} = 0$, resp. $\varpi_{n+1} \cdot \mathbf{d} = 0$. Similarly, if $j \in J_{f,k}$ is an (f,k)–junction (for some k) having a bar r — s and r^- and s^- are moving vertices in ϖ_{n+1}, then $\varpi_{n+1}.j = 0$ in $H^\mu_{f,k}$ for all $\mu \vdash (f-2k)$.

Proof. *(a)* Assume for the moment that the claim about $\delta_{n+1} \cdot \mathbf{d}$ is proved: then the one about $\mathbf{d} \cdot \delta_{n+1}$ follows at once applying Ω.

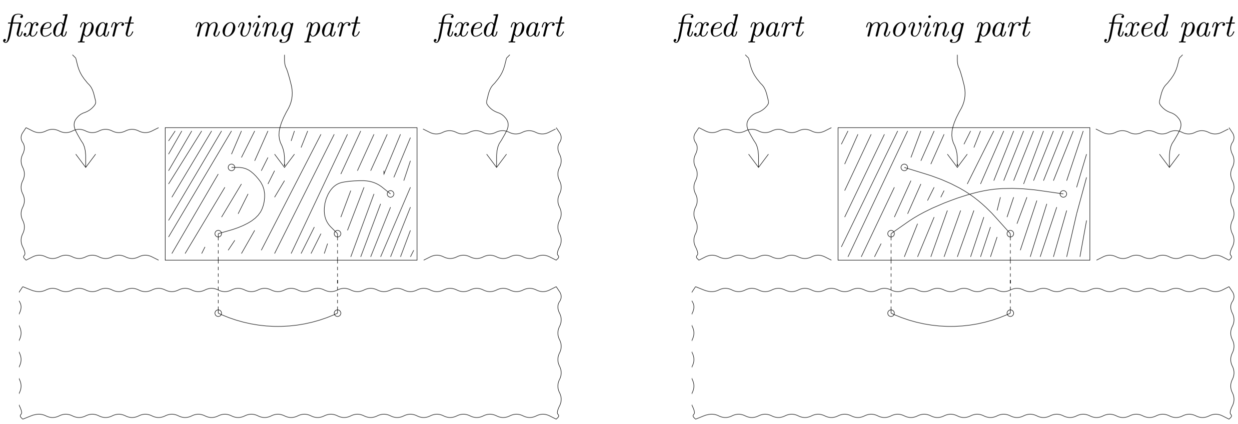
As for the claim about the junction j, it follows from the one about diagrams by thinking at j as $j = \mathrm{ubs}(\mathbf{d})$ for some f–diagram $\mathbf{d}$. Indeed, the definition of the action of $\mathcal{B}_f^{(x)}$ on $H_{f,k}$ is given in such a way that, if we pick any diagram $\mathbf{d} \in D_f$, then $\mathrm{ubs}\big([\mathbf{d}' \star \mathbf{d}]\big) = \mathbf{d}' \star \mathrm{ubs}(\mathbf{d})$, and $C\big(\mathbf{d}', \mathbf{d}\big) = C\big(\mathbf{d}', \mathrm{ubs}(\mathbf{d})\big)$ (with notation of §§2.2, 2.3, 2.11); therefore, for a given junction j we pick any diagram such that $j = \mathrm{ubs}(\mathbf{d})$: then $\delta_{n+1} \cdot \mathbf{d} = 0$ in $\mathcal{B}_f^{(n)}$ implies also $\delta_{n+1}.j = 0$ in $H_{f,k}^{\mu}$ for any μ, q.e.d.

The upshot is that we only have to show that $\delta_{n+1} \cdot \mathbf{d} = 0$.

Using Remark 2.7*(b)* we reduce to the case of $\mathbf{d} \in D_{f,1}$, that is $r^+ \,\text{---}\, s^+$ is the sole upper bar of $\mathbf{d}$. There are two cases to consider.

<u>*Case I*</u>: $\big|\{r^-, s^-\} \cap \big(I \cup J\big)\big| = 2$ with $\{r^-, s^-\} \subseteq I$ or $\{r^-, s^-\} \subseteq J$: in other words, r^- and s^- are both row (or column) vertices.

In this case, note that the diagrams $\mathbf{d}(\sigma) = \mathbf{d}[\sigma]$ (using notation of Remarks 3.5*(c)*) occurring in δ_{n+1} may be partitioned in $(n+1)!/2$ pairs, by pairing $\mathbf{d}[\sigma]$ with $\mathbf{d}[(r^- \; s^-)\,\sigma]$, where $(r^- \; s^-)$ is the transposition of r^- and s^-; then multiplying $\mathbf{d}[\sigma]$ or $\mathbf{d}[(r^- \; s^-)\,\sigma]$ with $\mathbf{d}$ gives exactly the same diagram (the picture below might be enligthening).



case of $\mathbf{d}[\sigma] * \mathbf{d}$ *case of* $\mathbf{d}[(r^- \; s^-)\,\sigma] * \mathbf{d}$

but $sgn\big((r^- \; s^-)\,\sigma\big) = -sgn(\sigma)$, so the two products above give to the sum expressing $\delta_{n+1} \cdot \mathbf{d}$ a like contribution with unlike sign: therefore adding up all the pairs we get at last $\delta_{n+1} \cdot \mathbf{d} = 0$.

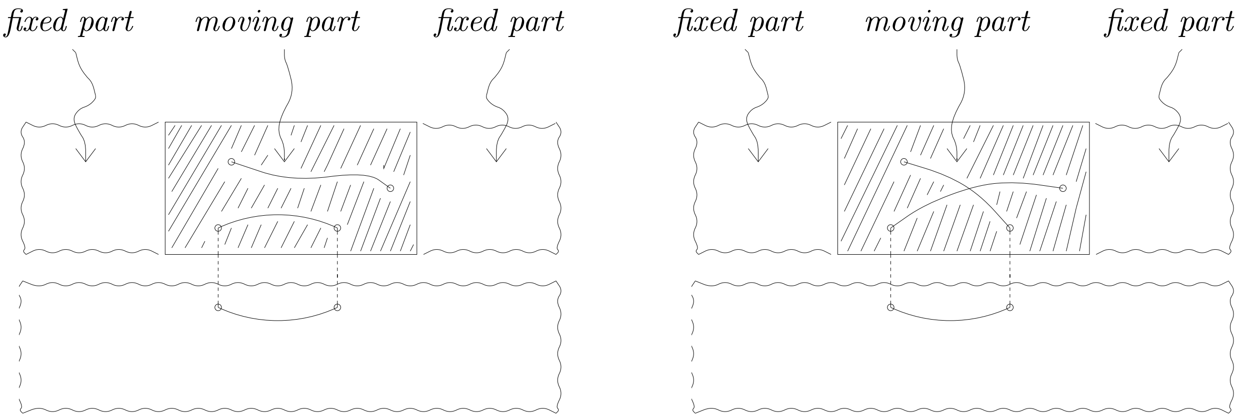
<u>*Case II*</u>: $\big|\{r^-, s^-\} \cap \big(I \cup J\big)\big| = 2$ with $r^- \in I$, $s^- \in J$ or $r^- \in J$, $s^- \in I$: in other words, both r^- and s^- are moved in δ_{n+1} and one of them is a row vertex whilst the other is a column vertex, say $r^- \in I$ and $s^- \in J$.

Consider a $\bar{\sigma} \in S_{n+1}$ such that r^- and s^- are joined in $\mathbf{d}\big[\bar{\sigma}\big] = \bar{\sigma}.\mathbf{d}[id]$: when computing the product $\mathbf{d}\big[\bar{\sigma}\big] \cdot \mathbf{d}$ the bar $r^+ \,\text{---}\, s^+$ in the upper row of $\mathbf{d}$ matches the bar $r^- \,\text{---}\, s^-$ in the lower row of $\mathbf{d}\big[\bar{\sigma}\big]$, so that $C\big(\mathbf{d}\big[\bar{\sigma}\big], \mathbf{d}\big) \geq 1$ (notation of §2.3), hence $\mathbf{d}\big[\bar{\sigma}\big] \cdot \mathbf{d} = n^z \mathbf{d}'$ for some $z \in \mathbb{N}_+$ and some $\mathbf{d}' \in D_f$.

Now fix in $\mathbf{d}\big[\bar{\sigma}\big]$ an edge $h \,\text{---}\, k$ in the moving part of δ_{n+1} which is different from $r^- \multimap s^-$, with $h \in I$, $k \in J$, say. Then look at the diagram $\mathbf{d}\big[(s^-\ k)\,\bar{\sigma}\big] = (s^-\ k)\,\bar{\sigma}.\mathbf{d}[id]$, which also occurs in the expression of δ_{n+1} as S_{n+1}–antisymmetric sum of type (3.3): this diagram is equal to $\mathbf{d}\big[\bar{\sigma}\big]$ but for the configuration on the four vertices r^-, s^-, h, k; in particular now r^- is joined to k and s^- is joined to h, so that we get

$$\mathbf{d}\big[(s^-\ k)\,\bar{\sigma}\big] * \mathbf{d} = \mathbf{d}\big[\bar{\sigma}\big] * \mathbf{d}\,, \qquad C\big(\mathbf{d}\big[(s^-\ k)\,\bar{\sigma}\big], \mathbf{d}\big) = C\big(\mathbf{d}\big[\bar{\sigma}\big], \mathbf{d}\big) - 1\,;$$

(the picture below illustrates the situation we are dealing with)



case of $\mathbf{d}\big[\bar{\sigma}\big] * \mathbf{d}$ *case of* $\mathbf{d}\big[(s^-\ k)\,\bar{\sigma}\big] * \mathbf{d}$

the upshot is that

$$\mathbf{d}\big[(s^-\ k)\,\bar{\sigma}\big] \cdot \mathbf{d} = n^{z-1}\mathbf{d}' = n^{-1}\,\mathbf{d}\big[\bar{\sigma}\big] \cdot \mathbf{d}\,, \qquad \text{or} \qquad \bar{\sigma}.\mathbf{d}[id] \cdot \mathbf{d} = n\,(s^-\ k)\,\bar{\sigma}.\mathbf{d}[id] \cdot \mathbf{d}\,;$$

in particular, this result is independent of the choice of $h \multimap k$. This operation can be done as many times as are the choices of the edge $h \multimap k$ in the moving part of δ_{n+1}, that is exactly n times; and each time, one has $sgn\big((s^-\ k)\,\bar{\sigma}\big) = -sgn\big(\bar{\sigma}\big)$.

Thus, when we expand the sum in right hand side of $\delta_{n+1} \cdot \mathbf{d} = \sum_{\sigma \in S_{n+1}} sgn(\sigma)\,\sigma.\mathbf{d}[id] \cdot \mathbf{d}$ in terms of the basis D_f of $\mathcal{B}_f^{(n)}$, if a diagram $\mathbf{d}'$ occurs then it occurs with a coefficient (actually, an integer number) which is a multiple of $\big(n - (\underbrace{1 + \cdots + 1}_{n})\big) = 0$; therefore we get $\delta_{n+1} \cdot \mathbf{d} = 0$, q.e.d.

(b) The proof resembles that of case *(a)*; in particular, it is enough to prove the statement about $\varpi_{n+1} \cdot \mathbf{d}$, for then applying Ω will give the other one too; and the claim involving junctions again follows from the one about diagrams, in the same way as in *(a)*.

Like for *(a)*, we can assume $\mathbf{d} \in D_{f,1}$, so $r^+ \multimap s^+$ is the sole upper bar of $\mathbf{d}$.

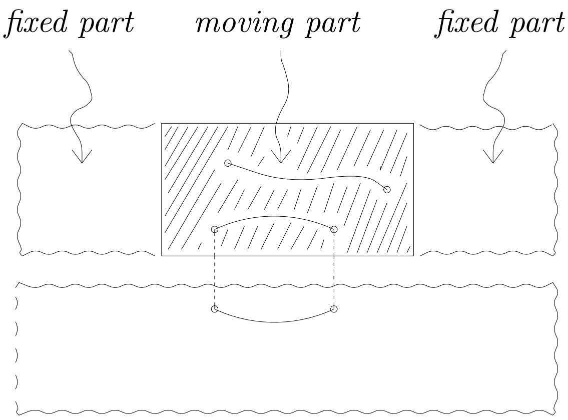
Let $r^- < s^-$, say. Among the diagrams in the expansion of $\varpi_{2(n+1)}$, there are some which contain the bar $r^- \multimap s^-$; pick one of these, call it $\mathbf{d}'$.

When making the product $\mathbf{d}' \cdot \mathbf{d}$ the two bars $r^+ \multimap s^+$ and $r^- \multimap s^-$ match each other to form a cycle, which gives a contribution $(-2n)$ to the coefficient $(-2n)^{C(\mathbf{d}', \mathbf{d})}$ in front of $\mathbf{d}' * \mathbf{d}$. Now, $\mathbf{d}'$ has exactly $n+1$ moving edges (i.e. edges which are not fixed in $\varpi_{2(n+1)}$): in particular there are exactly n moving edges different from $r^- \multimap s^-$. So let $h \multimap k$ be one of the latter edges; then among the diagrams in $\varpi_{2(n+1)}$

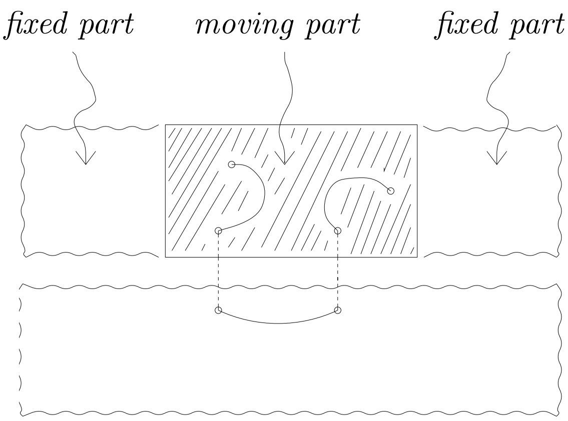
we find exactly two other diagrams — say $\mathbf{d}'_+$, $\mathbf{d}'_-$ — which have the same configuration as $\mathbf{d}'$ but on the four vertices r^-, s^-, h, k: one diagram, say $\mathbf{d}'_+$, has the pair of edges $\{h, r^-\}$, $\{k, s^-\}$, and the other, say $\mathbf{d}'_-$, has the pair of edges $\{h, s^-\}$, $\{k, r^-\}$ (note that we do not specify the relative positions of the four vertices involved); thus we have

$$\mathbf{d}'_+ * \mathbf{d} = \mathbf{d}'_- * \mathbf{d} = \mathbf{d}' * \mathbf{d}$$

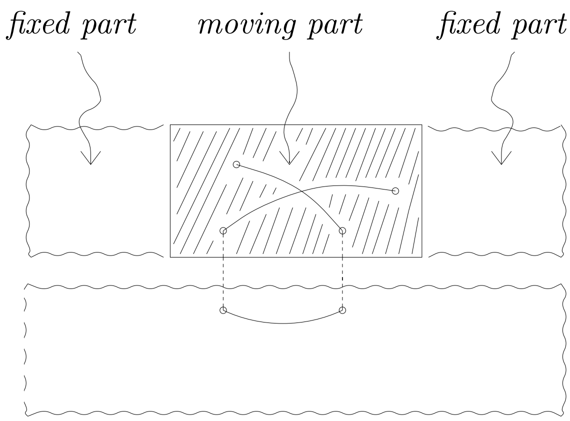
as the pictures below show:



case of $\mathbf{d}' * \mathbf{d}$



case of $\mathbf{d}'_+ * \mathbf{d}$



case of $\mathbf{d}'_- * \mathbf{d}$

Letting h k range among the n moving edges of $\mathbf{d}'$ different from r^- s^-, we find the same summand $\mathbf{d}' * \mathbf{d}$ in $\varpi_{2(n+1)}$ once with coefficient $-2n$ and exactly $2 \cdot n$ times with coefficient $+1$, so the final coefficient is zero. This operation takes care of all the diagrams occurring in $\varpi_{2(n+1)}$, hence we conclude that $\varpi_{2(n+1)} \cdot \mathbf{d} = 0$, q.e.d. $\square$

§4 The Littlewood's restriction rules

4.1 Schur's duality and multiplicities. When considering the $GL(U)$–action on $U^{\otimes f}$ (for a complex vector space U) by Schur's duality $U^{\otimes f}$ splits as

$$U^{\otimes f} \cong \bigoplus_{\substack{\lambda \vdash f \\ \lambda_1^t \leq dim(U)}} V_\lambda \otimes M_\lambda \tag{4.1}$$

as a $GL(U) \times End_{GL(U)}\big(U^{\otimes f}\big)$–module, where V_λ is the simple polynomial $GL(U)$–module attached to λ and M_λ is the simple $End_{GL(U)}\big(U^{\otimes f}\big)$–module attached to λ; it is known that the centralizer algebra $End_{GL(U)}\big(U^{\otimes f}\big)$ is $\mathbb{C}\big[S_f\big]$, thus M_λ is just the simple S_f–module we are used to consider. Similarly, Schur's duality yields a decomposition

$$V^{\otimes f} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2k) \\ \mu_1^t + \mu_2^t \leq n}} U_\mu \otimes N_\mu^+ \tag{4.2}$$

as an $O(V) \times End_{O(V)}\big(U^{\otimes f}\big)$–module, where U_μ is the simple $O(V)$–module attached to μ and N_μ^+ is the simple $End_{O(V)}\big(U^{\otimes f}\big)$–module attached to μ, and a decomposition

$$W^{\otimes f} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2k) \\ \mu_1^t \leq n}} W_\mu \otimes N_\mu^- \tag{4.3}$$

as an $Sp(W) \times End_{Sp(W)}\big(W^{\otimes f}\big)$–module, where W_μ is the simple $Sp(W)$–module attached to μ and N_μ^- is the simple $End_{Sp(W)}\big(W^{\otimes f}\big)$–module attached to μ. Notice that via π_V, resp. π_W, the modules N_μ^+, resp. N_μ^-, are also $\mathcal{B}_f^{(n)}$–modules, resp. $\mathcal{B}_f^{(-2n)}$–modules.

Lemma 4.2. *$\big[V_\lambda : U_\mu\big] = \big[N_\mu^+ : M_\lambda\big]$ and $\big[V_\lambda : W_\mu\big] = \big[N_\mu^- : M_\lambda\big]$. In other words, if*

$$V_\lambda \Big|_{O(V)}^{GL(V)} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\mu \vdash (f-2k)} D_\mu^\lambda \, U_\mu \qquad \textit{and} \qquad N_\mu^+ \Big|_{\mathbb{C}[S_f]}^{\mathcal{B}_f^{(n)}} \cong \bigoplus_{\lambda \vdash f} \hat{C}_{\lambda,\mu}^+ \, M_\lambda \ ,$$

$$V_\lambda \Big|_{Sp(W)}^{GL(V)} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\mu \vdash (f-2k)} E_\mu^\lambda \, W_\mu \qquad \textit{and} \qquad N_\mu^- \Big|_{\mathbb{C}[S_f]}^{\mathcal{B}_f^{(-2n)}} \cong \bigoplus_{\lambda \vdash f} \hat{C}_{\lambda,\mu}^- \, M_\lambda \ ,$$

then $D_\mu^\lambda = \hat{C}_{\lambda,\mu}^+$, $E_\mu^\lambda = \hat{C}_{\lambda,\mu}^-$ for all λ, μ.

Proof. This is standard. Comparing (4.1) with $U = V$ and (4.2) gives

$$\bigoplus_{\lambda,\mu} D_\mu^\lambda \, U_\mu \otimes M_\lambda \cong \bigoplus_\lambda V_\lambda \otimes M_\lambda \cong V^{\otimes f} \cong \bigoplus_\mu U_\mu \otimes N_\mu^+ \cong \bigoplus_{\mu,\lambda} U_\mu \otimes \hat{C}_{\lambda,\mu}^+ \, M_\lambda$$

where the indices λ and μ have to range in the proper sets; this forces $D_\mu^\lambda = \hat{C}_{\lambda,\mu}^+$, q.e.d. The like is for the other identity, using (4.1) with $U = W$ and (4.3). $\square$

Lemma 4.3. *As a $\mathbb{C}\big[S_f\big]$-module, $H^\mu_{f,k}$ splits into*

$$H^\mu_{f,k}\Big|^{\mathcal{B}_f^{(x)}}_{\mathbb{C}\,[S_f]} \cong \bigoplus_{\lambda\vdash f} C^\lambda_\mu \, M_\lambda \qquad \text{with} \quad C^\lambda_\mu = \sum_{\substack{\sigma\vdash 2k \\ \sigma \text{ has even rows}}} c^\lambda_{\mu,\sigma}$$

where $c^\lambda_{\mu,\sigma}$ is the Littlewood-Richardson coefficient expressing the multiplicity of M_λ in the decomposition of $Ind^{S_f}_{S_{f-2k}\times S_{2k}}\big(M_\mu \otimes M_\sigma\big)$.

Proof. A simple analysis of the definition shows that

$$H^\mu_{f,k}\Big|^{\mathcal{B}_f^{(x)}}_{\mathbb{C}\,[S_f]} \cong Ind^{S_f}_{S_{f-2k}\times S_{2k}}\Big(M_\mu \otimes H_{2k,k}\Big) \; ; \tag{4.4}$$

(where $H_{2k,k}$ is defined as in §2.2). On the other hand, we have an isomorphism of S_{2k}–modules

$$H_{2k,k} \cong Ind^{S_{2k}}_{S_2^{\times k}}\Big({M_{(2)}}^{\otimes k}\Big) \tag{4.5}$$

(where $M_{(2)}$ is the trivial representation of S_2): to realize such an isomorphism, one simply has to map the $(2k,k)$–junction  (as an element of $H_{2k,k}$) to any non-zero element of ${M_{(2)}}^{\otimes k}$. Now, it is known (cf. Proposition 1.5) that

$$Ind^{S_{2k}}_{S_2^{\times k}}\Big({M_{(2)}}^{\otimes k}\Big) \cong \bigoplus_{\substack{\sigma\vdash 2k \\ \sigma \text{ has even rows}}} M_\sigma$$

thus (4.4) and (4.5) together yield

$$H^\mu_{f,k}\Big|^{\mathcal{B}_f^{(x)}}_{\mathbb{C}\,[S_f]} \cong \sum_{\substack{\sigma\vdash 2k \\ \sigma \text{ has even rows}}} Ind^{S_f}_{S_{f-2k}\times S_{2k}}\Big(M_\mu \otimes M_\sigma\Big) \cong \bigoplus_{\lambda\vdash f} \sum_{\substack{\sigma\vdash 2k \\ \sigma \text{ has even rows}}} c^\lambda_{\mu,\sigma} \cdot M_\lambda$$

which gives the claim. □

To be short, from now on we use the notation $N'_\mu := H^\mu_{f,k}$ for all $k = 0, 1, \dots, [f/2]$ and all $\mu \vdash (f-2k)$.

The next result "locates" the (semi)simple quotient of N'_μ (cf. Corollary 2.15).

Proposition 4.4. *There exist a $\mathcal{B}_f^{(n)}$–module epimorphisms $\Theta : N'_\mu \longrightarrow\!\!\!\!\!\rightarrow N^+_\mu$, resp. a $\mathcal{B}_f^{(-2n)}$–module epimorphism $\Theta : N'_\mu \longrightarrow\!\!\!\!\!\rightarrow N^-_{\mu^t}$. In particular N^+_μ, resp. $N^-_{\mu^t}$, is the unique simple $\mathcal{B}_f^{(n)}[k;\mu]$–module, resp. $\mathcal{B}_f^{(-2n)}[k;\mu]$ (for the proper k).*

Proof. For the proof we need to describe $N^\pm_\mu$: for this we can resume the analysis of [GP].

Introduce the following subspaces of $V^{\otimes f}$ (for all $k \in \big\{0, 1, \dots, [f/2]\big\}$)

$$T^0\big(V^{\otimes f}\big) := \bigcup_{p\neq q} Ker\big(\Phi_{p,q}\big) \, , \quad T^k\big(V^{\otimes f}\big) := \sum_{i_1<j_1,\dots,i_k<j_k} \Psi_{i_1,j_1}\Psi_{i_2,j_2}\cdots\Psi_{i_k,j_k}\Big(T^0\big(V^{\otimes(f-2k)}\big)\Big)$$

Then it is known that $T^0\big(V^{\otimes f}\big)$, resp. $T^0\big(W^{\otimes f}\big)$, splits into

$$T^0\big(V^{\otimes f}\big) \cong \bigoplus_{\substack{\mu \vdash f \\ \mu_1^t + \mu_2^t \leq n}} U_\mu \otimes M_\mu\,, \qquad \text{resp.} \quad T^0\big(W^{\otimes f}\big) \cong \bigoplus_{\substack{\mu \vdash f \\ \mu_1^t \leq n}} W_\mu \otimes M_\mu \tag{4.6}$$

as a module over $O(V) \times \mathcal{B}_f^{(n)}$, resp. $Sp(V) \times \mathcal{B}_f^{(-2n)}$.

Now consider the space of invariants $\Big(\big(V^{\otimes 2k}\big)^*\Big)^{O(V)}$: we have ${\psi_V}^{\otimes k} \in \Big(\big(V^{\otimes 2k}\big)^*\Big)^{O(V)}$, and in fact $\Big(\big(V^{\otimes 2k}\big)^*\Big)^{O(V)} = \mathbb{C}\big[S_{2k}\big].{\psi_V}^{\otimes k}$. Similarly, $\Big(\big(W^{\otimes 2k}\big)^*\Big)^{Sp(W)} = \mathbb{C}\big[S_{2k}\big].{\psi_W}^{\otimes k}$ in the symplectic case.

From definitions we get $T^k\big(V^{\otimes f}\big) = \mathbb{C}\big[S_f\big].T^0\big(V^{\otimes (f-2k)}\big)$: then using (4.6) gives

$$T^k\big(V^{\otimes f}\big) \cong \bigoplus_{\substack{\mu \vdash f \\ \mu_1^t + \mu_2^t \leq n}} U_\mu \otimes \left(\mathbb{C}\big[S_f\big].\left(M_\mu \otimes \Big(\big(V^{\otimes 2k}\big)^*\Big)^{O(V)}\right)\right),$$

resp.
$$T^k\big(W^{\otimes f}\big) \cong \bigoplus_{\substack{\mu \vdash f \\ \mu_1^t \leq n}} W_\mu \otimes \left(\mathbb{C}\big[S_f\big].\left(M_\mu \otimes \Big(\big(W^{\otimes 2k}\big)^*\Big)^{Sp(W)}\right)\right).$$

Now, it is also known that

$$V^{\otimes f} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2k) \\ \mu_1^t + \mu_2^t \leq n}} T^k\big(V^{\otimes f}\big)\,, \qquad W^{\otimes f} \cong \bigoplus_{k=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2k) \\ \mu_1^t \leq n}} T^k\big(W^{\otimes f}\big)\,,$$

hence comparing with (4.2) and (4.3) we find

$$N_\mu^+ \cong \mathbb{C}\big[S_f\big].\left(M_\mu \otimes \Big(\big(V^{\otimes 2k}\big)^*\Big)^{O(V)}\right), \quad N_\mu^- \cong \mathbb{C}\big[S_f\big].\left(M_\mu \otimes \Big(\big(W^{\otimes 2k}\big)^*\Big)^{Sp(W)}\right). \tag{4.7}$$

On the other hand, there exists a natural isomorphism of S_{2k}–modules $H_{2k,k} \cong A_f^O$ (just map each $(2k,k)$–junction to the unique monomial $x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_k j_k}$ (see §1) such that $i_h \frac{\qquad}{} j_h$ is a bar of the junction); by composing it with $\alpha_V : A_f^O \longrightarrow \Big(\big(V^{\otimes 2f}\big)^*\Big)^{O(V)}$ (cf. Proposition 1.6) we get an epimorphism

$$\theta : H_{2k,k} \longrightarrow \Big(\big(V^{\otimes 2f}\big)^*\Big)^{O(V)} \quad \text{given by} \quad \theta\big(\smile \ \smile \ \cdots\cdots \ \smile\big) := \psi_V^{\otimes f}$$

which is indeed one of S_{2k}–modules and also of $\mathcal{B}_{2k}^{(n)}$–modules.

The same construction works in the symplectic case, but for the following fact: the action of S_{2k} on $W^{\otimes 2k}$ *through* $\mathcal{B}_{2k}^{(-2n)}$ (via $S_{2k} \subset D_{2k,0} \subset \mathcal{B}_{2k}^{(-2n)}$) coincides with the standard permutation action *twisted* by the alternating representation $M_{\underbrace{(1,1,\dots,1)}_{2k}}$: so repeating the previous analysis yields an epimorphism of $\mathcal{B}_{2k}^{(-2n)}$–modules

$$\theta : H_{2k,k} \longrightarrow \Big(\big(W^{\otimes 2f}\big)^*\Big)^{Sp(W)} \otimes M_{\underbrace{(1,\dots,1)}_{2k}} \quad \text{given by} \quad \theta\big(\smile\ \smile\ \cdots\cdots\ \smile\big) := \psi_V^{\otimes f} \otimes \mathbf{1}$$

where $\mathbf{1}$ is a basis vector of the sign representation $M_{(1,1,\dots,1)}$.

Now we can define uniquely a morphism of $\mathbb{C}\big[S_f\big]$–modules by

$$\Theta : Ind_{S_{f-2k}\times S_{2k}}^{S_f} \Big(T^0\big(V^{\otimes(f-2k)}\big) \otimes H_{2k,k}\Big) \longrightarrow\!\!\!\!\to T^k\big(V^{\otimes f}\big)\,, \qquad v \otimes h \mapsto v \otimes \theta(h)$$

($v \in T^0\big(V^{\otimes(f-2k)}\big)$, $h \in H_{2k,k}$); this is indeed an epimorphism of $O(V) \times \mathcal{B}_f^{(n)}$–modules. Then using again (4.4), (4.6) and (4.7) we get that Θ induces an epimorphism of $\mathcal{B}_f^{(n)}$–modules

$$\mathit{\Theta} : N'_\mu \cong Ind_{S_{f-2k}\times S_{2k}}^{S_f} \Big(M_\mu \otimes H_{2k,k}\Big) \longrightarrow\!\!\!\!\to \mathbb{C}\big[S_f\big].\bigg(M_\mu \otimes \Big(\big(V^{\otimes 2k}\big)^*\Big)^{O(V)}\bigg) \cong N_\mu^+$$
$$\text{given by} \qquad \mathit{\Theta}\big(m \otimes h\big) := m \otimes \theta(h) \qquad (\forall\, m \in M_\mu\,,\ h \in H_{2k,k})$$

which fulfills the claim. The same argument — *mutatis mutandis* — in the symplectic case gives an epimorphism of $\mathcal{B}_f^{(-2n)}$–modules

$$\mathit{\Theta} : N'_\mu \cong Ind_{S_{f-2k}\times S_{2k}}^{S_f} \Big(M_\mu \otimes H_{2k,k}\Big) \longrightarrow\!\!\!\!\to \mathbb{C}\big[S_f\big].\bigg(M_{\mu^t} \otimes \Big(\big(W^{\otimes 2k}\big)^*\Big)^{Sp(W)}\bigg) \cong N_{\mu^t}^-$$
$$\text{given by} \qquad \mathit{\Theta}\big(m \otimes h\big) := m \otimes \theta(h) \qquad (\forall\, m \in M_\mu\,,\ h \in H_{2k,k})$$

where we consider M_μ and M_{μ^t} as sharing the same vector space as socle (for instance, we can fix any identification $M_{\mu^t} \cong M_{\underbrace{(1,1,\dots,1)}_{f}} \otimes M_\mu$ so that $m \cong 1 \otimes m$ for all $m \in M_\mu$). The proof is complete. $\square$

<u>*Remark*</u>*:* in the "stable case" $(n \geq f)$ the epimorphisms $\mathit{\Theta}$ in the previous Proposition are isomorphisms: more precisely, they are the inverse of the isomorphisms ϕ given in [GP], Theorem 7.5.

Finally, we are ready for the key step.

Theorem 4.5. *Retain notations of Lemma 4.2 and Lemma 4.3. Then*

(a) $\quad \hat{C}^+_{\lambda,\mu} = C^\lambda_\mu \quad$ *for all* $\ \lambda \vdash f\ $ *such that* $\ \lambda_1^t + \lambda_2^t \leq n\,.$

(b) $\quad \hat{C}^-_{\lambda,\mu} = C^{\lambda^t}_{\mu^t} \quad$ *for all* $\ \lambda \vdash f\ $ *such that* $\ \lambda_1^t \leq n\,.$

Proof. The idea of the proof is to show that the multiplicity of M_λ is the same in both sides of the epimorphism $\varTheta : N'_\mu \longrightarrow\!\!\!\!\!\rightarrow N^+_\mu$ or $\varTheta : N'_\mu \longrightarrow\!\!\!\!\!\rightarrow N^-_{\mu^t}$ in Proposition 4.4, for then the claim follows from Lemma 4.3; to this end, it is enough (together with an additional remark for case *(b)*) to prove that for all λ as in the claim the kernel of $\varTheta$ has no isotypical components — as a $\mathbb{C}\big[S_f\big]$–module — of type λ: in other words,

$$Ker\big(\varTheta : N'_\mu \longrightarrow\!\!\!\!\!\rightarrow N^+_\mu\big) \subseteq \bigoplus_{\substack{\lambda \vdash f \\ \lambda^t_1 + \lambda^t_2 > n}} C^\lambda_\mu \, M_\lambda \, , \quad Ker\big(\varTheta : N'_\mu \longrightarrow\!\!\!\!\!\rightarrow N^-_{\mu^t}\big) \subseteq \bigoplus_{\substack{\lambda \vdash f \\ \lambda_1 > n}} C^\lambda_\mu \, M_\lambda \, . \tag{4.8}$$

From Proposition 2.12 and Proposition 4.4 it follows that

$$Ker\big(\varTheta : N'_\mu \twoheadrightarrow N^+_\mu\big) = Ker(\pi_V).N'_\mu \, , \quad Ker\big(\varTheta : N'_\mu \twoheadrightarrow N^-_{\mu^t}\big) = Ker(\pi_W).N'_\mu \tag{4.9}$$

Indeed, we have $N'_\mu \big/ Ker\big(\varTheta : N'_\mu \twoheadrightarrow N^+_\mu\big) \cong N_\mu$ and the latter is a simple module over $End_{O(V)}\big(V^{\otimes f}\big)$: since $End_{O(V)}\big(V^{\otimes f}\big) \cong \mathcal{B}^{(n)}_f \big/ Ker\big(\pi_V\big)$ we have $Ker\big(\pi_V\big).N'_\mu \subseteq Ker\big(\varTheta : N'_\mu \twoheadrightarrow N^+_\mu\big)$; on the other hand, $N'_\mu \big/ Ker(\pi_V).N'_\mu$ is a module over $\mathcal{B}^{(n)}_f \big/ Ker(\pi_V)$ $\cong End_{O(V)}\big(V^{\otimes f}\big)$, hence it is semisimple: but then Corollary 2.15 forces it to be simple, which in turn implies $\big(N^+_\mu \cong \big)\, N'_\mu \big/ Ker\big(\varTheta : N'_\mu \twoheadrightarrow N^+_\mu\big) \cong N'_\mu \big/ Ker(\pi_V).N'_\mu$ and then also $Ker\big(\varTheta : N'_\mu \twoheadrightarrow N^+_\mu\big) = Ker(\pi_V).N'_\mu$, q.e.d. The symplectic case is entirely similar.

So we are reduced to study $Ker\big(\pi_V\big).N'_\mu$ for *(a)* and $Ker\big(\pi_W\big).N'_\mu$ for *(b)*.

(a) We know that $Ker(\pi_V)$ is spanned by the minors of order $(n+1)$. Let δ_{n+1} be one of these minors: then it has $2(n+1)$ moving vertices, say r of them in the upper row and $s\,\big(=2(n+1)-r\big)$ on the lower row: we have to distinguish the cases $r \geq s$ and $r < s$.

Assume that $r \geq s$: then $r \geq (n+1) > n$. Then applying (3.4) we get

$$\delta_{n+1}.N'_\mu \;=\; \Bigg(Alt_{I_\ell} \, Alt_{I_t} \cdot \sum_{j=0}^{m} \sum_{(p_{j,i},q_{j,i}) \in V_j} (-1)^j \, \mathbf{h}_{p_{j,1}q_{j,1}} \mathbf{h}_{p_{j,2}q_{j,2}} \cdots \mathbf{h}_{p_{j,j}q_{j,j}} \mathbf{d} \Bigg).N'_\mu \subseteq$$

$$\subseteq \bigoplus_{\substack{\lambda \vdash f \\ \lambda^t_1 + \lambda^t_2 \geq r}} I_\lambda.N'_\mu \subseteq \bigoplus_{\substack{\lambda \vdash f \\ \lambda^t_1 + \lambda^t_2 > n}} I_\lambda.N'_\mu = \bigoplus_{\substack{\lambda \vdash f \\ \lambda^t_1 + \lambda^t_2 > n}} \big(N'_\mu\big)_\lambda = \bigoplus_{\substack{\lambda \vdash f \\ \lambda^t_1 + \lambda^t_2 > n}} C^\lambda_\mu M_\lambda$$

where by $\big(Y\big)_\lambda$ we denote the isotypical component of type λ in any $\mathbb{C}\big[S_f\big]$–module Y. Therefore letting $\Delta^{r \geq s}_{n+1}$ be the span of all the minors of order $(n+1)$ with $r \geq s$ we conclude that $\Delta^{r \geq s}_{n+1}.N'_\mu$ is contained in the direct sum in the left-hand-side of (4.8), q.e.d.

Now assume that $r < s$: we shall prove that either we get trivial results — i.e. zero contributions to the $Ker\big(\pi_V\big)$ — or we can reduce to the previous case, that is $r \geq s$. More precisely, given a junction $j \in J_{f,k}$ (where k is such that $\mu \vdash (f-2k)$) and $m \in M_\mu$, we shall prove the claim by showing that $\delta_{n+1}.(m \otimes j) = 0$ or we can reduce to a smaller value of s, so that an inductive argument (on s) will permit to reduce to the case $r \geq s$, hence to conclude.

Suppose $k = 0$: if $\delta_{n+1} \in \mathcal{B}^{(n)}_f(1)$ then of course $\delta_{n+1}.j = 0$ in $H_{f,k}$: this implies $\delta_{n+1}.(m \otimes j) = 0$ in N'_μ, hence we are done. But the hypothesis $r < s$ "forces" δ_{n+1} to belong to $\mathcal{B}^{(n)}_f(1)$, so there's nothing else to do.

Then assume $k > 0$. We have several cases to consider.

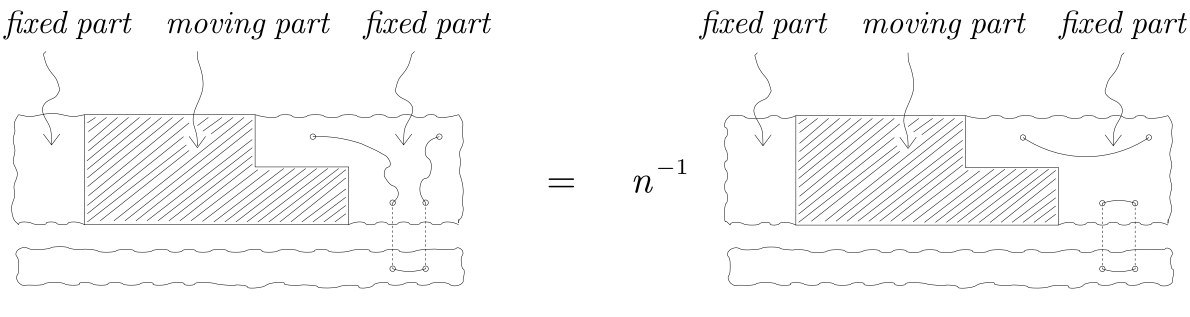
Case (a-1): Suppose that j has a bar $u \succ\!\!\!-\!\!\!-\!\!\circ\, v$ such that both u^- and v^- are moving in δ_{n+1}. Then Lemma 3.9*(a)* yields $\delta_{n+1}.j = 0$ in $H_{f,k}$, hence again $\delta_{n+1}.(m \otimes j) = 0$ in N'_μ, and we are done.

Case (a-2): Suppose that all bars of j match fixed vertices of δ_{n+1}.

If all the spots of the bars (k in number) of j match vertices in the lower row of δ_{n+1} which all belong to (fixed) bars, then $\delta_{n+1} \in \mathcal{B}_f^{(n)}(h)$ for some $h > k$: indeed, the previous assumption implies that δ_{n+1} has at least k bars in its fixed part — both in the upper and in the lower row — but since $r < s$ its fixed part is "bigger up than down", so it has strictly more bars up than down, whence the claim. But then $\delta_{n+1} \star j$ is an alternating sum of junctions which all belong to $J_{f,k'}$ with $k' \geq h > k$, hence $\delta_{n+1}.j = 0$ in $H_{f,k}$, so we can finish like above.

Similarly, if for each bar of j the (fixed) vertices (in the lower bar of δ_{n+1}) matched by those of this bar belong either both to bars (maybe one single bar for both vertices) — as above — or one to a bar and the other to a vertical edge, then $\delta_{n+1}.j = 0$ again. Indeed, the bars whose vertices both match bars are to be treated as before; as for the others, they can be grouped collecting together those which belong to a like path in $\Gamma(\delta_{n+1}, j)$ (notation having the obvious meaning). Fix one such path Π, and let t be the total number of bars of j involved in this path: if Π links a fixed upper vertex of δ_{n+1} with a spot of j, then Π also involves exactly t fixed bars of the lower row of δ_{n+1}, hence there are exactly t "corresponding" fixed bars in the upper row of δ_{n+1} which in turn provide t bars in $\delta_{n+1} \star j$ (notation having the obvious meaning); otherwise, i.e. if Π links two fixed upper vertices of δ_{n+1}, then Π also involves exactly $t-1$ fixed bars of the lower row of δ_{n+1}, which correspond to $t-1$ fixed bars in the upper row providing $t-1$ fixed bars in $\delta_{n+1} \star j$: but in addition the path Π itself yields a t^{th} bar in $\delta_{n+1} \star j$. This shows that the junctions occurring in $\delta_{n+1} \star j$ all have at least k' bars with $k' \geq k$; finally, since $r < s$ we can conclude like above that $k' > k$, whence $\delta_{n+1}.j = 0$ and $\delta_{n+1}.(m \otimes j) = 0$ as before.

Therefore we are left with the case when there is at least one bar $u \succ\!\!\!-\!\!\!-\!\!\circ\, v$ of j such that u^- and v^- (fixed, in δ_{n+1}) belong to vertical edges: then we proceed as follows. Let u^- and v^- be joined respectively to p^+ and q^+; then define $\delta'_{n+1} := h_{p,q} \cdot \delta_{n+1}$. A moment thought shows that $\delta_{n+1}.j = n^{-1} \cdot \delta'_{n+1}.j$, as the pictures below show:



(picture of $\delta_{n+1}.j$) *(picture of $\delta'_{n+1}.j$)*

Therefore we can switch to deal with δ'_{n+1} instead of δ_{n+1}; by iteration of this procedure, we are reduced to consider the case when no bar of j matches two vertices in our minor which both belong to vertical edges, that is we fall within the previous situation.

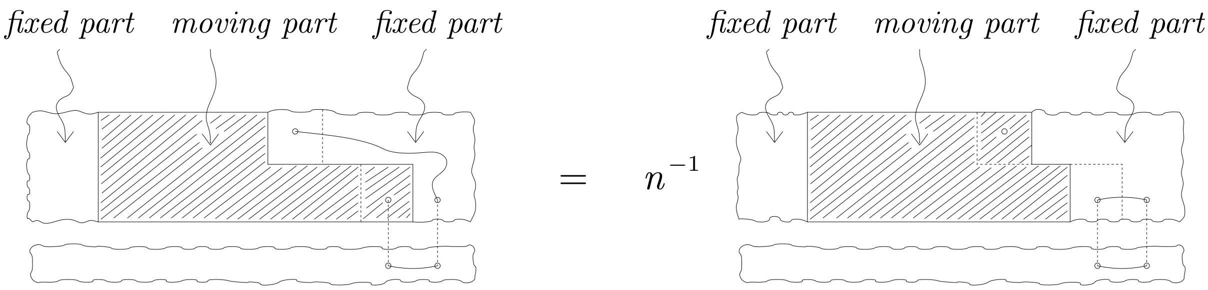
Case (a-3): Thanks to the previous analysis, we can restrict to consider the case in which at least one bar $u \multimap\!\!\!-\!\!\circ\, v$ of j has one vertex — say u — matching a moving vertex of δ_{n+1} and the other — v for us — matching a fixed vertex of δ_{n+1}.

Suppose that there are *two* bars $\hat{u} \multimap\!\!\!-\!\!\circ\, \hat{v}$ and $\tilde{u} \multimap\!\!\!-\!\!\circ\, \tilde{v}$ in j enjoying the previous property, and that the fixed vertices $\hat{v}^-$ and $\tilde{v}^-$ are joined by a fixed bar in δ_{n+1}; then when computing $\delta_{n+1}.j$ a path appears in $\Gamma(\delta_{n+1}.j)$ which links $\hat{v}$ and $\tilde{v}$: so the situation is the same as if the bar $\hat{v} \multimap\!\!\!-\!\!\circ\, \tilde{v}$ were in j, hence Lemma 3.9*(a)* gives again $\delta_{n+1}.j = 0$, whence we conclude in the usual way.

The possibilities allowed now are the following: each bar of j has a vertex matching a moving vertex m of δ_{n+1} and another one matching a fixed vertex w, but if the latter belongs to a bar (of δ_{n+1}) then the other bars of j do not match the vertex of δ_{n+1} joined to w.

Suppose that each bar of j meets — via some vertex w — a fixed bar of δ_{n+1}: the previous assumption implies that all these bars must be different; then we can do the same analysis as in *Case (a-2)*, but this time we have to proceed separately for each diagram in the expansion of δ_{n+1} (for now also the moving part is involved). Thus again we find that each of these diagrams has at least k bars in its lower row, so like in *Case (a-2)* we conclude that $\delta_{n+1}.(m \otimes j) = 0$.

By the last step, we can assume that at least one bar $u \multimap\!\!\!-\!\!\circ\, v$ of j meets a fixed vertex belonging to a (fixed) vertical edge of δ_{n+1}. Then one easily sees that $\delta_{n+1}.j = n^{-1}\, \delta'_{n+1}.j$, where δ'_{n+1} is a new minor of order $(n+1)$ whose fixed part has "sizes" $r' = r + 1$ and $s' = s - 1$: the following picture illustrates the situation:



(picture of $\delta_{n+1}.j$) *(picture of $\delta'_{n+1}.j$)*

Thus we are reduced to the case of a greater value of r, so applying a recursive procedure we can end with the case $r \geq s$, that we have considered (and solved) at the beginning.

(b) We can repeat almost step by step the prove we made for *(a)*: whenever a property of minors was required (e.g. Lemma 3.9*(a)*), the analogous property of Pfaffians (in the example, Lemma 3.9*(b)*) holds and works as well. Here we explicit the starting point.

Let $\varpi_{2(n+1)}$ be a Pfaffian of order $2(n+1)$, let it have r, resp. s, moving vertices in the upper, resp. lower, row, and assume $r \geq s$; thus $r \geq (n+1) > n$ too. From (3.5) we get

$$\varpi_{n+1}.N'_{\mu^t} = \bigg(Sym_{I_t} \cdot \sum_{j=0}^{m} \sum_{(p_{j,i},q_{j,i}) \in V_j} \big((h+j)!\, 2^{h+j} \big)^{-1} \mathbf{h}_{p_{j,1}q_{j,1}} \mathbf{h}_{p_{j,2}q_{j,2}} \cdots \mathbf{h}_{p_{j,j}q_{j,j}} \mathbf{d} \bigg).N'_{\mu^t} \subseteq$$
$$\subseteq \bigoplus_{\substack{\lambda \vdash f \\ \lambda_1 \geq r}} I_\lambda.N'_{\mu^t} \subseteq \bigoplus_{\substack{\lambda \vdash f \\ \lambda_1 > n}} I_\lambda.N'_{\mu^t} = \bigoplus_{\substack{\lambda \vdash f \\ \lambda_1 > n}} \big(N'_{\mu^t}\big)_\lambda = \bigoplus_{\substack{\lambda \vdash f \\ \lambda_1 > n}} C^\lambda_{\mu^t} M_\lambda$$

Thus, if $\Pi^{r \geq s}_{2(n+1)}$ is the span of all the Pfaffians of order $2(n+1)$ with $r \geq s$ we conclude that $\Pi^{r \geq s}_{2(n+1)}.N'_\mu$ is contained in the direct sum in the right-hand-side of (4.8), q.e.d.

A second remark is necessary. As we saw during the proof of Proposition 4.4 the action of S_f on $W^{\otimes f}$ *through* $\mathcal{B}_f^{(-2n)}$ (via $S_f \subset D_{f,0} \subset \mathcal{B}_f^{(-2n)}$) coincides with the standard permutation action *twisted* by the alternating representation: hence the isotypical components of type λ (for all λ) for the S_f–action through $\mathcal{B}_f^{(-2n)}$ are indeed isotypical components of type λ^t with respect to the standard S_f–action, and viceversa. Thus the multiplicity of M_λ (in N^-_μ) with respect to one action is equal to the multiplicity of M_{λ^t} with respect to the other action: therefore the multiplicity $\big[N^-_{\mu^t} : M_\lambda\big]$ when we consider on $N^-_{\mu^t}$ the standard S_f–action (that is the one we are interested in) is equal to the multiplicity $\big[N^-_{\mu^t} : M_{\lambda^t}\big]$ when we consider on $N^-_{\mu^t}$ the S_f–action via $\mathcal{B}_f^{(-2n)}$ (i.e. the twisted one); by the previous analysis, if $\lambda^t_1 \leq n$ the latter multiplicity is exactly the same as in N'_μ, and we can conclude. $\square$

By the way, we notice that, thanks to Theorem 2.10 and Lemma 2.13, a simple reformulation of the above proof of Theorem 4.5 yields the following

Corollary 4.6. *(a) Let $\mu \vdash (f-2k)$ be such that $\mu^t_1 + \mu^t_2 \leq n$. Then the radical of the $\mathcal{B}_f^{(n)}$–module $H^\mu_{f,k}$ is contained in the sum of all isotypical components (of $H^\mu_{f,k}$ as an S_f–module) of type λ with $\lambda \vdash f$ such that $\lambda^t_1 + \lambda^t_2 > n$. Similarly, the radical of the algebra $\mathcal{B}_f^{(n)}[k;\mu]$ is contained in the sum of all isotypical components (of $\mathcal{B}_f^{(n)}[k;\mu]$ as an $S_f \times S_f$–module) of type $({}_1\lambda, {}_2\lambda)$ with ${}_i\lambda \vdash f$ $(i=1,2)$ such that ${}_1\lambda^t_1 + {}_1\lambda^t_2 > n$ or ${}_2\lambda^t_1 + {}_2\lambda^t_2 > n$.*

(b) Let $\mu \vdash (f-2k)$ be such that $\mu^t_1 \leq n$. Then the radical of the $\mathcal{B}_f^{(-2n)}$–module $H^\mu_{f,k}$ is contained in the sum of all isotypical components (of $H^\mu_{f,k}$ as an S_f–module) of type λ with $\lambda \vdash f$ such that $\lambda^t_1 > n$. Similarly, the radical of the algebra $\mathcal{B}_f^{(-2n)}[k;\mu]$ is contained in the sum of all isotypical components (of $\mathcal{B}_f^{(-2n)}[k;\mu]$ as an $S_f \times S_f$–module) of type $({}_1\lambda, {}_2\lambda)$ with ${}_i\lambda \vdash f$ $(i=1,2)$ such that ${}_1\lambda^t_1 > n$ or ${}_2\lambda^t_1 > n$. $\square$

At last, our efforts are rewarded.

Corollary 4.7 (Littlewood's Restriction Rules).

$$\text{(a)}\quad \big[V_\lambda : U_\mu\big] = \sum_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even rows}}} c^\lambda_{\mu,\sigma} \qquad \textit{for all} \quad \lambda \vdash f \quad \textit{such that} \quad \lambda^t_1 + \lambda^t_2 \leq n \; ;$$

$$\text{(b)}\quad \big[V_\lambda : W_\mu\big] = \sum_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even columns}}} c^\lambda_{\mu,\sigma} \qquad \textit{for all} \quad \lambda \vdash f \quad \textit{such that} \quad \lambda^t_1 \leq n \; .$$

Proof. We simply have to collect all previous results. For *(a)*, just patch together Lemma 4.2, Theorem 4.5*(a)*, and Lemma 4.3. For *(b)*, do the same with *(b)* instead of *(a)*: then

$$\big[V_\lambda : W_\mu\big] = E^\lambda_\mu = \hat{C}^-_{\lambda,\mu} = C^{\lambda^t}_{\mu^t} = \sum_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even rows}}} c^{\lambda^t}_{\mu^t,\sigma}$$

for all $\lambda \vdash f$ such that $\lambda^t_1 \leq n$; thus

$$\big[V_\lambda : W_\mu\big] = \sum_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even rows}}} c^{\lambda^t}_{\mu^t,\sigma} = \sum_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even columns}}} c^{\lambda^t}_{\mu^t,\sigma^t} = \sum_{\substack{\sigma \vdash 2k \\ \sigma \text{ has even columns}}} c^\lambda_{\mu,\sigma}$$

for all $\lambda \vdash f$ such that $\lambda^t_1 \leq n$, q.e.d. $\square$

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